TOPOLOGICAL EVOLUTION OF CLASSICAL ELECTROMAGNETIC FIELDS AND THE PHOTON

R. M. KIEHN
Physics Department, University of Houston,
Houston, TX, 77004, *USA,*
E-mail: *jjkrmk@swbell.net*

The theory of classical electromagnetism is constructed in terms of two exterior differential systems, $F - dA = 0$, and $J - dG = 0$, which act as topological constraints on the variety of independent variables $\{x, y, z, t\}$. These two fundamental constraints lead to two other independent concepts of topological torsion, $A\^F$, and topological spin, $A\^G$, which are explicitly dependent upon the potentials, A. The exterior derivative of these 3-forms creates the two familiar Poincare deformation invariants of an electromagnetic system, valid in the vacuum or plasma state. When the Poincare invariants vanish, the closed integrals of $A\^F$ and $A\^G$ exhibit topological invariant properties similar to the "quantized" chiral and spin properties of a photon. The possible evolution of these and other topological properties is studied with respect to classes of processes that can be defined in terms of singly parameterized vector fields. Non-zero values of the Poincare invariants are the source of topological change and non-equilibrium thermodynamics.

Introduction

In the language of exterior differential systems [1] it becomes evident that classical electromagnetism is equivalent to a set of topological constraints on a variety of independent variables. Certain integral properties of such an electromagnetic system are deformation invariants with respect to *all* continuous evolutionary processes that can be described by a singly parameterized vector field. These deformation invariants (or topological properties) lead to the fundamental topological conservation laws described in the physical literature as the conservation of charge-current and the conservation of flux. The object of this article is to examine other objects that can behave as deformation invariants relative to certain equivalence classes of continuous processes, and yet exhibit topological change with respect to other classes of continuous processes. Recall the definitions:

Definition 1: *A continuous process is a map from an initial state with a topology $T_{initial}$ into a final state with perhaps a different topology T_{final} such that the limit points of the initial state are permuted among the limit points of the final state.* [2]

Definition 2: *A deformation invariant is an integral over a closed manifold $\int \bullet \bullet \int_{closed} \omega$ such that the Lie derivative of the closed integral with respect to a singly parameterized vector field, βV^k, vanishes, for any choice of the parametrization function, β.*

It is also important to recall that a given variety of independent variables can support more than one topology. In classical electromagnetism, experience indicates that there are topological concepts related to the Field Intensities, $(\mathbf{E},\mathbf{B})$, and forces, which are thermodynamically distinct from the topological concepts related to Field Excitations $(\mathbf{D},\mathbf{H})$, and sources. The Field Intensities have functional components which transform as a covariant tensor, while the Field Excitations are quantities with components that transform as a contravariant tensor density. These thermodynamic distinctions of intensities and quantities [3] are often masked by the imposition of a metric structure, or a limitation to self dual systems or volume preserving (often non-dissipative) evolutionary processes.

The Postulate of Potentials

Herein, the assumption that classical electromagnetic systems are defined by the topological constraint that the 2-form F is exact will be called the

$$\textit{Postulate of Potentials} \qquad F - dA = 0. \qquad \#$$

The postulate of potentials can be recognized as an exterior differential system constraining the topology of the independent variables. The constraint requires that the (2 dimensional) domain of support for F cannot be compact without boundary. The torus and the Klein bottle are the only exceptions. When the 2-form F is exact, the Poincare lemma, $ddA = dF = 0$, implies that the partial differential equations so generated [4] form a nested set independent from the dimension of the independent variables. Using the classic definitions of Field Intensities

$$\mathbf{E} = -\partial\mathbf{A}/\partial t - grad\phi, \qquad \mathbf{B} = curl\ \mathbf{A} \equiv \partial A_k/\partial x^j - \partial A_j/\partial x^k, \qquad \#$$

the closure of the exterior differential system, $dF = 0$, generates the Maxwell-Faraday partial differential equations:

$$\{curl\ \mathbf{E} + \partial\mathbf{B}/\partial t = 0, \quad div\,\mathbf{B} = 0\}. \qquad \#$$

These equations therefor are universal and applicable to all physical systems that support a 1 form of Action on a domain of four dimensions or more. By using Cartan's magic formula [5] it is possible to prove that the closed integral of F is a deformation invariant for *any* continuous evolutionary process that can be defined in terms of a singly parameterized vector field. This result is known as the conservation of flux.

The postulate of potentials is an essential point of departure from other theoretical developments, because, straight away, physical meaning is associated with topological equivalence classes of potentials. In fact, topological evolution will be observed most often when the potentials evolve from one equivalence class to another. A theory of topological evolution can not be gauge invariant, nor can it make unique predictions, because the prediction of the functional form of tensor fields is impossible if the Jacobian matrix does not have and inverse [6]. Yet when couched in the language of differential forms, a theory of continuous topological evolution can be retrodictively deterministic. It is the closed components of A that are not exact that determine many of the topological, multiply connected, features of the electromagnetic system.

