COMPACT DISSIPATIVE FLOW STRUCTURES WITH TOPOLOGICAL COHERENCE, EMBEDDED IN EULERIAN ENVIRONMENTS

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Abstract

Large scale secondary flow structures with compact boundary surfaces can be generated in the interior of a .Navler Stokes fluid These deformable. topologically coherent structures have topological features different from their environment, and may have a long metastable lifetime even in viscous media. An analytic example is given in terms of a parametric variation of an exact closed form solution to the Navier-Stokes equation that exhibits a saddle-node Hopf bifurcation. As the flow parameter Is varied, the initially unidirec-tional flow develops a reentrant secondary flow, or torsion defect, with the visual appearance of a large scale structure confined within a compact ellipsoidal surface. As the secondary flow defect is created the surface of null helicity density undergoes a topological phase transition. The visual effect is reminiscent of the "Vortex Bursting" problem, but for the example flow, the vorticity of the flow is an absolute invariant of the flow parameter. A theory of continuous topological evolution is presented.

1 INTRODUCTON

1.1 A Coherent Structure as a Topological Entity.

At the August 1989 IUTAM meeting in Cambridge on Topological FIuid Mechanics, it was suggested by this author [1] that a coherent structure in a fluid be defined as a compact connected deformable domain of invariant topological properties embedded in perhaps an open, or non-compact, domain of different topology. The epitome of such a flow was given long ago by Hill who showed that the Hill spherical vortex [2], (a domain with vorticity, without helicity) could be embedded in an irrotational potential flow. The topological property of Pfaff dimension has a value of 2 in the interior domain the Hill spherical vortex. If the flow is embedded in an exterior irrotational fIuid the Pfaff dimension of the exterior flow domain is 1. (For the concept of Pfaff dimension, see reference 1.)

The Hill spherical vortex is a compact defect in the sense that the interior flow contained within a compact surface; the interior streamlines never penetrate the bounding surface. The Hill spherical vortex solution is a special solution of the Euler equations, as well as the Navier-Stokes equations, where the vorticity field is harmonic. Contributions due to the kinematic viscosity produce no irreversible forces over the surface or in the interior of the spherical defect. The streamlines for the Hill spherical vortex are displayed in Figure 1.

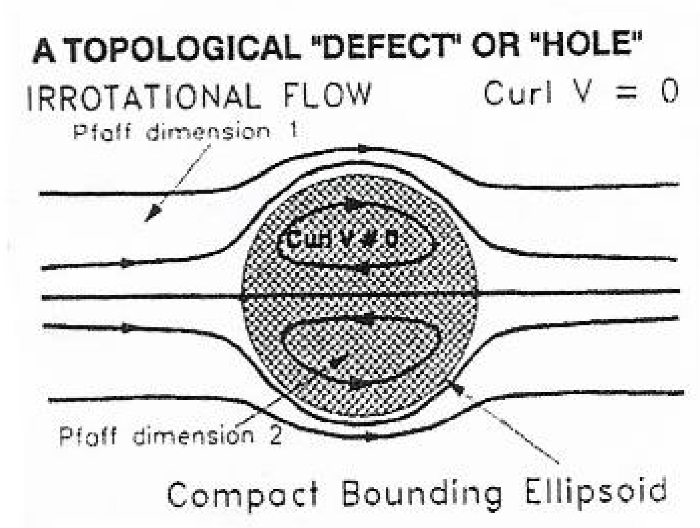


Fig. 1 Hill's Spherical Vortex

The question of how such domains are created was not addressed by the early authors. As the external flow is irrotational, the velocity field produces zero drag, and the spherical symmetry produces zero lift on the bounding surface. Distorted shapes in a harmonic velocity field can produce lift, but still will not produce viscous drag. The interest herein is to extend the notions of the Hill spherical vortex flow to other bounded viscous flow solutions to the Navier-Stokes equations for which the **vorticity** field is not only harmonic but has a Pfaff dimension of 3. Such flows support topological torsion. (The Hill spherical vortex is free of topological torsion.) Part of the pressure gradient for such flows is contributed by viscous forces, and part by the kinematics. Such flows can exhibit both drag and lift over a compact surface, but the associated forces are of the potential variety, or at worst of the cyclic variety. This means that the work done during such flow processes is reversible in the first case, or is quantized to the integers in the second case.

1.2 The Dynamical Persistence of a Coherent Structure.

A dynamical analysis of the Hill spherical vortex solution gives a clue as to how such large scale structures can be formed and how they can be maintained. The fixed points (stagnation points) of the spherical vortex flow coincide with the pole points of the spherical axis of rotational symmetry. See Figure 2.

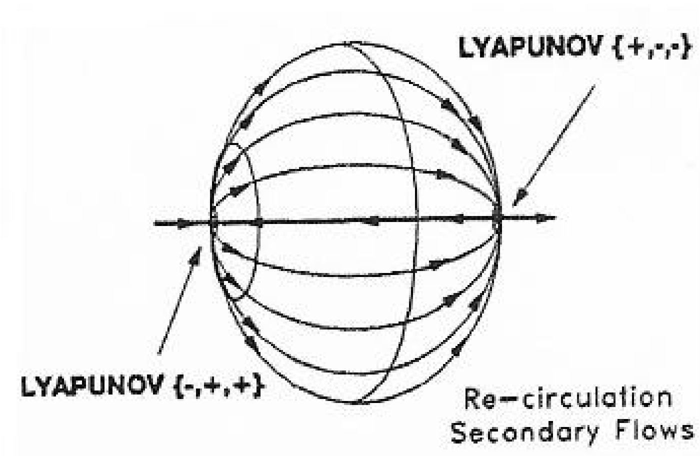


Fig. 2 Tertiary Saddle Node

A linear stability analysis about these fixed points dlemonstrates that both fixed points are unstable, locally. The upstream fixed point has two positive and one negative Lyapunov exponents, while the downstream fixed point has two negative and one positive Lyapunov exponent. Yet the solution exhibits global stability, where the positive exponents of one neighborhood drive the negative exponents of the other neighborhood. This basic feature of global stability is offered as an intuitive suggestion on how large scale coherent structures can persist. Their persistence is due to a global synergetic coupling of one locally unstable neighborhood interacting with another unstable neighborhood. The unstable stagnation points are isolated "fixed" point singularities on a boundary formed by a compact manifold. This bounding surface is the 3-dimensional analog of the separatrix in the plane.

1.3 The Formation of Coherent Structures.

If a swirl component is added to the Hill's Spherical vortex solution, then the Lyapunov exponents of the same sign can be put into correspondence with a Hopf (rotational) bifurcation. From truncated approximations, it has long been suspect that a tertiary Hopf bifurcation is among the solution sets to the Navier-Stokes equations, but a closed form solution to the Navier-Stokes equations that exhibits tertiary Hopf bifurcation has been discovered only recently [3]. An example of the streamlines for such a torsion solution is given in Figure 3.

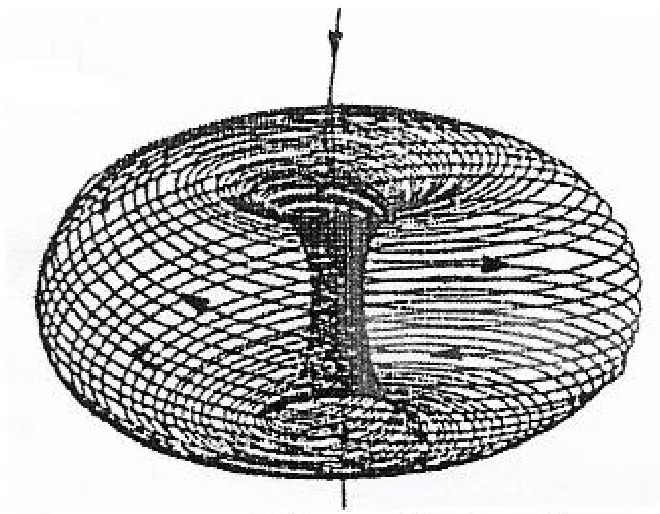


Fig. 3 A Flow with Torsion

It would appear that the creation of a large scale structure can be associated with a parametrically induced Hopf bifurcation occurring in synergetic pairs. An example of this topological creation event will be giver below. The large scale structure has its birth as a topological defect. The torsion mode solution given In Figure 3 was chosen deliberately because of its Hurricane-like appearance, and the fact that its primitive fixed points have correspondence with the tertiary Hopf bifurcation obtained by swirling the trajectories of Figure 2 about the flow axis, and then rotating the figure by 90 degrees. Hurricanes are perhaps the largest coherent structures observable or the earth's surface, and are embedded in a more or less laminar thin sheath of atmosphere between the earth's surface and the tropopause.

