A Topological Theory of the Physical Vacuum

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ABSTRACT

This article examines how the physical presence of field energy and particulate matter could influence the topological properties of space time. The theory is developed in terms of vector and matrix equations of exterior differential forms. The topological features and the dynamics of such exterior differential systems are studied with respect to processes of continuous topological evolution. The theory starts from the sole postulate that field properties of a Physical Vacuum (a continuum) can be defined in terms of a vector space domain, of maximal rank, infinitesimal neighborhoods, that supports a Basis Frame as a 4 x 4 matrix of C2 functions with non-zero determinant. The basis vectors of such Basis Frames exhibit differential closure. The particle properties of the Physical Vacuum are defined in terms of topological defects (or compliments) of the field vector space defined by those points where the maximal rank, or non-zero determinant, condition fails. The topological universality of a Basis Frame over infinitesimal neighborhoods can be refined by particular choices of a subgroup structure of the Basis Frame, [B]. It is remarkable that from such a universal definition of a Physical Vacuum, specializations permit the deduction of the field structures of all four forces, from gravity fields to Yang Mills fields, and associate the origin of topological charge and topological spin to the Affine torsion coefficients of the induced Cartan Connection matrix [C] of 1-forms.

Part I

The Physical Vacuum

1 Preface

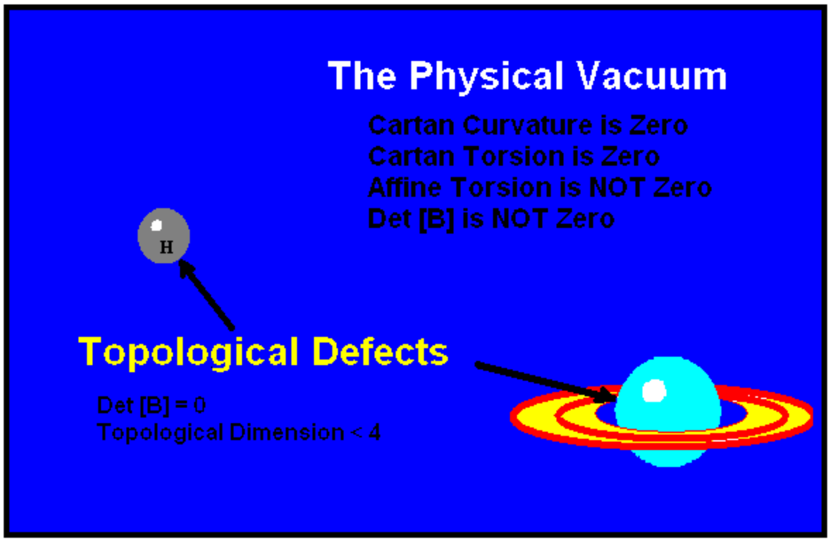
In 1993-1998, Gennady Shipov [29] presented his pioneering concept of a "Physical Vacuum" as a modified A_n space of "Absolute Parallelism". Shipov implied that a "Physical

Vacuum" could have geometrical structure specifically related to the concept of "affine" torsion. Such non-Riemannian structures are different from those induced by Gauss curvature effects associated with gravitational fields, for the A_n space defines a domain of zero Gauss curvature.

This article, however, examines how the physical presence of field energy and particulate matter could influence the *topological* properties (not only the *geometrical* properties) of space time to form a "Physical Vacuum". The point of departure in this article consists of three parts:

I Shipov's vision, that a "Physical Vacuum" is a space of Absolute Parallelism, is extended to include a larger set of admissible systems. The larger set is based on the sole requirement that *infinitesimal* neighborhoods of a "Physical Vacuum" are elements of a vector space. The additional (*global*) constraint of "Absolute Parallelism" is not utilized (necessarily). The sole requirement implies that the field points $\{y^a\}$ of the "Physical Vacuum" as a continuum support a matrix of C2 functions, with a non-zero determinant. This matrix of functions is defined as the Basis Frame, $[\mathbb{B}(y^a)]$, for the "Physical Vacuum", and represents the vector space properties of infinitesimal neighborhoods. The Basis Frame maps vector arrays of perfect differentials into vector arrays of 1-forms. The basis vectors that make up the Basis Frame for *infinitesimal* neighborhoods exhibit topological, differential closure. The set of admissible Basis Frames for the vector space of infinitesimal neighborhoods is larger than the set of basis frames for global neighborhoods, for the infinitesimal maps need not be integrable.

II In certain domains the Frame Matrix $[\mathbb{B}]$ is singular, and then one or more of its four eigenvalues is zero. These singular domains (or objects) may be viewed as topological defects of 3 (topological) dimensions or less embedded in the field domain of a 4 dimensional "Physical Vacuum". They can be thought of as condensates or particles or field discontinuities. The major theme of this article examines the field properties of the "Physical Vacuum", which is the domain free of singularities of the type $\det[\mathbb{B}] = 0$. The Basis Frame Matrix $[\mathbb{B}]$ will be assumed to consist of C2 functions, but only C1 differentiability is required for deriving a linear connection that defines infinitesimal differential closure. If the functions are not C2, singularities can occur in second order terms, such as curvatures (and accelerations).



The 4D Physical Vacuum with 3D topological defects

Although more complicated, the singular sets admit analysis, for example, in terms of propagating discontinuities and topologically quantized period integrals [18]. These topics will be considered in more detail in a subsequent article.

III It is recognized that topological coherent structures (fields, and particles, along with fluctuations) in a "Physical Vacuum" can be put into correspondence with the concepts of topological thermodynamics based on continuous topological evolution [32] [22]. Perhaps surprising to many, topology can change continuously in terms of processes that are not diffeomorphic. For example, a blob of putty can be deformed continuously into a cylindrical rope, and then the ends can be "pasted" together to create a non-simply connected object from a simply connected object. Topological continuity requires only that the limit points of the initial state topology be included in the closure of the topology of the final state. Such continuous maps are not necessarily invertible; it is important to remember that topology need not be conserved by such continuous processes. Diffeomorphic processes require continuity of the map and its inverse and therefor are specializations of homeomorphisms which preserve topology. This observation demonstrates why tensor constraints cannot be applied to problems of irreversible evolution and topological change [16].

The mathematical elements used herein are expressed in terms of vector and matrix arrays of Cartan's exterior differential forms. The matrix elements of such arrays are exterior differential p-forms which are automatically covariant with respect to diffeomorphisms. The combinatorial rules of multiplication will be that of the classic row-column product associated with the matrix multiplication of functions (0-forms). However, if the matrix elements are p-forms, then the matrix "dot" product becomes the matrix "exterior" product of differential forms, and the order of elemental factors is preserved. For example:

$$\begin{aligned}\begin{bmatrix} C^1_{1\mu}dx^\mu & C^1_{2\mu}dx^\mu \\ C^2_{1\mu}dx^\mu & C^2_{2\mu}dx^\mu \end{bmatrix} \hat{}\left| \begin{matrix} \sigma^1_\nu dx^\nu \\ \sigma^2_\nu dx^\nu \end{matrix} \right\rangle &= \left| \begin{matrix} (C^1_{1\mu}dx^\mu)\hat{}\sigma^1_\nu dx^\nu + (C^1_{2\mu}dx^\mu)\hat{}\sigma^2_\nu dx^\nu \\ C^2_{1\mu}dx^\mu)\hat{}\sigma^1_\nu dx^\nu + (C^2_{2\mu}dx^\mu)\hat{}\sigma^2_\nu dx^\nu \end{matrix} \right\rangle \\ &= \left| \begin{matrix} \{(C^1_{1\mu}\sigma^1_\nu - C^1_{1\nu}\sigma^1_\mu) + (C^1_{2\mu}\sigma^2_\nu - C^1_{2\nu}\sigma^2_\mu)\}dx^\mu\hat{}dx^\nu \\ \{(C^2_{1\mu}\sigma^1_\nu - C^2_{1\nu}\sigma^1_\mu) + (C^2_{2\mu}\sigma^2_\nu - C^2_{2\nu}\sigma^2_\mu)\}dx^\mu\hat{}dx^\nu \end{matrix} \right\rangle\end{aligned} \tag{1}$$

Topological properties are inherent in the differential systems defined by such mathematical structures. For example, a cohomology is defined when the difference between two non exact p-forms is a perfect differential. A classic example of cohomology is given by the first law of thermodynamics, where the difference between the two non-exact 1-forms of Heat, Q, and Work, W, is a perfect differential of the internal energy function:

$$Q - W = dU. \tag{2}$$

As another example of the topology encoded by an exterior differential system, recall that most exact 2-forms, which satisfy the exterior differential system, $F - dA = 0$, cannot be compact without boundary. Such topological features cast doubt on the universality of "self-dual" systems.

As all p-forms can be constructed from exterior products of 1-forms, the Pfaff topological dimension of the basis 1-forms yields pertinent topological information. For example, if any 1-form is of Pfaff Topological dimension[1] 3 or greater, the Cartan topology defined by such a 1-form is a "disconnected" topology [2]. For thermodynamic physical systems that can be encoded in terms of a single 1-form of Action, the Pfaff topological dimension can be used to distinguish between equilibrium and non equilibrium systems and adiabatic reversible and irreversible processes [32]. Topological evolution of such systems can be described in terms of Cartan's magic formulation of the Lie differential [11].

The topological properties of such structures and their evolutionary dynamics can be described in terms of the Lie directional differential (which admits topological change), but not in terms of the infamous "Covariant" differential of tensor analysis. If the Lie directional differential, representing a process, V, is applied to a 1-form of Action, A, that encodes a thermodynamic system, the result is to produce a 1-form Q of Heat.

[1] Thr Pfaff Topological dimension, or Class, of a 1-form determines the minimum number of functions required to encode its topological structure.

$$L_{(V)}A = i(V)dA + d(i(V)A) \tag{3}$$
$$= W + dU = Q. \tag{4}$$

If

$$L_{(V)}A = 0, \tag{5}$$

then the Cartan Topology is preserved. For integrals of n-volume forms, the equation is related to the constraint of the variational calculus. If the Lie derivative produces a non-zero result proportional to nA then the process is said to be homogeneous of degree n. Such a result encodes the Sommerfeld approach to quantum mechanics, for if n is an integer:

$$\text{Sommerfeld Quantum Mechanics:} \quad L_{(V)} \oint A = n \oint A, \tag{6}$$

and a quantum transition is related to an integer.

The Covariant differential is defined in terms of a "special linear connection" which (although constructed to produce a tensor from a tensor) does not admit irreversible topological change, a conclusion consistent with a constraint of tensor equivalence under diffeomorphism. The important point is that the covariant directional differential cannot describe non adiabatic irreversible evolutionary processes; the Lie differential can. The Lie differential can defined with respect to a directional field, V^k, where the classic kinematic assumption fails.

$$\omega^k \approx dx^k - V^k dt \neq 0. \tag{7}$$

Note that the statement $\omega^k = dx^k - V^k dt = 0$ is in the form of an exterior differential system, and imposes topological constraints on the functions that encode the classic concept of kinematics. It if $\omega^k \neq 0$, it follows that $\omega^k \hat{} d\omega^k = 0$, and the Pfaff topological dimension of ω^k must be 2 or less. Processes, $V^k(t)$, defined by 1 parameter group (kinematics) are always of Pfaff Topological dimension 1, as $d\omega^k = dV^k \hat{} dt = 0$. In general, however, evolutionary processes can change topological properties, such as the Pfaff topological dimension, continuously.

In this article, be aware that the dogmatic use of the covariant differential as an evolutionary operator is minimized in favor of the Lie differential. It is useful to remark that the difference between these two concepts of propagation by differential displacement has been described in terms of the Higgs field [1]. It is also important to realize that the same variety of sets can support more that 1-topological structure [23]. One topological structure can be composed of different topological substructures. Recall that a topological structure is essentially enough information to define continuity. A process possibly could be continuous relative to some given topological structure, but the same process might not be continuous relative to a different topological structure or substructure on the same variety of sets. These are not metrical types of ideas, and do not depend upon scales, necessarily.

2 Topological Structure of a Physical Vacuum

2.1 The Fundamental Postulate:

Assume the existence of a matrix array of 0-forms (C2 functions), $[\mathbb{B}] = [\mathbb{B}^{\text{row}}_{\text{col}}(y)] = \left[\mathbb{B}^{k}_{a}(y)\right]$, on a 4D variety of points $\{y^a\}$. The domain for which the determinant of $[\mathbb{B}]$ is not zero will be defined as a Physical Vacuum field, and in such regions, there exists an inverse Frame, $[\mathbb{B}]^{-1}$. The compliment of the Physical Vacuum field domain is defined as the singular domain, which is the realm of topologically coherent defect structures (such as particles and field discontinuities) which will be described later. From this sole topological assumption, that the physical vacuum field is a continuous collection of infinitesimal neighborhoods that form a vector space, the concepts displayed below are deducible results, based on the Cartan Calculus of exterior differential forms. The methods do not impose the constraints of a metric, or a linear connection, or a coframe of 1-forms, as a starting point, as is done in MAG and other theories [9] [10], but instead these concepts are deduced in terms of topological refinements of the single hypothesis:

Definition 1 *The Physical Vacuum is a maximal rank vector space on a 4D domain, which, as a differential ideal, supports topological differential closure.*

2.2 Constructive Results

By applying algebraic and exterior differential processes developed by E. Cartan to an element, $[\mathbb{B}]$, of given

equivalence class of Basis Frame of C2 functions, the following concepts are **derived, not postulated,** results. It is possible to frame these results in terms of existence theorems and constructive proofs. The constructions consist of:

1. A "flat" Cartan Connection of 1-forms $[\mathbb{C}]$ leading to structural equations of curvature and torsion which are zero, but which can support the concept of "affine" torsion. This derived Cartan Connection is algebraically compatible with the Basis Frame, and is differentially unique.

2. A symmetric quadratic congruence of functions, deduced by algebraic methods from Basis Frame, that can play the role of a metric $[g]$ compatible with the Cartan Connection.

The quadratic form can be used to generate another connection, $[\Gamma]$, not equal to the Cartan Connection, by using the Christoffel method. The Christoffel connection is free of "affine" torsion, but need not be flat.

3. A vector array of 1-forms, $\left|A^k\right\rangle$, which can be put into correspondence with the classical concept of potentials used to describe electromagnetic, hydrodynamic, and thermodynamic continua. The 1-forms A^k, if not zero, can have Pfaff Topological dimensions 1 to 4. Exterior differentiation of $\left|A^k\right\rangle$ produces a vector array of 2-forms, $\left|A^k\right\rangle \Rightarrow d\left|A^k\right\rangle = \left|F^k\right\rangle$, that can be put into correspondence with gauge invariant field intensities (think E and B). This vector $\left|F^k\right\rangle$ of 2-forms is "pair" and is an invariant of differential mappings that need not be diffeomorphisms.

4. Another vector array of 2-forms $\left|G^k\right\rangle$ can be deduced by exterior differentiation of the formula that describes infinitesimal mappings of differentials $d\{[\mathbb{B}]\circ\left|dx^k\right\rangle\} = d\left|A^k\right\rangle = \left|F^k\right\rangle$. This vector array $\left|G^k\right\rangle = [\mathbb{C}]\circ\left|dx^k\right\rangle$ is an "impair" structure that defines coefficients of field excitations (think D and H). The vector array $\left|G^k\right\rangle$, unlike the vector array of 2-forms, $\left|F^k\right\rangle$, is not an invariant of differential mappings that are not diffeomorphisms. It is remarkable that the coefficients of $\left|G^k\right\rangle$ are those that have been described (historically) as the coefficients of "Affine Torsion". It is further remarkable that the theory associates the Affine Torsion of the Cartan Connection with the Field intensities of electromagnetism. Without Affine Torsion, relative to a Cartan Connection, there is no D and H excitations, no charge current densities, and no Field intensities, E and B. It has been conjectured that Affine Torsion is related to Spin, but the theory of the physical vacuum goes much deeper.

Remark 2 *Affine torsion relative to a Cartan Connection for a Physical Vacuum is the source of topological charge and topological spin.*

5. By algebraic decomposition, the Cartan Connection $[\mathbb{C}]$ can be uniquely split into a Christoffel part $[\Gamma]$ and a Residue part $[\mathbb{T}]$. The Affine Torsion antisymmetry is entirely contained in the Residue part. As the equations of structure built on the Cartan Connection vanish, a constraint is placed upon the contributions to the exterior differential curvature equations of each component of the decomposition. The result can be interpreted as a global equivalence principle relating to curvatures induced by a Christoffel metric connection and the curvature 2-forms of interaction between the Christoffel components and the Residue components of the Cartan Connection.

6. 3-forms of Topological Torsion, $A\hat{}F$ and Topological Spin, $A\hat{}G$, can be constructed leading to quantized period integrals, independent from macro or microscopic scales. The concept of the Lorentz force is a derivable expression created by use of the Lie differential, not the covariant differential, to describe continuous topological evolution.

7. The Bianchi identities are demonstrated to be statements defining the cohomological structure of certain matrices of 3 forms, similar to the cohomological structure of thermodynamics in terms of 1-forms.

