Electromagnetism and Differential Forms:

The Topology of Forces (E,B) vs The Topology of Sources (D,H)

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Abstract: In Maxwell's theory of Electromagnetism, the concept of force is related to the fields of **E** and **B,** which are the components of an exact 2-form $\mathbb{F}$=d$\mathbb{A}$ on 4D space time. The concept of source is related to the fields of **D** and **H**, which are the components of an inexact N-2=2 form density, $\mathbb{G}$, on space time. The presence of matter induces geometrical constraints in the form of constitutive relations between $\mathbb{F}$ and $\mathbb{G}$. However, the symplectic topology $\mathbb{F}$^$\mathbb{F}$ is open, while the symplectic topology of $\mathbb{G}$^$\mathbb{G}$ is compact. The Maxwell system supports three fundamental 3-forms, $\mathbb{A}$^$\mathbb{F}$, $\mathbb{A}$^$\mathbb{G}$, and $\mathbb{J}$=d$\mathbb{G}$. The ratios of the integrals of $\mathbb{F}$^$\mathbb{F}$ to $\mathbb{G}$^$\mathbb{G}$ is the radiation impedance of the free space. The ratio of the closed integrals of $\mathbb{A}$^$\mathbb{F}$ to $\mathbb{A}$^$\mathbb{G}$ is the rational Hall impedance. The cross ratio of the four quantities is twice the fine structure constant, a topological invariant.

(Engineers do it right. Physicists do it wrong.)

Cartan's Theory of Exterior Differential Calculus

operates on differential p-forms, A, B, ... (integrands of multiple integrals)

Two non-conventional ideas:

1. **^ : The exterior product.** $A \wedge B = -B \wedge A$, $A \wedge A = 0$
2. **d : The exterior derivative.** $F = dA$, $ddA = dF \Rightarrow 0$

Algebraic Closure
and
Differential Closure

For Odd p-forms, $A \wedge A = 0$
For even p-forms, the 0 result depends on the topological dimension.

On a space of three topological dimensions, the fundamental 2-form of electromagnetism, F = dA, satisfies $F \wedge F = 0$, always.

On a space of four (or more) topological dimensions, the fundamental 2-form of electromagnetism, F = dA, need not satisfy the constraint $F \wedge F = 0$. In fact, irreversible processes are to be associated with the fact that on 4D

$$F \wedge F \neq 0$$

Moral: Must use 4D representation of differential forms for electromagnetism.

Differential Forms
carry
Topological Information

Concept of Closure focuses Attention on Topological Aspects of Cartan theory

What about other topological features?

Pfaff Topological Dimension

Defines the irreducible number of functions required to describe an exterior differential system.

Period Integrals and Harmonic forms

Every p-form can be decomposed into THREE parts. (deRham)

A = exact + harmonic + co-exact
$\Rightarrow$ grad ϕ + γ + curl Z

In a euclidean topology, harmonic part, γ, does not exist.
There is a unique γ for every topological hole or handle.

The closed (period) integrals of
{ d ϕ + γ} = 0 + { γ}
are evolutionary invariants:

{ γ} = integers (times a constant)

Harmonic vector fields are not dissipative, and generate minimal surfaces and "stationary" states.

{ d ϕ + γ } are the "gauge" terms

Topological Defects
and
Differential Forms

If a form is exact, then it can have a domain of support which is compact, or compact with boundary cycles or “defects” representing zeros of the form.

If the Harmonic part of the form has zero divergence, then the form has a handle?

If a form is closed, then it can have a domain of support which is compact, or compact with boundary cyclic “defects” which must come in equal and opposite pairs.
(Charge from D and H)

If an exact form is never zero
then its domain of support
cannot be compact without boundaries.
(E and B)

Maxwell's Electromagnetism is a set of topological statements.

Independent from metric, connection, dimension!

These ideas started with Van Dantzig about 1934

Start with an ordered array:

$$\{1, 2, 3, 4, \dots .\}$$

Assert the existence of ordered set of independent variables.

$$\{x, y, z, t \dots .\}$$

Construct a set of C2 functions of {x, y, z, t ... .}

$$\{A_x, A_y, A_z, \phi \dots .\}$$

Construct the 1-form of Action

$$\mathbb{A} = A_k dx^k - \phi dt$$

Use rules of Cartan Calculus to construct the 2-form of Field Intensities

$$\mathbb{F} = d\mathbb{A} = B_z dx \wedge dy + \dots + E_x dx \wedge dt + \dots$$

with $E = -\text{grad}\ \phi - \partial A/\partial t$ and $B = \text{curl}\ A$
(Field Intensities do not depend upon choice of Gauge)

Use Poincare lemma for second exterior derivative and derive First Maxwell pair.

$$d\mathbb{F} = dd\mathbb{A} = 0$$

which is equivalent to

$$\text{Curl}\ E + \partial B/\partial t = 0, \qquad \text{div}\ B = 0$$

Any set of symbols ! EM, Hydrodynamics, ... Economics!
Always get First Maxwell set of (topological) equations.

