TOPOLOGICAL EVOLUTION OF CLASSICAL ELECTROMAGNETIC FIELDS AND THE PHOTON

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The theory of classical electromagnetism is constructed in terms of two exterior differential systems, $F - dA = 0$, and $J - dG = 0$, which act as topological constraints on the variety of independent variables $\{x, y, z, t\}$. These two fundamental constraints lead to two other independent concepts of topological torsion, A^F, and topological spin, A^G, which are explicitly dependent upon the potentials, A. The exterior derivative of these 3-forms creates the two familiar Poincare deformation invariants of an electromagnetic system, valid in the vacuum or plasma state. When the Poincare invariants vanish, the closed integrals of A^F and A^G exhibit topological invariant properties similar to the "quantized" chiral and spin properties of a photon. The possible evolution of these and other topological properties is studied with respect to classes of processes that can be defined in terms of singly parameterized vector fields. Non-zero values of the Poincare invariants are the source of topological change and non-equilibrium thermodynamics.

1 Introduction

In the language of exterior differential systems [1] it becomes evident that classical electromagnetism is equivalent to a set of topological constraints on a variety of independent variables. Certain integral properties of such an electromagnetic system are deformation invariants with respect to *all* continuous evolutionary processes that can be described by a singly parameterized vector field. These deformation invariants (or topological properties) lead to the fundamental topological conservation laws described in the physical literature as the conservation of charge-current and the conservation of flux. The object of this article is to examine other objects that can behave as deformation invariants relative to certain equivalence classes of continuous processes, and yet exhibit topological change with respect to other classes of continuous processes. Recall the definitions:

> ***Definition* 1**: *A continuous process is a map from an initial state with a topology $T_{initial}$ into a final state with perhaps a different topology T_{final} such that the limit points of the initial state are permuted among the limit points of the final state*. [2]

***Definition* 2**: *A deformation invariant is an integral over a closed manifold* $\int \bullet \bullet \int_{closed} \omega$ *such that the Lie derivative of the closed integral with respect to a singly parameterized vector field,* βV^k, *vanishes, for any choice of the parametrization function*, β.

It is also important to recall that a given variety of independent variables can support more than one topology. In classical electromagnetism, experience indicates that there are topological concepts related to the Field Intensities, $(\mathbf{E}, \mathbf{B})$, and forces, which are thermodynamically distinct from the topological concepts related to Field Excitations $(\mathbf{D}, \mathbf{H})$, and sources. The Field Intensities have functional components which transform as a covariant tensor, while the Field Excitations are quantities with components that transform as a tensor density. These thermodynamic distinctions are often masked by the imposition of a metric structure, or a limitation to volume preserving (often non-dissipative) evolutionary processes [3].

The idea of a deformation invariant comes from the Cartan concept of a tube of trajectories as applied to Hamiltonian mechanics. On the odd dimensional state space of variables, $\{p, q, t\}$, Cartan evaluated the closed integral of a given 1-form of Action, $\oint_C A(p, q, t) = \oint_C pdq - H(p, q, t)dt$, on a curve, C, that encloses a tube of (possible) trajectories. The objective was to determine which trajectories (evolutionary processes) leave the closed integral invariant, as the points of C curve are arbitrarily deformed or transported along the trajectories. The only restriction was that a point of the curve C on a given trajectory stays on the same trajectory. Cartan proved that the direction field that defines such a tube of trajectories is unique, and has the usual classic Hamiltonian representation [4]. Cartan's proof is not restricted to state space, but instead applies to any 1-form of Action whose Pfaff dimension, or class, is odd. Such Action 1-forms always admit a Hamiltonian representation for the evolutionary vector field. Moreover, for arbitrary physical systems that can be defined by a C2 differentiable 1-form of Action, A, by using Cartan's magic formula [5], it is possible to prove:

***Theorem* 1**. *The closed integral of the derived* 2*-form* $F = dA$ *is an evolutionary deformation invariant with respect to all continuous processes that can be defined by a singly parameterized vector field*.

Proof: $L_{(\beta V)} \iint_{closed} F = \iint_{closed} \{i(\beta V)dd F + d(i(\beta V)F)\} \;=\; 0 + 0.$

The integration domain is, in this case, a two dimensional closed two surface, which need not be a boundary. This concept is at the basis of the Helmholtz theorems in hydrodynamics, and the conservation of flux in classical electromagnetism. The necessary condition that a 2-form be an evolutionary deformation invariant for all continuous processes is that the 2-form be closed, $dF = 0$. This requirement is satisfied by the constraint of the exterior differential system, $F - dA = 0$, and C2 differentiability; the 2-form F is said to be exact. The domains of support for an exact 2-form are usually either open (and extend to infinity) or are compact with boundary. The only exceptions are the Torus and the Klein bottle. In this article it is subsumed that classical electromagnetism is defined by an exact 2-form, F. On the other hand, the domain of support for the Field Excitations, G, can be compact without boundary, a fact that leads to the intuitive concepts of particles.

1.1 *The Postulate of Potentials*

Herein, the assumption that classical electromagnetic systems are defined by the topological constraint that the 2-form F is exact will be called the *Postulate of Potentials*. The postulate is an essential point of departure from other theoretical developments, because physical meaning is associated with topological equivalence classes of potentials. When written as the equation, $F - dA = 0$, the postulate of potentials can be recognized as an exterior differential system constraining the topology of the independent variables. When the 2-form F is exact, the Poincare lemma, $ddA = dF = 0$, implies that the partial differential equations so generated form a nested set independent from the dimension of the independent variables. The Maxwell-Faraday equations are therefor universal and applicable to all physical systems that support a 1 form of Action on a domain of four dimensions or more.

The concepts developed in this article subsume that the potentials, A, have physical meaning in a topological sense of equivalence classes. In fact, topological evolution will be observed most often when the potentials evolve from one equivalence class to another. A theory of topological evolution can not be gauge invariant, nor can it make unique predictions. Yet when couched in the language of differential forms, a theory of continuous topological evolution can be retrodictively deterministic [6]. It is the closed components of A that are not exact that determine many of the topological, multiply connected, features of the electromagnetic system.