8. The idea of a Higgs function is demonstrated to be related to internal energy of a thermodynamic system.

3 Differential Closure

3.1 Cartan Connections for infinitesimal neighborhoods

3.1.1 The Derivation of the Cartan Right Connection

From the identity $[\mathbb{B}] \circ [\mathbb{B}]^{-1} = [\mathbb{I}]$, use exterior differentiation to *derive* the (right) Cartan Connection $[\mathbb{C}]$ as a matrix of 1-forms.

$$\text{Right Cartan Connection}: \ [\mathbb{C}]$$

$$d[\mathbb{B}] \circ [\mathbb{B}]^{-1} + [\mathbb{B}] \circ d[\mathbb{B}]^{-1} = d[\mathbb{I}] = 0 \tag{8}$$

$$\text{hence } d[\mathbb{B}] = [\mathbb{B}] \circ [\mathbb{C}], \tag{9}$$

$$\text{where } [\mathbb{C}] = -d[\mathbb{B}]^{-1} \circ [\mathbb{B}] \tag{10}$$

$$= +[\mathbb{B}]^{-1} \circ d[\mathbb{B}] \tag{11}$$

$$= [C^b_{am}(y)dy^m] \tag{12}$$

As each of the a columns of the Basis Frame are linearly independent, they may be considered as Basis Vectors, $\mathbf{e}^k_{(a)}$, for a vector space at the point $\{y\}$. In terms of the Cartan Connection, differential of any specific basis vector $\mathbf{e}^k_{(a)}$ is a linear combination of the basis vectors $\mathbf{e}^k_{(b)}$ of the set, thereby exhibiting differential closure:

$$: \text{Differential Closure}$$

$$d\mathbf{e}^k_{(a)} = \Sigma_b \mathbf{e}^k_{(b)} \{\Sigma_m C^b_{am} dy^m\}. \tag{13}$$

The Cartan Connection is a matrix of 1-forms; but the Basis Frame is a matrix of functions (0-forms), not a matrix of 1-forms. The literature is a bit confusing on this point, for often the tensor analysis machinery is used to describe a "tetrad" by the symbols ε^k_b in some places, and discuss its determinant properties as if it was an array of functions. Then, in other places, the same symbol is used to describe a matrix "coframe" of 1-forms. The matrix display used herein gets rid of this confusion; to repeat, the Basis Frame is a matrix of functions, not 1-forms. The Cartan Connection $[\mathbb{C}]$ is a matrix array of 1-forms.

3.1.2 Infinitesimal Neighborhoods

The fundamental assumption of differential closure implies that the Basis Frame $[\mathbb{B}(y)]$ can be used to map vector arrays of exact differentials $|dy^a\rangle$ into infinitesimally nearby vector arrays of 1-forms $|\sigma^k\rangle$ which are not necessarily exact differentials $|dx^k\rangle$:

$$\text{A set of Infinitesimal Basis Frames } : [\mathbb{B}^{\text{row}}_{\text{col}}(y)] \tag{14}$$

$$\text{Maps 1-forms } \quad \left[\mathbb{B}^k_a(y)\right] \circ |dy^a\rangle \;\Rightarrow\; \left|\sigma^k\right\rangle. \tag{15}$$

It is important to realize that the set of such Basis Frames, $[\mathbb{B}(y)]$, over infinitesimal neighborhoods is larger than the set of "global" basis frames $[\mathbb{F}(y)]$ which linearly map finite vectors into finite vectors,

$$\text{A set of Finite Basis Frames } : [\mathbb{F}(y)]$$

$$\text{Maps 0-forms } \quad [\mathbb{F}(y)] \circ |y^a\rangle \;\Rightarrow\; \left|x^k\right\rangle. \tag{16}$$

The set of Basis Frames, $[\mathbb{B}(y)]$, does not generate necessarily a set of diffeomorphisms.

Also note that $\left|\sigma^k\right\rangle$ is a vector array of 1-forms, and is not the same as a "coframe" matrix of 1-forms which makes up a cornerstone of Metric-Affine-Gravity theories [10] [9]. The vector array of 1-forms $\left|\sigma^k\right\rangle$ can be closed, exact, integrable, or non integrable. These are topological properties of the Pfaff Topological dimension (Class) of each of the 1-forms, σ^k.

$$Exact : \sigma^k = dy^k, \qquad \text{Pfaff dimension 1} \tag{17}$$

$$d\sigma^k = d(dy^k) = 0, \tag{18}$$

$$\oint \sigma^k = 0 \tag{19}$$

$$Closed\ but\ not\ exact : d\sigma^k = 0, \;\; \text{Homogeneous degree 0} \tag{20}$$

$$\oint \sigma^k = 2\pi \neq 0 \;\; \text{A Period Integral} \tag{21}$$

$$\text{example:}\sigma^k = |xdy - ydx\rangle \,/\, \left(x^2 + y^2\right) \tag{22}$$

$$Integrable\ Not\ Closed : d\sigma^k \neq 0,\; \sigma^k \hat{}\, d\sigma^k = 0 \qquad \text{Pfaff dimension 2} \tag{23}$$

$$Not\ integrable : \sigma^k \hat{}\, d\sigma^k \neq 0,\; d\sigma^k \hat{}\, d\sigma^k = 0 \;\; \text{Pfaff dimension 3} \tag{24}$$

$$Irreversible : d\sigma^k \hat{}\, d\sigma^k \neq 0 \qquad \text{Pfaff dimension 4} \tag{25}$$

An important topological fact is that when the Pfaff topological dimension of any 1-form is 3 or greater, the Cartan topology associated with such a system is a disconnected topology [32]. Moreover such systems of Pfaff Topological dimension greater than 2 have the thermodynamic properties of non equilibrium systems. Examples of these different topological possibilities are given Part 2.

3.1.3 The Cartan Left Connection

It is also possible to define a left Cartan Connection as a matrix of 1-forms, $[\mathbf{\Delta}]$,

$$\text{Left}: \text{ Cartan Connection } [\mathbf{\Delta}] \tag{26}$$

$$d[\mathbb{B}] = [\mathbf{\Delta}] \circ [\mathbb{B}], \tag{27}$$

$$[\mathbf{\Delta}] = -[\mathbb{B}] \circ d[\mathbb{B}]^{-1} \tag{28}$$

$$= +d[\mathbb{B}] \circ [\mathbb{B}]^{-1}. \tag{29}$$

The coefficients that make up the matrix of 1-forms, $[\mathbf{\Delta}]$, can be associated with what have been called the Weitzenboch connection coefficients..

The Right and Left Cartan connections are not (usually) identical. They are equivalent in terms of the similarity transformation:

$$[\mathbb{C}] = [\mathbb{B}]^{-1} \circ [\mathbf{\Delta}] \circ [\mathbb{B}]. \tag{30}$$

The left Cartan Connection, in general, is *not* the same as the transpose of the right Cartan Connection.

Also note that inverse matrix also enjoys differential closure properties.

$$d[\mathbb{B}]^{-1} = [\mathbb{B}]^{-1} \circ [-\mathbf{\Delta}], \tag{31}$$

$$= [-\mathbb{C}] \circ [\mathbb{B}]^{-1}. \tag{32}$$

3.2 Electromagnetic 2-forms from a topological structure

3.2.1 Vectors of Intensity 2-forms $|F^k\rangle$

It is remarkable that from the sole assumption that defines the physical vacuum, exterior differentiation leads to the topological formalism of electromagnetism. Both field structures of field intensities (analogous to E and B) and the field structures of the field excitations (analogous to D and H) are deduced by successive applications of the exterior derivative. Both the Maxwell Faraday equations and the Maxwell Ampere equations appear as necessary consequences of the definition of the "Physical Vacuum".

In fact, if the 1-forms $|\sigma^k\rangle$ were written to include a constant factor of physical dimensions, $\hbar/e$, the resulting k individual 1-forms are formally equivalent to a set of (up to 4) electromagnetic 1-forms of Action (the vector potentials), $|A^k\rangle$. Suppose for k = 1 to 3, $d\sigma^k = dA^k \Rightarrow 0$, then the single remaining 1-form can represent (formally) the classical development of electromagnetic field, $F = dA = d\sigma^4$ [35]. This equation defines topological features of an exterior differential system. An example is given in Section 3, below, where the Basis Frame is presumed to be developed from the Hopf map, which has the properties described above:

$$dA^1 = dA^2 = dA^3 = 0; \quad dA^4 = F. \tag{33}$$

The fact that, in general, there can be 4 such electromagnetic-like structures is a new development deduced from the single assumption used to define the "Physical Vacuum". These multiple field structures under suitable constraints of topological refinement can lead to Yang Mills theory, and perhaps more importantly to its generalizations.

For purposes of more rapid comprehension - based on the assumption of familiarity with electromagnetic theory - the notation for the Basis Frame infinitesimal mapping formula is rewritten as:

$$\left[\mathbb{B}_a^k(y)\right] \circ |dy_k\rangle \;\Rightarrow\; \left|A^k(y, dy)\right\rangle. \tag{34}$$

The notation is specific, but the formalism is universal and is valid for any continuum structures, such as found in fluids and plasmas.

Exterior differentiation of the infinitesimal mapping equation eq(34) generates a vector of (exact) 2-forms $\left|F^k\right\rangle$ which is formally equivalent (for each index, k) to the (pair, or even) 2-form of $\mathbf{E}, \mathbf{B}$ field intensities of electromagnetic theory.

$$\text{Vector } \left|F^k\right\rangle : \text{ of intensity 2-forms}$$

$$[\mathbb{B}(y)] \circ [\mathbb{C}] \,\hat{}\, |dy^a\rangle \;\Rightarrow\; \left|dA^k\right\rangle = \left|F^k\right\rangle, \tag{35}$$

$$\text{Field Intensity 2-forms } : \left|F^k(y, dy)\right\rangle = \left|dA^k\right\rangle, \tag{36}$$

The field intensity 2-forms (think E and B coefficients) are true tensor invariants relative to diffeomorphisms.

$$\text{By diffeomorphic substitution } : \text{of } \vartheta^b \Rightarrow\; y^k = \left|f^k(\vartheta^b\rangle\right), \tag{37}$$

$$\text{and} \left|dy^k\right\rangle = [\mathbb{J}] \circ \left|d\vartheta^k\right\rangle, \text{ into eq(34)} \tag{38}$$

$$\text{producing } [\mathbb{B}(\vartheta)] \Leftarrow [\mathbb{B}(y)] \circ [\mathbb{J}] \tag{39}$$

$$\text{with } \quad [\mathbb{B}(\vartheta)] \circ |d\vartheta\rangle \quad \Leftarrow \quad [\mathbb{B}(y)] \circ |dy\ \rangle \tag{40}$$

$$\text{yields } \left|A^k(\vartheta, d\vartheta)\right\rangle = \left|A^k(y, dy)\right\rangle, \tag{41}$$

$$\text{and } \left|F^k(\vartheta, d\vartheta)\right\rangle = \left|F^k(y, dy)\right\rangle. \tag{42}$$

3.2.2 Vectors of Excitation 2-forms: $\left|G^b\right\rangle$

On the other hand, another vector array of two forms $\left|G^b\right\rangle$ (think D and H coefficients) can be constructed from the formalism:

$$\text{Vector } \left|G^b\right\rangle : \text{ of excitation 2-forms}$$

$$[\mathbb{B}(y)] \circ [\mathbb{C}] \hat{} \left|dy^a\right\rangle = [\mathbb{B}(y)] \circ \left|G^b\right\rangle , \tag{43}$$

$$\text{Field Excitation 2-forms} = \left|G^b\right\rangle = \left[\mathbb{C}^b_{am} dy^m\right] \hat{} \left|dy^a\right\rangle \tag{44}$$

$$= \left|2\mathbb{C}^b_{[am]} dy^m \hat{} dy^a\right\rangle . \tag{45}$$

The "vector array" $\left|G^b\right\rangle$ of excitation 2-forms, unlike the vector array of intensity 2-forms, $\left|F^k\right\rangle$, is not a true "tensor" invariant with respect to diffeomorphisms. It transforms under pullback in terms of the inverse Jacobian mapping.

$$\text{By diffeomorphic : substitution of } y^a = \left|f^a(\vartheta^k)\right\rangle , \tag{46}$$

$$\text{with } dy^a = [J_{acobian}] \circ \left|d\vartheta^k\right\rangle \tag{47}$$

$$[\mathbb{B}(\vartheta)] \circ \left|G^b(\vartheta, d\vartheta)\right\rangle , \Leftarrow [\mathbb{B}(y)] \circ \left|G^k(y, dy)\right\rangle , \tag{48}$$

$$\text{yields } \left|G^b(\vartheta, d\vartheta)\right\rangle = [J_{acobian}]^{-1} \circ \left|G^k(y, dy)\right\rangle \tag{49}$$

Conclusion 3 *Counter to current tensor dogma, the bottom line is that, in general, the vector array of field excitations* $\left|G^b\right\rangle$ *is an array of "impair" 2-forms, and depends upon a choice of diffeomorphic coordinates. The vector array of field intensities* $\left|F^k\right\rangle$ *is an array of "pair" 2-forms, and does not upon the choice of diffeomorphic coordinates. This fact implies that charge is a pseudo-scalar, not a scalar [14].*

3.3 Affine Torsion is the source of Excitation 2-forms (EM Sources)

The vector of Excitation Torsion 2-forms $\left|G^b\right\rangle = \left[\mathbb{C}^b_a\right] \hat{} \left|dy^a\right\rangle$ has coefficients that are in 1-1 correspondence with what has been defined historically as the coefficients of "Affine" torsion. It is important to realize that the use of the words "Affine torsion" to describe the antisymmetric coefficients of a Cartan connection is unfortunate, and has nothing to do with whether or not the Basis Frame matrix is a member of the Affine group, or one of its subgroups. Classically, the affine group is a *transitive* group of 13 parameters in 4D, (see p.162 in Turnbull [30]). The anti-symmetry concept related to torsion is described by the same formula that defines the vector of excitation 2-forms, $\left|G^b\right\rangle = \left[\mathbb{C}^b_a\right] \hat{} \left|dy^a\right\rangle$, for any Basis Frame $[\mathbb{B}(y)]$. For example, the torsion formula holds equally well for Basis Frames which are elements of the 15 parameter projective group, which is not affine.

$$\text{(Affine) Torsion 2-forms } \left|G^b\right\rangle = [\mathbb{C}] \hat{} \left|dy^a\right\rangle , \tag{50}$$

$$\text{Vector of Field Excitation 2-forms} = \left|G^b\right\rangle \tag{51}$$

$$= [\mathbb{B}]^{-1} \circ \left|F^k\right\rangle , \tag{52}$$

The vector of 2-forms $\left|G^b\right\rangle$ is formally equivalent (for each index b) to the (impair, or odd) 2-form (density) of the field excitations (D and H) in electromagnetic theory [35]. In the notation of electromagnetism, the source of field excitations (and, consequently, topological charge and and topological spin) is due to the Affine Torsion components of the Cartan Connection.

Note that the matrix $[\mathbb{B}]^{-1}$ plays the role of the Constitutive map between $\mathbf{E},\mathbf{B}$ and $\mathbf{D},\mathbf{H}$.

$$\left|G^b\right\rangle = [\mathbb{B}]^{-1} \circ \left|F^k\right\rangle, \tag{53}$$
$$[\mathbb{B}]^{-1} \approx \text{a Constitutive map} \tag{54}$$

If the global (integrability) assumption, $[\mathbb{F}(y)] \circ |y^a\rangle \Rightarrow \left|x^k\right\rangle$, is imposed, then it is possible by exterior differentiation to show that a constraint must be established between the excitation 2-forms and the Cartan Curvature 2-forms constructed from the globally integrable Basis Frames, $[\mathbb{F}(y)]$:

$$[\mathbb{F}(y)] \circ \{[\mathbb{C}_{\mathbb{F}}] \circ |y^a\rangle + |y^a\rangle = \left|dx^k\right\rangle, \tag{55}$$
$$\text{such that } [\mathbb{C}_{\mathbb{F}}] \circ |dy^a\rangle = -\{d[\mathbb{C}_{\mathbb{F}}] + [\mathbb{C}_{\mathbb{F}}]\hat{\ }[\mathbb{C}_{\mathbb{F}}]\} \circ |y^a\rangle . \tag{56}$$

Hence as the vector of excitation 2-forms $\left|G^b\right\rangle$ has been defined in terms of the Cartan Connection, two different results are obtained for the two different types of Basis Frames:

$$\left|G^b\right\rangle_{\mathbb{B}} = [\mathbb{C}_{\mathbb{B}}]\hat{\ } |dy^a\rangle \tag{57}$$
$$\left|G^b\right\rangle_{\mathbb{F}} = [\mathbb{C}_{\mathbb{F}}]\hat{\ } |dy^a\rangle . \tag{58}$$

The integrability condition places a constraint on the Cartan Curvature and the Affine torsion coefficients of the Cartan Connection, $[\mathbb{C}_{\mathbb{F}}]$, which is not equivalent to the constitutive map (eq 53) given above:

$$\left|G^b\right\rangle_{\mathbb{F}} = -\{d[\mathbb{C}] + [\mathbb{C}]\hat{\ }[\mathbb{C}]\} |y^a\rangle \tag{59}$$
$$\neq [\mathbb{B}]^{-1} \circ \left|F^k\right\rangle . \tag{60}$$

The result demonstrates that the set of infinitesimal Basis Frames is much different from the global set of Basis Frames.

None of this development depends upon the explicit specification of a metric, reinforcing the fact that Maxwell's theory of Electrodynamics is a topological, not a geometric theory [35]. Again, remember that the electromagnetic notation is used as a learning crutch to emphasize the universal ideas of the Physical Vacuum. The formulas are valid topological descriptions of the field structures of all continuum "fluids".