The Postulate of Conserved Currents

It is also stipulated that the classic electromagnetic system requires a second topological constraint to be imposed upon the domain of independent variables. This postulate will be called the

$$\textit{Postulate of Conserved Currents}. \qquad J - dG = 0. \qquad \#$$

The electromagnetic domain not only supports the 1-form A, but also supports an N-1=3 form, J, which is exact. The equivalent differential system, $J - dG = 0$, requires that the (N-1 dimensional) domain of support for J cannot be compact without boundary. However, using Cartan's magic formula it is possible to prove that the closed integrals of J are deformation invariants for *any* continuous evolutionary process that can be defined in terms of a singly parameterized vector field. This result is otherwise known as the conservation of charge-current density.

This second postulate, the Postulate of Conserved Currents, on a four dimensional domain of independent variables, assumes the existence of a N-2 form density given by the expression (in terms of the Field Excitations),

$$G = G^{34}(x,y,z,t)dx\hat{}dy.. + G^{12}(x,y,z,t)dz\hat{}dt... = \mathbf{D}^z dx\hat{}dy.. + \mathbf{H}^z dz\hat{}dt... \qquad \#$$

Exterior differentiation produces an N-1 form,

$$J = \mathbf{J}^z(x,y,z,t)dx\hat{}dy\hat{}dt... - \rho(x,y,z,t)dx\hat{}dy\hat{}dz. \qquad \#$$

Matching the coefficients of the exterior expression $dG = J$ leads to the Maxwell-Ampere partial differential equations,

$$curl\,\mathbf{H} - \partial\mathbf{D}/\partial t = \mathbf{J} \quad \textit{and} \quad div\,\mathbf{D} = \rho. \qquad \#$$

The fact that J is exact leads to the charge conservation law, $dJ = ddG = 0$, or

$$\partial\mathbf{J}^x/\partial x + \partial\mathbf{J}^y/\partial y + \partial\mathbf{J}^z/\partial z + \partial\rho/\partial t = 0. \qquad \#$$

It is important to note that the domain of support for G (not F) can be compact without boundary, and idea that leads to the intuitive concept of charged particles.

The Torsion and Spin 3-forms

The two fundamental postulates of an electromagnetic system require the existence of four fundamental exterior differential forms, $\{A, F, G, J\}$, which form a differential ideal. The elements of the ideal can be used to construct the complete Pfaff sequence of forms

$$Pfaff\ Sequence = \{A, F = dA, G, J = dG, A\hat{}F, A\hat{}G, A\hat{}J, F\hat{}F, G\hat{}G\}. \qquad \#$$

by the processes of exterior differentiation and exterior multiplication. A (Cartan) topology constructed on this system of forms has the useful feature that the exterior derivative may be interpreted as a limit point, or closure, operator in the sense of Kuratowski [7]. It is important to note that the complete Maxwell system of differential forms (which assumes the existence of the potentials, A) also generates two other exterior differential systems.

$$d(A\hat{}G) - (F\hat{}G - A\hat{}J) = 0, \qquad d(A\hat{}F) - F\hat{}F = 0. \qquad \#$$

These equations introduce the (apparently novel to many researchers) 3-forms of Spin Current density, $A\hat{}G,$ [8] and Topological Torsion-Helicity, $A\hat{}F$ [9].

For an electromagnetic system, the Action 1-form, A, which has the physical dimensions of the flux quantum, h/e. The 2-form, G, has the physical dimensions of charge, e. The 3-form, $A\hat{}G$, has the physical dimensions of spin, h, and the 3-form $A\hat{}F,$ has the physical dimensions of spin multiplied by the Hall impedance, $(h/e)^2 = h(h/e^2) = hZ_{hall}$.

By direct evaluation of the exterior product on a domain of 4 independent variables, each 3-form will have 4 components that can be symbolized by the 4-vector arrays,

$$Spin - Current : \mathbf{S}_4 = [\mathbf{A} \times \mathbf{H} + \mathbf{D}\phi, \mathbf{A} \circ \mathbf{D}] \equiv [\mathbf{S}, \sigma], \qquad \#$$

$$Torsion - vector : \mathbf{T}_4 = [\mathbf{E} \times \mathbf{A} + \mathbf{B}\phi, \mathbf{A} \circ \mathbf{B}] \equiv [\mathbf{T}, h], \qquad \#$$

which are to be compared with the charge current 4-vector density:

$$\text{Charge-Current} : \mathbf{J}_4 = [\mathbf{J}, \rho], \qquad \#$$

Note that the ubiquitous helicity density is merely the fourth component of $A\hat{}F$.

The 3-forms then can be defined by the equivalent contraction processes

$$Spin\,Current\ 3 - form = A\hat{}G = i(\mathbf{S}_4)dx\hat{}dy\hat{}dz\hat{}dt. \qquad \#$$

$$Torsion\,(helicity)\ 3 - form = A\hat{}F = i(\mathbf{T}_4)dx\hat{}dy\hat{}dz\hat{}dt \qquad \#$$

The vanishing of $A\hat{}G$ is a topological constraint on the domain that defines topologically transverse electric (TTE) waves: the vector potential, $\mathbf{A}$, is orthogonal to $\mathbf{D}$, in the sense that $\mathbf{A} \circ \mathbf{D} = 0$. The vanishing of $A\hat{}F$ is a topological constraint on the domain that defines topologically transverse magnetic (TTM) waves: the vector potential, $\mathbf{A}$, is orthogonal to $\mathbf{B},$ in the sense that $\mathbf{A} \circ \mathbf{B} = 0$. When both 3-forms vanish, the topological constraint on the domain defines topologically transverse (TTEM) waves. For classic real fields this double constraint would require that the vector potential, $\mathbf{A}$, is collinear with the field momentum, $\mathbf{D} \times \mathbf{B}$, and in the direction of the wave vector, $\mathbf{k}$. Such constraints permit the definition of singular solutions of propagating discontinuities, or electromagnetic ”signals” [10].