As will be demonstrated below, the formation of these domains of different topology embedded in simple, Eulerian, environments can take place contlnuously. Most perceptions of topological change are associated with discontlnuous transformations, but continuity alone is not sufficient to preserve all topological properties. The idea that such topological modifications can take place continuously, but irreversibly, was also suggested at the Cambridge conference [1]. Continuous transformations may not be reversible in the sense that either the inverse transformation does not exist, or if it exists, it is not continuous. If an evolutionary transformation is both continuous and reversible, then it is a homeomorphism [4]. Recall that topological properties, such as orientability, compactness, connectivity, hole count, and Pfaff dimension, are invariants of homeomorphisms. The production of a large scale structure involves topological evolution, hence if the process is continuous, it cannot be reversible. As will be discussed below, if a transformation is continuous and differentiable, but not reversible, then the even-dimensional topological features are invariants of the transformation, but the odd-dimensional topological features can change. (technically, the even dimensional cohomology groups are invariant, but the odd dimensional cohomology groups are not.) For continuous media and hydrodynamic systems, this result implies that the 1-dlmensional features of circulation and the 3-dimensional features of topological torsion can change continuously, but the 2-dimensional features of vorticity, and the 4-dimensional features of topological parity remain constant, with respect to continuous but irreversible transformations

1.4 Continuous evolution

In order to sensitize the reader to continuous transformations that admit topological change, a number of transformations are presented graphically in Figure 4:

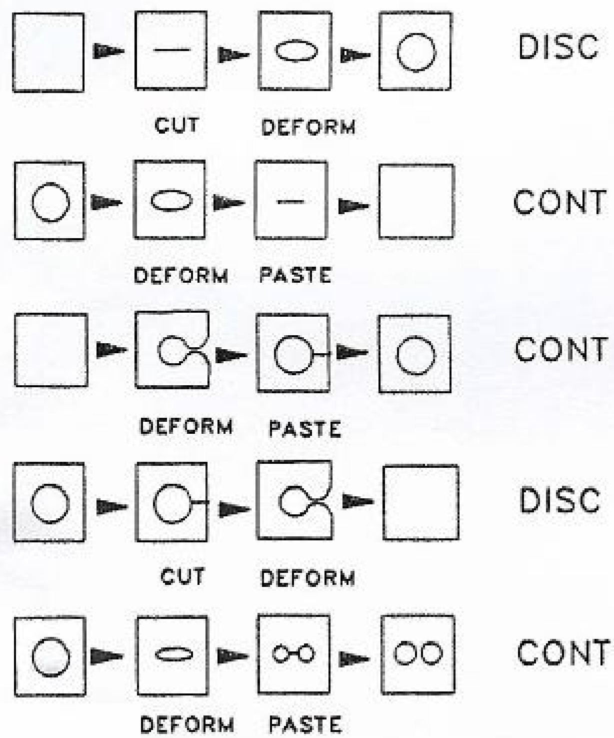


Fig. 4 Topological Evolution

Some transformations are continuous and some are not. In all cases, the topology of the system changes because the hole count in the initial state is not the same as the hole count in the final state. It is remarkable that the production or destruction of holes can be accomplished in either a continuous or a discontinuous manner. Figure 5

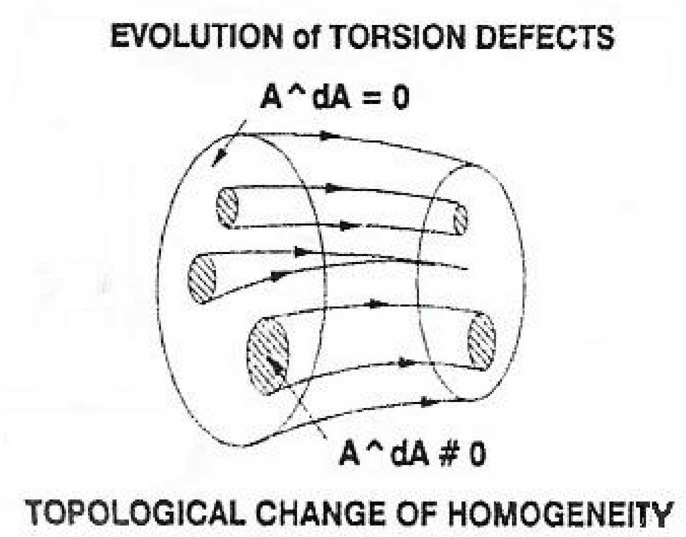
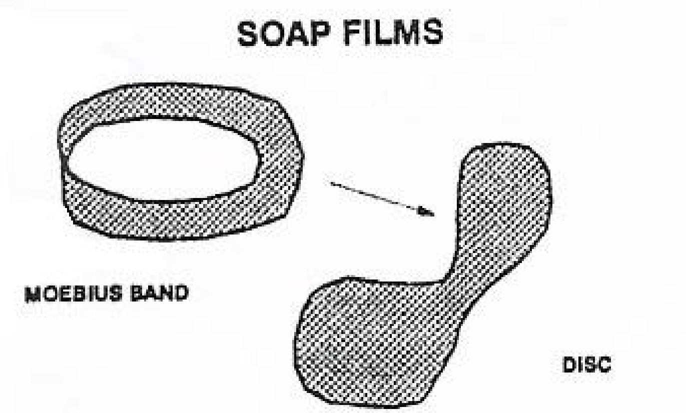


Fig. 5 Evolution of Helicity

is a display of a hydrodynamic continuous process in which the "holes" are represented by domains that have helicity density (and Pfaff dimension 3), and which are embedded in an environmental flow which has Pfaff dimension 2 or less. It was this figure that originally stimulated the author to investigate the topological features of the transition to turbulence in terms of a topological change of Frobenius integrability [5]. A 1-form in any number of dimensions which is of Pfaff dimension 2 (A^dA $\neq 0$) or less has a null set that satisfies the Frobenius conditions of complete integrability.

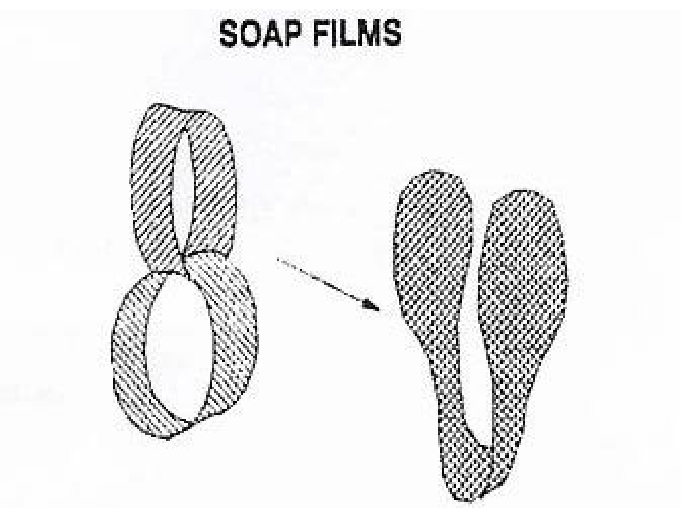
Physical exhibitions of continuous and discontinuous transformations can be achieved through the deformations of a soap film attached to a wire frame boundary. Figure 6



TOPOLOGICAL CHANGE OF ORIENTABILITY

Fig. 6 Evolution or Orientability

demonstrates the continuous deformation of a soap film which involves a change of the topological property of orientability.