Herein, for simplicity, it is assumed that all functions of the Basis Frame are at least C2. However, note that the definitions of the matrix of connection 1-forms $[\mathbb{C}]$ and the vector of 2-forms $|F\rangle$ only require C1 functions.

3.3.1 Cartan Torsion ≠ Affine Torsion

It is possible to use the left matrix of Cartan Connection 1-forms to define another vector of torsion 2-forms. Exterior differentiation of the infinitesimal mapping equation eq. (34) produces the equation:

$$\left[\mathbf{\Delta}_m^k\right] \hat{} ([\mathbb{B}_a^m(y)] \circ |dy^a\rangle) \Rightarrow \left[\mathbf{\Delta}_m^k\right] \hat{} |\sigma^m\rangle = \left|d\sigma^k\right\rangle. \tag{61}$$

The algebraic combination given below, is defined as the Cartan vector of Torsion 2-forms, $\left|\Sigma^k\right\rangle$. It is not the same as the vector of excitation ("affine") Torsion 2-forms defined by $\left|G^b\right\rangle_{\mathbb{B}} = [\mathbb{C}_{\mathbb{B}}]\hat{}\,|dy^a\rangle$

$$\text{Cartan : vector of Torsion 2-forms } \left|\Sigma^k\right\rangle \tag{62}$$
$$\left|\Sigma^k\right\rangle = \left|d\sigma^k\right\rangle - \left[\mathbf{\Delta}_m^k\right] \hat{} |\sigma^m\rangle \neq \left|G^k\right\rangle, \tag{63}$$
$$\left|\Sigma^k\right\rangle \Rightarrow 0. \tag{64}$$

The Cartan vector of Torsion 2-forms, $\left|\Sigma^k\right\rangle$, is always Zero for "Physical Vacuums", while the vector of excitation Torsion 2-forms[2], $\left|G^k\right\rangle$ is not necessarily zero. The $\left|\Sigma^k\right\rangle$ are often said to define one of Cartan's equations of structure. Another equation of structure, based upon curvature, will be deduced below. Cartan's equations of structure are equal to zero for the Physical Vacuum.

3.3.2 Maxwell Faraday and Maxwell Ampere Equations

The exterior derivative of the vector of field intensity 2-forms $\left|F^k\right\rangle$ vanishes, resulting in k sets of partial differential equations of the Maxwell Faraday [35] type:

$$\left|dF^k\right\rangle \Rightarrow 0 \tag{65}$$
$$= \text{Maxwell Faraday PDE,s.} \tag{66}$$

The concept requires the functions that define the potentials $\left|A^k\right\rangle$ be C2 differentiable.

[2] Whose coefficients are the classical "affine" torsion coefficients.

The exterior derivative of the vector of 2-form excitations $\left|G^b\right\rangle$ produces a vector of 3-forms (formally) equivalent (for each index b) to the conserved electromagnetic charge-current density for C2 functions.

$$\left|dG^b\right\rangle = \left|J^b\right\rangle, \tag{67}$$

$$d\left|J^b\right\rangle = 0, \tag{68}$$

$$= \text{Maxwell Ampere PDE,s} \tag{69}$$

It is extraordinary that the format of both the Maxwell Faraday and the Maxwell Ampere equations occur naturally from exterior differential processes applied to the fundamental postulate of a Physical Vacuum. If all but one of the four 1-form components of $\left|A^k\right\rangle$ are closed, then the formalism encodes the classical theory of Electromagnetic fields. A non trivial example constructed from the Hopf map will be given in Part 2. Moreover, if the totality of the four 1-form components of $\left|A^k\right\rangle$ are not closed, the same starting point encodes the fields that are utilized by Yang Mills theory. Each of these specializations is a topological refinement.

3.4 The Lorentz Force and the Lie differential

The Lorentz force is a *derived, universal,* concept in terms of the thermodynamic cohomology. It is generated by application of Cartan's Magic formula [11] to the 1-form A that that encodes all or part of a thermodynamic system. The system, A, can be interpreted as an electromagnetic system, or a hydrodynamic system, or any other system that supports continuous topological evolution. [32]. For the purposes herein apply Cartan's magic formula to the formula for infinitesimal mapping produced by the matrix multiplication of a vector of perfect (exact) differentials of the base variables given by eq(34).

$$\left[\mathbb{B}^k_a(y)\right] \circ \left|dy_k\right\rangle \;\Rightarrow\; \left|A^k(y,dy)\right\rangle \tag{70}$$

$$L_{(V)}\{\left[\mathbb{B}^k_a(y)\right] \circ \left|dy_k\right\rangle\} = L_{(V)}\left|A^k(y,dy)\right\rangle. \tag{71}$$

Recall that the exterior derivative of any specific 1-form, A^k, if not zero, can be defined as a 2-form with coefficients of the type

$$F = dA = \{\partial A_k/\partial x^j - \partial A_j/\partial x^k\} dx^j\hat{}dx^k = F_{jk} dx^j\hat{}dx^k \tag{72}$$
$$= \mathbf{B}_z dx\hat{}dy + \mathbf{B}_x dy\hat{}dz + \mathbf{B}_y dz\hat{}dx + \mathbf{E}_x dx\hat{}dt + \mathbf{E}_y dy\hat{}dt + \mathbf{E}_z dz\hat{}dt.$$

The specialized notation for the coefficients used above is that often used in studies of electromagnetism, but the topological 2-form concepts are universal, independent from the notation.

Given any process that can be expressed in terms of a vector direction field, $V = \rho[\mathbf{V}, 1]$, and for a physical system, or component of a physical system, that can be encoded in terms of a 1-form of Action, A, the topological evolution of the 1-form relative to the direction field can be described in terms the Lie differential:

$$\begin{aligned} L_{(V)}A &= i(V)dA + d(i(V)A) && (73)\\ &= i(V)F + d(i(V)A) && (74)\\ &= \rho\{\mathbf{E} + \mathbf{V} \times \mathbf{B})_k dx^k - \rho\{\mathbf{E} \cdot \mathbf{V}\}dt && (75)\\ &+ d(\rho\mathbf{A} \cdot \mathbf{V} - \rho\phi) && (76)\\ &= \text{Work due to Lorentz force - dissipative power} && (77)\\ &+ \text{change of internal interaction energy.} && (78) \end{aligned}$$

Note that if the notation is changed (such that the vector potential is designated as the velocity components of a fluid), then the "Lorentz force" represents the classic expression to be found in the formulation of the hydrodynamic Lagrange Euler equations of a fluid [34]. A fluid, based upon a 1-form of Action of Pfaff Topological dimension 2 (or greater) obeys the topological equivalent of a Maxwell Faraday induction law!

Remark 4 *The universal concept of a Lorentz force is derived from the properties of a "Physical Vacuum", and does not require a separate postulate of existence.*

3.5 3-forms

3.5.1 Topological Torsion and Topological Spin 3-forms

The development above indicates that certain vectors of 3 forms are pertinent to the theory of conserved currents. However, there are other important 3-forms that can be deduced from the single postulate that defines a Physical Vacuum. These 3-forms have utilization in the understanding of both non equilibrium thermodynamic phenomena and, remarkably enough, the concept of Topological Spin.

Construct the (pair - even) Topological Torsion [17] [35] scalar of 3-forms, defined as

$$\begin{aligned} \textbf{Topological Torsion:}\quad H &= \left\langle A^k \right| \hat{}\, \left| F^k \right\rangle && (79)\\ &= A^1\hat{}\,F^1 + A^2\hat{}\,F^2 + A^3\hat{}\,F^3 + A^4\hat{}\,F^4. && (80) \end{aligned}$$

Note that the topological dimensions of H are that of the Flux Quantum, $\hbar/e$. The object H is composed of 3-forms that are tensor invariants relative to diffeomorphisms.

Each of the four 3-form elements for a fixed index k, $A^k\hat{}F^k$, if not zero, will indicate that the Pfaff Topological dimension of the specific 1-form, A^k, is 3 or more. The idea is that it takes at least 3 independent functions (of the 4 space time variables) to describe the topological features of the 1-form of given index, k. As equilibrium thermodynamic phenomena require two independent functions at most [4], it follows that systems where the Pfaff Topological dimension is 3 or more are non equilibrium systems.

Construct the (impair-odd) Topological Spin [19] [35] (pseudo) scalar of 3-forms defined as

$$\textbf{Topological Spin:} \quad S = \langle A^m | \hat{} \, | G^m \rangle \, , \tag{81}$$

$$= A^1\hat{}G^1 + A^2\hat{}G^2 + A^3\hat{}G^3 + A^4\hat{}G^4. \tag{82}$$

For electromagnetic systems, the physical units of S are $\hbar$ (Planck's constant of angular momentum). The concept of Topological Spin depends upon the fact that the vector of excitation 2-forms, $|G\rangle$, is not zero. Consequently the coefficients of Affine torsion must not be zero. "Affine Torsion" is necessary, but not sufficient to produce Topological Spin. Examples of such properties will be given in Part 2. Note that the individual 3-form components, $A\hat{}G$, of the Topological Spin composite do not transform as a tensor invariant relative to diffeomorphisms. Each term does depend on the Inverse Jacobian matrix of the diffeomorphism. Each term has a pullback proportional to the adjoint of the Jacobian matrix, divided by the determinant of the transformation. If the diffeomorphisms are limited to elements of the special orthogonal (or unitary) group[3] then a dualism can be defined between the vector of intensities and the vector of excitations. However, this self (or anti-self) dualism is a very special case (assumed in the Yang-Mills formalism, but not herein).

3.5.2 Poincare Invariants

Exterior differentiation of these two composite 3-forms, H and S, produces the Poincare invariants

$$dH = \left\langle F^k \right| \hat{} \left| F^k \right\rangle = \sum_k 2(\mathbf{E}\cdot\mathbf{B})^k \quad = \textbf{Poincare II} \tag{83}$$

$$dS = \left\langle F^k \right| \hat{} \left| G^k \right\rangle - \left\langle A^k \right| \hat{} \left| J^k \right\rangle \qquad = \textbf{Poincare I.} \tag{84}$$

$$= \sum_l [\{(\mathbf{B}\cdot\mathbf{H})^k - (\mathbf{D}\cdot\mathbf{E})^k\} - \{(\mathbf{A}\cdot\mathbf{J})^k - (\rho\phi)^k\}]. \tag{85}$$

[3] Such as SO3, or SO2.

The notation in terms of $\mathbf{E}, \mathbf{B}, \mathbf{D}, \mathbf{H}$ is symbolic, and refers only to the formal equivalence of the formulas to electromagnetic theory. If the Poincare invariants vanish, the closed integrals of the closed 3-forms could become topologically quantized through the concept of deRham period integrals. There are many ways that this result could happen, due to the fact that there are four (Yang Mills type) components to the excitation 2-forms, $\left|G^b\right\rangle$. Note that if there exists only 1-component of $\left|G^b\right\rangle$, say G^4, and it is closed, $dG^4 = 0$, then, formally, the integral over a closed 2D integration chain of G^4 defines the Quantized Charge, e, of classical electromagnetic theory.

$$\iint_{closed} G^4 \Rightarrow e. \tag{86}$$

The multi-component Field intensity and Field excitation equations are *extensions* of a Yang Mills theory, as $\pm$ self duality is not imposed on the system.

3.6 Period Integrals and Topological Quantization.

Although the main interest of this article is associated with the field properties of the Physical Vacuum, a few words are appropriate about the topological defect structures (that will be treated in more detail in another article). Of specific interest are those topological structures represented by closed, but not exact, differential forms. Such exterior differential forms are homogeneous of degree zero expressions. Such closed structures can lead to deRham period integrals [18], whose values have rational ratios, when the integration chain, z1, is also closed, and not equal to a boundary. For example, the Flux Quantum of EM theory is given by the closed integral of the (electrodynamic) 1-form of Action:

$$\text{Flux quantum} = n \oint_{z1} A \tag{87}$$

$$\text{In domains where } F = dA \Rightarrow 0 \tag{88}$$

This period integral is NOT dependent upon the electromagnetic field intensities, F. Stokes theorem does not apply if the integration chain is not a boundary. All integrals of exact forms over boundaries would yield zero.

As another example, the Period integral of quantized charge is given by the expression,

$$\text{Charge quantum} = n \iint_{z2} G \tag{89}$$

$$\text{In domains where } J = dG \Rightarrow 0. \tag{90}$$

The integration is over a closed chain, z2, which is not a boundary. Again, the quantized Period integral does not depend upon the charge current 3-form, J, and the expression is valid only in domains where $J \Rightarrow 0$. These same properties of topological quantization are universal ideas independent from the notation, or some topological refinement to a specific types of physical systems. Note that the concept of the Charge quantum depends upon the existence of G which in turn depends upon the existence of Affine Torsion of the Cartan Connection $[\mathbb{C}]$ based on $[\mathbf{B}]$.

3.7 Quadratic Congruents and Metrics

Starting from the existence of a Linear Basis Frame, it is remarkable that symmetric properties of $[\mathbb{B}]$ can be deduced in terms of a quadratic congruence (see p. 36 in Turnbull and Aitken [31]). The quadratic congruence is related to the concept of strain in elasticity theory, and is quite different from the linear definition of matrix symmetries in terms of the sum of a matrix $[\mathbb{B}]$ and its transpose $[\mathbb{B}]^T$. The algebraic quadratic congruence will be used to define compatible symmetric (metric) qualities in terms of the structure of the Basis Frames, $[\mathbb{B}]$:

$$[g] = [\mathbb{B}]^T \circ [\eta] \circ [\mathbb{B}]. \tag{91}$$

The matrix $[\eta]$ is a (diagonal) Sylvestor signature matrix whose elements are ± 1. Recall that in projective geometry the congruence transformation based upon $[\mathbb{B}]^T$ defines a correlation, where the similarity transformation based upon $[\mathbb{B}]^{-1}$ defines a collineation. Note that the ubiquitous choice of an orthonormal basis frame, where $[\mathbb{B}]^T = [\mathbb{B}]^{-1}$, limits the topological generality of the concept of a Physical Vacuum.

In a later section it will be demonstrated how the similarity invariants of the Basis Frame find use in representing thermodynamic phase functions appropriate to the Physical Vacuum.

Remark 5 *Note that this quadratic (multiplicative) symmetry property is not the equivalent to the (additive) symmetry property defined by the linear sum of the matrix* $[\mathbb{B}]$ *and its transpose.*

However, from the Basis Frame, it is also possible to construct a topological exterior differential system that defines a quadratic form of the law of differential closure. It then follows that, for $d[\eta] = 0$,

$$d[g] = [\widetilde{\mathbb{C}}_r] \circ [g] + [g] \circ [\mathbb{C}_r]. \tag{92}$$

This deductive result leads to the metricity condition for a Physical Vacuum, in terms of the right Cartan connection $[\mathbb{C}_r]$:

$$\textbf{Metricity condition:} \;\; d[g] - [\widetilde{\mathbb{C}}_r] \circ [g] - [g] \circ [\mathbb{C}_r] \Rightarrow 0. \tag{93}$$

This equation is an exterior differential system, and therefor defines topological properties.

Compute the Christoffel connection, and its matrix of 1-forms, $[\Gamma]$, from the quadratic "metric" matrix $[g]$, using the Levi-Civita-Christoffel formulas.

$$\text{Coefficients : Christoffel Connection} \tag{94}$$

$$\Gamma^b_{ac}(\xi^c) = g^{be}\{\partial g_{ce}/\partial\xi^a + \partial g_{ea}/\partial\xi^c - \partial g_{ac}/\partial\xi^e\}, \tag{95}$$

$$[\Gamma] = \left[\Gamma^b_{ac} dy^c\right] \quad \text{as a matrix of 1-forms} \tag{96}$$

The Christoffel Connection also satisfies the metricity condition,

$$\textbf{Metricity condition:} \;\; d\,[g] - [\Gamma] \circ [g] - [g] \circ [\Gamma] \Rightarrow 0. \tag{97}$$

3.7.1 Decomposition of the Cartan connection

The linear properties of matrices permits the Cartan Connection matrix $[\mathbb{C}]$ to be decomposed into a term, $[\Gamma]$, based on its quadratic congruent symmetries, and a residue $[\mathbb{T}]$ that will contain any asymmetries due to non zero Affine Torsion coefficients. Decompose the Cartan Connection matrix of 1-forms as follows:

$$[\mathbb{C}] = [\mathbf{\Gamma}] + [\mathbb{T}] \tag{98}$$

As the topological metricity condition is automatically satisfied for Christoffel Connection $[\mathbf{\Gamma}]$, as well as for the Cartan Connection, it must also be satisfied for residue matrix of 1-forms, $[\mathbb{T}]$.

3.7.2 Matrices of Curvature 2-forms

Construct the matrix of Cartan Curvature 2-forms, $[\mathbf{\Phi}]$, derived from the second exterior differentiation of the Basis Frame, $dd\,[\mathbb{B}]$, and based on the Cartan Connection, $[\mathbb{C}]$:

$$dd\,[\mathbb{B}] = d\,[\mathbb{B}] \circ [\mathbb{C}] + [\mathbb{B}] \circ d\,[\mathbb{C}]\,, \tag{99}$$

$$= [\mathbb{B}] \circ \{d\,[\mathbb{C}] + [\mathbb{C}] \,\hat{}\, [\mathbb{C}]\}, \tag{100}$$

$$= [\mathbb{B}] \circ [\mathbf{\Phi}] \Rightarrow 0, \tag{101}$$

$$\text{Curvature 2-forms } [\mathbf{\Phi}] = \{d\,[\mathbb{C}] + [\mathbb{C}] \,\hat{}\, [\mathbb{C}]\}, \tag{102}$$

$$[\mathbf{\Phi}] = dd\,[\mathbb{B}] \Rightarrow 0. \tag{103}$$

The zero result, based upon the Poincare lemma, $dd\,[\mathbb{B}] \Rightarrow 0$, requires C2 differentiability of the Basis Frame functions.