Note that if the 2-form F was not exact, such topological concepts of transversality would be without distinct meaning, for the 3-forms of Topological Spin and Topological

Torsion depend explicitly upon the existence of the 1-form of Action. For future developments, observe that the torsion vector $\mathbf{T}_4$ and the Spin vector $\mathbf{S}_4$ are associated vectors to the 1-form of Action, in the sense that

$$i(\mathbf{T}_4)A = 0 \;\; and \;\; i(\mathbf{S}_4)A = 0. \qquad \#$$

The two distinct concepts of Spin Current and the Torsion vector have had almost no utilization in applications of classical electromagnetic theory, for they are explicitly dependent upon the potentials, A. Examples, both novel and well-known, of vacuum and plasma solutions to the electromagnetic system which satisfy (and which do not satisfy) these topological constraints are given elsewhere [11].

The Poincare Invariants

The exterior derivatives of the 3-forms of Spin and Torsion produce two 4-forms, $F\hat{}G - A\hat{}J$ and $F\hat{}F$, whose closed integrals are deformation invariants for *any* continuous evolutionary process that can be defined in terms of a singly parameterized vector field. These topological objects are related to the conformal invariants of a Lorentz system as discovered by Poincare and Bateman [12]. Note that their topological properties are valid even in the plasma domain of dissipative charge currents and radiation, as well as in the vacuum. In the format of independent variables $\{x, y, z, t\}$, the exterior derivative corresponds to the 4-divergence of the 4-component Spin and Torsion vectors, $\mathbf{S}_4$ and $\mathbf{T}_4$.

$$\begin{aligned} Poincare\ 1 &= d(A\hat{}G) = F\hat{}G - A\hat{}J \qquad \# \\ &= \{div_3(\mathbf{A} \times \mathbf{H} + \mathbf{D}\phi) + \partial(\mathbf{A} \circ \mathbf{D})/\partial t\} dx\hat{}dy\hat{}dz\hat{}dt \\ &= \{(\mathbf{B} \circ \mathbf{H} - \mathbf{D} \circ \mathbf{E}) - (\mathbf{A} \circ \mathbf{J} - \rho\phi)\} dx\hat{}dy\hat{}dz\hat{}dt \end{aligned}$$

$$\begin{aligned} Poincare\ 2 &= d(A\hat{}F) = F\hat{}F \qquad \# \\ &= \{div_3(\mathbf{E} \times \mathbf{A} + \mathbf{B}\phi) + \partial(\mathbf{A} \circ \mathbf{B})/\partial t\} dx\hat{}dy\hat{}dz\hat{}dt \\ &= \{-2\mathbf{E} \circ \mathbf{B}\} dx\hat{}dy\hat{}dz\hat{}dt \end{aligned}$$

For the vacuum state, defined by $J = 0$, zero values of the Poincare invariants require that the magnetic energy density is equal to the electric energy density $(1/2\mathbf{B} \circ \mathbf{H} = 1/2\mathbf{D} \circ \mathbf{E})$, and, respectively, that the electric field is orthogonal to the magnetic field $(\mathbf{E} \circ \mathbf{B} = 0)$. Note that these constraints often are used as elementary textbook definitions of what is meant by electromagnetic waves. Consider the definitions:

Definition 3: Spin is defined as the closed integral of the 3-form $A\hat{}G$

$$Spin = \iiint_{closed} A\hat{}G \qquad \#$$

Definition 4: Chirality is defined as the closed integral of the 3-form $A\hat{}F$

$$Chirality = \iiint_{closed} A \wedge F \qquad \#$$

By using Cartan's magic formula it is possible to prove

Theorem 1: If the First Poincare Invariant vanishes, Spin is an evolutionary deformation invariant with values whose ratios are rational.

Theorem 2. If the second Poincare Invariant vanishes, Chirality is an evolutionary deformation invariant with values whose ratios are rational.

The quantized (integer) ratios comes from the deRham cohomology theorems on closed integrals of closed p-forms [13].

It is important to realize that these topological conservation laws are valid in a plasma as well as in the vacuum, subject to the conditions of zero values for the Poincare invariants. On the other hand, topological transitions between "quantized" states of Spin or Chirality require that the respective Poincare invariants are not zero.