1.2 *The Postulate of Conserved Currents*

Herein, it is stipulated that the classic electromagnetic system requires a second topological constraint to be imposed upon the domain of independent variables. This postulate will be called the *Postulate of Conserved Currents*. The electromagnetic domain not only supports the 1-form A, but also supports an N-1=3 form, J, which is exact. The equivalent differential system, $J - dG = 0$, requires that the (N-1 dimensional) domain of support for J cannot be compact without boundary. However, using Cartan's magic formula it is possible to prove (see Theorem 1):

> ***Theorem* 2**: *The closed integrals of J are deformation invariants for any continuous evolutionary process that can be defined in terms of a singly parameterized vector field*.

For the 3-forms of charge current, a similar argument indicates that the compact domains of support are limited to those of zero Euler characteristic. The classic example is the three sphere, S^3. The three sphere (that will support currents without zeros) has a famous map to a compact two sphere. Hence, there can exist domains of field excitations on compact two spheres, such that the induced current, $J = dG$, resides on the three sphere. The image is the Hopf map, which can have torsion. Such currents are in the direction of the torsion vector, $A\hat{}dA = A\hat{}F$, and have extraordinary properties, as will be shown below.

In section 2, it will be demonstrated explicitly that the classic formalism of electromagnetism is a consequence of a system of two fundamental topological constraints

$$F - dA = 0, \qquad J - dG = 0. \tag{1}$$

defined on a domain of four independent variables. The theory requires the existence of four fundamental exterior differential forms, $\{A, F, G, J\}$, which form a differential ideal. The elements of the differential ideal can be used to construct the complete Pfaff sequence of forms

$$\textit{Pfaff Sequence} = \{A, F = dA, G, J = dG, A\hat{}F, A\hat{}G, A\hat{}J, F\hat{}F, G\hat{}G\}. \tag{2}$$

by the processes of exterior differentiation and exterior multiplication. On a domain of four independent variables, the complete Pfaff sequence contains three 3-forms: the classic 3-form of charge current density, J, and the (apparently novel to many researchers) 3-forms of Spin Current density, $A\hat{}G$, [7] and Topological Torsion-Helicity, $A\hat{}F$ [8]. For an electromagnetic system, the Action 1-form, A, has the physical dimensions of the flux quantum, h/e. The 2-form, G, has the physical dimensions of charge, e. The 3-form, $A\hat{}G$, has the physical dimensions of spin, h, and the 3-form $A\hat{}F$, has the physical dimensions of spin multiplied by the Hall impedance, $(h/e)^2 = h(h/e^2) = hZ_{hall}$. These last two 3-forms are explicitly dependent upon postulate of potentials, and demonstrate the physical significance of the vector and scalar potentials. [9]

As the charge current 3-form, J, is a deformation invariant by construction, it is of interest to determine topological refinements or constraints for which the 3-forms of Spin Current and Topological Torsion will define physical topological conservation laws in the form of deformation invariants. The additional constraints are equivalent to the topological statement that the closure (exterior derivative) of each of the three forms is empty (zero). It will be demonstrated in section 3 that these closure conditions define the two classic Poincare invariants (4-forms) as deformation invariants, and when each of these invariants vanish the corresponding 3-form generates a topological quantity (Spin or Torsion respectively) which is also a deformation invariant. The possible values of the topological quantities, as deRham period integrals [10], form rational ratios.

The two distinct concepts of Spin Current and the Torsion vector have had almost no utilization in applications of classical electromagnetic theory, for they are explicitly dependent upon the potentials, A. Hence, if the additional constraint of gauge invariance is placed upon the theory of electromagnetism, then the construction of such 3-forms is contrived, if not impossible. The constraint of gauge invariance is NOT subsumed in this article, as an understanding of topological evolution is the desired goal. Just as the vanishing of the 3-form of charge current, $J = 0$, defines the topological domain called the vacuum, the vanishing of the two other 3-forms, $A\hat{}G$ and $A\hat{}F$, will refine the fundamental topology of the Maxwell system. Such constraints permit a definition of transversality to be made on topological (rather than geometrical) grounds. If both $A\hat{}G$ and $A\hat{}F$ vanish, the vacuum state supports topologically transverse modes only (TTEM). A topologically transverse magnetic (TTM) mode corresponds to the topological constraint that $A\hat{}F = 0$. A topologically transverse electric mode (TTE) corresponds to the topological constraint that $A\hat{}G = 0$. Examples, both novel and well-known, of vacuum solutions to the electromagnetic system which satisfy (and which do not satisfy) these topological constraints are given elsewhere [11].

In section 3, evolutionary processes will be studied in terms of the Lie derivative (with respect to a vector field) acting as a propagator on each element of the Pfaff sequence. The evolutionary processes (as vector fields) will be put into equivalence classes according to certain topological refinements that they impose on the physical electromagnetic system, as described by the elements of the Pfaff sequence. For example, a plasma process (which is to be distinguished from a Hamiltonian process) will be restricted to those evolutionary vector fields which leave the closed integrals of G a deformation invariant. (Compare to the Cartan definition that a Hamiltonian process is a restriction on arbitrary processes such that the closed integrals of A are deformation invariants.) A plasma process need not conserve energy. A *perfect* plasma process is a plasma process which is also a Hamiltonian process. Again, the three forms, J, A^G and A^F are of particular interested because their tangent manifolds define direction fields, or "lines", in the 4-dimensional variety of space and time. Relative to plasma processes, the topological evolution associated with such lines, and their entanglements or knots, is of utility in understanding solar corona and plasma instability. [12]

2. The Fundamental Exterior Differential Systems.

The use of differential forms should not be viewed as just another formalism of fancy. The technique goes beyond the methods of tensor calculus, and admits the study of topological evolution. Recall that if an exterior differential system is valid on a final variety of independent variables $\{x, y, z, t\}$, then it is also true on any initial variety of independent variables that can be mapped onto $\{x, y, z, t\}$. The map need only be differentiable, such that the Jacobian matrix elements are well defined *functions*. The Jacobian matrix does not have to have an inverse, so that the exterior differential system is not restricted to the equivalence class of diffeomorphisms. The field intensities on the initial variety are functionally well defined by the pullback mechanism, which involves algebraic composition with components of the Jacobian matrix transpose, and the process of functional substitution. This independence from a choice of independent variables (or coordinates) for Maxwell's equations was first reported by Van Dantzig [13]. It follows that the Maxwell differential system is well defined in a covariant manner for both Galilean transformations as well as Lorentz transformations, or any other diffeomorphism. (The singular solution sets to the equations do not enjoy this universal property [14]).