TOPOLOGICAL CHANGE OF "HOLES and HANDLES

Fig. 7 Evolution of Handles

In Figure 7 the continuous evolution of the number of handles (or holes) Is

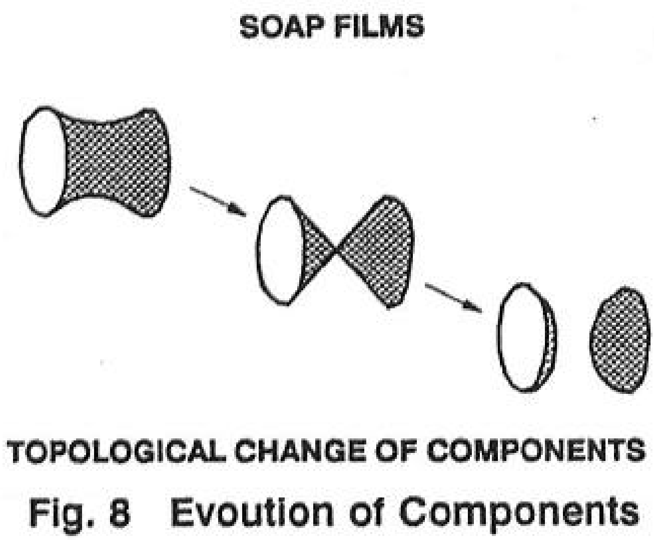


Figure 1:

also emulated by a continuous deformation of a wire frame supporting a soap film.

In Figure 8, the evolution of the number of components is studied in terms of the dynamics of a soap film stretched between two rings. As the rings move further apart, the soap film stretches, still forming a catenoid of revolution. However, when the separation to radius ratio exceeds a critical value, the topological system becomes unstable, and without further perturbation from the environment, the soap film distorts until the original hyperbola of one sheet forms a conical surface (with a critical point at the conical apex) and then separates to form a hyperbola of two sheets which ultimately contracts to form two isolated discs. The originally connected minimal surface undergoes a topological change to where it becomes two disconnected (minimal) surfaces. An example of this topological transition in the surface of null helicity density will be described below, in conjunction with the parametric saddle node Hopf bifurcation of a Navier-Stokes flow.

1.5 Deformations and embeddings

It is possible to demonstrate that Hill's Spherical Vortex may be deformed homeomorphically into an ellipsoid of revolution and still be embedded in an environmental flow which is irrotational. However if the Hill Spherical vortex is deformed continuously, but not uniformly, the environmental flow may no longer be described globally as a potential flow (which produces no lift) but instead requires the global modification of the potential flow into a potential flow with circulation. The terms that induce the circulation are harmonic velocity solutions (to Laplace's equation) that have finite meaning only in non-simply connected topological domains. The total flow about the "hole" or obstruction is still without vorticity, but now the interaction between the potential flow and the harmonic circulation induces a directional force on the ellipsoid which is called aerodynamic lift. This irrotational flow about the single defect ellipsoid does NOT produce aerodynamic drag! The defect in the environmental irrotational flow experiences a force that does no work. The net force on the defect is perpendicular to the velocity of the defect. The harmonic contribution, whose synergetic interaction with the potential flow produces lift, is a perfect physical example of the topological theory of cyclic cohomology at work in a practical sense.

1.6 Harmonic Vector Fields

The idea that a flow without vorticity can have finite circulation is a topological idea that cannot exist in a domain with euclidean topology (which is simply connected). If the total flow has a bounded interior defect (or large scale structure), then the flow topology is no longer euclidean (it is no longer simply connected). The flow interior to the defect can have vorticity, while the flow outside the defect may be vorticity free. An irrotational flow vector field representing the environment exterior to the embedded flow must consist of (only) two parts, the first part is a pure potential flow part and can be globally represented by a gradient field. The second part of the flow cannot be represented by a gradient field, but has zero curl. This contribution to the vector field was called harmonic field by deRham [6]. A vector field that consists of a gradient field and a harmonic vector field is at most of Pfaff dimension 2, and is said to closed with respect to exterior differentiation.

A simple example of the harmonic vector field contribution is given by the pression:

$$\mathbf{V} = \{y\mathbf{i} - x\mathbf{j}\}/(x2 + y2), \tag{1}$$

which is a specialization of the flow,

$$\mathbf{V} = \{\Phi\ grad\Psi - \Psi\ grad\Phi\}/(\Phi^2 + \Psi^2), \tag{2}$$

where Φ and Ψ are arbitrary functions of $\{x, y, z, t\}$. For each representation where the integration cycle, C, encircles the 1-dimensional obstruction or world line formed by the intersection of the surfaces $\Phi = 0$ and $\Psi = 0$, the clrculation is finite, $\Gamma = \int_C \mathbf{V} \circ d\mathbf{l}$, but the vorticity is zero, $curl\ \mathbf{V} = 0$.

It is the harmonic **velocity** field contribution to a closed flow that produces aerodynamic lift. Such notions are at the heart of the Joukowski airfoil theory and focus attention on cyclic cohomology theory. Note that the boundary conditions for closed velocity fields do not permit the specification of a constant zero value of the velocity field everywhere on a boundary. The no-slip boundary condition is not compatible with closed velocity fields.

The idea of topological torsion is to extend these concepts of cyclic cohomology to 3.dimensional obstructions and flows which are not irrotational, and have Pfaff dimension of 3 or greater. The flow velocity field may still have potential and harmonic components, but now a curl component is also admitted, and it this curl field that is the ultimate source of dissipation and drag. The question now is to examine those situations where the vorticity field, $curl\ \mathbf{V}$, exists, but the vorticity field is either harmonic or integrable in the sense of Frobenious. As will be explained below, such velocity fields are uniformly continuous.

When a harmonic field is added to the potential flow of the environment, the Pfaff dimension of the environment does not change, but the Betti number of the environment changes for there is now (at least) one hole that cannot mapped away in a uniformly continuous manner. The concept of Pfaff dimension represents one topological property, and the Betti numbers represent a different topological property.

For Stokes flows (where the vorticity field is harmonic, $curlcurlcurl\ \mathbf{V} = 0$) a no-slip boundary condition is admissible (the boundary can be a domain where the velocity vector is zero at all points of the boundary, and not just at isolated points). For the Navier-Stokes fluid, the implication is that the vector field representing the pressure and viscous forces (not the velocity field) is closed, and therefore (according to the DeRham decomposition theorem) consists of a perfect gradient and a harmonic part. In other words, the velocity field is of Pfaff dimension 3 or more, but the force field is of Pfaff dimension 2 or less, for Stokes flows.I

These Stokes solutions can have drag forces that act on the bounding surface. The classical example is that given by Stokes in which the inertial terms of the Navier-Stokes equations are ignored, and the velocity field is assumed to satisfy Stokes equation, such that the curl of the vorticity is a gradient field. Such flows have drag. A somewhat less stringent condition is that the vorticity field be harmonic. As mentioned above, the cyclic work is zero if the viscous contribution to the force field is a pure gradient, and is "quantized" if the viscous contribution to the force field is harmonic. The first case implies that the "rotational" energy induced in the fluid is recoverable, and the latter case will be of interest to the problem of vortex shedding.

2 BOUNDED SOLUTIONS TO THE NAVIER.STOKES EQUATIONS.

2.1 Tertiary Hopf Bifurcations

The Saddle-Node Hopf closed form solution to the Navier-Stokes equations in rotating coordinate systems suggests how closed, reentrant, secondary flows with coherent topological structures may be constructed and embedded in environmental surroundings of different topology. Of particular interest to this author are those flows with compact boundary which have domains with topological torsion and a Pfaff dimension > 2. These domains have finite helicity density, and their streamlines have Frenet torsion. The streamlines may be either ergodic or closed within the bounded domain. These structures have received little exploitation in the hydrodynamic literature.