Topological Evolution and Thermodynamics

Topological evolution can be studied in terms of the Lie derivative (with respect to a vector field) acting as a propagator on each element of the Pfaff sequence. No metric or connection is necessary. Evolutionary processes (as vector fields) can be put into equivalence classes according to certain topological (invariance) refinements that they impose on the elements of the Pfaff sequence. For example, a plasma process (which is to be distinguished from a Hamiltonian process) will be defined as an element of the equivalence class of evolutionary vector fields which leave the closed integrals of G a deformation invariant. Cartan defined an extremal Hamiltonian process as an element of an equivalence class of processes such that the closed integrals of A are deformation invariants. [14]. A plasma process need not conserve energy. A *perfect* plasma process is a plasma process which is also a Hamiltonian process. The three forms, J, $A \wedge G$ and $A \wedge F$ are of particular interest, because their tangent manifolds define direction fields, or "lines", in the 4-dimensional variety of space and time. Relative to plasma processes, the topological evolution associated with such lines, and their entanglements or knots, is of utility in understanding solar corona and plasma instability. [15]

Topological Thermodynamics and Evolutionary Processes

The basic tool for studying topological evolution is Cartan's magic formula for the action of the Lie derivative on exterior differential forms [5]. It is presumed that a physical system can be described, minimally, by a 1-form of Action, A, and that a physical process can be represented by the direction field of a contravariant vector field, $\mathbf{V}$. It is important to realize that Cartan's magic formula of topological evolution is equivalent to the first law of thermodynamics.

$$L_{(\mathbf{V})}A = i(\mathbf{V})dA + d(i(\mathbf{V})A) = W + dU = Q. \qquad \#$$

A is the "Action" 1-form that describes the hydrodynamic or electromagnetic system. $\mathbf{V}$ is the vector field that defines the evolutionary process. W is the 1-form of (virtual) work. Q is the 1-form of heat. The basic idea behind this formalism (which is at the foundation of the Cartan-Hilbert variational principle) is that the postulate of potentials is valid: $F - dA = 0$. Herein, the base manifold will be the 4-dimensional variety $\{x, y, z, t\}$ of engineering practice, but no metrical features, or constitutive properties, are presumed a priori. The fundamental formula can be applied to integral properties as well. For example, if (relative to V) $L_{(\mathbf{V})}\int A = \int Q \Rightarrow 0$, then $\int A$ is said to be an integral invariant of the evolution generated by $\mathbf{V}$. In thermodynamics such processes are said to be adiabatic.

From the point of view of differential topology, a key idea is that the Pfaff dimension, or class [16], of the 1-form of Action specifies topological properties of the system. Given the Action 1-form, A, the Pfaff sequence, $\{A, dA, A\hat{}dA, dA\hat{}dA, ...\}$ will terminate at an integer number of terms equal to, or less than, the number of dimensions of the domain of definition. On a 4 dimensional domain, the top Pfaffian, $dA\hat{}dA$, will define a volume element with a density function whose singular zero set (if it exists) reduces the symplectic domain to a contact manifold of Pfaff dimension 3. This (defect) contact manifold supports a unique extremal field that leaves the Action integral "stationary", and leads to the Hamiltonian conservative representation for the Euler flow in hydrodynamics. The irreversible regime will be on an irreducible symplectic manifold of Pfaff dimension 4, where $dA\hat{}dA \neq 0$. Topological defects (or coherent structures) appear as singularities of lesser Pfaff (topological) dimension, $dA\hat{}dA = 0$.

Reversible Processes

Classical hydrodynamic and electromagnetic processes can be represented by certain nested categories of vector fields, $\mathbf{V}$. Three distinct classes of processes are defined by the constraints on the evolutionary vector field such that

$$Extremal - (unique\ Hamiltonian\ generator) \qquad i(\mathbf{V})dA = 0 \qquad \#$$

$$Bernoulli - Casimir - (\Theta\ is\ the\ generator) \qquad i(\mathbf{V})dA = d\Theta \qquad \#$$

$$Helmholtz - Stokes - -Symplectic \qquad di(\mathbf{V})dA = 0. \qquad \#$$

The vector fields defined by first two constraints have generators that create a Hamiltonian flow. This Hamiltonian flow is uniquely defined, in the extremal case, on a contact manifold of odd dimensions as the null eigen vector of the matrix of coefficients of the 2-form, dA. In the $Bernoulli - Casimir$ (maximal rank) case, the manifold is of even dimensions and symplectic, so there does not exist a unique null eigenvector direction field. The evolutionary field depends upon the choice of the $Bernoulli - Casimir$ function Θ. Such

Bernoulli processes can correspond to energy dissipative symplectic processes, but they, as well as all symplectic processes, are reversible in the thermodynamic sense described below. The mechanical energy need not be constant, but the Bernoulli-Casimir function(s), Θ, are evolutionary invariant(s), and may be used to describe non-unique "stationary" thermodynamic state(s).

The equations, above, are in effect constraints on the topological evolution of any physical system represented by an Action 1-form, A. The Pfaff dimension of the 1-form of virtual work, defined as $W = i(\mathbf{V})dA$ is 2 or less for each of the three categories. The extremal constraint can be used to generate the Euler equations of hydrodynamics for a incompressible fluid. The Bernoulli-Casimir constraint can be used to generate the equations for a barotropic compressible fluid. The Helmholtz constraint can be used to generate the equations for a Stokes (not Navier-Stokes) flow. However as will be shown below, all such processes are thermodynamically reversible.

An important idea is that domains of Pfaff dimension 3, or more, with attendant properties of non-uniqueness, envelopes, regressions, and projectivized tangent bundles, are required to yield the concepts of Spin and Torsion-Helicity. As demonstrated below, it takes systems of Pfaff dimension 4 to accommodate processes which are thermodynamically irreversible.