2.1 *The Maxwell-Faraday exterior differential system*.

The Maxwell-Faraday equations are a consequence of the exterior differential system

$$F - dA = 0, \tag{3}$$

where A is a 1-form of Action, with twice differentiable coefficients (potentials proportional to momenta) which induce a 2-form, $F,$ of electromagnetic intensities ($\mathbf{E}$ and $\mathbf{B}$, related to forces). The exterior differential system is a topological constraint that in effect defines field intensities in terms of the potentials. On a four dimensional space-time of independent variables, (x, y, z, t) the 1-form of Action (representing the postulate of potentials) can be written in the form

$$A = \Sigma_{k=1}^{3} A_k(x,y,z,t)dx^k - \phi(x,y,z,t)dt = \mathbf{A} \circ d\mathbf{r} - \phi dt. \qquad (4)$$

Subject to the constraint of the exterior differential system, the 2-form of field intensities, F, becomes:

$$F = dA = \{\partial A_k/\partial x^j - \partial A_j/\partial x^k\}dx^j\hat{}dx^k \qquad (5)$$
$$= F_{jk}dx^j\hat{}dx^k = \mathbf{B}_z dx\hat{}dy... + \mathbf{E}_x dx\hat{}dt...$$

where in usual engineering notation,

$$\mathbf{E} = -\partial\mathbf{A}/\partial t - grad\phi, \qquad \mathbf{B} = curl\ \mathbf{A} \equiv \partial A_k/\partial x^j - \partial A_j/\partial x^k. \qquad (6)$$

The closure of the exterior differential system, $dF = 0$,

$$dF = ddA = \{curl\ \mathbf{E} + \partial\mathbf{B}/\partial t\}_x dy\hat{}dz\hat{}dt - .. + .. - div\,\mathbf{B}dx\hat{}dy\hat{}dz\} \Rightarrow 0, \qquad (7)$$

generates the Maxwell-Faraday partial differential equations.:

$$\{curl\ \mathbf{E} + \partial\mathbf{B}/\partial t = 0, \quad div\,\mathbf{B} = 0\}. \qquad (8)$$

The component functions ($\mathbf{E}$ and $\mathbf{B}$) of the 2-form, F, transform as a covariant tensor of rank 2. The topological constraint that F is exact, implies that the domain of support for the field intensities cannot be compact without boundary, unless the Euler characteristic vanishes. These facts distinguish classical electromagnetism from Yang-Mills field theories. Moreover, the fact that F is subsumed to be exact and C1 differentiable excludes the concept of magnetic monopoles from classical electromagnetic theory on topological grounds. By Theorem 1, the integral of the 2-form F over any closed 2-manifold is a deformation (topological) invariant of any evolutionary process that can be described by a singly parameterized vector field.

2.2 *The Maxwell Ampere exterior differential system*

The Maxwell Ampere equations are a consequence of a second exterior differential system,

$$J - dG = 0, \qquad (9)$$

where G is an N-2 form *density* of field excitations ($\mathbf{D}$ and $\mathbf{H}$, related to sources), and J is the N-1 form of charge-current densities. The partial differential equations equivalent to the exterior differential system are precisely the Maxwell-Ampere equations. This second postulate, the Postulate of Conserved Currents, on a four dimensional domain of independent variables, assumes the existence of a N-2 form density given by the expression,

$$G = G^{34}(x,y,z,t)dx\wedge dy.. + G^{12}(x,y,z,t)dz\wedge dt... = \mathbf{D}^z dx\wedge dy.. + \mathbf{H}^z dz\wedge dt... \qquad (10)$$

Exterior differentiation produces an N-1 form,

$$J = \mathbf{J}^z(x,y,z,t)dx\wedge dy\wedge dt... - \rho(x,y,z,t)dx\wedge dy\wedge dz. \qquad (11)$$

Matching the coefficients of the exterior expression $dG = J$ leads to the Maxwell-Ampere partial differential equations,

$$curl\,\mathbf{H} - \partial\mathbf{D}/\partial t = \mathbf{J} \quad and \quad div\,\mathbf{D} = \rho. \qquad (12)$$

The fact that J is exact leads to the charge conservation law, $dJ = ddG = 0$, or

$$\partial\mathbf{J}^x/\partial x + \partial\mathbf{J}^y/\partial y + \partial\mathbf{J}^z/\partial z + \partial\rho/\partial t = 0. \qquad (13)$$

From Theorem 2, the closed integral $\iiint_{closed} J$ is a topological deformation invariant of any process that can be described by a singly parameterized vector field. Note that the domains of support for G (but not F) can be compact without boundary.

2.3 *The Torsion and Spin* 3-*forms*

As mentioned above, the method of exterior differential forms goes beyond the domain of classical tensor analysis, for it admits of maps from initial to final state that are without inverse. (Tensor analysis and coordinate transformations require that the Jacobian map from initial to final state has an inverse - the method of exterior differential forms does not.) Hence the theory of electromagnetism expressed in the language of exterior differential forms admits of topological evolution, at least with respect to continuous processes without Jacobian inverse. With respect to such non-invertible maps, both tensor fields and differential forms are not functionally well defined in a predictive sense [15]. Given the functional forms of a tensor field on an initial state, it is impossible to predict uniquely the functional form of the tensor field on the final state unless the map between initial and final state is invertible. However differential forms are functionally well defined in a retrodictive sense, by means of the pullback. Covariant anti-symmetric tensor fields pull back retrodictively with respect to the transpose of the Jacobian matrix (of functions) and functional substitution, and contravariant tensor densities pullback retrodictively with respect to the adjoint of the Jacobian matrix, and functional substitution. The transpose and the adjoint of the Jacobian exist, even if the Jacobian inverse does not.