A generalization of the Saddle Node Hopf solution to the Navier-Stokes equations which exhibits compact domains Is given by the vector field $\mathbf{V} = [u, v, w]$, where

$$\begin{aligned} u &= C\ x\ \partial f(z)/\partial z - \Omega y \\ v &= C\ y\ \partial f(z)/\partial z + \Omega x \\ w &= +A\ f(z) - T\ h(r^2). \end{aligned} \quad (3)$$

The constraint of incompressibility requires that 2C = A. The polynomial functions, f(z) and h(r), are arbitrary, but polynomial expressions of the quartic variety are of interest to the problem of wakes. The representation that fits the simple saddle-node Hopf solution, and which is studied in detail below, is given by the quadratic polynomial formulas:

$$Af(z) = F - D\ z^2 \quad and \quad h(r) = r^2 + 10. \quad (4)$$

The simpler quadratic polynomial solutions are of interest to single defect struc tures, and it is easy to show that for all quadratic flows the Navier-Stokes viscous dissipation term is representable (locally) by a gradient field,

$$grad\ div\ \mathbf{V} - curl\ curl\ \mathbf{V} \ \Rightarrow\ grad\ \Psi. \quad (5)$$

It follows that the pressure and viscous forces associated with such flows not produce cyclic work, hence the systems are conservative. Energy of rotation induced by viscous torques is recoverable! It is also true that the vorticity of a quadratic flow is globally harmonic,

$$curl\ curl\ curl\ \mathbf{V} = 0, \quad div\ curl\ curl\ \mathbf{V} = 0. \quad (6)$$

Recall that harmonic vorticity implies conservation of angular momentum, the closure criteria required to produce harmonic vorticity does not imply neccessarily that a gradient representation for the viscous force term is globally valid.

It is interesting to note that the degree of the polynomials, f and h, is twice number of distinct defects that the flow represents. As the quadratic velelocity field does not produce irreversible dissipation in a Stokes fluid (the Stokes term is a gradient field) then it becomes apparent that the number of defects must exceed two if entropy is to increase.

2.2 The Bounding Surface

In all cases the bounding "ellipsoidal" surface to the secondary flow generated by equations (3) will be given by the zero set of the function Φ(x,y,z), such that

$$\mathbf{V}\cdot\ grad\ \Phi = \lambda\ \Phi(x, y, z). \tag{7}$$

Note that this requirement forces the velocity field to be "tangent" to the bounding surface. The bounding function, Φ, is a conformal invariant of the flow, $\mathbf{V}$, with conformality factor, λ. When the conformality factor vanishes, the, bounding ellipsoid is an absolute invariant of the flow. These two cases correspond to (asymptotic) continuity and uniform continuity, respectively. Further note that the "renormalized" velocity field, $\Phi\ \mathbf{V}$, satisfies the "no-slip" condition on the bounding surface. This important result states that the streamlines of a no-slip flow and the streamlines of a flow that leaves the bounding surface a conformal invariant are the same. This correspondence between Eulerian flow and no-slip flow around obstacles is a curiosity that has appeared in the literature before, but the association to a conformally invariant bounding surface is apparently novel to this article. The representation of a no-slip surface by the conformal constraint may be utilized to show that the Euler characteristic of the flow manifold is completely determined by the zeros of $\mathbf{V}$ that are simultaneously "singular" points of the bounding surface.

In the general polynomial case, if the function h(r) is written as a polynomial

$$h(r) = \sum h_n r^n, \tag{8}$$

then the bounding "ellipsoid" may be constructed as the zero set of

$$\Phi(r, z) = G(r) + D\ F(z), \tag{9}$$

where

$$G(r) = \sum\ T\ D\ h_n r^n/(D - nC). \tag{10}$$

It follows that

$$\mathbf{V}\cdot\ grad\ \Phi = \{D\ \partial f(z)/\partial z\ \}\ \Phi(x, y, z). \tag{11}$$

All such bounded compact flows are solutions to the Navier-Stokes equations in a rotating frame of reference.

(Note Feb 26, 2003: The bounding surface is therefor a zero-form that is homogeneous of degree $\{D\ \partial f(z)/\partial z\ \}$*. This implies that the bounding surface can be fractal.)*

2.3 The Saddle Node Hopf Solution.

The simple Saddle Node Hopf flow field as given by (3) and (4) will be studied as a function of the mean flow parameter, F. The study as presented in Figures 9 through 12 will demonstrate that a torsion bubble defect, containing a bounded secondary reverse flow, can be produced as a result of a parametrically induced Hopf bifurcation. The topological creation and persistence of a large scale structure in a Navier-Stokes flow is thereby demonstrated by this example.

The bounded secondary flow can be embedded in a closed flow representing the environment.

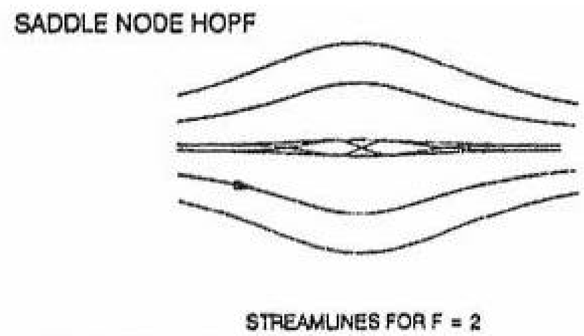


Fig. 9 Flow < Critical Value

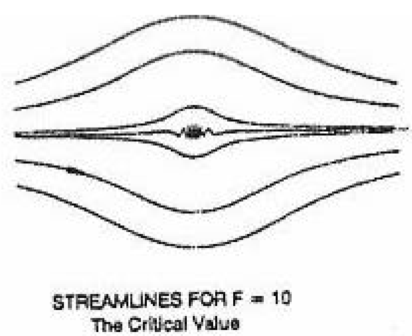


Fig. 10 Flow = Critical Value

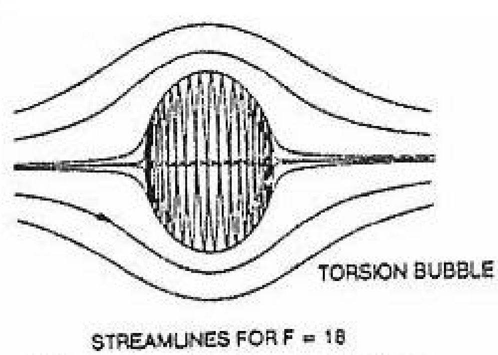


Fig. 11 Flow > Critical Value

A mentioned above, a specialization of the two polynomials to the quadratic forms given by equation (4) leads to a flow with two fixed points of the saddle node Hopf variety. In the vicinity of $\{z, r\} = \{+((F - 10T)/D)^{1/2}, 0\}$ the fixed point has two positive and one negative Lyapunov exponents {+, + ,-}. while the fixed point at $\{z, r\} = \{-((F - 10T)/D)^{1/2}, 0\}$ has two negative and one positive exponents, {-,-, +}. Both regions are locally unstable, but the flow in the neighborhood of one fixed point is coupled to the flow in the neighborhood of the second fixed point to produce a globally stable structure. See Figure 2.

Expressions for the vorticity and the helicity density of the SNH flow may be analytically computed. In particular it is to be noted that the vorticity is independent of the parameter F, but the helicity density depends on F. For low flow rates (F = 2) the streamlines appear as in Figure 9, and the surface of zero helicity density is a hyperbola of revolution consisting of one sheet. (See Figure 12.) The regions of negative helicity density are connected, as are the regions of positive helicity density.

For the parameters chosen, and such that $div\mathbf{V} = 0 \;\; (T = 1, \; D = C)$, when the flow parameter $F = 10$, another critical point is reached, for now the paired Lyapunov exponents can become complex. A locus of Hopf points forms a circle of radius, $r = \{+((F - 10T)/T)^{1/2}\}$ in the z = 0 plane. See Figure 10. At this critical parameter point, the surface of zero helicity density becomes a cone. Except for those streamlines in the vicinity of the origin, the swirl component has been suppressed in the figures: only the envelope of the swirling streamlines is displayed, for reasons of visual clarity.

When the flow parameter F exceeds 10, a defect torsion bubble will form with an axial dimension given by $z = +[(F - 10T)/D]^{1/2}$ and with the radius of the bounding surface given by $r = \{+(2(F - 10T)/T)^{1/2}$. The Hopf circle is always contained by the bounding ellipsoid. Note however that the surface of null helicity density becomes a hyperbola of two sheets, and a "helicity gap" develops (along with the torsion bubble) separating domains of negative helicity density. For F = 18, the streamlines are presented in Figure 11. Again, only the envelope of the streamlines outside the defect are plotted, while the swirl is detailed for trajectories within the defect bubble. Note that the secondary flow has a reverse velocity component within the bubble defect, and that all streamlines are tangent to the bounding surface. The topological transition of the surface of null helicity density is described qualitatively as a function of the flow parameter, F , in Figure 12.