Irreversible Processes

Although there does not exist a unique extremal process on a symplectic manifold of Pfaff dimension 4, remarkably there does exist a unique (conformal) vector field whose direction field depends only upon the functional form of the 1-form, A, that is used to define the physical system. The direction field on the four dimensional domain is defined by the 3-form of topological torsion, $A\hat{}dA$, as discussed in Section 2.3. This unique (to within a factor) vector field is defined in component form as the Torsion Current, $\mathbf{T}_4$, and satisfies (on the 4 dimensional manifold) the equation,

$$i(\mathbf{T}_4)dx\hat{}dy\hat{}dz\hat{}dt = A\hat{}dA \qquad \#$$

This (four component) vector field, $\mathbf{T}_4$, has a non-zero divergence almost everywhere, for if the divergence is zero, then the 4-form $dA\hat{}dA$ vanishes, and the domain is no longer a symplectic 4-manifold! The Torsion vector, $\mathbf{T}_4$, can be used to generate a dynamical system that will decay to the stationary states ($div_4(\mathbf{T}_4) \Rightarrow 0$) starting from arbitrary initial conditions. As shown below these processes are irreversible in the thermodynamic sense. It is remarkable that this unique evolutionary vector field, $\mathbf{T}_4$, is completely determined (to within a factor) by the physical system itself; e.g., the components of the 1-form, A, determine the direction field of the Torsion vector.

To understand what is meant by thermodynamic irreversibility, recall that in the theory of classical thermodynamics, a process is defined to be irreversible when the heat 1-form Q does not admit an integrating factor.

Definition 5: An irreversible (non-equilibrium) process is one for which the Heat

1-form Q does not admit an integrating factor [17].

From the Frobenius theorem, the lack of an integrating factor implies that $Q \hat{} dQ \neq 0$. Hence a simple (non-statistical) test may be made for any process, **V**, relative to a physical system described by an Action 1-form, A:

Theorem 3:

$$\textit{If}\ \ L_{(\mathbf{v})}A \hat{} L_{(\mathbf{v})}dA \neq 0 \ \textit{then the process is irreversible.} \qquad \#$$

Proof: Using Cartan's magic formula yields $L_{(\mathbf{v})}A = Q$ and $L_{(\mathbf{v})}dA = dQ$. Hence the requirement that an integrating factor does not exist is
$Q \hat{} dQ \Rightarrow L_{(\mathbf{v})}A \hat{} L_{(\mathbf{v})}dA \neq 0$.

This topological definition implies that the three categories (above) of Symplectic evolutionary processes, **S**, are reversible in a thermodynamic sense (as $L_{(\mathbf{S})}dA = dQ = 0$). However, for evolution in the direction of the Torsion vector, $\mathbf{T}_4$, direct computation demonstrates that the fundamental equations lead to a conformal evolutionary process:

$$L_{(\mathbf{T}_4)}A = \sigma A \quad \textit{and} \quad i(\mathbf{T}_4)A = 0 \qquad \#$$
$$\textit{with} \quad \sigma \sim div_4(\mathbf{T}_4) \sim d(A \hat{} dA)$$

Theorem 4: Evolution in the direction of the Torsion Vector is irreversible.
Proof: The direction field associated with $\mathbf{T}_4$ is uniquely determined by the functional form of the 1-form of Action that defines the physical system on the four dimensional variety. By direct evaluation,

$$L_{(\mathbf{T}_4)}A \hat{} L_{(\mathbf{T}_4)}dA = Q \hat{} dQ = \sigma A \hat{} (\sigma dA + d\sigma \hat{} A) = \{div_4(\mathbf{T}_4)\}^2 A \hat{} dA. \qquad \#$$

As the domain is of Pfaff dimension 4, it follows that $A \hat{} dA$ is not zero, and $dA \hat{} dA \sim div_4(\mathbf{T}_4)$ is not zero. Therefore, the RHS is not zero, and the irreversibility result follows from theorem 3.

Explicit evaluations are carried out in the next section for electromagnetic systems of Pfaff dimension 4.

Applications to Electromagnetism

All of the development above will carry over to the electromagnetic system, which also assumes the postulate of potentials. The topological torsion 3-form, $A \hat{} dA$, induces the torsion current

$$\mathbf{T}_4 = \{(\mathbf{E} \times \mathbf{A} + \mathbf{B}\phi); \mathbf{A} \circ \mathbf{B}\} \equiv \{\mathbf{S}, h\}. \qquad \#$$

If $div_4\mathbf{T} = -2\ \mathbf{E} \circ \mathbf{B} \neq 0$, the electromagnetic 1-form, A, defines a domain of Pfaff dimension 4. Such domains cannot support topologically transverse magnetic waves $(as\ A \hat{} F \neq 0)$. Evolutionary processes (including plasma currents) that are proportional to the Torsion current are thermodynamically irreversible, if $\sigma = -\mathbf{E} \circ \mathbf{B} \neq 0$. However, the conformal properties of evolution in the direction of the Torsion current lead to extraordinary properties when the plasma current is in the direction of the Torsion vector. Using the notation of an electromagnetic system

$$L_{(\mathbf{T}_4)}A = \sigma A = -(\mathbf{E} \circ \mathbf{B})A \quad and \quad L_{(\mathbf{T}_4)}(A\hat{}F) = 2\sigma A = -2(\mathbf{E} \circ \mathbf{B})A\hat{}F. \qquad \#$$

Hence, it follows that motion along the direction of the torsion vector freezes-in the lines of the torsion vector in space time, but the process is irreversible unless the second Poincare invariant is zero.