The formula given by eq(102) is called Cartan's second (sometimes the first) structural equation and is based upon a matrix of curvature 2-forms. The formula can be applied to arbitrary connections (not defined in terms of the Basis Frame of the Physical Vacuum) and will yield non zero values. Similar remarks may be made about Cartan's first equation of structure based on Cartan's vector of Torsion 2-forms given by eq(63). Cartan's matrix of curvature 2-forms, and Cartan's vector of Torsion 2-forms are both zero for the "Physical Vacuum". The "Affine" torsion coefficients of the Cartan Matrix need not be zero for the "Physical Vacuum".

3.7.3 Bianchi Identities and Matrices of 3-forms.

Note that exterior differentiation of the Cartan structure matrix of curvature 2-forms is equivalent to the Bianchi identity:

$$[d\mathbf{\Phi}] + [d\mathbb{C}] \hat{} [\mathbb{C}] - [\mathbb{C}] \hat{} [d\mathbb{C}] = \tag{104}$$

$$[d\mathbf{\Phi}] + [\mathbf{\Phi}] \hat{} [\mathbb{C}] - [\mathbb{C}] \hat{} [\mathbf{\Phi}] \Rightarrow 0. \tag{105}$$

This concept of a Bianchi identity is valid for all forms of the Cartan structure equations. The Bianchi statements are essentially definitions of cohomology, in that the difference between two non-exact p-forms is equal to a perfect differential (an exterior differential system). In this case the Bianchi identity describes the cohomology established by two matrices of 3-forms, $[J2] - [J1]$.

$$[J2] - [J1] = [d\mathbf{\Phi}]\,, \tag{106}$$

$$\text{where } [J1] = [d\mathbb{C}] \hat{} [\mathbb{C}] \tag{107}$$

$$\text{and } [J2] = [\mathbb{C}] \hat{} [d\mathbb{C}]\,. \tag{108}$$

3.8 The Higgs vector of zero forms (Internal Energy)

Once again consider the Lie differential with respect to a direction field V, operating on the formula for differential closure

$$L_{(V)}([\mathbb{B}(y)] \circ |dy^a\rangle) = L_{(V)}(|A^a\rangle) = i(V)d\,|A^a\rangle + d(i(V)\,|A^a\rangle) \tag{109}$$

$$= i(V)\,|F^a\rangle + d(i(V)\,|A^a\rangle) = \tag{110}$$

$$= |W^a\rangle + d\,|h^a\rangle\,. \tag{111}$$

From Koszul's theorem, $|W^a\rangle = i(V)d\,|A^a\rangle$ is a covariant differential based on some (abstract) connection (for each a). Hence, the difference between the Lie differential and the

Covariant differential is the exact term, $d(i(V)\,|A^a\rangle)$:

$$L_{(V)}(|A^a\rangle) - i(V)d\,|A^a\rangle = d(i(V)\,|A^a\rangle) = d\,|h^a\rangle\,. \tag{112}$$

This equation is another statement of Cohomology, another exterior differential system, where the difference of two non-exact objects is an exact differential.

From the topological formulation of thermodynamics [32] in terms of Cartan's magic formula [11],

$$\text{Cartan's Magic Formula } L_{(\rho\mathbf{V}_4)}A = i(\rho\mathbf{V}_4)dA + d(i(\rho\mathbf{V}_4)A) \tag{113}$$

$$\text{First Law } : W + dU = Q, \tag{114}$$

$$\text{Inexact Heat 1-form } \; Q = W + dU = L_{(\rho\mathbf{V}_4)}A \tag{115}$$

$$\text{Inexact Work 1-form } W = i(\rho\mathbf{V}_4)dA, \tag{116}$$

$$\text{Internal Energy } \; U = i(\rho\mathbf{V}_4)A, \tag{117}$$

Now consider particular process paths (defined by the directional field $\rho\mathbf{V}_4$), and deduce that in the direction of the process path

$$i(\rho\mathbf{V}_4)W = 0, \tag{118}$$

$$\text{Work } : \text{ is transversal;} \tag{119}$$

$$i(\rho\mathbf{V}_4)Q = i(\rho\mathbf{V}_4)dU \;\neq 0 \tag{120}$$

$$\text{Heat } : \text{is not transversal;} \tag{121}$$

$$\text{but if } \; i(\rho\mathbf{V}_4)Q = 0, \tag{122}$$

$$\text{the process } : \text{is adiabatic.} \tag{123}$$

It is the non-adiabatic components of a thermodynamic process that indicate that there is a change of internal energy and hence an inertial force in the direction of a process. This implies that the non-adiabatic processes are inertial effects, and could be related to changes in mass.

Now to paraphrase a statements and ideas from Mason and Woodhouse, (see p. 49 [12]) and [1] :

"... then there is a Higgs field.... which measures the difference between the Covariant differential along V and the Lie differential along V."

It becomes apparent that the

$$|W^a\rangle = \text{Vector of Work 1-forms. (transversal)} \tag{124}$$
$$|h^a\rangle = \text{Higgs potential as vector of 0-forms (Internal Energy)} \tag{125}$$
$$d\,|h^a\rangle = \text{Higgs vector of 1-forms.} \tag{126}$$
$$i(V)d\,|h^a\rangle = \text{vector of longitudinal inertial accelerations} \tag{127}$$
$$= \text{non adiabatic components of a process} \tag{128}$$

The method of the "Physical Vacuum" and its sole assumption leads to inertial properties and the Higgs field, all from a topological perspective and without "quantum" fluctuations.

3.9 A Strong Equivalence Principle

At this point, there has been no indication that the problem being investigated has anything to do with the Gravitational Field. The gravity issue is to be encoded into how the quadratic congruent symmetries of $[\mathbb{B}]$, and its topological group structures, are established. In general, different choices for the group structure of the Basis Frame will strongly influence the application to any particular physical system of fields and particles.

Without the Einstein Ansatz, it appears that the concept of a Physical Vacuum can lead to a Strong Equivalence principle. Substitute $[\Gamma]+[\mathbb{T}]$ for $[\mathbb{C}]$ in the definition of the matrix of curvature 2-forms, and recall that for the Physical Vacuum the Cartan matrix of curvature 2-forms, $[\mathbf{\Phi}]$, is zero.

$$[\mathbf{\Phi}_{\mathbb{C}}] = \{d\,[\mathbb{C}] + [\mathbb{C}]\,\hat{}\,[\mathbb{C}]\} \Rightarrow 0, \tag{129}$$
$$= \{d([\Gamma]+[\mathbb{T}]) + ([\Gamma]+[\mathbb{T}])\hat{}([\Gamma]+[\mathbb{T}])\} \tag{130}$$
$$= \{d[\Gamma] + [\Gamma]\hat{}[\Gamma]\} + \{[\mathbb{T}]\,\hat{}\,[\Gamma] + [\Gamma]\,\hat{}\,[\mathbb{T}]\} + \{d[\mathbb{T}] + [\mathbb{T}]\hat{}[\mathbb{T}]\}, \tag{131}$$

Separate the matrices of 2-forms into the metric based (Christoffel) curvature 2-forms, defined as

$$[\mathbf{\Phi}_{\mathbf{\Gamma}}] = \{d\,[\Gamma] + [\Gamma]\,\hat{}\,[\Gamma]\} = [Field\ metric\ 2-forms]\,, \tag{132}$$

and the remainder, defined as

$$[-\mathbf{\Phi}_{Inertial}] = [\mathbf{\Phi}_{\mathbb{C}}] - [\mathbf{\Phi}_{\mathbf{\Gamma}}] \tag{133}$$
$$= \{[\mathbb{T}]\,\hat{}\,[\Gamma] + [\Gamma]\,\hat{}\,[\mathbb{T}]\} + \{d[\mathbb{T}] + [\mathbb{T}]\hat{}[\mathbb{T}]\} \tag{134}$$
$$= \{interaction\_2-forms\} \quad + \quad \{[\mathbf{\Phi}_{\mathbb{T}}]\} \tag{135}$$

. The decomposition leads to the strong equivalence equation,

$$\text{Principle of : Strong Equivalence} \tag{136}$$

$$[\text{Metric Field curvature 2-forms}] = [\text{Inertial curvature 2-forms}], \tag{137}$$

$$[\mathbf{\Phi}_{\mathbf{\Gamma}}] = [-\mathbf{\Phi}_{Inertial}] \tag{138}$$

3.10 The Source of electromagnetic Charge and Spin

A number of years ago, it became apparent to me that the origin of charge was to be associated with topological structures of space time [26]. The concepts exploited in this study assumed the existence of an impair 2-form of field intensities G. Based on the development above, it can be stated:

Remark 6 *The theory of a Physical Vacuum asserts that the existence of charge is dependent upon the Affine Torsion of the Cartan connection* $[\mathbb{C}]$.

This result is based upon the formal correspondence between equations of electromagnetic field excitations $|G\rangle$ and the "Affine torsion" coefficients (not the Cartan torsion coefficients) deduced for the Cartan Connection matrix of 1-forms $[\mathbb{C}]$.

$$\text{Excitation 2-forms } |G\rangle = [\mathbb{C}] \hat{} |dy\rangle \text{ Affine Torsion 2-forms.} \tag{139}$$

The impair 2 forms that compose the elements of $|G\rangle$ formally define the field excitations in terms of the coefficients of "Affine torsion". The closed period integrals of those (closed but not exact) components of $|G\rangle$, which are homogeneous of degree 0, lead to the deRham integrals with rational, quantized, ratios. Such excitation 2-forms G do not exist if the coefficients of Affine torsion are zero.

The topological features of the Physical Vacuum are determined by the structural properties of the Basis Frames, $[\mathbb{B}]$, and the derived Cartan Connection matrix of 1-forms, $[\mathbb{C}]$.

The Topological Structure of the Physical Vacuum

in terms of $[\mathbb{B}]$ and $[\mathbb{C}]$

Mass	Non Zero Cartan Curvature of of $[\Gamma] = [\mathbb{B}]^T \circ [\eta] \circ [\mathbb{B}]$	Cartan Curvature of $[\mathbb{C}]$ is ZERO
Charge	Non Zero Affine Torsion of $[\mathbb{C}]$ $\lvert G\rangle = [\mathbb{C}] \hat{} \lvert dy\rangle$	Cartan Torsion of $[\mathbb{C}]$ is ZERO

Where the presence of mass is recognized in terms of the Riemannian curvature of the quadratic congruences of a Physical Vacuum, the presence of charge is recognized in terms of the coefficients of Affine Torsion of a Physical Vacuum.

4 Remarks

This universal set ideas enumerated in Part I startles me. There is only ONE fundamental assumption, and the rest of the concepts are derived, following the rules of the Cartan exterior calculus. The results appear to be universal rules. Under topological refinements, the rules are specialized into domains that are recognizable as having the features of the four forces of physics. Not only do the long range concepts (fields) of gravity and electromagnetism have the same base, but so also do the short range concepts (fields) of the nuclear and weak force have the same base. Earlier, related, thoughts about the topological and differential geometric ideas associated with the four forces appeared in [15]. Now the topological theory of a Physical Vacuum as presented above re-enforces the earlier work. In addition, particulate concepts also appear from the same fundamental postulate of a Physical Vacuum. They appear as topologically coherent defect structures in the fields. Quantization occurs in a topological manner from the deRham theorems as period integrals. The quantized topologically coherent structures form the basis of macroscopic quantum states.

Part II

Examples

5 Example 1. The Schwarzschild Metric embedded in a Basis Frame, $[\mathbb{B}]$, as a 10 parameter subgroup of an affine group.

5.1 The Metric - a Quadratic Congruent symmetry

The algebra of a quadratic congruence can be used to deduce the metric properties of a given Basis Frame. These deduced metric features may be used to construct a "Christoffel" or metric compatible connection, different from the Cartan Connection. The Christoffel connection constructed from the quadratic congruence of the Basis Frame may or may not generate a "Riemannian" curvature. By working backwards, this example will demonstrate how to construct the Basis Frame, given a metric. The method is algebraic and exceptionally simple for all metrics that represent a 3+1 division of space-time. The important result is that given a Basis Frame of a given matrix group structure, a metric and a compatible Christoffel Connection can be deduced.

Remark 7 *All 3+1 metric structures are to be associated with the 10 parameter subgroup of the 13 parameter Affine groups.*

In this example, it will be demonstrated how the isotropic form of the Schwarzschild metric can be incorporated into the Basis Frame for a Physical Vacuum, [$\mathbb{B}$]. The technique is easily extendable for diagonal metrics. However, the symmetry properties of the Cartan Connection are not limited to metrics of the "gravitational" type. Once the Schwarzschild metric is embedded in to the Basis Frame, then the universal methods described above will be applied to the representative Basis Frame, and each important result will be evaluated.

The isotropic Schwarzschild metric is a diagonal metric of the form,

$$(\delta s)^2 = -(1+m/2r)^4\{(dx)^2+(dy)^2+(dz)^2\}+\frac{(2-m/r)^2}{(2+m/r)^2}(dt)^2 \tag{140}$$

$$= -(\alpha)^2\{(dx)^2+(dy)^2+(dz)^2\}+(\beta)^2(dt)^2 \tag{141}$$

$$\text{with } r = \sqrt{(x)^2+(y)^2+(z)^2}, \tag{142}$$

As Eddington [5] points out, the isotropic form is palatable with the idea that the speed of light is equivalent in any direction. That is not true for the non-isotropic Schwarzschild metric, where transverse and longitudinal null geodesics do not have the same speed.

For the isotropic Schwarzschild example, the metric $[g_{jk}]$ can be constructed from the triple matrix product:

$$[g_{jk}] = [\widetilde{f}]\circ[\eta]\circ[f]\,, \tag{143}$$

$$\text{where } f = \begin{bmatrix} \alpha & 0 & 0 & 0 \\ 0 & \alpha & 0 & 0 \\ 0 & 0 & \alpha & 0 \\ 0 & 0 & 0 & \beta \end{bmatrix}, \tag{144}$$

$$\text{and} \quad \alpha = (1+m/2r)^2 \; = (\gamma/2r)^2, \quad \beta = \frac{(2-m/r)}{(2+m/r)} \; = \delta/\gamma, \tag{145}$$

$$\text{and} \quad \eta = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}. \tag{146}$$

At first glance it would appear that the Schwarzschild metric forms a quadratic form constructed from the congruence of a 4 parameter (diagonal) matrix. It is not obvious that this may be a special case of a 10 parameter group with a fixed point. In order to admit the 10 parameter Poincare group (which is related to the Lorentz group), a map from spherical 3+1 space to Cartesian 3+1 space will be perturbed by the matrix $[f]$.

5.1.1 The Diffeomorphic Jacobian Basis Frame

At first, consider the diffeomorphic map ϕ^k from spherical 3+1 space to Cartesian 3+1 coordinates:

$$\{y^a\} = \{r, \theta, \varphi, \tau\} \Rightarrow \{x^k\} = \{x, y, z, t] \tag{147}$$

$$\phi^k : [r\sin(\theta)\cos(\varphi), r\sin(\theta)\sin(\varphi), rcos(\theta), \tau] \Rightarrow [x, y, z, t] \tag{148}$$

$$\{dy^a\} = \{dr, d\theta, d\phi.d\tau\}. \tag{149}$$

The Jacobian of the diffeomorphic map ϕ^k can be utilized as an integrable Basis Frame matrix $[\mathbb{B}]$ which is an element of the 10 parameter F-Affine group (The Affine subgroup with a fixed point):

$$[\mathbb{B}] = \begin{bmatrix} \sin(\theta)\cos(\varphi) & r\cos(\theta)\cos(\varphi) & -r\sin(\theta)\sin(\varphi) & 0 \\ \sin(\theta)\sin(\varphi) & r\cos(\theta)\sin(\varphi) & r\sin(\theta)\cos(\varphi) & 0 \\ \cos(\theta) & -r\sin(\theta) & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}. \tag{150}$$

The infinitesimal mapping formula based on $[\mathbb{B}]$ yields

$$\left[\mathbb{B}^k_a(y)\right] \circ \left| \begin{matrix} dr \\ d\theta \\ d\varphi \\ dt \end{matrix} \right\rangle \Rightarrow \left|\sigma^k\right\rangle = \left| \begin{matrix} dx \\ dy \\ dz \\ dt \end{matrix} \right\rangle, \tag{151}$$

such that all of the 1-forms $\left|\sigma^k\right\rangle$ are exact differentials.