The differential forms that make up the complete Pfaff sequence, and their unions, may be used to form a topological base on the domain of independent variables. The Cartan topology constructed on this system of forms has the useful feature that the exterior derivative may be interpreted as a limit point, or closure, operator in the sense of Kuratowski [16]. The exterior differential systems that define the Maxwell-Ampere and the Maxwell-Faraday equations above are essentially topological constraints of closure. Note that the complete Maxwell system of differential forms (which assumes the existence of A) also generates two other exterior differential systems.

$$d(A\hat{}G) - (F\hat{}G - A\hat{}J) = 0, \tag{14}$$

$$d(A\hat{}F) - F\hat{}F = 0. \tag{15}$$

The two objects, $A\hat{}G$ and $A\hat{}F$ are three forms, not usually found in discussions of classical electromagnetism. The closed components of the first 3-form (density) were called topological spin [17] and the closed components of the second 3-form were called topological torsion (or helicity or chirality) [18]. By direct evaluation of the exterior product, and on a domain of 4 independent variables, each 3-form will have 4 components that can be symbolized by the 4-vector arrays

$$Spin-Current\ :\ \mathbf{S}_4 = [\mathbf{A}\times\mathbf{H}+\mathbf{D}\phi, \mathbf{A}\circ\mathbf{D}] \equiv [\mathbf{S}, \sigma], \tag{16}$$

$$Torsion-vector\ :\ \mathbf{T}_4 = [\mathbf{E}\times\mathbf{A}+\mathbf{B}\phi, \mathbf{A}\circ\mathbf{B}] \equiv [\mathbf{T}, h], \tag{17}$$

which are to be compared with the charge current 4-vector density:

$$\text{Charge-Current}\ :\ \mathbf{J}_4 = [\mathbf{J}, \rho], \tag{18}$$

The 3-forms then can be defined by the equivalent contraction processes

$$Topological\ Spin\ 3-form \doteq A\hat{}G \tag{19}$$
$$= i(\mathbf{S}_4)dx\hat{}dy\hat{}dz\hat{}dt = \mathbf{S}^x dy\hat{}dz\hat{}dt..... - \sigma dx\hat{}dy\hat{}dz$$

$$Topological\ Torsion-helicity\ 3-form \doteq A\hat{}F \tag{20}$$
$$= i(\mathbf{T}_4)dx\hat{}dy\hat{}dz\hat{}dt = \mathbf{T}^x dy\hat{}dz\hat{}dt..... - h dx\hat{}dy\hat{}dz.$$

The vanishing of $A\hat{}G$ is a topological constraint on the domain that defines topologically transverse electric (TTE) waves: the vector potential, $\mathbf{A}$, is orthogonal to $\mathbf{D}$, in the sense that $\mathbf{A}\circ\mathbf{D} = 0$. The vanishing of $A\hat{}F$ is a topological constraint on the domain that defines topologically transverse magnetic (TTM) waves: the vector potential, $\mathbf{A}$, is orthogonal to

$\mathbf{B},$ in the sense that $\mathbf{A} \circ \mathbf{B} = 0.$ When both 3-forms vanish, the topological constraint on the domain defines topologically transverse (TTEM) waves. For classic real fields this double constraint would require that the vector potential, $\mathbf{A},$ is collinear with the field momentum, $\mathbf{D} \times \mathbf{B},$ and in the direction of the wave vector, $\mathbf{k}.$ Such constraints permit the definition of singular solutions of propagating discontinuities, or electromagnetic "signals" [14].

The geometric notion of distinct transversality modes of electromagnetic waves is a well known concept experimentally, but the association of transversality to topological issues is novel herein. For certain examples it is apparent that the concepts of geometric transversality and topological transversality are the same. In the classic case, often considered in fiber optic theory of wave guides with open boundaries, it is known that the TEM modes do not transmit power. However, it is possible to construct vacuum wave solutions which satisfy the geometric concept of transversality, but the mode radiates, because it does not satisfy the topological concept of transversality. The conjecture obtained from examples is that a TTEM solution does not radiate.

Note that if the 2-form F was not exact, such topological concepts of transversality would be without distinct meaning, for the 3-forms of Topological Spin and Topological Torsion depend explicitly upon the existence of the 1-form of Action. For future developments, observe that the torsion vector $\mathbf{T}_4$ and the Spin vector $\mathbf{S}_4$ are associated vectors to the 1-form of Action, in the sense that

$$i(\mathbf{T}_4)A = 0 \quad and \quad i(\mathbf{S}_4)A = 0. \tag{21}$$

2.4 *The Poincare Invariants*

The exterior derivatives of the 3-forms of Spin and Torsion produce two 4-forms, $F\hat{}G - A\hat{}J$ and $F\hat{}F,$ whose closed integrals are deformation invariants for the Maxwell system. (The deformation invariance follows from Cartan's magic formula and the fact that the 4-forms are exact). These topological objects are related to the conformal invariants of a Lorentz system as discovered by Poincare and Bateman [19]. Note that their topological properties are valid even in the plasma domain of dissipative charge currents and radiation, as well as in the vacuum. In the format of independent variables $\{x, y, z, t\},$ the exterior derivative corresponds to the 4-divergence of the 4-component Spin and Torsion vectors, $\mathbf{S}_4$ and $\mathbf{T}_4.$

$$Poincare\ 1 = d(A \hat{} G) = F \hat{} G - A \hat{} J \tag{22}$$
$$= \{div_3(\mathbf{A} \times \mathbf{H} + \mathbf{D}\phi) + \partial(\mathbf{A} \circ \mathbf{D})/\partial t\} dx \hat{} dy \hat{} dz \hat{} dt$$
$$= \{(\mathbf{B} \circ \mathbf{H} - \mathbf{D} \circ \mathbf{E}) - (\mathbf{A} \circ \mathbf{J} - \rho\phi)\} dx \hat{} dy \hat{} dz \hat{} dt$$

$$Poincare\ 2 = d(A \hat{} F) = F \hat{} F \tag{23}$$
$$= \{div_3(\mathbf{E} \times \mathbf{A} + \mathbf{B}\phi) + \partial(\mathbf{A} \circ \mathbf{B})/\partial t\} dx \hat{} dy \hat{} dz \hat{} dt$$
$$= \{-2\mathbf{E} \circ \mathbf{B}\} dx \hat{} dy \hat{} dz \hat{} dt$$