Recall that the definition of a plasma current, J, is equivalent to an evolutionary process such that the number of charges is an evolutionary invariant, $L_{(J)}G = 0$. Consider a plasma current which is also in the direction of the Torsion vector. Then

$$L_{(J)}A\hat{}G = (L_{(J)}A)\hat{}G + A\hat{}L_{(J)}G = (L_{(\gamma\mathbf{T}_4)}A)\hat{}G + 0 = -\gamma \bullet (\mathbf{E} \circ \mathbf{B})\ A\hat{}G \qquad \#$$

Hence for plasma motions in the direction of the (possibly dissipative) torsion vector, both the ”lines” of the Spin vector are ”frozen in” and the lines of the Torsion vector are ”frozen in”. Such ”frozen in” objects [18] can be used to give a topological definition of deformable coherent structures in a plasma. Moreover, as the evolutionary process causes the ”frozen in” structures to deform and decay, it is conceivable that evolution could proceed to form stationary (not stagnant) states (where $\mathbf{E} \circ \mathbf{B} \Rightarrow 0$), such that the ”frozen in” field line structures become local deformation invariants, or topological defects. Electromagnetic coherent structures are evolutionary deformable (and perhaps decaying) domains of Pfaff dimension 4, which form stationary states of topological defects (including the null state) in regions of Pfaff dimension 3, where $\mathbf{E} \circ \mathbf{B} = 0$.

The Classical Photon with Chirality and Spin

An empirical property of the photon is its quantized spin. As it is not commonly appreciated that the classical Maxwell electromagnetic field can have quantized spin and torsion properties, three examples are presented below which demonstrate spin and torsion properties of the field. In the first example, the spin radiation is related to the back reaction of a Lorentz force on a rotating plasma, to produce an accretion disk. The first example has zero torsion but finite spin. The second example models a plasma which has finite helicity, but zero spin. In the third example, a time dependent vacuum wave solution is presented. The vector potentials exhibit a lack of time reversal invariance, and the fields are not transverse, as $\mathbf{E} \circ \mathbf{B} \neq 0$. The third example has both finite torsion and finite spin. A fourth example (without torsion) demonstrates a solution where the Poynting vector is in the direction of the Spin vector, indicating that radiation is related to the spin qualities of the photon. Moreover, in the third example, the closed integrals of the Spin 3-form are quantized in the sense that their values have integer ratios.

A rotating plasma with an accretion disk (Finite Spin current - Zero Torsion)

A very interesting time independent set of functions that satisfy the Maxwell system is given by the Potentials,

$$\mathbf{A} = \{f(x,y,z,t)\}Kelvin = \{z/\sqrt{(\delta z^2 + x^2 + y^2)}\}[-y,x,0]/(x^2 + y^2) \qquad \#$$

which generates the Hedge-Hog B field,

$$\mathbf{B} = -[x,y,z]/(\delta z^2 + x^2 + y^2)^{3/2}, \qquad \#$$

and (assuming the Lorentz constitutive relations) the London-like Current density,

$$\mathbf{J} = \Lambda(x,y,z,t)\mathbf{A} = -3(1-\delta)(x^2 + y^2)/(\delta z^2 + x^2 + y^2)^2\mathbf{A}. \qquad \#$$

It is apparent that the Helicity density $\mathbf{A} \circ \mathbf{B}$ vanishes identically, as do all components of the Topological Torsion tensor: $A\hat{}F \Rightarrow 0$. There exists a Lorentz force,

$$\mathbf{J} \times \mathbf{B} = \{3(1-\delta)/(\delta z^2 + x^2 + y^2)^4\}[yz^2, xz^2, -z(x^2 + y^2)]/\mu \qquad \#$$

which has the remarkable features that for an ”oblate” situation ($\delta < 1$), the ”plasma” is forced away from the rotation z axis, but is attracted to the $z = 0$ plane to form an accretion disk. There are no Amperian currents and no Lorentz force unless the system is anisotropic (oblate or prolate).

When the 3-form of topological Spin is evaluated, $A\hat{}G \neq 0$, it is remarkable that the Lorentz force is proportional to the Spin current

$$\mathbf{J} \times \mathbf{B} = -3(1-\delta)\mathbf{S}/(x^2 + y^2). \qquad \#$$

In this example, the spin current is to be considered as a back reaction to the Lorentz force. The usual interpretation in MHD theory is to resolve $\mathbf{J} \times \mathbf{B}$ in terms of a ”magnetic pressure and a magnetic tension”:

$$\mathbf{J} \times \mathbf{B} = -\nabla(\mathbf{B} \circ \mathbf{H})/2 + \mathbf{H} \circ \nabla \mathbf{B} = \text{”}\mathit{magnetic\ pressure} \ + \ \mathit{magnetic\ tension}\text{”}. \qquad \#$$

However, the Spin interpretation would imply that the effects are more related to rotational deformations, rather than to translational deformations.