5.1.2 The Perturbed Basis Frame with a Congruent Symmetry

Theorem 8 *The effects of a diagonal metric $[g_{jk}]$ can be absorbed into a re-definition of the Frame matrix:*

$$[\widehat{\mathbb{B}}] = [f] \circ [\mathbb{B}]. \tag{152}$$

The integrable Jacobian Basis Frame matrix given above will be perturbed by multiplication on the left by the diagonal matrix, $[f]$. The perturbed Basis Frame becomes

$$[\widehat{\mathbb{B}}] = [f] \circ [\mathbb{B}] \quad \text{the Schwarzschild Cartan Basis Frame.} \tag{153}$$

$$= \begin{bmatrix} \sin(\theta)\cos(\varphi)\gamma^2/4r^2 & \cos(\theta)\cos(\varphi)\gamma^2/4r & -\sin(\theta)\sin(\varphi)\gamma^2/4r & 0 \\ \sin(\theta)\sin(\varphi)\gamma^2/4r^2 & \cos(\theta)\sin(\varphi)\gamma^2/4r & \sin(\theta)\cos(\varphi)\gamma^2/4r & 0 \\ \cos(\theta)\gamma^2/4r^2 & -\sin(\theta)\gamma^2/4r & 0 & 0 \\ 0 & 0 & 0 & \delta/\gamma \end{bmatrix}. \tag{154}$$

$$\gamma = (2r + m), \quad \delta = (2r - m) \tag{155}$$

Use of the congruent pullback formula based on the perturbed Basis Frame, $[\widehat{\mathbb{B}}]$, yields,

$$[g_{jk}] = [\widehat{\mathbb{B}}_{transpose}] \circ \eta \circ [\widehat{\mathbb{B}}], \tag{156}$$

$$[g_{jk}] = \begin{bmatrix} -(\gamma^2/4r^2)^2 & 0 & 0 & 0 \\ 0 & -(\gamma^2/4r)^2 & 0 & 0 \\ 0 & 0 & -(\gamma^2/4r)^2\sin^2(\theta) & 0 \\ 0 & 0 & 0 & +(\delta/\gamma)^2 \end{bmatrix}, \tag{157}$$

$$\gamma = (2r + m), \quad \delta = (2r - m) \tag{158}$$

which agrees with formula given above for the isotropic Schwarzschild metric in spherical coordinates. It actually includes a more general idea, for the coefficients, α,and β, can be dependent upon both r and τ.

The infinitesimal mapping formula based on $[\widehat{\mathbb{B}}]$ yields

$$\left[\mathbb{B}_a^k(y)\right] \circ \left| \begin{matrix} dr \\ d\theta \\ d\varphi \\ dt \end{matrix} \right\rangle \Rightarrow \left|\sigma^k\right\rangle = \left| \begin{matrix} (\gamma^2/4r^2)dx \\ (\gamma^2/4r^2)dy \\ (\gamma^2/4r^2)dz \\ (\delta/\gamma)dt \end{matrix} \right\rangle, \tag{159}$$

such that all of the 1-forms $\left|\sigma^k\right\rangle$ are NOT exact differentials.

5.2 The Schwarzschild-Cartan connection.

The Schwarzschild-Cartan (right) Connection $[\widehat{\mathbb{C}}]$, as a matrix of 1-forms relative to the perturbed Basis Frame $[\widehat{\mathbb{B}}]$, becomes

$$[\widehat{\mathbb{C}}] = [\widehat{\mathbb{B}^{-1}}] \circ d[\widehat{\mathbb{B}}], \tag{160}$$

$$[\widehat{\mathbb{C}}] = \begin{bmatrix} -2mdr/r\gamma & -rd\theta & \sin^2(\theta)rd\phi & 0 \\ d\theta/r & \delta dr/\gamma & -\cos(\theta)\sin(\theta)d\phi & 0 \\ d\phi/r & \cot(\theta)d\phi & \cot(\theta)d\theta + \delta dr/\gamma & 0 \\ 0 & 0 & 0 & 4mdr/(\gamma\delta) \end{bmatrix}. \tag{161}$$

$$\gamma = (2r+m), \quad \delta = (2r-m) \tag{162}$$

Perturbed Cartan Connection from Maple

It is apparent that the Cartan Connection matrix of 1-forms is again a member of the 10 parameter matrix group, a subgroup of the 13 parameter affine matrix group.

5.3 Vectors of Torsion 2-forms

Surprisingly, for the perturbed Basis Frame $[\widehat{\mathbb{B}}]$ which contains a the square root of a congruent metric field of a massive object, the vector of excitation torsion 2-forms, based on the 10 parameter affine subgroup with a fixed point, is not zero, and can be evaluated as:

$$|\widehat{G}\rangle = [\widehat{\mathbb{C}}]\hat{}\,|dy^a\rangle \tag{163}$$

$$\text{Torsion 2-forms : of the Affine subgroup with a fixed point} \tag{164}$$

$$|\widehat{G}\rangle = \left| \begin{array}{c} 0 \\ (2m/r\gamma)(d\theta\hat{}dr) \\ (2m/r\gamma)(d\phi\hat{}dr) \\ (4m/\gamma\delta)(dr\hat{}d\tau) \end{array} \right\rangle \text{"Schwarzschild Excitations"} \tag{165}$$

$$\gamma = (2r+m), \quad \delta = (2r-m) \tag{166}$$

The unexpected result is that the isotropic Schwarzschild metric admits coefficients of "affine torsion" relative to the Cartan Connection matrix, $[\widehat{\mathbb{C}}]$.

Similarly the vector of 2-form of field intensities $|\widehat{F}\rangle$ can be evaluated in terms of the perturbed Basis Frame as:

$$\widehat{|F\rangle} = d([\widehat{\mathbb{B}}] \circ |dy^a\rangle) = \left|dA^k\right\rangle \qquad \text{"Schwarzschild Intensities"} \tag{167}$$

$$\text{: Intensity 2-forms of the Affine subgroup with a fixed point} \tag{168}$$

$$\widehat{|F\rangle} = \left| \begin{matrix} +(m\gamma/2r^2)\{(\sin(\phi)\cos(\theta)d\theta\hat{}dr) - (\sin(\theta)\cos(\phi)d\phi\hat{}dr) \\ +(m\gamma/2r^2)\{(\sin(\phi)\cos(\theta)d\theta\hat{}dr) + (\sin(\theta)\cos(\phi)d\phi\hat{}dr) \\ -(m\gamma/2r^2)(\sin(\theta)d\theta\hat{}dr) \\ (4m/\gamma^2)(dr\hat{}d\tau) \end{matrix} \right\rangle. \tag{169}$$

$$\gamma = (2r + m), \qquad \delta = (2r - m) \tag{170}$$

The constitutive map relating the field intensities and the field excitations,

$$\widehat{|G\rangle} = [\widehat{\mathbb{B}}]^{-1} \circ \widehat{|F\rangle}, \tag{171}$$

is determined by the inverse of the perturbed Basis Frame, $[\widehat{\mathbb{B}}]^{-1}$:

$$\textbf{Schwarzschild Constitutive map from Maple} \tag{172}$$

$$[\widehat{\mathbb{B}}]^{-1} = 4r/\gamma^2 \begin{bmatrix} r\sin(\theta)\cos(\phi) & r\sin(\theta)\sin(\phi) & \cos(\theta) & 0 \\ \cos(\theta)\cos(\phi) & \cos(\theta)\sin(\phi) & -\sin(\theta) & 0 \\ -\frac{\sin(\phi)}{\sin(\theta)} & \frac{\cos(\phi)}{\sin(\theta)} & 0 & 0 \\ 0 & 0 & 0 & \gamma^3/4r\delta \end{bmatrix} \tag{173}$$

$$\gamma = (2r + m), \qquad \delta = (2r - m) \tag{174}$$

5.3.1 Vectors of 3-forms

The exterior derivative of the vector of excitations is zero, hence there are no current 3-forms for the perturbed Basis Frame that encodes the Schwarzschild metric as a Congruent symmetry:

$$\text{Charge Current 3-form} \quad |J\rangle = d\,|G\rangle = 0. \tag{175}$$

The Topological Torsion for the Schwarzschild example vanishes: $H = \langle A| \hat{}\, |F\rangle \Rightarrow 0$. The implication is that the system is of Pfaff dimension 2 (and therefor is an equilibrium thermodynamic system).

Both Poincare invariant 4-forms vanish, but the Topological Spin 3-form is NOT zero.

$$\text{Topological Spin 3-form} \qquad S = \langle A|\,\hat{}\,|G\rangle \qquad dA = 0. \tag{176}$$
$$= (-m\gamma\sin(\theta)/2r^2)\{\cos(\phi)+1)(dr\hat{}d\theta\hat{}d\phi)\}. \tag{177}$$

The Spin 3-form depends upon the "mass" coefficient, m, and the 2-forms of Affine torsion.

5.3.2 The three Connection matrices

The three matrices of Connection 1-forms are presented below for each (perturbed) connection, $[\Gamma]$, $[\mathbb{C}]$, $[\mathbb{T}]$

$$[\widehat{\mathbf{\Gamma}}] = \begin{bmatrix} -2mdr/r\gamma & -\delta rd\theta/\gamma & \delta\sin^2(\theta)rd\phi/\gamma & 64\delta md\tau/\gamma^7 \\ \delta d\theta/r\gamma & \delta dr/r\gamma & -\cos(\theta)\sin(\theta)d\phi & 0 \\ \delta d\phi/r\gamma & \cot(\theta)d\phi & \cot(\theta)d\theta+\delta dr/\gamma & 0 \\ 4md\tau/(\gamma\delta) & 0 & 0 & 4mdr/(\gamma\delta) \end{bmatrix}. \tag{178}$$

$$[\widehat{\mathbb{C}}] = \begin{bmatrix} -2mdr/r\gamma & -rd\theta & \sin^2(\theta)rd\phi & 0 \\ d\theta/r & \delta dr/\gamma & -\cos(\theta)\sin(\theta)d\phi & 0 \\ d\phi/r & \cot(\theta)d\phi & \cot(\theta)d\theta+\delta dr/\gamma & 0 \\ 0 & 0 & 0 & 4mdr/(\gamma\delta) \end{bmatrix}. \tag{179}$$

$$[\widehat{\mathbb{T}}] = \begin{bmatrix} 0 & -2mrd\theta/\gamma & 2m\sin^2(\theta)rd\phi/\gamma & -64mr^4\delta d\tau/\gamma^7 \\ 2md\theta/r\gamma & 0 & 0 & 0 \\ 4md\phi/r\gamma & 0 & 0 & 0 \\ -4md\tau/\delta\gamma & 0 & 0 & 0 \end{bmatrix}. \tag{180}$$

$$\gamma = (2r+m), \qquad \delta = (2r-m) \tag{181}$$

Schwarzschild Perturbed Connections

The matrix of (metric) curvature 2-forms, $[\Phi_\Gamma]$, based on the formula

$$[\Phi_\Gamma] = d[\Gamma] + [\Gamma]\hat{}[\Gamma], \tag{182}$$

is computed to be:

Curvature 2-forms for the Schwarzschild Christoffel Connection (183)

$$[\Phi_\Gamma] = 4m/\gamma^2 \begin{bmatrix} 0 & -rdr\hat{}d\theta & r\sin^2(\theta)dr\hat{}d\phi & -32r^4dr\hat{}d\tau/\gamma^6 \\ 2mdr\hat{}d\theta/r\gamma & 0 & -2\sin^2(\theta)d\theta\hat{}d\phi & 16\delta^2r^3d\theta\hat{}d\tau/\gamma^6 \\ 4mdr\hat{}d\phi/r\gamma & -2rd\theta\hat{}d\phi & 0 & 16\delta^2r^3d\phi\hat{}d\tau/\gamma^6 \\ -4mdr\hat{}d\tau/\delta\gamma & 2rd\theta\hat{}d\tau & -2\sin^2(\theta)d\phi\hat{}d\tau & 0 \end{bmatrix} \tag{184}$$

By the Strong Equivalence Principle,

$$: \{d[\Gamma] + [\Gamma]\hat{}[\Gamma]\} \quad + \{[\mathbb{T}]\,\hat{}\,[\Gamma] + [\Gamma]\,\hat{}\,[\mathbb{T}]\} \quad + \{d[\mathbb{T}] + [\mathbb{T}]\hat{}[\mathbb{T}]\} \tag{185}$$

$$= \quad \{[\mathbf{\Phi}_\Gamma]\} \quad + \{[Interaction\ 2 - forms]\} + \quad \{[\mathbf{\Phi_T}]\} \Rightarrow 0, \tag{186}$$

as well as the other formulas of the general theory of the physical vacuum can be checked using Maple programs which can be found at

http://www22.pair.com/csdc/pdf/mapleEP1-isotropicSchwartz.pdf,
http://www22.pair.com/csdc/pdf/mapleEP1b-nonsotropicSchwartz.pdf
and will be published on a CD rom, [37].

5.4 Summary Remarks

The idea that has been exploited is that the arbitrary Basis Frame (a linear form), without metric, can be perturbed algebraically to produce a new Basis Frame that absorbs the properties of a quadratic congruent metric system. This result establishes a constructive existence proof that compatible metric features of a Physical Vacuum can be derived from the structural format of the Basis Frame. The Basis Frame is the starting point and the congruent metric properties are deduced.

For the Schwarzschild example, another remarkable feature is that the 1-forms $\left|\sigma^k\right\rangle$ constructed according to the formula

$$[\widehat{\mathbb{B}}] \circ |dy^a\rangle \Rightarrow \left|\sigma^k\right\rangle = \left|A^k\right\rangle, \tag{187}$$

are all integrable (as the Topological Torsion term is Zero), but the coefficients of affine torsion are not zero. The symbol $|dy^a\rangle$ stands for the set $[dr, d\theta, d\varphi, d\tau]$ (transposed into a column vector), and $[\widehat{\mathbb{B}}]$ is the "perturbed" Basis Frame which contains the Schwarzschild metric as a congruent symmetry. The integrability condition means that there exist integrating factors $\lambda^{(k)}$ for each σ^k such that a new Basis Frame can be constructed from $[\widehat{\mathbb{B}}]$ algebraically. Relative to this new Basis Frame, the vector of torsion 2-forms is zero, $\left|d\sigma^k\right\rangle = \left|dA^k\right\rangle = \left|F^k\right\rangle = 0!$ The "Coriolis" acceleration which is related to the 2-form of torsion 2-forms $\left|F^k\right\rangle$ can be eliminated algebraically.! Although this result is possible algebraically, it is not possible diffeomorphically.

Of course, this algebraic reduction is impossible if any of the 1-forms, σ^k, is of Pfaff dimension 3 or more. The Basis Frame then admits Topological Torsion, which is irreducible.

6 Example 2. $[\mathbb{B}]$ as a 13 parameter group

6.1 The Intransitive "Wave -Affine" Connection

The next set of examples considers the structure of those 4 x 4 Basis Frames that admit a 13 parameter group in 4 geometrical dimensions of space-time. There are 3 interesting types of 13 parameter group structures. This first example utilizes the canonical form of the 13 parameter "Wave Affine" Basis Frame. These Basis Frames will have zeros for the first 3 elements of the right-most column. Wave Affine Basis Frames exhibit closure relative to matrix multiplication. All products of Wave Affine Basis Frames have 3 zeros on the right column. From a projective point of view, these matrices are not elements of a transitive group. They are intransitive and have fixed points. The true affine group in 4 dimensions is a transitive group (without fixed points) and is discussed in the next subsection..

For simplicity in display, the 9 parameter space-space portions of the Basis Frame will be assumed to be the 3×3 Identity matrix, essentially ignoring spatial deformations and spatially extended rigid body motions. In the language of projective geometry, this intransitive system has a fixed point. The first 3 elements in the bottom row can be identified (formally) with the components of a vector potential in electromagnetic theory. The 4th (space-time) column will have three zeros, and the $\mathbb{B}_4^4$ component will be described in terms of a function $\phi(x, y, z, t)$.

$$\left[\mathbb{B}_{wave\_affine}\right] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ Ax & Ay & Az & -\phi \end{bmatrix}. \tag{188}$$

The projected 1-forms become

$$[\mathbb{B}_{wave\_affine}] \circ |dy^a\rangle = |A^k\rangle \tag{189}$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ A_x & A_y & A_z & -\phi \end{bmatrix} \circ \begin{vmatrix} dx \\ dy \\ dz \\ dt \end{matrix} \right\rangle \Rightarrow \begin{vmatrix} dx \\ dy \\ dz \\ Action \end{matrix} \right\rangle = |A^k\rangle, \tag{190}$$

$$Action = A_x dx + A_y dy + A_z dz - \phi dt \tag{191}$$

The exterior derivative of the vector of 1-forms, $|A^k\rangle$, produces the vector of 2-forms representing the field intensities $|F^k\rangle$ of electromagnetic theory,

$$\textbf{Intensity 2-forms: } d\left|A^k\right\rangle = \left|F^k\right\rangle = \left| \begin{array}{c} dx \\ dy \\ dz \\ d(Action) \end{array} \right\rangle . \tag{192}$$

Note that the Action 1-form produced by the wave affine Basis Frame is precisely the format of the 1-form of Action used to construct the Electromagnetic field intensities in classical EM theory.

The Cartan right Connection matrix of 1-forms based upon the Basis Frame, $\left[\mathbb{B}_{wave\_affine}\right]$, given above is given by the expression

$$\left[\mathbb{C}_{wave\_affine\_right}\right] = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -d(A_x)/\phi & -d(A_y)/\phi & -d(A_z)/\phi & d(\ln\phi) \end{bmatrix} . \tag{193}$$

The general theory permits the vector of excitation 2-forms, $|G\rangle$, to be evaluated as:

$$\left|\Sigma_{W-Affine\_torsion}\right\rangle = \left[\mathbb{C}_{wave\_affine\_right}\right] \hat{}\ |dy^m\rangle \simeq |G\rangle \tag{194}$$

$$\textbf{Excitation 2-forms } |G\rangle = \left| \begin{array}{c} 0 \\ 0 \\ 0 \\ -(F)/\phi \end{array} \right\rangle = \left| \begin{array}{c} 0 \\ 0 \\ 0 \\ -d(Action)/\phi \end{array} \right\rangle \tag{195}$$

$$\neq \left|\Sigma_{Car\tan\_torsion}\right\rangle \tag{196}$$

The coefficients of the vector of excitation 2-forms are precisely those ascribed to the coefficients of "Affine Torsion", even though the Basis Frame used as the example is not a member of the transitive affine group.