For the vacuum state, defined by $J = 0$, zero values of the Poincare invariants require that the magnetic energy density is equal to the electric energy density ($1/2\mathbf{B} \circ \mathbf{H} = 1/2\mathbf{D} \circ \mathbf{E}$), and, respectively, that the electric field is orthogonal to the magnetic field ($\mathbf{E} \circ \mathbf{B} = 0$). Note that these constraints often are used as elementary textbook definitions of what is meant by electromagnetic waves. Consider the definitions:

***Definition* 3**: *Spin is defined as the closed integral of the* 3*-form* $A \hat{} G$

$$Spin = \iiint_{closed} A \hat{} G \tag{24}$$

***Definition* 4**: *Torsion-Helicity is defined as the closed integral of the* 3*-form* $A \hat{} F$

$$Torsion\text{-}Helicity = \iiint_{closed} A \hat{} F \tag{25}$$

By using Cartan's magic formula it is possible to prove

***Theorem* 3**: *If the First Poincare Invariant vanishes, the Spin is an evolutionary deformation invariant with values whose ratios are rational.*

***Theorem* 4**. *If the second Poincare Invariant vanishes, the Torsion Helicity is an evolutionary deformation invariant with values whose ratios are rational.*

The quantized (integer) ratios comes from the deRham cohomology theorems on closed integrals of closed p-forms.

It is important to realize that these topological conservation laws are valid in a plasma as well as in the vacuum, subject to the conditions of zero values for the Poincare invariants.

On the other hand, topological transitions between ”quantized” states of Spin or Torsion require that the respective Poincare invariants are not zero.

3. Thermodynamics

3.1 *Topological Thermodynamics and Irreversibility*

The basic tool for studying topological evolution is Cartan's magic formula [5], in which it is presumed that a physical system can be described adequately by a 1-form of Action, A, and that a physical process can be represented by the direction field of a contravariant vector field, $\mathbf{V}$. The application of Cartan's magic formula yields

$$L_{(\mathbf{V})}\int A = \int L_{(\mathbf{V})}A = \int\{i(\mathbf{V})dA + d(i(\mathbf{V})A)\} \;\; = \int\{W + d(U)\} \;\; = \int Q. \qquad (26)$$

The basic idea behind this formalism (which is at the foundation of the Cartan-Hilbert variational principle) is that the postulate of potentials is valid: $F - dA = 0$. The base manifold will be the 4-dimensional variety $\{x,y,z,t\}$ of engineering practice, but no metrical features are presumed a priori. If relative to the process, V, the RHS is zero, $\int Q \Rightarrow 0$, then $\int A$ is said to be an integral invariant of the evolution generated by $\mathbf{V}$. In thermodynamics such processes are said to be adiabatic.

From the point of view of differential topology, the key idea is that the Pfaff dimension, or class [20], of the 1-form of Action specifies topological properties of the system. Given the Action 1-form, A, the Pfaff sequence, $\{A, dA, A\hat{}dA, dA\hat{}dA, ...\}$ will terminate at an integer number of terms equal to, or less than, the number of dimensions of the domain of definition. On a 4 dimensional domain, the top Pfaffian, $dA\hat{}dA$, will define a volume element with a density function whose singular zero set (if it exists) reduces the symplectic domain to a contact manifold of Pfaff dimension 3. This (defect) contact manifold supports a unique extremal field that leaves the Action integral ”stationary”, and leads to the Hamiltonian conservative representation for the Euler flow in hydrodynamics. The irreversible regime will be on an irreducible symplectic manifold of Pfaff dimension 4, where $dA\hat{}dA \neq 0$. Topological defects (or coherent structures) appear as singularities of lesser Pfaff (topological) dimension, $dA\hat{}dA = 0$.

3.2 *Reversible Processes*

Classical hydrodynamic and electromagnetic processes can be represented by certain nested categories of vector fields, $\mathbf{V}$. Three distinct classes of processes are defined by the constraints on the evolutionary vector field such that

$$Extremal - (unique\ Hamiltonian\ generator) \qquad i(\mathbf{V})dA = 0 \qquad 27a)$$
$$Bernoulli - Casimir - (\Theta\ is\ the\ generator) \qquad i(\mathbf{V})dA = d\Theta \qquad (27b)$$
$$Helmholtz - Stokes - -Symplectic \qquad di(\mathbf{V})dA = 0. \qquad (27c)$$

The vector fields defined by first two constraints have generators that create a Hamiltonian flow. This Hamiltonian flow is uniquely defined, in the extremal case, on a contact manifold of odd dimensions, as the null eigen vector of the matrix of coefficients of the 2-form, dA. In the $Bernoulli - Casimir$ (maximal rank) case, the manifold is of even dimensions and symplectic, so there does not exist a unique null eigenvector direction field. The evolutionary field depends upon the choice of the $Bernoulli - Casimir$ function Θ. Such Bernoulli processes can correspond to energy dissipative symplectic processes, but they, as well as all symplectic processes, are reversible in the thermodynamic sense described below. The mechanical energy need not be constant, but the Bernoulli-Casimir function(s), Θ, are evolutionary invariant(s), and may be used to describe non-unique ”stationary” thermodynamic state(s).

The equations, above, are in effect constraints on the topological evolution of any physical system represented by an Action 1-form, A. The Pfaff dimension of the 1-form of virtual work, $W = i(\mathbf{V})dA$, is 2 or less for each of the three categories. The extremal constraint can be used to generate the Euler equations of hydrodynamics for a incompressible fluid. The Bernoulli-Casimir constraint can be used to generate the equations for a barotropic compressible fluid. The Helmholtz constraint can be used to generate the equations for a Stokes (not Navier-Stokes) flow. However as will be shown below, all such processes are thermodynamically reversible.