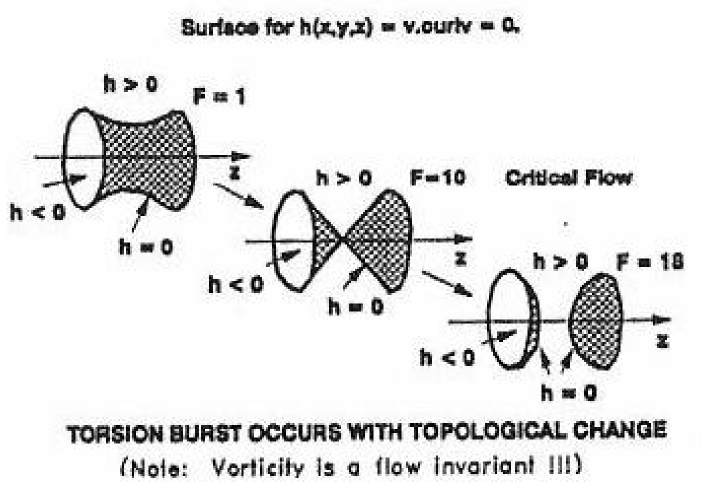


Fig. 12 Evolution of Helicity density

Recall that although the visual defect appears to be associated with a "vortex burst", such is not the case. During this parametric creation (only the mean flow parameter F has been changed) of a large scale structure in a viscous media, the lines of vorticity are the same for all Figures 9 through 12. The vorticity of the saddle node Hopf flow is an absolute invariant of the flow itself, and its value is independent of the mean flow parameter, F. It is suggested that this visual effect is better described by the words, "torsion burst", than by the words "vortex burst". To make a more direct contact with the experimental exposes of the "torsion bursting" problem, a slight modification to the SNH solution is presented below. A small cubic addition to the f polynomial changes topological features of the solution.

2.4 The Hysteretic Hopf Solution

If the polynomial, t, is modified from the SNH quadratic to include a term cubic in z (add .05 z^3 to the SNH for-mula for the z component of velocity) then the streamlines for low values of F will appear as given in Figure 13. The near central streamline flows in the direction of the single real fixed point, and the diverges in a spiral fashion around the surface generated by the cubic polynomial. If the parameter F is increased to 26, then again a torsion bubble is formed upstream from and before the spiral instability is encountered. See Figure 14. The streamlines give a visual appearance that may be compared to the experimental data presented for the "vortex bursting" problem [7].

In the numerical evaluations of the streamlines, if a small $div\ \mathbf{V} \neq 0$ term is permitted, then the central streamlines will enter into the torsion bubble in the vicinity of the downstream "fixed" point. The streamline remains trapped

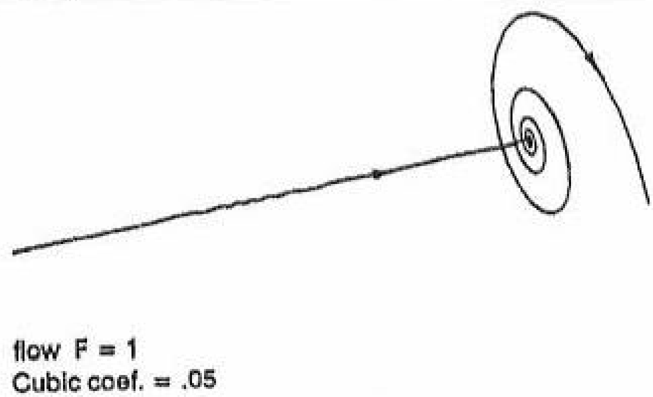


Fig. 13 Hysteretic Torsion Burst
Flow < Critical Value

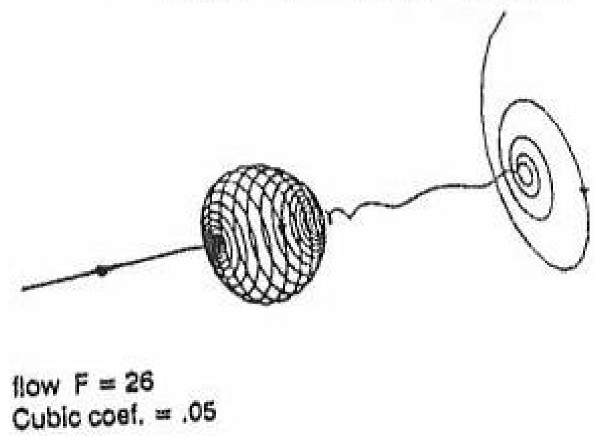


Fig. 14 Hysteretic Torsion Burst

Flow > Critical Value

Figure 2:

within the "bubble" for long periods of computing time. This result mimics the experimental dye results presented in reference 7, and is suggestive of the Arnold diffusion process in phase space when the KAM (bounding) surface becomes non-compact and has non-Isolated boundary points or holes through which the phase space trajectories can "diffuse".

3 THE FALACO VERTEX PAIR

Another experimental example of a large scale structure is given by the "Falaco" effect. A long lived torsional soliton state can be formed in the free surface of water, and was .brought to this author's attention accidentally while visiting an old friend (hence the name -the Falaco effect) in Rio de Janeiro [8]. See Figure 15. The properties of this globally stabilized torsional state have topological similarities to the analytic solutions described above.

Briefly, a pair of concave Rankine vortices (generated in the free surface of a swimming pool by slowly sweeping a large half-submerged circular disk perpendicular to its planform) to produce a long-lived pair of con-rting convex dimpled surface states of negative Gaussian curvature. The original Rankine surface is predominantly of positive Gaussian curvature. Each convex singular surfac may be observed easily by means of the Snell law of refraction that produces a black disc (or absencew of light) surrounded by a bright ring (or halo) on the bottom of the swimming pool. As the convex surface is near a minimal surface, the Snell projection is a conformal mapping such that the image on the bottom of the pool is circular, even though the source of illumination (the sun) is not directly overhead. The Falaco effect black spots have a striking long lifetime (many minutes in a still pool) and exhitiit many of the phase coherent features associated with numerical solutions to soliton scattering problems. A photograph of the "Falaco Spots" is given in Figure 15 and the Snell explanation of the optics is given in Figure 16.

What is not obvious unless dye is injected into the water is that there exists a topological singular "thread" In the form of a circular arc that connects one convex vertex to the other, and transverse torsion waves can propagate from vertex to the other guided by this central thread. See Figure 17. If the thread is "severed", the paired surface defects do not diffuse away, they apparently explode away. It is remarkably easy to experimentally create such a pair of 2-dimensional topological surface defects, globally connected and apparently globally stabilized by a 1-dimensional topological defect. Moreover, the system supports torsional waves. What more could someone interested in applied toopology and topological torsion want? Although exact solutions to this problem have yet to be found, the experiments associated with the Falaco Effect stimulated the work described in the preceding sections.

Fig. 15 The FALACO effect.
Rotational stabilized solitons,
a few minutes after formation.