The exterior derivative of the vector of two forms $|G\rangle$ produces the vector of 3-form currents $|J\rangle$. The result is

$$|J\rangle = \left| \begin{array}{c} 0 \\ 0 \\ 0 \\ d(-F)/\phi \end{array} \right\rangle = \left| \begin{array}{c} 0 \\ 0 \\ 0 \\ +d\phi\hat{}F/\phi^2 \end{array} \right\rangle \tag{197}$$

Note that for this example, the current 3-form is closed producing a conservation law:

$$d\,|J\rangle = |0\rangle \tag{198}$$

The **wave affine group** of Basis Frames supports a vector of "Affine Torsion" 2-forms that are abstractly related to excitation 2-forms $|G\rangle$ representing the fields (the **D** and **H** fields) generated by the sources in classical EM theory. The charge-current 3-form need not vanish. The 3-forms of Topological Spin and Topological Torsion are proportional to one another for the simple example, and are not zero if the Pfaff Topological Dimension of the 1-form of Action is 3 or more. Such systems are not in thermodynamic equilibrium.

$$\begin{aligned} \text{Topological Torsion: } & A\hat{}F \\ \text{Topological Spin: } & A\hat{}G = -A\hat{}F/\phi \end{aligned}$$

If the Poincare invariants are to vanish it is necessary that the 4-form of Topological Parity is zero, $F\hat{}F \Rightarrow 0$.

These results for the 13 parameter intransitive "Wave Affine group" are to be compared to the results for the 13 parameter transitive "Particle Affine group" given in the next example. Maple programs for both types are available.

6.2 The Transitive "Particle - Affine" Connection

The next example utilizes the canonical form of the 13 parameter "Particle Affine" Basis Frame, which will have zeros for the first 3 elements of the fourth row. For simplicity, the 9 parameter space-space portions of the Basis Frame will be assumed to be the 3×3 Identity matrix, essentially ignoring spatial deformations and spatial extended rigid body motions. In the language of projective geometry, the transitive system has no fixed point. The 4th (space-time) column will consist of components that can be identified (formally) with a velocity field. The products of particle-affine matrices exhibit structural closure relative to matrix multiplication. The bottom row always will have three zeros for the first three matrix elements, and the $\mathbb{B}_4^4$ component will be described in terms of a function $\psi(x, y, z, t)$.

$$\left[\mathbb{B}_{Particle\_affine}\right] = \begin{bmatrix} 1 & 0 & 0 & -V^x \\ 0 & 1 & 0 & -V^y \\ 0 & 0 & 1 & -V^z \\ 0 & 0 & 0 & \psi \end{bmatrix}. \tag{199}$$

The projected 1-forms become

$$[\mathbb{B}] \circ |dy^a\rangle = \left|\sigma^k\right\rangle \tag{200}$$

$$\begin{bmatrix} 1 & 0 & 0 & -V^x \\ 0 & 1 & 0 & -V^y \\ 0 & 0 & 1 & -V^z \\ 0 & 0 & 0 & \psi \end{bmatrix} \circ \left| \begin{array}{c} dx \\ dy \\ dz \\ dt \end{array} \right\rangle \Rightarrow \left| \begin{array}{c} \sigma^x \\ \sigma^y \\ \sigma^z \\ \omega \end{array} \right\rangle \tag{201}$$

$$= \left| \begin{array}{c} dx - V^x dt \\ dy - V^y dt \\ dz - V^z dt \\ \psi dt \end{array} \right\rangle = \left| \begin{array}{c} \Delta x \\ \Delta y \\ \Delta z \\ \psi dt \end{array} \right\rangle, \tag{202}$$

where the terms $\Delta x,\ \Delta y, \Delta z$, if not zero, can be considered as the topological fluctuations in "kinematic perfection". The exterior derivative of the vector of 1-forms, $|A^k\rangle$, produces the vector of 2-forms representing the field intensities $|F^k\rangle$ of electromagnetic theory,

$$\text{Intensity 2-forms:}\ \ d\left|A^k\right\rangle = \left|F^k\right\rangle = \left| \begin{array}{c} -dV^x\hat{}dt \\ -dV^y\hat{}dt \\ -dV^z\hat{}dt \\ d\psi\hat{}dt \end{array} \right\rangle = \left| \begin{array}{c} d(\Delta x) \\ d(\Delta y) \\ d(\Delta z) \\ d\psi\hat{}dt \end{array} \right\rangle. \tag{203}$$

If the fluctuations about kinematic perfection are closed, or the velocity field is a function of the single parameter of time, and if the function ψ is not dependent upon the spatial variables, then the vector of field intensities is zero; **E** and **B** fields do not exist in this refined topology of kinematic perfection and universal time, based upon the transitive affine group. If the potential function ψ is not spatially uniform, then there can exist the equivalent of an Electric Field, but there never appear Magnetic fields (for the transitive affine Basis Frame without spatial deformations).

The Cartan right Connection matrix of 1-forms for this refined Basis Frame is given by the expression

$$\left[\mathbb{C}_{Particle\_affine\_right}\right] = \begin{bmatrix} 0 & 0 & 0 & -dV^x + V^x d(\ln\psi) \\ 0 & 0 & 0 & -dV^y + V^y d(\ln\psi) \\ 0 & 0 & 0 & -dV^z + V^z d(\ln\psi) \\ 0 & 0 & 0 & d(\ln\psi) \end{bmatrix} \tag{204}$$

The Connection coefficients can be computed and exhibit components of non zero "P-Affine Torsion ", especially if spatial deformations exist, or the potential, ψ, is spatially dependent. The vector of "Affine Torsion" 2-forms is given by the expressions:

$$\left|\Sigma_{P-Affine\_torsion}\right\rangle = \mathbb{C}\hat{}\ |dy^m\rangle \simeq |G_{Particle}\rangle \tag{205}$$

$$\textbf{field excitations}: |G_{Particle}\rangle = \left| \begin{array}{c} -d(V^x)\hat{}d(t) + V^x d(\ln\psi)\hat{}d(t)) \\ -d(V^y)\hat{}d(t) + V^y d(\ln\psi)\hat{}d(t)) \\ -d(V^z)\hat{}d(t) + V^z d(\ln\psi)\hat{}d(t)) \\ d(\ln\psi)\hat{}d(t) \end{array} \right\rangle \tag{206}$$

$$\neq \left|\Sigma_{\text{Cartan\_torsion}}\right\rangle \tag{207}$$

If the velocity field is a function of time only, $V = V(t)$, then total differential of the velocity field leads to the classic kinematic concept of accelerations, and the "affine" torsion 2-forms depend only on the potential ψ. If the potential function ψ is such that its total differential is zero, or a function of time (and not dependent on the spatial coordinates), then all of the affine torsion coefficients vanish (for this example). Moreover, under the kinematic assumption, the 3-forms of currents vanish:

$$|J\rangle = d\,|G_{Particel}\rangle \Rightarrow 0. \tag{208}$$

Now consider the case of topological fluctuations, Δx^k, about kinematic perfection. That is suppose

$$dx^k - V^k dt = \Delta x^k, \tag{209}$$

$$\text{or} \quad \left| \begin{array}{c} \boldsymbol{\sigma}^k \\ \omega \end{array} \right\rangle = \left| \begin{array}{c} \Delta \mathbf{x}^k \\ \psi dt \end{array} \right\rangle \tag{210}$$

$$\text{such that} \quad \left| \begin{array}{c} d\boldsymbol{\sigma}^k \\ d\omega \end{array} \right\rangle = \left| \begin{array}{c} d(\Delta \mathbf{x}^k) \\ d\psi\hat{}dt \end{array} \right\rangle. \tag{211}$$

The algebra of this example based on topological fluctuations about kinematic perfection are discussed in the Maple programs which can be found at

http://www22.pair.com/csdc/pdf/mapleEP2-waveaffine.pdf,
http://www22.pair.com/csdc/pdf/mapleEP2-particleaffine.pdf
and will be published on a CD rom, [37].

The Maple programs construct the more tedious algebraic results (that are related to curvatures and the Bianchi identities) developed in Section 2 above.

The **particle affine** group of Basis Frames has a connection that is compatible with the concept of kinematic evolution of massive particles.

7 Example 3. $[\mathbb{B}]$ as an element of the Lorentz Group

In the theory of Electromagnetic Signals, it was an objective of Fock [6] to establish the equivalence class of coordinates, or diffeomorphic systems of reference, where by if one observer in one system of reference claims to "see" an electromagnetic signal (defined as a propagating discontinuity), then another observer in a different reference system would make a similar statement with regard to the same physical discontinuity phenomenon. Both observers claim to see a "signal"; they both see a propagating discontinuity in field strength. The propagating discontinuity is defined by the solution to a non-linear first order partial differential equation, known as the Null Eikonal equation [35] that describes the point set upon which the solutions to the Maxwell system of PDE's are not unique. The Null Eikonal equation is a quadratic sum of partial differentials with the Minkowski signature:

$$\textbf{Null Eikonal} \quad (\pm\partial\varphi/\partial x)^2 \pm (\partial\varphi/\partial y)^2 \pm (\partial\varphi/\partial z)^2 \mp 1/c^2(\partial\varphi/\partial t)^2 = 0. \tag{212}$$

To preserve the discontinuity, the Null Eikonal equation must be preserved under the diffeomorphisms that relate one observer to another.

The Null Eikonal equation can be formulated as a congruent product of a vector of 1-forms.

$$\left|(d\varphi)^k\right\rangle = \begin{bmatrix} (\partial\varphi/\partial x) & 0 & 0 & 0 \\ 0 & (\partial\varphi/\partial y) & 0 & 0 \\ 0 & 0 & (\partial\varphi/\partial z) & 0 \\ 0 & 0 & 0 & (\partial\varphi/c\partial t) \end{bmatrix} \circ \left| \begin{matrix} dx \\ dy \\ dz \\ dt \end{matrix} \right\rangle \tag{213}$$

$$\text{Null Eikonal Equation}: 0 = \langle d\varphi| \circ [\eta] \circ |d\varphi\rangle \Rightarrow \langle d\varphi| \circ [\mathbb{L}]^T \circ [\eta] \circ [\mathbb{L}] \circ |d\varphi\rangle = 0. \tag{214}$$

If the infinitesimal vector of 1-forms $\left|(d\varphi)^k\right\rangle$ is transformed to a new set of 1-forms by the Basis Matrix $[\mathbb{L}]$, it is apparent that the Null Eikonal equation is invariant in form, when $[\mathbb{L}]$ is a Lorentz transformation:

$$\text{Lorentz automorphism:} \quad [\mathbb{L}]^T \circ [\eta] \circ [\mathbb{L}] = [\eta]. \tag{215}$$

Fock established that the only linear transformation that would preserve this equivalence was the Lorentz transformation. However, there is a larger non-linear class of transformations that also preserves the propagating discontinuity as a discontinuity. The key requirement such that the propagating field amplitude *discontinuity* is preserved is that the null line element (the zero set) must be preserved. It is not the quadratic metric form that must be preserved, it is only the zero set of the quadratic form that must be preserved. The zero set of the Null Eikonal equation is also satisfied relative to the conformal extensions of the Lorentz transformation.:

$$\textbf{Extended} \; : \textbf{Lorentz transformation} \tag{216}$$

$$[\lambda\mathbb{L}]^T \circ [\eta] \circ [\lambda\mathbb{L}] = \lambda^2 [\eta] , \tag{217}$$

$$\langle d\varphi| \circ [\eta] \circ |d\varphi\rangle \Rightarrow 0 \supset \lambda^2 \langle d\varphi| \circ [\eta] \circ |d\varphi\rangle \Rightarrow 0. \tag{218}$$

Fock demonstrated that the extended Conformal (non linear) Lorentz transformations also preserve the the propagating electromagnetic discontinuity. In this example of the "Physical Vacuum" conformal non-linear Lorentz transformations will be studied as a class of Basis Frames suitable for description of a specialized "Physical Vacuum". The surprise is that the method leads to a better understanding not only of the theory of electromagnetic signals as propagating amplitude discontinuities, but also to the theory of propagating tangential discontinuities such as Wakes, in a field continuum, or a fluid.

7.0.1 Harmonic Wakes

From the topological point of view it is remarkable how often flow instabilities and wakes take on one or another of two basic scroll patterns. The first scroll pattern is epitomized by the Kelvin-Helmholtz instability (Figure 1a) and the second scroll pattern is epitomized by the Raleigh-Taylor instability (Figure 1b)

Figure 1a. The pattern of the Kelvin-Helmholtz instability.

Figure 1b. The pattern of the Rayleigh-Taylor Instability.

The repeated occurrence of these two scroll patterns (one similar to a replication of a Cornu Spiral and the other similar to a replication of a Mushroom Spiral), often deformed but still recognizable and persistent, even in a dissipative environment, suggests that a basic simple underlying topological principle is responsible for their creation. The essential questions are: Why do these spiral patterns appear almost universally in wakes? Why do they persist for such substantial periods of time? Why are they so sharply defined? What are the details of their creation? As H.K.Browand has said (p. 117 in [3]) "There does not exist a satisfactory theoretical explanation for these wake patterns."

The mushroom pattern is of particular interest to this author, who long ago was fascinated by the long lived ionized ring that persists in the mushroom cloud of an atomic explosion. Although the mushroom pattern appears in many diverse physical systems (in the Frank-Reed source of crystal growth, in the scroll patterns generated in excitable systems, in the generation of the wake behind an aircraft,...), no simple functional description of the deformable mushroom pattern as a topological property was known to me in 1957. My first exposure to clothoids was about 1995 [7]. Classical geometric analysis applied to the equations of hydrodynamics failed to give a satisfactory description of these persistent structures, so often observed in many different situations. It is remarkable that Frenet theory concepts not only yields a quantitative picture of the odd and even scroll patterns, but also gives closed form solutions for the creation of the limit set figures above.

An argument made in (1985-1990) for the existence of long lived wakes was extracted from fluids described phenomenologically in terms of the Navier-Stokes equations. For incompressible fluid dynamics, with $div\mathbf{V} = 0$, an initial (perhaps turbulent) fluid velocity distribution would decay by shear viscosity processes, such as those encoded by the Navier-Stokes term, $\nu\nabla^2\mathbf{V} \neq 0$. However, any components of the initial velocity field that are harmonic, $\nabla^2\mathbf{V} = 0$, will not decay, and it was argued that these components of the initial velocity distribution are those that form a residue or wake. The residue velocity fields admit a Hamiltonian representation, and therefor have a persistent existence. The important fundamental observation of differential geometry was that a harmonic vector field describes a minimal surface. Hence wakes should be related (somehow) to flows that describe minimal surfaces in velocity space. The first analytic example is discussed in Chapter 8.2.4 of [34].

The observable features of hydrodynamic wakes can be put into correspondence with those characteristic surfaces of tangential discontinuities upon which the solutions to the evolutionary equations of hydrodynamics are not unique. Only the robust minimal surface subset, associated with a harmonic vector field, will be persistent and of minimal dissipation. Surprisingly, those minimal surfaces if generated by iterates of complex holomorphic curves in four dimensions are related to fractal sets.

The clue that wave fronts, representing propagating limit sets of tangential discontinuities, are the basis for the two basic spiral (or scroll) instability patterns in fluid dynamics came to this author during a study of Cartan's methods of differential topology and the Frenet-Cartan concept of the Repere Mobile as applied to the production of defects and topological torsion in dynamical systems [21] [24] [25].

At first consider Frenet space curves in the plane, which are generated by a pair of differential equations of the form,

$$dx/ds = \sin(Q(s)), \tag{219}$$

$$dy/ds = \cos(Q(s)), \tag{220}$$

The RHS of these equations is always a unit vector, which encourages the identification of this formalism with the Frenet formulas for motion in a plane.

$$d\mathbf{R}/ds = \left| \begin{array}{c} dx/ds \\ dy/ds \end{array} \right\rangle = \mathbf{t}(s) = \left| \begin{array}{c} sin(Q(s)) \\ cos(Q(s)) \end{array} \right\rangle \tag{221}$$

By constructing the differential of the unit vector, $\mathbf{t}(s)$, with respect to the "arclength" s leads to the classic expression for the Frenet curvature:

$$d\mathbf{t}(s)/ds = \kappa\mathbf{n}(s) = \{dQ/ds\} \left| \begin{array}{c} \cos(Q(s)) \\ -\sin(Q(s)) \end{array} \right\rangle, \tag{222}$$

$$\kappa = \{dQ/ds\}. \tag{223}$$

For $\kappa = 1$, integration yields the space curve which is a circle. For $\kappa = 1/s$, the resulting space curve is the logarithmic spiral in the x-y plane. For $\kappa = s$, the resulting image in the x-y plane is the Cornu spiral. These facts have been known for more than 100 years to differential geometers.

However, a simple sequence is to be recognized :

$$: \quad \kappa = s^{-1}, \qquad \kappa = s^{0}, \qquad \kappa = s^{1}... \tag{224}$$

: Log spiral, circle, Cornu-Fresnel spiral...