An important idea is that it takes domains of Pfaff dimension 3, or more, with attendant properties of non-uniqueness, envelopes, regressions, and projectivized tangent bundles, to yield the concepts of Spin and Torsion-Helicity. It takes systems of Pfaff dimension 4 to accommodate processes which are thermodynamically irreversible.

3.3 *Irreversible Processes*

Although there does not exist a unique extremal process on a symplectic manifold of Pfaff dimension 4, remarkably there does exist a unique (conformal) vector field whose direction field depends only upon the functional form of the 1-form, A, that is used to define the physical system. The direction field on the four dimensional domain is defined by the 3-form of topological torsion, $A\hat{}dA$, as discussed in Section 2.3. This unique (to within a factor) vector field is defined in component form as the Torsion Current, $\mathbf{T}_4$, and satisfies (on the 4 dimensional manifold) the equation,

$$i(\mathbf{T}_4)dx\hat{}dy\hat{}dz\hat{}dt = A\hat{}dA \tag{28}$$

This (four component) vector field, $\mathbf{T}_4$, has a non-zero divergence almost everywhere, for if the divergence is zero, then the 4-form $dA\hat{}dA$ vanishes, and the domain is no longer a symplectic 4-manifold! The Torsion vector, $\mathbf{T}_4$, can be used to generate a dynamical system that will decay to the stationary states ($div_4(\mathbf{T}_4) \Rightarrow 0$) starting from arbitrary initial conditions. As shown below these processes are irreversible in the thermodynamic sense. It is remarkable that this unique evolutionary vector field, $\mathbf{T}_4$, is completely determined (to within a factor) by the physical system itself; e.g., the components of the 1-form, A, determine the direction field of the Torsion vector.

To understand what is meant by thermodynamic irreversibility, realize that Cartan's magic formula of topological evolution is equivalent to the first law of thermodynamics.

$$L_{(\mathbf{v})}A = i(\mathbf{V})dA + d(i(\mathbf{V})A) = W + dU = Q. \tag{29}$$

A is the "Action" 1-form that describes the hydrodynamic or electromagnetic system. $\mathbf{V}$ is the vector field that defines the evolutionary process. W is the 1-form of (virtual) work. Q is the 1-form of heat. From classical thermodynamics, a process is irreversible when the heat 1-form Q does not admit an integrating factor.

> ***Definition* 5**: *An irreversible (non-equilibrium) process is one for which the Heat 1-form Q does not admit an integrating factor* [21].

From the Frobenius theorem, the lack of an integrating factor implies that $Q\hat{}dQ \neq 0$. Hence a simple (non-statistical) test may be made for any process, $\mathbf{V}$, relative to a physical system described by an Action 1-form, A:

> ***Theorem* 5**:
>
> $$\text{If } L_{(v)}A\hat{}L_{(v)}dA \neq 0 \text{ then the process is irreversible.} \tag{30}$$
>
> *Proof*: *Using Cartan's magic formula yields $L_{(v)}A = Q$ and $L_{(v)}dA = dQ$. Hence the requirement that an integrating factor does not exist is $Q\hat{}dQ \Rightarrow L_{(v)}A\hat{}L_{(v)}dA \neq 0$.*

This topological definition implies that the three categories (above) of Symplectic $\supset$ Hamiltonian $\supset$ Extremal evolutionary processes, $\mathbf{S}$, are reversible in a thermodynamic sense (as $L_{(\mathbf{S})}dA = dQ = 0$).

However, for evolution in the direction of the Torsion vector, $\mathbf{T}_4$, direct computation demonstrates that the fundamental equations lead to a conformal evolutionary process:

$$L_{(\mathbf{T}_4)}A = \sigma A \quad \textit{and} \quad i(\mathbf{T}_4)A = 0 \qquad (31)$$
$$\textit{with} \quad \sigma \sim div_4(\mathbf{T}_4) \sim d(A\hat{}dA)$$

***Theorem* 6**: *Evolution in the direction of the Torsion Vector is irreversible.*

Proof: *The direction field associated with* $\boldsymbol{T}_4$ *is uniquely determined by the functional form of the* 1*-form of Action that defines the physical system on the four dimensional variety. By direct evaluation,*

$$\begin{aligned} L_{(\boldsymbol{T}_4)}A\hat{}L_{(\boldsymbol{T}_4)}dA &= Q\hat{}dQ = \sigma A\hat{}(\sigma dA + d\sigma\hat{}A) \qquad (32)\\ &= \sigma^2 A\hat{}dA = \{div_4(\boldsymbol{T}_4)\}^2 A\hat{}dA. \end{aligned}$$

As the domain is of Pfaff dimension 4, *it follows that* $A\hat{}dA$ *is not zero, and* $dA\hat{}dA \sim div_4(\boldsymbol{T}_4)$ *is not zero. Therefore, the RHS is not zero, and the irreversibility result follows from theorem* 5.

Explicit evaluations are carried out in the next section for electromagnetic systems of Pfaff dimension 4.

3.4 *Applications to Electromagnetism*

All of the development above will carry over to the electromagnetic system, which also assumes the postulate of potentials. The topological torsion 3-form, $A\hat{}dA$, induces the torsion current

$$\mathbf{T}_4 = \{(\mathbf{E} \times \mathbf{A} + \mathbf{B}\phi); \mathbf{A} \circ \mathbf{B}\} \equiv \{\mathbf{S}, h\}. \qquad (33)$$

If $div_4\mathbf{T} = -2\,\mathbf{E} \circ \mathbf{B} \neq 0$, the electromagnetic 1-form, A, defines a domain of Pfaff dimension 4. Such domains cannot support topologically transverse magnetic waves (*as* $A\hat{}F \neq 0$). Evolutionary processes (including plasma currents) that are proportional to the Torsion current are thermodynamically irreversible, if $\sigma = -\mathbf{E} \circ \mathbf{B} \neq 0$. However, the conformal properties of evolution in the direction of the Torsion current lead to extraordinary properties when the plasma current is in the direction of the Torsion vector. Using the notation of an electromagnetic system

$$L_{(\mathbf{T}_4)}A = \sigma A = -(\mathbf{E} \circ \mathbf{B})A \quad and \quad L_{(\mathbf{T}_4)}(A\hat{}F) = 2\sigma A = -2(\mathbf{E} \circ \mathbf{B})A\hat{}F. \tag{34}$$

Hence, it follows that motion along the direction of the torsion vector freezes-in the lines of the torsion vector in space time, but the process is irreversible unless the second Poincare invariant is zero.