A plasma with helicity density. (but Zero Spin)

Consider the Beltrami potentials (related to a Heisenberg exterior differential system of Pfaff dimension 3)

$$\mathbf{A} = [-y, x, -a]/(a^2 + x^2 + y^2), \qquad \#$$

which generate the Fields, and the Amperian currents,

$$\mathbf{B} = 2a[-y, x, -a]/(a^2 + x^2 + y^2)^2, \qquad \#$$

$$\mu\mathbf{J} = 4a[-2ay, 2ax, -a(-a^2 + x^2 + y^2)]/(a^2 + x^2 + y^2)^3. \qquad \#$$

The Torsion vector has one component (the helicity density) and the Spin vanishes:

$$\mathbf{T}_4 = -2a[0,0,0,1]/(a^2 + x^2 + y^2)^2, \qquad \mathbf{S}_4 = [0,0,0,0] \quad . \qquad \#$$

A time dependent irreversible vacuum wave (with $\mathbf{E} \circ \mathbf{B} \neq \mathbf{0}$).

Modifications of the Hopf map suggest consideration of the system of potentials given by the equations

$$\mathbf{A} = [+y, -x, +ct]/\lambda^4 \;, \quad \phi = cz/\lambda^4, \quad where\ \lambda^2 = -c^2t^2 + x^2 + y^2 + z^2, \qquad \#$$

which yield the real field intensities,

$$\mathbf{E} = [-2(cty - xz), +2(ctx + yz), -(c^2t^2 + x^2 + y^2 - z^2)]2c/\lambda^6 \qquad \#$$

$$\mathbf{B} = [-2(cty + xz), +2(ctx - yz), -(c^2t^2 + x^2 + y^2 - z^2)]2/\lambda^6. \qquad \#$$

Subject to the dispersion relation, $\varepsilon\mu c^2 = 1$ and the Lorentz constitutive conditions, these time dependent wave functions satisfy the homogeneous Maxwell equations without charge currents, and are therefore acceptable vacuum solutions, $\mathbf{J}_4(+t) = 0$.

The extensive algebra involved in these and other computations in this article were checked with a Maple symbolic mathematics program [23].

It is to be noted that when the substitution $t \Rightarrow -t$ is made in the functional forms for the potentials, the fields computed from the new functional forms fail to satisfy the vacuum Lorentz conditions for zero charge-currents, $\mathbf{J}_4(-t) \neq 0$. The $\mathbf{E}_{(-t)}$ field calculated from the potentials $\mathbf{A}_4(-t)$ is not equal to the $\mathbf{E}_{(+t)}$ field computed from $\mathbf{A}_4(+t)$, and exhibits a non-zero divergence; *e.g.*,

$$\mathbf{E}_{(-t)} = [-2(cty - xz), +2(ctx + yz), (c^2t^2 + z^2)]2c/\lambda^6 \qquad \#$$

$$\rho/\varepsilon = div_3\{\mathbf{E}_{(-t)}\} = -8cz(x^2 + 5c^2t^2 + y^2 + z^2)/\lambda^8. \qquad \#$$

In this sense, the valid vacuum $(+t)$ solution presented above is not time-reversal invariant. (Of course the differential form for the Action transforms properly).

The Spin current density for this first non-transverse wave $(+t)$ example is evaluated as:

$$Spin : \mathbf{S}_4 = [x(3\lambda^2 - 4y^2 - 4x^2), y(3\lambda^2 - 4y^2 - 4x^2), z(\lambda^2 - 4y^2 - 4x^2),$$
$$t(\lambda^2 - 4y^2 - 4x^2)](2/\mu)/\lambda^{10}, \qquad \#$$

and has zero divergence. Hence its global integral (Spin) is quantized. The Torsion current may be evaluated and leads to:

$$Torsion : \mathbf{T}_4 = -[x, y, z, t]2c/\lambda^8 \qquad Poincare\ 2 = -2\mathbf{E} \circ \mathbf{B} = +8c/\lambda^8. \qquad \#$$

The solution has magnetic helicity as $\mathbf{A} \circ \mathbf{B} \neq 0$ and is radiative in the sense that the Poynting vector, $\mathbf{E} \times \mathbf{H} \neq 0$, and the wave is not transverse. The Chirality integral is not quantized.

It is to be noted that the example solution given above is but one of a class of vacuum wave solutions that have similar non transverse properties. As a second example, consider the fields that can be constructed from the potentials,

$$\mathbf{A} = [+ct, -z, +y]/\lambda^4 , \quad \phi = cx/\lambda^4, \quad where\ \lambda^2 = -c^2t^2 + x^2 + y^2 + z^2. \qquad \#$$

These potentials will generate the field intensities,

$$\mathbf{E} = [+(-c^2t^2 + x^2 - y^2 - z^2), +2(ctz + yx), -2(cty - zx)]2c/\lambda^6 \qquad \#$$
$$\mathbf{B} = [+(-c^2t^2 + x^2 - y^2 - z^2), +2(-ctz + yx), +2(cty + zx)]2/\lambda^6. \qquad \#$$

As before, these fields satisfy the Maxwell-Faraday equations, and the associated excitations satisfy the Maxwell-Ampere equations without producing a charge current 4-vector. However, it follows by direct computation that the second Poincare invariant, and the Torsion 4-vector are of opposite signs to the values computed for the first example:

$$Torsion : \mathbf{T}_4 = +[x, y, z, t]2c/\lambda^8 , \qquad -2\mathbf{E} \circ \mathbf{B} = -8c/\lambda^8 . \qquad \#$$

A radiating vacuum wave with Quantized Spin (finite Poynting Vector - Zero Chirality)