Figure 3:

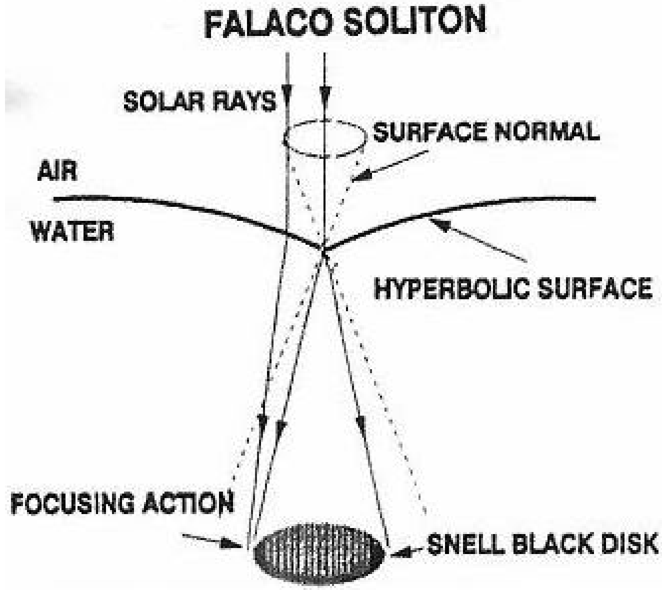


Fig. 16 Snell's Law formation of the Falaco Black Spots

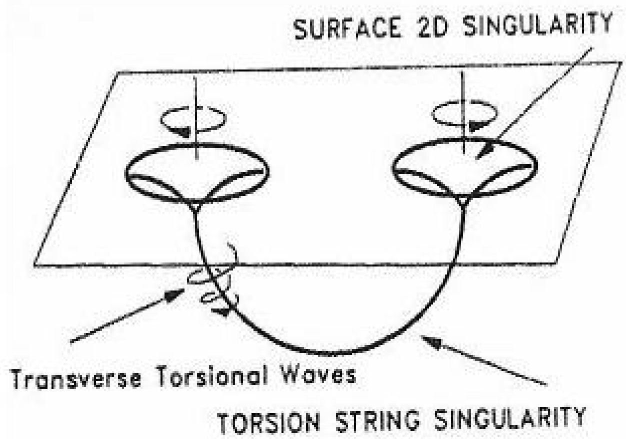


Fig. 17 Torsion Waves

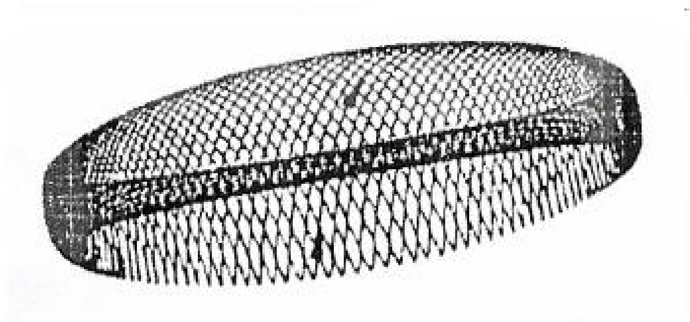


Fig. 18 A distorted bounding ellipsoid for a toroidal SNH flow with the z (flow) axis horizontal.

Figure 4:

4 PHASE TRANSITIONS IN DYNAMICAL SYSTEMS

As suggested by this author at the IUTAM conference on Topological Hydro Mechanics, a dynamical system should be expected to exhibit thermodynamic phase transitions induced by parametric variations. Recall that phase transitions imply a change in topology. If the Saddle Node Hopf solution to the Navier-Stokes equations is modified slightly to include an anisotropic perturbation parameter, R, then a study of the parametric change of topology in the bounding surface of the secondary flow can be made analytically [9]. In brief, for the example equations presented below, the ellipsoidal bounding surface of the modified SNH flow admits two isolated singular points for small values of the anisotropy parameter (Figure 18). The radial "size" of the bounding ellipsoid is a measure of the topological coherence in the defect, and will be defined as an order parameter for the flow.

Confined within the interior of the bounding ellipsoidal surface is a closed curve of Hopf singular points in the form of an ellipse. As the perturbation parameter is increased, the Hopf ellipse grows until the Hopf ellipse penetrates the bounding ellipsoidal surface, such that the bounding surface now admits 6 singular points (Figure 19). The radial size of the bounding ellipsoid, which acts as an "order" parameter, decreases with increasing anisotropic perturbations. The interior solution streamline. trajectories are always confined by the bounding surface, arid are rotationally guided by "or attracted to. the Hopf ellipse. The trajectories, representing evolving streamlines, sweep out paths that are confined within a torus for small values of the perturbation parameter (Figure 18).

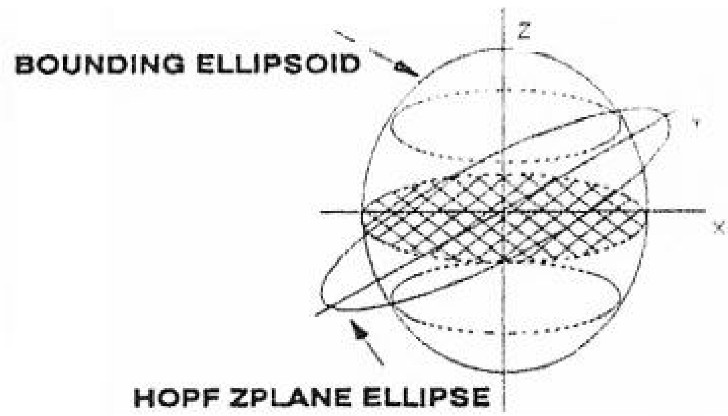


Fig. 19 Hopf Ellipse penetrating the bounding surface.

Figure 5:

As the perturbation parameter is changed, a topological phase transition takes place when the Hopf ellipse penetrates the bounding ellipsoid, and thereafter the stream lines are confined to a surface with the appearance of a topological button with 2 handles (Figure 20).

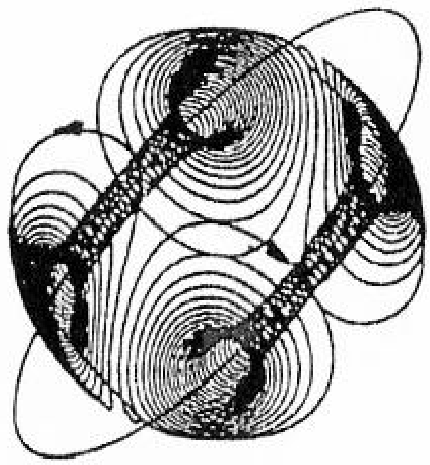


Fig. 20 A Chaotic Streamline guided by the Hopf Ellipse.

At a certain point, further increase of the perturbation parameter, R, destroys the dynamical stability of the system, and the order parameter goes to zero (or becomes imaginary). The variation of the order parameter is associated with a change of topology, and a corresponding change of phase (Figure 21).

The streamlines in the torus phase numerically seem to be non-chaotic (in that the topological handle along the z axis maintains its "size"). However, the streamlines in the button phase exhibit "Lagrangian" chaos. The throat size of each of the two Hopf induced handles varies intermittently, and the Poincare return map fills large domains of the surface of section. A portion of a single stream line Is presented in Figure 20, demonstrating the topological structures of the flow solution in the button phase.

As mentioned above, an example of a parametrically induced phase transition in a dynamical system was presented at the Cambridge IUTAM conference by the present author. At the same conference the author was stimulated by the numerical work of Bajer, et. al. 10], which was presented for a particular (and Ingeniously chosen) stretch-twist-fold vector field. The present author realized that the stretch-twist-fold vector field of Moffatt was a special case of the recently dis-covered Saddle Node Hopf solution to the Navier- Stokes equations, and should exhibit a topological phase transition. The problem was formulated analytically with results, as presented above and extend the Bayer numerical results. The detailed equations for the analytical modified SNH flow are given in the next section.

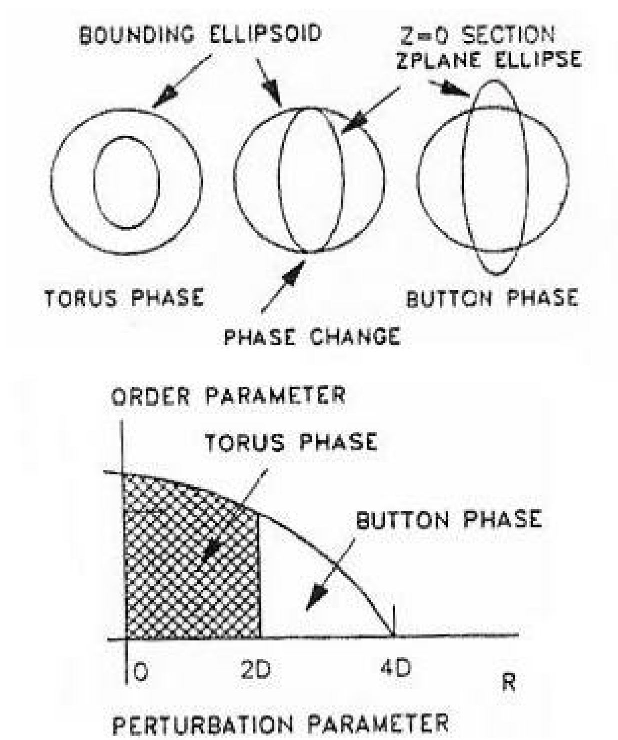


Fig. 21 Change of Phase in a modified SNH flow.