An extension of the sequence raises the question: what are the characteristic shapes of intrinsic space curves for which the curvature is proportional to an arbitrary power of the arc length, $\kappa = g(s) = s^n$, for all positive and negative integers (or rational fractions)? Through the power of the PC these questions may be answered quickly by integrating the Harmonic Ordinary Differential equations.

Suppose the argument function is given by the expression,

$$Q = s^{n+1}/(n+1), \text{ such that} \tag{225}$$
$$dQ/ds = \kappa = s^n. \tag{226}$$

The results of the numerical integrations are presented in Figure 2 for n = 1 and n = 2. A most remarkable result is that the Cornu-Fresnel spiral of Figure 2a is the deformable equivalent for all odd-integer n > 0, and the Mushroom spiral of Figure 2b is the deformable equivalent for all even-integer n > 0.

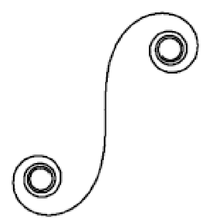


Figure 2a. The Cornu spiral is a harmonic function for which the phase function, $Q = s^2$.

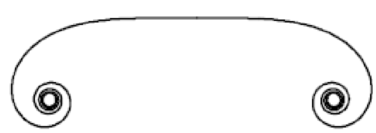


Figure 2b. The Mushroom spiral is the harmonic function for which the phase function, $Q = s^3$.

Not only has the missing analytic description of the mushroom spiral been found, but also a *raison d'être* has been established for the universality of the two spiral patterns. They belong to the even and odd classes of arc length exponents describing plane curves in terms of the formula, $\kappa = s^n$. The Cornu-Fresnel spiral is the first odd harmonic function that maps the infinite interval into a bounded region of the plane, and the mushroom spiral is the first even function that maps the infinite interval into a bounded region of the plane. Note that the zeroth harmonic function maps the infinite interval into the bounded region of the plane, but the trajectory is unique in that it is not only bounded but it is closed; i.e., a circle. A similar sequence can be generated for the half-integer exponents with n = +3/2, 7/2, 11/2... giving the Mushroom spirals and 5/2,9/2,13/2... giving the Cornu spirals. In numeric and experimental studies of certain shear flows, the Cornu spiral appears to dominate the motion in the longitudinal direction, while the Mushroom spiral appears in the transverse direction.

Remark 9 *It is important to remember that the Cornu spirals are related to diffraction of waves.*

Periodic patterns can be obtained by examining phase functions of various forms. For example, the Kelvin Helmholtz instability of Figure 1a is homeomorphic to the choice Q(s) = $1/\cos^2(s)$. Similarly, the Rayleigh-Taylor instability of Figure 1b is homeomorphic to the case Q(s) = tan(s)/cos(s).

Figure 1a'. The pattern of the Kelvin-Helmholtz instability. Q(s) = $1/\cos^2(s)$.

Figure 1b'. The pattern of the Rayleigh-Taylor Instability. Q(s) = tan(s)/cos(s).

These shapes were found by numerical integration (on a 100Mhz PC) of the Frenet space curve equations in a plane. The integration technique made use of the arclength s as the fundamental integration parameter.

Remark 10 *What has all this to do with Conformal Lorentz transformations? As will be developed below, the harmonic forms Q(s) generating the harmonic wake patterns also appear as the canonical matrix elements in extended Lorentz transformations!*

7.1 Conformal Lorentz Transformations

The problem is reduced to the study of those transformations $[\mathbb{L}]$ acting on the differential position vectors $|dx^a\rangle$ that preserve a certain quadratic form (the square of the infinitesimal arclength) as an invariant of the transformation, to within a factor. By definition of the Physical Vacuum, a Basis Frame $[\mathbb{B}]$ acting on a vector array of perfect differentials $|dx^a\rangle$ will produce a vector array of 1-forms, $\left|\sigma^k\right\rangle$. These 1-forms need not be exact, and not even closed.

$$\left[\mathbb{B}_a^k\right] \circ |dx^a\rangle = [\mathbb{B}] \circ \left| \begin{matrix} dx \\ dy \\ dz \\ cdt \end{matrix} \right\rangle \Rightarrow \left|\sigma^k\right\rangle. \tag{227}$$

The concept of an infinitesimal arclength squared can be defined in terms of a congruent quadratic measure defined with respect to a diagonal Sylvestor signature matrix:

$$(\delta s)^2 = \langle dx^a| \circ [\mathbb{B}]^{Transpose} \circ [\eta] \circ [\mathbb{B}] \circ |dx^a\rangle \tag{228}$$

$$= \langle dx^a| \circ [g] \circ |dx^a\rangle = \left\langle \sigma^k\right| \circ [\eta] \circ \left|\sigma^k\right\rangle \tag{229}$$

$$= \mathbf{B}_m^1 dx^m \eta_{11} \mathbb{B}_n^1 dx^n + \mathbf{B}_m^2 dx^m \eta_{22} \mathbb{B}_n^2 dx^n .... \tag{230}$$

The infinitesimal arclength, δs, can be exact, not closed, or even a non-integrable 1-form. A null length is given by the expression, $(\delta s)^2 = 0$.

Now restrict the group of Basis Frames $[\mathbb{B}]$ to be those (to within a factor) that belong to the Lorentz group $[\mathbb{L}]$ such that the quadratic congruence is invariant about the identity. The idea is that

$$[\eta] \Leftarrow [\mathbb{L}]^{Transpose} \circ [\eta] \circ [\mathbb{L}] = [\eta], \tag{231}$$

$$(\delta s)^2 = \left\langle dx^k\right| \circ [\eta] \circ \left|dx^k\right\rangle = \tag{232}$$

$$= \left\langle dx^k\right| \circ [\mathbb{L}]^{Transpose} \circ [\eta] \circ [\mathbb{L}] \circ \left|dx^k\right\rangle = (\delta s')^2 \tag{233}$$

$$= \mp(dx)^2 \mp (dy)^2 \mp (dz)^2 \pm c^2(dt)^2. \tag{234}$$

The Lorentz transformation is a correlation automorphism with respect to the Minkowski metric, $[\eta]$.

However, to preserve the zero set of the line element, such that

$$(\delta s')^2 \Rightarrow 0 \supset (\delta s)^2 \Rightarrow 0, \tag{235}$$

all that is required is that the transformation matrix be an element of the group of automorphisms (relative to $[\eta]$) to within a factor, λ :

$$[\mathbb{L}]^{transpose} \circ [\eta] \circ [\mathbb{L}] = \lambda^2(x, y, z, t)\, [\eta]. \tag{236}$$

The factor λ can be an arbitrary function of the variables $\{x, y, z, t\}$.

The fundamental concept that a signal remains a signal is equivalent to the idea that the Minkowski metric (or better said, the quadratic form of the Eikonal equation) is to be preserved, *to within a factor*, by the congruent mapping presented in the equation above. Such mappings, $\lambda [\mathbb{L}]$,which need not have constant matrix elements, are defined herein as "extended" or non-linear Lorentz transformations. The conventional Lorentz transformations are the restricted subset of the extended Lorentz transformations, where the constraints require that $\lambda = \pm 1$ and the matrix array is a set of constants. Such restrictions will not be the general case studied herein; constant matrix elements are considered to be the exception. Extended Lorentz transformations define an equivalence class of coordinate systems (frames of reference). It has been conventional to describe the restricted class of Lorentz systems as "inertial" frames of reference. This concept will be examined (and extended) below.

7.2 Integrable vs. Non-Integrable Lorentz transformations

Without exhibiting the functional form of the extended Lorentz transformations, it is to be observed that the vector array of 1-forms, $\left|\sigma^k\right\rangle$,

$$\left|\sigma^k\right\rangle = [\mathbb{L}] \circ \left|dx^k\right\rangle \tag{237}$$

need not be an array of perfect differentials. The group requirement is fixed by the constraint that

$$[\mathbb{L}]^{transpose} \circ [\eta] \circ [\mathbb{L}] = \lambda^2(x, y, z, t) [\eta] . \tag{238}$$

If the mapping $[\mathbb{L}]$ produces exact differentials, then the exterior derivative of vector array, $\left|\sigma^k\right\rangle$, is zero, component by component.

$$\text{Vector array of Closure 2-forms} : d \left|\sigma^k\right\rangle = (d [\mathbb{L}]) \circ \left|dx^k\right\rangle \Rightarrow 0. \tag{239}$$

If the mapping functions of the matrix $[\mathbb{L}]$ are constants, then $d [\mathbb{L}] = 0$, and the induced 1-forms $\left|\sigma^k\right\rangle$ are always closed. Each component of $\left|\sigma^k\right\rangle$ then satisfies the Frobenius integrability condition and each new differential coordinate can be reduced globally to the exact differential of a single function (case1).

However, the components of the extended transformations $\lambda [\mathbb{L}]) \circ \left|dx^k\right\rangle$ need not be constants, and yet the new 1-forms created by the mapping may be integrable over a large but perhaps limited domain. The new 1-forms are closed, but not exact (case 2), leading to interesting topological situations. In other situations, the components of $\left|\sigma^k\right\rangle$ may not be closed, but might admit integrating factors which would make them exact or closed (case 3). The vector array of closure 2-forms (see the next paragraph below) need not be zero.

In addition, when the matrix elements of $[\mathbb{L}]$ are not constants, it may be true that the components of $\left|\sigma^k\right\rangle$ are not uniquely integrable at all (case 4). Each of these four cases should be studied separately. Each of the 4 cases defines a different topological set of Lorentz transformations.

From the general theory of the Physical Vacuum, it is to be remarked that if a subset of the extended Lorentz transformations (with inverse) are used as a Basis Frame at a point p of $\{x, y, z, t\}$, then it is possible to define a connection in terms of the right Cartan matrix $[\mathbb{C}]$ of 1-forms created by exterior differentiation of the extended Lorentz matrix $[\mathbb{L}] \Leftrightarrow [\mathbb{B}]$. The induced vector array of closure 2-forms becomes

$$d\left|\sigma^k\right\rangle = (d\,[\mathbb{L}]) \circ |dx^m\rangle = [\mathbb{L}] \circ [C]\,\hat{}\,|dx^m\rangle \tag{240}$$

$$= L_b^k \circ \left|C_{[mn]}^b dx^m\,\hat{}\,dx^n\right\rangle = L_b^k \circ \left|C^b\right\rangle \tag{241}$$

The vector of 2-forms $d\left|\sigma^k\right\rangle$ are formally equivalent to the field intensities of electromagnetism $\left|F^k\right\rangle$. The vector of 2-forms $\left|C^b\right\rangle$ is formally equivalent to the field excitations of electromagnetic theory. The coefficients $C_{[mn]}^b$ are precisely those used in tensor analysis to define the concept of "Affine Torsion". Without "Affine Torsion" there does not exist electromagnetic field components.

An example of a classic Lorentz transformation (often called a "boost") is given by the matrix of translational shears,

$$[\mathbb{L}(\beta)_{xt}] = \begin{bmatrix} \frac{1}{\sqrt{1-\beta^2}} & 0 & 0 & \frac{\beta}{\sqrt{1-\beta^2}} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \frac{\beta}{\sqrt{1-\beta^2}} & 0 & 0 & \frac{1}{\sqrt{1-\beta^2}} \end{bmatrix}, \tag{242}$$

where β is defined as $\mathbf{v}^x/c$, the ratio of the x component of "velocity" and the speed of light, and is conventionally interpreted as a constant.

As another example of a different Lorentz transformation, consider the matrix (with unit determinant) representing shears of rotation and/or expansion.

$$[\mathbb{L}(\theta(s)_{xt}] = \begin{bmatrix} \frac{1}{\cos(\theta)} & 0 & 0 & \pm\frac{\sin(\theta)}{\cos(\theta)} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \pm\frac{\sin(\theta)}{\cos(\theta)} & 0 & 0 & \frac{1}{\cos(\theta)} \end{bmatrix}, \tag{243}$$

Substitute this matrix into the equation defining the Lorentz automorphism, compute the matrix products to show that, indeed, the matrix $[\mathbb{L}(\theta(s))_{xt}]$ preserves the Minkowski metric:

$$[\mathbb{L}(\theta)_{xt}]^{transpose} \circ [\eta] \circ [\mathbb{L}(\theta)_{xt}] = [\eta]\,. \tag{244}$$

Remark 11 *Note that the dimensionless function θ can be an arbitrary function of all the independent variables,*

$$\theta(s) = \theta(x, y, z, t). \tag{245}$$

For example, $\theta(x, y, z, t)$ could be equivalent to the arclength divided by a constant scale length, $\theta(x, y, z, t) \Rightarrow s/\lambda$.

Use the Lorentz matrix $[L(\theta)_{xt}]$ to compute the differentials:

$$\begin{bmatrix} \frac{1}{\cos\theta} & 0 & 0 & \pm\frac{\sin\theta}{\cos\theta} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \pm\frac{\sin\theta}{\cos\theta} & 0 & 0 & \frac{1}{\cos\theta} \end{bmatrix} \circ \left| \begin{matrix} dx \\ dy \\ dz \\ cdt \end{matrix} \right\rangle = \left| \begin{matrix} \frac{1}{\cos\theta}dx \pm \frac{\sin\theta}{\cos\theta}cdt \\ dy \\ dz \\ \frac{\sin\theta}{\cos\theta}dx \pm \frac{1}{\cos\theta}cdt \end{matrix} \right\rangle = \left|\sigma^k\right\rangle \tag{246}$$

In general, all of the components of $\left|\sigma^k\right\rangle$ are not perfect differentials. For example, consider

$$\sigma^1 = \frac{1}{\cos\theta}dx \pm \frac{\sin\theta}{\cos\theta}cdt. \tag{247}$$

When the angular function θ is a constant, then a new coordinate, X,is well defined by integration, and its differential yields the expression above.

$$X = \frac{1}{\cos\theta}x \pm \frac{\sin\theta}{\cos\theta}ct, \qquad \sigma^1 \Rightarrow dX \quad and \quad d\sigma^1 = 0 \tag{248}$$

However, in the general case, the vector of closure 2 forms $d\left|\sigma^k\right\rangle$ of field intensities do not vanish:

$$\text{Field Intensities} \;\; d\left|\sigma^k\right\rangle = \left| \begin{matrix} -\frac{\sin\theta}{\cos^2\theta}dx\hat{}d\theta - \frac{1}{\cos^2\theta}dt\hat{}d\theta \\ 0 \\ 0 \\ -\frac{\sin\theta}{\cos^2\theta}dt\hat{}d\theta - \frac{1}{\cos^2\theta}dx\hat{}d\theta \end{matrix} \right\rangle \neq 0. \tag{249}$$

For the example given, the 4 forms of the type $d\sigma^k\hat{}d\sigma^k$ do not vanish (the Topological Torsion is zero as there are only 3 independent differentials, $dx\hat{}d\theta\hat{}dt$.

Now consider the extended Lorentz transformations given by the matrix

$$[\mathbf{P}(\theta)_{xt}] = \lambda\,[\mathbb{L}(\theta)_{xt}] = (1/\cos\theta)\,[\mathbb{L}(\theta)_{xt}] \tag{250}$$

$$[\mathbf{P}(\theta)_{xt}] = \begin{bmatrix} \frac{1}{\cos^2\theta} & 0 & 0 & \pm\frac{\sin\theta}{\cos^2\theta} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \pm\frac{\sin\theta}{\cos^2\theta} & 0 & 0 & \frac{1}{\cos^2\theta} \end{bmatrix} \tag{251}$$

It is remarkable that the two universal Rayleigh-Taylor_Kelvin-Helmholtz fluid instability patterns discussed above are related to the matrix elements of the non-linear Lorentz transformations. The two instabilities are generated by the functions $Q = 1/\cos^2(\theta)$ and $Q = \sin(\theta)/\cos^2(\theta)$. These functions have another surprising property: they can be put into correspondence with non-linear, conformal, Lorentz transformations that generate rotations and expansions.

For consider the angular coordinate being proportional to arclength to yield the extended Lorentz transformation: The conformality factor is $\lambda^2 = 1/\cos^2\theta$ and the extended Lorentz transformation $[\mathbb{P}]$ is such that

$$[\mathbb{P}]^T \circ \left[\eta_{\mu\nu}\right] \circ [\mathbb{P}] \Rightarrow \lambda^2 \cdot \left[\eta_{\mu\nu}\right]. \tag{252}$$

The result implies that the zero set Eikonal equation is preserved invariantly by the Poincare, or extended Lorentz transformations $[P(\theta)_{xt}]$. In electromagnetic theory this means that the propagating discontinuities remain invariant under conformal Lorentz transformations of rotation and expansion, $[\mathbb{P}]$;

Remark 12 *The bottom line is that signals, defined as propagating discontinuities [6] [8], remain signals to all conformally equivalent electromagnetic observers, not just Lorentz equivalent observers.*

It is well known that the null quadratic form can also be related to Spinors, treated as complex mappings of 4D to 4D. Such objects can be composed of complex combinations of Hopf maps. The implication is that the conformal Lorentz transformations, which are directly related to the instability functions $Q = 1/\cos^2(s)$ and $Q = \sin(s)/\cos^2(s)$ of hydrodynamic wakes, indicate that Spinors are not just artifacts of electromagnetic theory, but also apply to hydrodynamic systems. This is a relatively unexplored area of hydrodynamics and continuous topological evolution. The bottom line of this result is that the relationship between the hydrodynamic instabilities and non-linear Lorentz rotations can not be considered as an accident. Real fluids have finite wave propagation velocities and are not perfectly incompressible. These formal clues can be used to associate Spinors to wake and instability formation, in terms of the known relationship between Spinors, Harmonic vector fields and Minimal surfaces.