When the two examples above are combined by addition (or subtraction), the resulting wave is transverse magnetic (in the topological sense that $\mathbf{A} \circ \mathbf{B} = 0$). Not only does the second Poincare invariant vanish under superposition, but so also does the Torsion 4 vector. Conversely, the examples above show that there can exist transverse magnetic waves which can be decomposed into two non-transverse waves. A notable feature of the superposed solutions is that the Spin 4 vector current does not vanish, hence the example superposition is a wave that is not transverse electric (in the topological sense that $\mathbf{A} \circ \mathbf{D} \neq 0$). For the superposed example, the first Poincare invariant vanishes, which implies that the Spin integral remains a conserved topological quantity, with values proportional to the integers. The non-zero Spin current density for the combined examples is given by the formula:

$$Spin : \mathbf{S}_4 = [-2x(y+ct)^2, (y+ct)(x^2-y^2+z^2-2cty-c^2t^2), -2z(y+ct)^2,$$
$$-(y+ct)(x^2+y^2+z^2+2cty+c^2t^2)](4/\mu)/\lambda^{10}, \qquad \#$$

while the Torsion current is a zero vector, $A\hat{}F \Rightarrow 0$.

In addition, for the superposed example, the spatial components of the Poynting vector are equal to the Spin current density vector multiplied by γ, such that

$$\mathbf{E}\times\mathbf{H} = \gamma\,\mathbf{S}, \quad with\ \gamma = -(x^2+y^2+z^2+2cty+c^2t^2)/2c(y+ct)\lambda^2. \qquad \#$$

These results seem to give classical credence to the Planck assumption that the vacuum state of Maxwell's electrodynamics supports quantized angular momentum, and that the energy flux must come in multiples of the spin quanta. In other words, these combined solutions to classical electrodynamics have some of the experimental qualities of the quantized photon.

Summary

Many of the chiral and spin features of the quantized photon have their basis in the topological properties of classical electromagnetism, and the 3-forms $A\hat{}F$ and $A\hat{}G$.

References

[1] R. L. Bryant, S. S. Chern, R. B. Gardner, H. L. Goldschmidt, and P. A. Griffths, Exterior Differential Systems (Springer Verlag, 1991).
[2] S. Lipschutz, General Topology (Schaum, New York, 1965) p. 88.
[3] A. Sommerfeld, Electrodynamics (Academic Press, New York, 1952).
J. A. Stratton, Electromagnetic Theory (McGraw Hill, NewYork, 1941).
Sommerfeld carefully distinguishes between intensities and excitations on thermodynamic grounds.
[4] E. Cartan, Lecons sur les invariants integrauxs (Hermann, Paris, 1958).
[5] J. E. Marsden and T. S. Riatu, Introduction to Mechanics and Symmetry (Springer-Verlag, 1994) p.122
[6] R. M. Kiehn, ”Retrodictive Determinism”, Int. J. of Eng. Sci. **14**, 749 (1976)
[7] R. M. Kiehn and J. F. Pierce, ”An Intrinsic Transport Theorem” Phys. Fluids **12**, 1971 (1969)
[8] R. M. Kiehn, ”Topological Torsion, Pfaff Dimension and Coherent Structures”, in: H. K. Moffatt and T. S. Tsinober eds, Topological Fluid Mechanics (Cambridge University Press, Cambridge, 1990) p. 449 .
[9] R. M. Kiehn, ”Are there three kinds of superconductivity” Int. J. Mod. Phys B **5** 1779 (1991)
[10] G. deRham, Varietes Differentiables (Hermann, Paris, 1960).
[11] http://www.uh.edu/~rkiehn
[12] G. Hornig and K. Schindler, K. ”Magnetic topology and the problem of its invaraint definition” Physics of Plasmas, **3**, p.646 (1996).
[13] D. Van Dantzig, Proc. Cambridge Philos. Soc. **30**, 421 (1934). Also see:

D. Van Dantzig, ”Electromagnetism Independent of metrical geometry”, Proc. Kon. Ned. Akad. v. Wet. 37 (1934).
[14] R. M. Kiehn, G. P. Kiehn, and R. B. Roberds, ”Parity and Time-reversal Symmetry Breaking, Singular Solutions”, Phys Rev A **43**, 5665
(1991).
[15] Ibid 6
[16] K. Kuratowski, Topology (Warsaw, 1948), Vol. I.
[17] R. M. Kiehn, ”Periods on manfolds, quantization and gauge”, J. of Math Phys **18** no. 4, p. 614 (1977).
[18] Ibid 8.
[19] H. Bateman, Electrical and Optical Wave Motion, (Dover, New York, 1914, 1955) p.12.
[20] J. A. Schouten, and W. Van der Kulk, Pfaff’s Problem and its Generalizations (Oxford Clarendon Press, 1949).
[21] P. M. Morse, Thermal Physics, (Benjamin,1964) p. 60.
[22] N. E. Kochin, I. A. Kibel, N. N. Roze, Theoretical Hydrodynamics, (Interscience, NY 1964) p.157.
R. M. Kiehn, ”Intrinsic hydrodynamics with applications to space time Fluids”, Int. J. Engng Sci **13**, 941 (1975).
[23] http://www.uh.edu/~rkiehn/pdf/emhopf.pdf