The First Maxwell Set (Forces)

As F = dA is exact and non-zero,
from Stokes theorem
$$\int_{boundary} A = \iint F > 0$$
If the left side is compact, then it has no boundary, and the integral is zero, in disagreement with the right side. Therefore domain of support for F is NOT COMPACT.

For evolution, use Cartan's Magic Formula:

$$L_{(\rho V)} A = i(\rho V)dA + d(i(\rho V)A) = W + dU \Rightarrow Q$$
Topological equivalent of the First Law of Thermodynamics

Virtual Work $\Rightarrow i(\rho V)dA = f_L \bullet dr + Pdt$
where

$f_L = \rho[E + V \times B]$ **the Lorentz force!**

1. **Virtual Work** $\Rightarrow 0$ **: implies Extremal process (Hamiltonian)**
Yields The Perfect Plasma equations:
$$f_L = \rho[E + V \times B] = 0$$

2. **d(Virtual Work)** $\Rightarrow 0$ **: implies Symplectic process (Helmholtz)**
Yields The Master Equation:
$$\partial B/\partial t - \mathrm{curl}\,(V \times B) = 0.$$

Both of these processes are Thermodynamically Reversible !!!

Many uses of CARTAN's Magic Formula:

$$L_{(\rho V)}A = i(\rho V)dA + d(i(\rho V)A) = Q$$

where 4 vector $(J,\rho) = \rho(V,1)$

1. Describes topological evolution

2. Describes "stationary" paths if $Q = 0$

3. Is equivalent to the First Law of Thermodynamics.

$$L_{(\rho V)}A = W + d(U) = Q$$

$$W = i(\rho V)dA \equiv \quad \text{1-form of virtual Work}$$

Axiom T: A Thermodynamic Process is irreversible when the Heat 1-form Q does not admit an integrating factor. (P. M. Morse)

Differential Form Criteria for existence of Integrating Factor is the Frobenius Condition $\equiv$ $Q\hat{}dQ = 0$.

Therefore

$$L_{(\rho V)}A \,\hat{}\, L_{(\rho V)}dA = Q\hat{}dQ \neq 0$$

implies

Process ρV is irreversible

on Physical System represented by A the 1-form of Action.

HOW TO FIND EXAMPLES of IRREVERSIBLE PROCESSES ?

Given Electromagnetic Action, A
compute

TOPOLOGICAL TORSION 3-form

$$i(T)dx\wedge dy\wedge dz\wedge dt = A\wedge dA$$

4D TORSION CURRENT

$$T = \{E\times A + B\phi,\ A\bullet B\}$$

$$4\mathrm{Div}\ T = -2\ E\bullet B$$

By Direct Computation

$$L_{(T)}A \wedge L_{(T)}dA = \Gamma^2\ A\wedge dA \neq 0$$

$$\Gamma \approx E\bullet B$$

Processes in the direction of the Torsion Current are irreversible, when DivT $\neq 0$.

IMPLICATION

$dA \wedge dA = -2\, E \bullet B\, dx \wedge dy \wedge dz \wedge dt \neq 0$

$\supset$ 4D Symplectic Manifold

(No unique extremal processes exist on a symplectic manifold.)
Hamiltonian Processes can exist on a symplectic manifold, but they are not unique.
They are dissipative in "energy" but not in "angular momentum";
hence they are reversible in a thermodynamic sense.

Moral:
To Minimize Irreversible Dissipation in a Plasma, Minimize or Eliminate Domains where $E \bullet B \neq 0$.

When $E \bullet B = 0$, domain is a contact manifold of Pfaff Topological dimension 3 and admits a unique non-dissipative Hamiltonian Evolution as an extremal process.

The Second Maxwell Set (Sources)
Subsume there exists a
N-2 form G,
such that the N-1 form, J
is exact

Then $dG = J$

is equivalent to: $\mathrm{curl}\, H + \partial D/\partial t = j$
$\mathrm{div}\, D = \rho$

Conservation Law:
$ddG = dJ = 0$

is equivalent to: $\mathrm{div}\, j + \partial\rho/\partial t = 0$

Duality relations:

A is dual to J
F is dual to G

Many duality mechanisms

The Complete Maxwell System (on 4D)

{A, F, G, J, A^G, A^F, A^J, F^F, F^G}

Exact Components:

Physical Dimension h/e	$dA=F$
Physical Dimension e	$dG= J,$
Physical Dimension $(h/e)^2$	$d(A\wedge F) = F\wedge F,$
Physical Dimension h	$d(A\wedge G) = F\wedge G-A\wedge J$

When RHS = 0, then $\int_{closed}(LHS) \approx$ integers n

Flux quantum h/e	$\int_{c1}A = n\ h/e$
Charge quantum e	$\int_{c2}G = n\ e,$
Topological Torsion $(h/e)^2$	$\int_{c3} (A\wedge F) = n\ (h/e)^2,$
Spin quantum h	$\int_{c3} (A\wedge G) = n\ h$

Note that it is possible to define 3
IMPEDANCES
which are ratios of period integrals and therefore are not dissipative.

$$Z_{Hall} = h/e^2 = 25{,}812.81491 \text{ ohms}$$

Superconductivity Type II

Physical Dimension h/e^2

$$Z_1 = \int_{c1} A / \int_{c2} G$$

finite in domain interior: $\{A,\phi,D,H\}$
null in domain interior: $\{E,B,J,\rho\}$

Superconductivity (Hi-TC? Anyon?)

Physical Dimension h/e^2

$$Z_2 = \int_{c3} A\wedge G/\left(\int_{c2} G\right)^2$$

finite in domain interior: $\{A,\phi,E,B,D,H,E\bullet B\}$
null in domain interior: $\{J,\rho,(B\bullet H-D\bullet E)\}$

Rational Fraction Quantum Hall effect

Physical Dimension h/e^2

$$Z_3 = \int_{c3} A\wedge F/\int_{c3} A\wedge G$$

finite in domain interior: $\{A,\phi,E,B,D,H,J,\rho\}\}$
null in domain interior: $\{E\bullet B,(B\bullet H-D\bullet E)-(A\bullet J-\rho\phi)\}$

Dissipative Radiation Impedance of Free Space:

$$Z_0 = \mu/\varepsilon = 376.730313 \text{ ohms}$$

Fine Structure constant

$$\alpha^2 \bullet R_{\text{Bohr Orbit}} = \alpha \bullet \lambda_{\text{Compton}} = r_{\text{classical radius of electron}}$$

NOTE

$$\alpha = \tfrac{1}{2}(Z_0/Z_{\text{hall}}) = 1/137.0360411$$

Implies Z_0 must have topological significance.

$$(Z_0)^2 = \iint\iint F\wedge F / \iint\iint G\wedge G \quad = \quad \mu/\varepsilon$$

or

$$(Z_0)^2 = 4\,(Z_{\text{hall}})^2 \{(\quad G)^2 / \iint\iint G\wedge G\}$$

$$(Z_0)^2 = 4\,(Z_{\text{hall}})^2 \{\alpha^2\}$$

Topological Result #1:

**If $E \bullet B \neq 0$ over a domain,
then the topological dimension is 4
creating a symplectic manifold which
is NOT compact without boundary.**

Topological Result #2:

**If $E \bullet B = 0$, but $E \times A + B\phi \neq 0$
then topological dimension is 3
creating a contact manifold,
With a unique (Hamiltonian) evolutionary field.**

Topological Result #3:

**If $E \times A + B\phi = 0$,
then topological dimension is 2
creating a symplectic manifold.**

OTHER TOPICS
in terms of
DIFFERENTIAL FORMS

1. Quaternion TORSION Waves

10^{18} Signal to Noise in Dual Polarized Ring Lasers

2. FTL Propagation of Singularities (> C)

V. Fock proves Conformal solutions indicate FTL for Topological Singularities

3. Casimir ΔP and Symplectic ΔT

4. Duality Conditions - Yang Mills theory.

5. Needle Radiation and Torsion Defects

Collapse of the Wave Function

6. Minimal Surfaces as Defects in EM waves (Electromagnetic Wakes!)

Anti-Stealth Technology ?

7. Planck Eq. Is Topological Formula

(Hence must be derivable from Exterior Differential Form arguments)

8. MagnetoDynamic Alpha effect

Symplectic Evolution ($E \bullet B \neq 0$) implies

$$\rho[E + V \times B] = -\text{grad}\ \Theta$$

and

$$B \bullet \text{grad}\ \Theta \neq 0.$$

Implies gradient of Temperature ($\Theta = kT$) along B field lines,
and
Ohmic Dissipation in the direction of the B field lines.

$$\rho[E \bullet V] = -\text{grad}\ \Theta \bullet V$$

∴ Along B field Lines

$$\rho E = -\text{grad}(kT)$$

Using $J = \sigma E$ leads to

Thompson thermal power formula:

$$J = -\text{grad}(kT)/\rho\sigma$$

Conjecture:

Magneto Hydrodynamic Acceleration is due to Thermal gradient along B field lines around rotating neutron stars.