Recall that the definition of a plasma current, J, is equivalent to an evolutionary process such that the number of charges is an evolutionary invariant, $L_{(J)}G = 0$. Consider a plasma current which is also in the direction of the Torsion vector. Then

$$L_{(J)}A\hat{}G = (L_{(J)}A)\hat{}G + A\hat{}L_{(J)}G = (L_{(\gamma\mathbf{T}_4)}A)\hat{}G + 0 = -\gamma \bullet (\mathbf{E} \circ \mathbf{B})\ A\hat{}G \tag{35}$$

Hence for plasma motions in the direction of the (possibly dissipative) torsion vector, both the ”lines” of the Spin vector are ”frozen in” and the lines of the Torsion vector are ”frozen in”. Such ”frozen in” objects [22] can be used to give a topological definition of deformable coherent structures in a plasma. Moreover, as the evolutionary process causes the ”frozen in” structures to deform and decay, it is conceivable that evolution could proceed to form stationary (not stagnant) states (where $\mathbf{E} \circ \mathbf{B} \Rightarrow 0$), such that the ”frozen in” field line structures become local deformation invariants, or topological defects. Electromagnetic coherent structures are evolutionary deformable (and perhaps decaying) domains of Pfaff dimension 4, which form stationary states of topological defects (including the null state) in regions of Pfaff dimension 3, where $\mathbf{E} \circ \mathbf{B} = 0$.

4. The Classical Photon

An empirical property of the photon is its quantized spin. As it is not commonly appreciated that the classical Maxwell electromagnetic field can have quantized spin an torsion properties, three examples are presented below which demonstrate spin and torsion properties of the field. In the first example, the spin radiation is related to the back reaction of a Lorentz force on a rotating plasma, to produce an accretion disk. The first example has zero torsion but finite spin. In the second example, a time dependent vacuum wave solution is presented. The vector potentials exhibit a lack of time reversal invariance, and the fields are not transverse, as $\mathbf{E} \circ \mathbf{B} \neq 0$. The second example has both finite torsion and finite spin. A third example (without torsion) demonstrates a solution where the Poynting vector is in the direction of the Spin vector, indicating that radiation is related to the spin qualities of the photon. Moreover, in the third example, the closed integrals of the Spin 3-form are quantized in the sense that their values have integer ratios.

4.1 *A rotating plasma with an accretion disk*

A very interesting time independent set of functions that satisfy the Maxwell system is given by the Potentials,

$$\mathbf{A} = \{f(x,y,z,t)\}Kelvin = \{z/\sqrt{(\delta z^2 + x^2 + y^2)}\}[-y,x,0]/(x^2+y^2) \tag{36}$$

which generates the Hedge-Hog B field,

$$\mathbf{B} = -[x,y,z]/(\delta z^2 + x^2 + y^2)^{3/2}, \tag{37}$$

and (assuming the Lorentz constitutive relations) the London-like Current density,

$$\mathbf{J} = \Lambda(x,y,z,t)\mathbf{A} = -3(1-\delta)(x^2+y^2)/(\delta z^2 + x^2 + y^2)^2\mathbf{A}. \tag{38}$$

It is apparent that the Helicity density $\mathbf{A} \circ \mathbf{B}$ vanishes identically, as do all components of the Topological Torsion tensor: $A\hat{}F \Rightarrow 0$. There exists a Lorentz force,

$$\mathbf{J} \times \mathbf{B} = \{3(1-\delta)/(\delta z^2 + x^2 + y^2)^4\}[yz^2, xz^2, -z(x^2+y^2)]/\mu \tag{39}$$

which has the remarkable features that for an "oblate" situation ($\delta < 1$), the "plasma" is forced away from the rotation z axis, but is attracted to the $z = 0$ plane to form an accretion disk. There are no Amperian currents and no Lorentz force unless the system is anisotropic (oblate or prolate).

When the 3-form of topological Spin is evaluated, $A\hat{}G \neq 0$, it is remarkable that the Lorentz force is proportional to the Spin current

$$\mathbf{J} \times \mathbf{B} = -3(1-\delta)\mathbf{S}/(x^2+y^2). \tag{40}$$

In this example, the spin current is to be considered as a back reaction to the Lorentz force. The usual interpretation in MHD theory is to resolve $\mathbf{J} \times \mathbf{B}$ in terms of a "magnetic pressure and a magnetic tension":

$$\mathbf{J} \times \mathbf{B} = -\nabla(\mathbf{B} \circ \mathbf{H})/2 + \mathbf{H} \circ \nabla\mathbf{B} = \text{"}\textit{magnetic pressure} + \textit{magnetic tension}\text{"}. \tag{41}$$

However, the Spin interpretation would imply that the effects are more related to rotational deformations, rather than to translational deformations.

4.2 *A time dependent irreversible vacuum wave with* $\boldsymbol{E} \circ \boldsymbol{B} \neq \mathbf{0}$.

Modifications of the Hopf map suggest consideration of the system of potentials given by the equations

$$\mathbf{A} = [+y, -x, +ct]/\lambda^4 \ , \quad \phi = cz/\lambda^4, \quad where\ \lambda^2 = -c^2t^2 + x^2 + y^2 + z^2, \tag{42}$$

which yield the real field intensities,

$$\mathbf{E} = [-2(cty - xz), +2(ctx + yz), -(c^2t^2 + x^2 + y^2 - z^2)]2c/\lambda^6 \tag{43}$$

$$\mathbf{B} = [-2(cty + xz), +2(ctx - yz), -(c^2t^2 + x^2 + y^2 - z^2)]2/\lambda^6. \tag{44}$$

Subject to the dispersion relation, $\varepsilon\mu c^2 = 1$ and the Lorentz constitutive conditions, these time dependent wave functions satisfy the homogeneous Maxwell equations without charge currents, and are therefore acceptable vacuum solutions, $\mathbf{J}_4(+t) = 0$.