7.3 The Cartan Connection matrix of 1-forms.

Following the methods of section 2, above, the Cartan Connection is computed for the Basis Frame of the extended Lorentz transformation, $[\mathbf{P}(\theta(x,y,z,t))_{xt}]$.

$$\textbf{The Cartan Connection matrix } [\mathbb{C}] \textbf{ of 1-forms based on } [\mathbb{P}(\theta)_{xt}] \tag{253}$$

$$[\mathbb{C}]_{\mathbf{P}(\theta)} = \begin{bmatrix} \sin(\theta)d\theta/\cos(\theta) & 0 & 0 & d\theta/\cos(\theta) \\ 0 & \sin(\theta)d\theta/\cos(\theta) & 0 & 0 \\ 0 & 0 & \sin(\theta)d\theta/\cos(\theta) & 0 \\ d\theta/\cos(\theta) & 0 & 0 & \sin(\theta)d\theta/\cos(\theta) \end{bmatrix} \tag{254}$$

7.4 The Vectors of Torsion 2-forms

The rest of the presentation is extracted from Maple computations based upon $[\mathbb{P}(\theta)_{xt}]$. The function $\theta = \theta(x, y, z, t)$ is unspecified:

$$\textbf{The vector of 1-form Potentials} \tag{255}$$

$$\left|A^k\right\rangle = [\mathbb{P}(\theta)_{xt}] \circ |dx^a\rangle = \left| \begin{array}{c} \{dx + \sin(\theta)dt\}/\cos(\theta)^2 \\ dy/\cos(\theta) \\ dz/\cos(\theta) \\ \{\sin(\theta)dx + dt\}/\cos(\theta)^2 \end{array} \right\rangle \tag{256}$$

$$\textbf{The Vector of Intensity 2-forms} \tag{257}$$

$$\left|F^k\right\rangle = d\left|A^k\right\rangle = \left| \begin{array}{c} \{2d\theta\hat{\ }dx + (1+\sin(\theta)^2)d\theta\hat{\ }dt\}\sin(\theta)/\cos(\theta)^3 \\ \sin(\theta)\{d\theta\hat{\ }dy\}/\cos(\theta)^2 \\ \sin(\theta)\{d\theta\hat{\ }dz\}/\cos(\theta)^2 \\ \{(1+\sin(\theta)^2)d\theta\hat{\ }dx + 2d\theta\hat{\ }dt\}\sin(\theta)/\cos(\theta)^3 \end{array} \right\rangle \tag{258}$$

$$\textbf{The Vector of Excitation 2-forms} \quad \textbf{(affine torsion)} \tag{259}$$

$$|G^a\rangle = [\mathbb{C}]\,\hat{\ }\,|dx^m\rangle = \left| \begin{array}{c} \{\sin(\theta)d\theta\hat{\ }dx + d\theta\hat{\ }dt\}/\cos(\theta) \\ \sin(\theta)\{d\theta\hat{\ }dy\}/\cos(\theta) \\ \sin(\theta)\{d\theta\hat{\ }dz\}/\cos(\theta) \\ \{d\theta\hat{\ }dx + \sin(\theta)d\theta\hat{\ }dt\}/\cos(\theta) \end{array} \right\rangle \tag{260}$$

$$\textbf{The 4 Topological Torsion 3-forms} :(\mathrm{A\hat{\ }F})^k \tag{261}$$

$$\text{Topological Torsion} \left| \begin{array}{c} A^1\hat{\ }F^1 \\ A^2\hat{\ }F^2 \\ A^3\hat{\ }F^3 \\ A^4\hat{\ }F^4 \end{array} \right\rangle = \left| \begin{array}{c} \{dx\hat{\ }d\theta\hat{\ }dt\}/\cos(\theta)^3 \\ 0 \\ 0 \\ -\{dx\hat{\ }d\theta\hat{\ }dt\}/\cos(\theta)^3 \end{array} \right\rangle \tag{262}$$

It is remarkable that the individual components of Topological Torsion are not zero, but the composite Total Topological Torsion vanishes. The Total second Poincare Invariant vanishes.

$$\textbf{The 4 Topological Spin 3-forms} : (A\hat{}G)^k \tag{263}$$

$$\text{Topological Spin} \left| \begin{array}{c} A^1\hat{}G^1 \\ A^2\hat{}G^2 \\ A^3\hat{}G^3 \\ A^4\hat{}G^4 \end{array} \right\rangle = \left| \begin{array}{c} \{dx\hat{}d\theta\hat{}dt\}/\cos(\theta)^3 \\ 0 \\ 0 \\ -\{dx\hat{}d\theta\hat{}dt\}/\cos(\theta)^3 \end{array} \right\rangle \tag{264}$$

Also, it is remarkable that the individual components of Topological Spin are not zero, but the composite Total Topological Spin vanishes. The Total First Poincare Invariant vanishes.

Note that the Congruent quadratic form, or metric, is always the Minkowski metric, to within a factor. The Eikonal equation is an invariant of the Lorentz system. With a constant conformal factor, the Riemann curvature of the metric is zero.

However, if the conformal factor is not zero, then the conformal metric can exhibit curvature properties associated with expansions and rotations of space time.

The algebra of a Physical Vacuum in terms of a Basis Frame with the structural format $[\mathbb{P}(\theta)_{xt}]$ is quite extensive. The best place to see and study the results and derivations using the general theory are to be found in the Maple programs:

http://www22.pair.com/csdc/pdf/mapleEP3-Lorentzboost.pdf
http://www22.pair.com/csdc/pdf/mapleEP3-Lorentzwake.pdf

8 Example 4. [$\mathbb{B}$] as a Projection in terms of the Hopf map.

The Hopf map is a famous mapping from 4D space to 3D space. One representative of a Hopf mapping is given by the functions $\{X, Y, Z\}$and the 3 exact differentials $\{dX, dY, dZ\}$

$$\left| \begin{array}{c} X \\ Y \\ Z \end{array} \right\rangle = \left| \begin{array}{c} (xz + ys) \\ (xs - yz) \\ (x^2 + y^2)/2 - (z^2 + s^2)/2 \end{array} \right\rangle, \tag{265}$$

$$\left| \begin{array}{c} dX \\ dY \\ dZ \\ A_{Hopf} \end{array} \right\rangle = \left| \begin{array}{c} zdx + xdz + sdy + yds \\ sdx + xds - zdy - ydz \\ xdx + ydy - zdz - sds \\ -xdy + ydx - zds + sdz \end{array} \right\rangle. \tag{266}$$

The additional 1-form

$$\text{Hopf 1-form} \quad A_{Hopf} = -xdy + ydx - zds + sdz \tag{267}$$

is added to the set of 3 perfect differentials to form a vector of 1-forms that can be related to a Basis Frame of the type

$$[\mathbb{B}_{Hopf}] = \begin{bmatrix} z & s & x & y \\ s & -z & -y & x \\ x & y & -z & -s \\ -y & x & -s & z \end{bmatrix}, \tag{268}$$

$$[\mathbb{B}_{Hopf}] \circ \left| \begin{array}{c} dx \\ dy \\ dz \\ ds \end{array} \right\rangle = \left| \begin{array}{c} dX \\ dY \\ dZ \\ A_{Hopf} \end{array} \right\rangle \tag{269}$$

$$\det [\mathbb{B}_{Hopf}] = +(x^2 + y^2 + z^2 + s^2)^2 = (R^2)^2 \tag{270}$$

$[\mathbb{B}_{Hopf}]$ is in the format ready to be used for infinitesimal mappings into nearby neighborhoods of a Physical Vacuum. The Cartan Connection (based on $[\mathbb{B}_{Hopf}]$),as a matrix of 1-forms, becomes:

$$\textbf{Cartan Connection} \tag{271}$$

$$[\mathbb{C}]_{\mathbf{P}(\theta)} = \begin{bmatrix} C_1^1 & -C_1^2 & -C_1^3 & -C_1^4 \\ C_1^2 & C_1^1 & -C_1^4 & -C_2^4 \\ C_1^3 & C_1^4 & C_1^1 & -C_1^2 \\ C_1^4 & C_2^4 & C_1^2 & C_1^1 \end{bmatrix} \tag{272}$$

$$C_1^1 = d\ln(R^2)/2 \tag{273}$$

$$C_1^2 = (sdz - zds + ydx - xdy)/R^2 \tag{274}$$

$$C_1^3 = (sdy - yds \ + xdz - zdx)/R^2 \tag{275}$$

$$C_1^4 = (ydz - zdy \ + xds - sdx)/R^2 \tag{276}$$

$$C_2^4 = (yds - sdy \ + zdx - xdz)/R^2 \tag{277}$$

$$\textbf{Hopf map} : \textbf{The Congruent pullback metric} \tag{278}$$

$$[g_{jk}]_{Hopf} = [\mathbb{B}_{hopf\_transpose}] \circ \eta \circ [\mathbb{B}_{hopf}], \tag{279}$$

$$[g_{jk}]_{Hopf} = \begin{bmatrix} g_{11} & -2xy & 2ys & -2yz \\ -2xy & g_{22} & -2xs & 2xz \\ 2ys & -2xs & g_{33} & -2sz \\ -2yz & 2xz & -2sz & g_{44} \end{bmatrix}, \tag{280}$$

$$g_{11} = (-x^2 + y^2 - z^2 - s^2) \tag{281}$$

$$g_{22} = (+x^2 - y^2 - z^2 - s^2) \tag{282}$$

$$g_{33} = (-x^2 - y^2 - z^2 + s^2) \tag{283}$$

$$g_{44} = (-x^2 + y^2 + z^2 - s^2) \tag{284}$$

$$\textbf{Hopf Map} : \text{Vector Array of } \textbf{Affine Torsion 2-forms} \tag{285}$$

$$\text{Field Excitations} \left| \begin{array}{l} G^x \\ G^y \\ G^z \\ G^s \end{array} \right\rangle = 2/(R^2) \left| \begin{array}{c} ydy\hat{}dx + yds\hat{}dz \\ -xdy\hat{}dx - xds\hat{}dz \\ sdy\hat{}dx + sds\hat{}dz \\ -zdy\hat{}dx - zds\hat{}dz \end{array} \right\rangle, \ \text{(D and H)} \tag{286}$$

$$R^2 = x^2 + y^2 + z^2 + s^2. \tag{287}$$

$$\textbf{Hopf Map} \ : \textbf{Vector potentials} \ \left|A^k\right\rangle \tag{288}$$

$$\text{Potentials} \ \left|A^k\right\rangle = \left| \begin{matrix} zdx + sdy + xdz + yds \\ sdx - zdy - ydz + xds \\ xdx + ydy - zdz - sds \\ zds - sdz + xdy - ydx \end{matrix} \right\rangle \tag{289}$$

$$\textbf{Hopf Map} \ : \textbf{Intensities} \quad \left|F^k\right\rangle = d\left|A^k\right\rangle \tag{290}$$

$$\text{Intensities:} \quad \left|F^k\right\rangle = \left| \begin{matrix} 0 \\ 0 \\ 0 \\ 2dx\hat{\ }dy + 2dz\hat{\ }ds \end{matrix} \right\rangle \tag{291}$$

Note that first three 2-forms of field intensities are zero, $F^1 = F^2 = F^3 = 0$, but all four of the field excitations are not zero, $G^1 = G^2 = G^3 = G^4 \neq 0$. It is also remarkable that only the last Topological Torsion 3-form is not zero, but all 4 of the elements that make of the Total Topological Spin 3-form are not zero. For these results and others of more complicated algebraic display, see the Maple programs at

http://www22.pair.com/csdc/pdf/mapleEP5-Hopf.pdf,
which will be published on a CD rom, [37].

9 Thermodynamic Phase Functions from [$\mathbb{B}$]

As pointed out in my first volume[4] "Non Equilibrium Thermodynamics and Irreversible Processes", that, given any matrix, useful information (related to Mean and Gaussian Curvatures of the domain described by the matrix) can be obtained from the Cayley-Hamilton Characteristic Polynomial. This polynomial exists for any square matrix. In particular, for 4 x 4 matrices, if the matrix has no null eigenvalues (maximal rank), the domain is then Symplectic and the characteristic polynomial is of 4th degree in the polynomial variable (which represents the eigenvalues).

There exists a well known transformation of complex variable, $\xi = s + M/4$, which will reformulate the characteristic polynomial into a system where the "mean curvature" is zero:

[4] See http://www.lulu.com/kiehn

$$\Theta(x, y, z, t;\ \xi) = \xi^4 - X_M\xi^3 + Y_G\xi^2 - Z_A\xi^1 + T_K \Rightarrow 0, \quad (292)$$

becomes $$\Psi(x, y, z, t;\ s) = s^4 + gs^2 - as + k = 0. \quad (293)$$

Consider the reduced Phase formula, $\Psi = 0$, and its derivatives with respect to the family parameter, s.

$$\Psi = s^4 + gs^2 - as + k = 0, \quad (294)$$

$$\therefore k = -(s^4 + gs^2 - as), \quad (295)$$

$$\Psi_s = \partial\Phi/\partial s = 4s^3 + 2gs - a = 0 \quad (296)$$

$$\therefore a = 4s^3 + 2gs \quad (297)$$

$$\Psi_{ss} = \Phi_s = \partial^2\Phi/\partial s^2 = 12s^2 + 2g = 0 \quad (298)$$

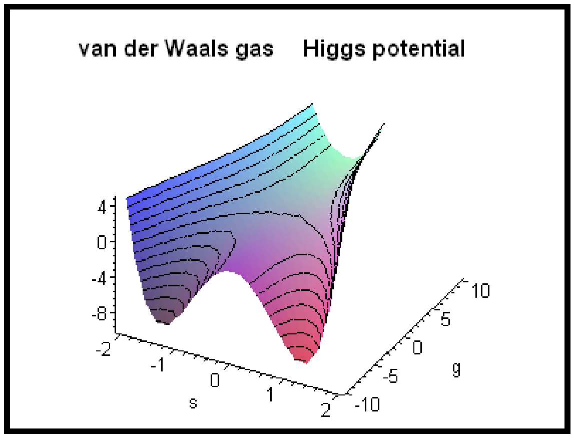
$$\therefore g = -6s^2 \quad (299)$$

9.1 The Higgs potential as an Envelope of the Thermodynamic Phase function.

Replacing the parameter a (from the envelope condition, $\Psi_s = 0$) in the equation for k yields

$$k = s^2(3s^2 + g). \quad (300)$$

A plot of this implicit surface appears below. The envelope of the reduced quartic polynomial yields a thermodynamic phase function that establishes an extraordinary correspondence between the binodal and spinodal lines of a van der Waals gas, and its critical point. I call this envelope, the Higgs Phase function. Of particular importance are the curves $s = 0$ and $\partial k/\partial s = 0$ on the Higgs phase function. These curve $s = 0$ defines the thermodynamic Binodal line as a pitch-fork bifurcation set, and the set $\partial k/\partial s = 0$ defines the Spinodal line as the limit of phase stability. The critical point is at $s = g = k = 0$. These topological concepts can be mapped to, and should be compared with, the historic features of a van der Waals gas.



The vertical axis is the Higgs function, k.
Singularities in $[\mathbb{B}]$ **occur when** $k = 0$ **or** $\partial k/\partial s = 0$.
The critical point is where $k = s = g = 0$.

The conjecture is that Higgs features (related to mass and inertia) have a basis in topological thermodynamics, for at the critical point it is known from chemical thermodynamics (and Lev Landau) that there are large fluctuations in density, as the material attempts to condense, and these fluctuations are correlated with a $1/\mathrm{r}^2$ force law. It should be noted that the Higgs function, k, is related to the determinant of the Basis Frame, and is an artifact of 4 topological dimensions. The s component is related to an abstract order parameter, or a molar (mass) density[5] in a thermodynamic interpretation, and g has the abstract features of a temperature. Details are to be found in [32]. An interesting application is in the form of a cosmological model, where it is presumed that the universe is a low density gas near its critical point [27].

It is remarkable to me that all of this starts from the sole assumption of a Basis Frame $[\mathbb{B}]$. The Higgs idea and the extended Yang Mills theory (and weak force) seem to be related to an irreducible Pfaff topological dimension 4, where evolutionary irreversible processes are possible and parity is not preserved.

I conjecture that if the Phase function has 1 null eigen value, then the space is related to a non equilibrium configuration of Pfaff dimension 3, for which parity is always preserved (the strong Force). The electromagnetic domain only requires Pfaff dimension 2, and the gravity domain is embedded in Pfaff topological dimension 1. This follows from an old I argument (based on differential geometry) that I presented in 1975 [15].

The idea is that spaces of Pfaff Topological dimension 2 or less (hence thermodynamically these domains are isolated or are in thermodynamic equilibrium) are topologically *connected*,

[5] The effective eigen value, s, is related to the molar density minus the energy of Mean curvature.

and interactions can range over the entire domain (long range forces - gravity and electromagnetism). However, spaces of Pfaff Dimension 3 or more are topologically *disconnected* domains of multiple components [32], and are not in thermodynamic equilibrium. Hence interactions and forces are short range (strong and weak forces, that may or may not preserve parity)

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