Figure 6:

4.1 Anisotropic Modifications of the SNH flow.

In the following example the function f(z) will be constrained to the quadratic form

$$A\ f(z) = (F - D\ z^2), \tag{12}$$

but the polynomial h(r), as well as the coefficients C will be modified to permit anisotropy in the xy plane:

$$h(r) \Rightarrow H(x,y) = S\ x^2 + T\ y^2. \tag{13}$$

The modified saddle node Hopf solution becomes the flow $\mathbf{V} = [u, v, w]$, where

$$\begin{aligned} u &= (C - R)\ x\ \ \partial f(z)/\partial z - \Omega y \\ v &= (C - R)\ y\ \partial f(z)/\partial z + \Omega x \\ w &= +D\ f(z) + S\ x^2 + T\ y^2. \end{aligned} \tag{14}$$

R is a measure of the "asymmetry" of the bounding ellipsoid in the z = 0 plane and will be defined as the "perturbation parameter". The solution has axial symmetry when R = O. The "order parameter" is defined as the mean radial "size" of the bounding ellipsoid.

The bounding ellipsoidal surface is given by the expression

$$\Phi(x,y,z) = D\ f(z) + \{SD/(D - 2C - 2R)\}x^2 + \{TD/(D - 2C)\}y^2. \tag{15}$$

It is possible to choose R, S, and T such that the bounding ellipsoid ellipsoid of revolution, or even a sphere.

The example flow is remarkable in that the two fixed points along the z axis form the polar points of the bounding ellipsoid, and are adjoined by a Hopf ellipse In the z = 0 plane. For various values of R < 2D, the Hopf ellipse is confined within the bounding ellipsoid and the resulting streamlines of motion lie on nested tori with a guiding center dictated by the Hopf ellipse. The streamlines are confined to toroidal surfaces of Euler characteristic zero (Figure 19). At larger values of R, 20 < R < 40, the Hopf ellipse penetrates the bounding ellipsoid, and the resulting streamlines are confined to a surface that I"!C1S a non-zero Euler characteristic. and the appearance of a topological button (Figure 20). A topological phase transformation takes place when R = 2D.

The streamlines of the flow in the torus phase exhibit Poincare sections that are indicative of periodic (section dimension 0) or doubly periodic (section dimension 1) motion. The streamlines of the flow in the burton phase exhibit Poincare sections that appear to be chaotic (section dimension > 1). It should be remarked that (3) is an exact solution of the Navier-Stokes equations in a rotating frame of reference, where (14) requires an anisotropic external force.

5 CONTINUOUS TOPOLOGICAL EVOLUTION

5.1 Introductory remarks

In this last section, a rudimentary theory of topological evolution is constructed. Such a theory is necessary to explain the creation of large scale structures in continuous media, for such creation processes involve topological change. Recall that conservative classical Hamiltonian mechanics is not applicable to problems that involve topological change. Topological evolution implies that the transition from an initial to a final state is NOT a homeomorphism. It follows that the transformation representing topological change is either 1.) NOT continuous, or 2.) irreversible, or both. Otherwise, topological properties would be invariant, as a continuous reversible transformation is a homeomorphism that preserves all topological properties.

In the development that follows (which will be limited to continuous processes) the third order tensor field of topological torsion will play a dominant role. The divergence of this tensor field leads to a fundamental equation of continuity with source, where the source term is the fourth rank tensor field of topological parity. These topological concepts are independent from a metric or other geometrical constraints (such as a connection) that may be imposed on the variety, {x,y,z,t}. The intuitive idea is that a large scale structure is a topological defect in an otherwise homogeneous domain.

A fundamental topological idea is that during a continuous process, all limit points of a set remain within the closure of the set [11]. This idea is more useful than the usual definitions of continuity which involve the concepts of an inverse image. A more restrictive version of the continuity concept would be the study of those transformations for which the limit points not only remain within the closure, but also are invariants of the transformation. This idea is at the heart of the notion of "uniform" continuity, and will occupy most of the discussion that follows.

5.2 Exterior Differential Systems

To study contInuous topological evolution, three basic concepts are required:

1. A method of defining time dependent topological properties over a domain of interest has to be developed.
2. The limit points relative to this time dependent topology must be computable.
3. The equations describing the evolution of these topological properties must be specified.

These requirements can be satisfied by means of Cartan's theory ofdifferential systems [12]. Given a system of differential forms, Σ, the closure may be constructed by adjoining to the original system all of the exterior derivatives, $d\Sigma$, of the initial system. These additional forms may be viewed as the "limit sets" of the differential system. The combined system of forms, or $\{\Sigma,\ d\Sigma\}$, forms a topological subbase on the variety over which the system is defined. By forming the intersection of all elements of the subbase, a topological basis, or Pfaff sequence, is formed: $\{\Sigma,\ d\Sigma, \Sigma\hat{}d\Sigma, d\Sigma\hat{}d\Sigma\}$. The Cartan exterior product, $\hat{}$, is used as a convenient "intersection" operator.

To test for topological evolution (with respect to an arbitrary vector field, V), each element of the topological base may be tested for invariance, or lack of invariance, relative to the flow defined by V. If all elements of the the base are invariant, then the selected vector field represents a homeomorphism, and all topological properties are invariant relative to V. If the limit sets (the exterior derivatives of the initial system of forms) are invariant, then the vector field is uniformly continuous, relative to the Cartan topology. The concept of the Lie derivative, L(V), relative to the field V acting on the differential forms [13] that make up the topological base will be used as a mechanism for testing for topological evolution. That is, for any 1-form A of the topological base, the action of the Lie derivative on A creates a new 1-form Q,

$$L(V)A = Q. \tag{16}$$

Similarly for a 3-form, $H = A\hat{}dA$, the action of the Lie derivative produces another 3-form, S:

$$L(V)H = S \tag{17}$$

If Q or S is zero then the form A or H is an invariant of the flow generatred by V.

These are topological ideas about neighborhoods, and do not depend upon geometrical constraints of metric or connection. The basic laws governing continuous evolution have the format of the equations (16) and (17). As will become evident from the discussion below, equation (16) describes the evolution of energy and is equivalent to Newton's law of motion. Equation (16) is to be read as "The Lie derivative of A with respect to V is equal to the 1-form Q, or better, "The topological evolution of the action, A, is determined by the inexact 1-form of heat, Q".

Equation (17) describes the evolution of torsion defects (entropy) and is the novel contribution of the topological method. For systems for which the velocity field is completely integrable (in the sense of Frobenius), equation (17) is empty; but otherwise, equation (17) is to be interpreted as the fundamental equation for the evolution of defects.

5.3 The Kinematic Topological Base

For continuous media in space-time, the key idea is that the exterior differential system consists of a Pfaff sequence constructed from a single 1-form of Action A, plus (perhaps) some additional constraints. The work of Arnold (and others) [14] has established that the singular points (zero's) of a global 1-form carry topological information. This idea is to be extended to the singular points of all elements of the Pfaff sequence, or topological base.