The extensive algebra involved in these and other computations in this article were checked with a Maple symbolic mathematics program [23].

It is to be noted that when the substitution $t \Rightarrow -t$ is made in the functional forms for the potentials, the fields computed from the new functional forms fail to satisfy the vacuum Lorentz conditions for zero charge-currents, $\mathbf{J}_4(-t) \neq 0$. For example, the $\mathbf{E}_{(-t)}$ field calculated from the potentials $\mathbf{A}_4(-t)$ is not equal to the $\mathbf{E}_{(+t)}$ field computed from $\mathbf{A}_4(+t)$ and exhibits a non-zero divergence; *e.g.*,

$$\mathbf{E}_{(-t)} = [-2(cty - xz), +2(ctx + yz), (c^2t^2 + z^2)]2c/\lambda^6 \tag{45}$$

$$\rho/\varepsilon = div_3\{\mathbf{E}_{(-t)}\} = -8cz(x^2 + 5c^2t^2 + y^2 + z^2)/\lambda^8. \tag{46}$$

In this sense, the valid vacuum ($+t$) solution presented above is not time-reversal invariant. (Of course the differential form for the Action transforms properly).

The Spin current density for this first non-transverse wave ($+t$) example is evaluated as:

$$\begin{aligned} Spin : \mathbf{S}_4 = [&x(3\lambda^2 - 4y^2 - 4x^2), y(3\lambda^2 - 4y^2 - 4x^2), z(\lambda^2 - 4y^2 - 4x^2), \\ &t(\lambda^2 - 4y^2 - 4x^2)](2/\mu)/\lambda^{10}, \end{aligned} \tag{47}$$

and has zero divergence. Hence its global integral (Spin) is quantized. The Torsion current may be evaluated and leads to:

$$Torsion : \mathbf{T}_4 = -[x, y, z, t]2c/\lambda^8 \qquad Poincare\ 2 = -2\mathbf{E} \circ \mathbf{B} = +8c/\lambda^8. \tag{48}$$

The solution has magnetic helicity as $\mathbf{A} \circ \mathbf{B} \neq 0$ and is radiative in the sense that the Poynting vector, $\mathbf{E} \times \mathbf{H} \neq 0$, and the wave is not transverse. The Torsion_Helicity integral is not quantized.

It is to be noted that the example solution given above is but one of a class of vacuum wave solutions that have similar non transverse properties. As a second example, consider the fields that can be constructed from the potentials,

$$\mathbf{A} = [+ct, -z, +y]/\lambda^4 , \quad \phi = cx/\lambda^4, \quad where\ \lambda^2 = -c^2t^2 + x^2 + y^2 + z^2. \tag{49}$$

These potentials will generate the field intensities,

$$\mathbf{E} = [+(-c^2t^2 + x^2 - y^2 - z^2), +2(ctz + yx), -2(cty - zx)]2c/\lambda^6 \tag{50}$$

$$\mathbf{B} = [+(-c^2t^2 + x^2 - y^2 - z^2), +2(-ctz + yx), +2(cty + zx)]2/\lambda^6. \tag{51}$$

As before, these fields satisfy the Maxwell-Faraday equations, and the associated excitations satisfy the Maxwell-Ampere equations without producing a charge current 4-vector. However, it follows by direct computation that the second Poincare invariant, and the Torsion 4-vector are of opposite signs to the values computed for the first example:

$$Torsion : \mathbf{T}_4 = +[x, y, z, t]2c/\lambda^8 , \qquad -2\mathbf{E} \circ \mathbf{B} = -8c/\lambda^8 . \tag{52}$$

4.3 *Quantized Spin and the Poynting vector*

When the two examples above are combined by addition (or subtraction), the resulting wave is transverse magnetic (in the topological sense that $\mathbf{A} \circ \mathbf{B} = 0$). Not only does the second Poincare invariant vanish under superposition, but so also does the Torsion 4 vector. Conversely, the examples above show that there can exist transverse magnetic waves which can be decomposed into two non-transverse waves. A notable feature of the superposed solutions is that the Spin 4 vector current does not vanish, hence the example superposition is a wave that is not transverse electric (in the topological sense that $\mathbf{A} \circ \mathbf{D} \neq 0$). For the superposed example, the first Poincare invariant vanishes, which implies that the Spin integral remains a conserved topological quantity, with values proportional to the integers. The non-zero Spin current density for the combined examples is given by the formula:

$$\begin{aligned} Spin : \mathbf{S}_4 = [&-2x(y + ct)^2, (y + ct)(x^2 - y^2 + z^2 - 2cty - c^2t^2), -2z(y + ct)^2, \\ &-(y + ct)(x^2 + y^2 + z^2 + 2cty + c^2t^2)](4/\mu)/\lambda^{10}, \end{aligned} \tag{53}$$

while the Torsion current is a zero vector, $A \hat{} F \Rightarrow 0$.

In addition, for the superposed example, the spatial components of the Poynting vector are equal to the Spin current density vector multiplied by γ, such that

$$\mathbf{E} \times \mathbf{H} = \gamma \mathbf{S}, \quad with\ \gamma = -(x^2 + y^2 + z^2 + 2cty + c^2t^2)/2c(y + ct)\lambda^2. \tag{54}$$

These results seem to give classical credence to the Planck assumption that the vacuum state of Maxwell's electrodynamics supports quantized angular momentum, and that the energy flux must come in multiples of the spin quanta. In other words, these combined solutions to classical electrodynamics have some of the experimental qualities of the quantized photon.

5. Summary

Many of the chiral and spin features of the quantized photon have their basis in the topological properties of classical electromagnetism, and the 3-forms $A\hat{}F$ and $A\hat{}G$.

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