Following the arguments given above for a general differential system, the com-plete Pfaff sequence constructed from a kinematical 1-form of action, A, forms a topological base. Explicitly, each kinematic situation is abstractly describable in terms of a 1-form, $A = A\mu dx^{\mu}$, of action. The exterior derivative of A produces a 2-form of closure points, F = dA, whose components are given by the ex-pression, $F_{\mu\nu}dx^{\mu}dx^{\nu}$. The combined set $\{A, F\}$ forms the closure of the set A, and acts as a primitive topological subbase for the domain {x,y,z,t}. All possible intersections of the subbase, { A, F, A^F, F^F }, form a (primitive) topological basis for the domain. The topological basis is defined as a Pfaff sequence:

$$\begin{array}{ccc} TOPOLOGICAL\ ACTION & A = & A\mu dx^{\mu} \\ TOPOLOGICAL\ VORTICITY & F = dA = & F_{\mu\nu}dx^{\mu}\hat{}dx^{\nu} \\ TOPOLOGICAL\ TORSION & H = A\hat{}dA = & H_{\mu\nu\rho}dx^{\mu}\hat{}dx^{\nu}\hat{}dx^{\rho} \\ TOPOLOGICAL\ PARITY & K = dA\hat{}dA = & K_{\mu\nu\rho\sigma}dx^{\mu}\hat{}dx^{\nu}\hat{}dx^{\rho}\hat{}dx^{\sigma} \end{array} \tag{18}$$

The "singular" null points of each of these elements of the topological base determine the kinematic topology induced on the domain, {x,y,z,t}.

For continuous media on a variety {x,y,z,t}, the Cartan Action, A, can be defined kinematically as:

$$A = \mathbf{v}.d\mathbf{r} - \{\mathbf{v}.\mathbf{v}/2\}dt. \tag{19}$$

The 2-form of topological vorticity, F = dA, has six components,

$$F = dA = \omega_z dx\hat{}dy + \omega_x dy\hat{}dz + \omega_y dz\hat{}dx + a_x dx\hat{}dt + a_y dy\hat{}dt + a_z dz\hat{}dt, \tag{20}$$

which can be designated as one 6-vector or two 3-vectors,

$$\boldsymbol{\omega} = curl\ \mathbf{v} \quad and \quad \mathbf{a} = -\partial\mathbf{v}/\partial t - grad\{\mathbf{v}.\mathbf{v}/2\}\} \tag{21}$$

These vector fields always satisfy the Poincare-Faraday induction equations, $dF = ddA = 0, or,$

$$curl\ \mathbf{a} - \partial\boldsymbol{\omega}/\partial t = 0, \quad div\ \boldsymbol{\omega} = 0. \tag{22}$$

The 3-form of Topological Torsion, H, constructed from the exterior product of A and dA, has four components, and can be written as,

$$H = A\hat{}dA = H_{\mu\nu\rho}dx^{\mu}\hat{}dx^{\nu}\hat{}dx^{\rho} \tag{23}$$

$$= T_x dy\hat{}dz\hat{}dt - T_y dx\hat{}dz\hat{}dt + T_z dx\hat{}dy\hat{}dt - h dx\hat{}dy\hat{}dz, \tag{24}$$

where **T** is the Topological Torsion axial vector current,

$$\mathbf{T} = \mathbf{a} \times \mathbf{v} + \{\mathbf{v}.\mathbf{v}/2\}\ curl\ \mathbf{v} = \mathbf{a} \times \mathbf{v} + \{\mathbf{v}.\mathbf{v}/2\}\ \boldsymbol{\omega} \tag{25}$$

and h is the helicity density.

$$h = \mathbf{v} \cdot curl\ \mathbf{v}. \tag{26}$$

The torsion vector, **T**, consists of two parts. The first term represents the shear of translational accelerations, and the second part represents the shear of rotational accelerations. The topological torsion tensor, $H_{\mu\nu\rho}$, is a third rank, completely anti-symmetric covariant tensor field, with four components on the variety {x,y,z,t}. Note that the helicity density is the fourth component of a third rank tensor field that transforms covariantly under all diffeomorphisms, including a Galilean translation.

The Topological Parity becomes

$$K = d(H) = -2(\mathbf{a} \circ \boldsymbol{\omega}) dx\hat{}dy\hat{}dz\hat{}dt. \tag{27}$$

The fundamental law for the local evolution of this set is given by the expression

$$div\ \mathbf{T} + \partial h/\partial t = K. \tag{28}$$

The pseudo scalar function, K, acts as the source for the divergence of the torsion vector, **T**, and the torsion or helicity density, h. When $K = 0$, the evolutionary "lines" associated with the torsion tensor never cross, implying that the system is free of defects in space time. If K is positive or negative, the defects in the system are either growing or decaying. Equation (26) is the fundamental new law of topological physics that governs the specific realizations of control-led processes that minimize or maximize defect evolution.

5.4 The Navler.Stokes Fluid

The kinematic topology is too course for direct application to a typical physical system. Additional topological constraints must be applied. For a Navier-Stokes fluid, the additional topological constraints on the admissible flow fields, $V = \{\mathbf{v}, 1\}$ take the form:

$$i(V)dA = -\{(gradP/\rho + \upsilon\ curl\ \boldsymbol{\omega})_i (dx^i - \mathbf{v}^i dt)\}. \tag{29}$$

Equation (26) is the differential form equivalent to the Navier-Stokes equations; such a constraint limits the class of all V to those V that are solutions to the Navier-Stokes partial differential equations.

The constraint given by (27) may be used evaluate the behavior of the topological base with respect to the evolution described by V. For example, the evolution of the Action is given by the expression,

$$L(V)A = i(V)dA + d\{i(V)A\} \quad (30)$$
$$= -\{(gradP/\rho + \upsilon\ curl\ \boldsymbol{\omega})_i(dx^i - \mathbf{v}^i dt)\} + d\{\mathbf{v}\cdot\mathbf{v}/2\} \quad (31)$$

The evolution of the limit sets is given by

$$L(V)dA = -d\{(gradP/\rho + \upsilon\ curl\ \boldsymbol{\omega})_i(dx^i - \mathbf{v}^i dt)\}. \quad (32)$$

If the flow V is uniformly continuous, then the RHS of (29) must vanish, making $F = dA$ a flow invariant. This result is an extension of the Helmholtz theoremon the conservation of vorticity. It would follow that the 4-form, $K = dA\hat{}dA$ is also a flow invariant, for uniformly continuous flows.

The conclusion is reached that the even dimensional topological properties $\{F, K\}$are invariants of a uniformly continuous flow. If topology is to change uniformly continuous manner, the only possibilities for topological change is to be associated with the 1- dimensional circulation, A, and the 3-dimensionesional torsion, H. For incompressible flows ($div\mathbf{v} = 0$), circulation defects must be associated with boundaries; however, if $K\#0$, then according to (26) torsion defects can occur within the bulk media. t is the author's perception the production of torsion defects is the key to the understanding of large scalestructures in continuous media.

In general, if the flow is continuous, then the limit sets dr must remain with closure of Σ, hence

$$L(V)d\Sigma = 0. \quad (33)$$

These concepts will be exploited in a later article.

5.5 The Torsion Current

In closing, it is to be noted that the Navier-Stokes constraint (27) may be to express the acceleration term, **a**, dynamically; i.e.,

$$\mathbf{a} = -[grad\{\mathbf{v}.\mathbf{v}/2\} + \partial\mathbf{v}/\partial t] = -\mathbf{v}\times curl\ \mathbf{v} + gradP/\rho + v\{curl\ curl\ \mathbf{v}\}. \quad (34)$$

By substituting this expression for **a** into the formula for the torsion vector current, a simple engineering representation is obtained for a Navier-Stokes fluid:

$$\mathbf{T} = \{h\mathbf{v} - \{\mathbf{v}\circ\mathbf{v}/2\}curl\ \mathbf{v}\} - \mathbf{v}\times gradPl\rho - v\{\mathbf{v}\times(curl\ curl\ \mathbf{v})\} \quad (35)$$

Note that the torsion axial vector current persists even for Euler flows with zero vorticity, $\omega = 0$. The measurement of the components of the Torsion vector have completely ignored by experimentalists (and theorists) in hydrodynamics.

By a similar substitution, the topological parity pseudo-scalar becomes expressible in terms of engineering quantities as,

$$K = -2gradP/\rho \circ curl\ \mathbf{v} - 2v\{curl\ \mathbf{v}.(curl\ curl\ \mathbf{v})\}. \quad (36)$$

From this expression it is apparent that even in the limit of zero viscosity (high Reynolds number), it is still possible to produce torsion defects when the pressure gradient and the vorticity, ω, are not orthogonal. Moreover, If the vorticity field is integra ble in the sense of Frobenius, then viscosity does NOT contribute to the creation of torsion defects. The integral of K over {x,y,z,t} gives the Euler Index of the flow.

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