A Topological Perspective of Plasmas

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Abstract

A plasma is an electromagnetic physical system that is not a thermodynamic isolated equilibrium system (which must be of Pfaff topological dimension of 1), but can appear as an isolated physical system (of Pfaff topological dimension 2), or as a chaotic closed physical system (of Pfaff topological dimension 3), or, in general, as an irreversibly dissipative turbulent physical system (of Pfaff topological dimension 4). The topological properties of closed or open plasma systems are encoded in terms of two exterior differential 3-forms of topological spin, $A\hat{}G$, and topological torsion, $A\hat{}F$. These 3-forms are composed from the set of exterior differential forms $\{A, F = dA, G, J = dG\}$ used to define an electromagnetic system. These 3-forms are identically zero in equilibrium systems of Pfaff dimension 1. The topological limit sets (exterior derivative) of these 3-forms define the topological Poincare Invariants. When the Poincare invariants vanish, the 3-forms have properties similar to the conservative charge current 3-form $J = dG$, and can exhibit topological quantization in terms of deRham period integrals as topologically coherent sructures. However when the Poincare invariants do not vanish they become the source of topological evolution, irreversible changes of phase, topological defects, and stationary states far from equilibrium.

1. Introduction

Perhaps the most formidable problem in the study of plasmas is to recognize that the plasma is a thermodynamic system NOT in equilibrium. Most plasmas are

OPEN thermodynamic systems exchanging energy and mass with their environment. Such electromagnetic systems are best described as turbulent dissipative plasmas, evolving by processes that induce topological change and which are thermodynamically irreversible. However, a plasma also can support "stationary" or "excited" states of coherent, but deformable and measurable topological structure, with long apparent lifetimes (modulo topological fluctuations). That is, the evolutionary processes for the "excited states" are not turbulent, and can preserve the recognizable topological coherent properties of the physical system. However, such long lived excited states are NOT equilibrium states in a thermodynamic sense. Such states are locally equivalent to *CLOSED* thermodynamic systems, where radiation is exchanged with the environment, but mass is not. In the topological development presented below, the *OPEN* Turbulent Irreversible plasma state will have a minimum topological Pfaff dimension of 4; the *CLOSED* "Stationary" states will have a Pfaff Topological dimension of 3, and can appear as topological defects in the Turbulent Irreversible plasma. Both sets of states are "far from equilibrium", for the equilibrium system is a subset of an *ISOLATED* thermodynamic system which requires a Pfaff topological dimension of 2. The fully equilibrium state will be of Pfaff topological dimension 1. The concept if Pfaff topological dimension is discussed in [Baldwin 1991] and in Section 4.2 below.

What makes the non-equilibrium thermodynamic system so formidable is that the historical concepts and geometric techniques, which impose a dogmatic search for unique predictions from given initial data, fail. The dynamics of non-equilibrium systems requires a mathematics which can describe processes that incorporate topological, not merely geomtrical, change. Hamiltonian mechanics fails, for it will not describe changes of the Cartan topology constructed from an exterior differential 1-form of Action used to encode the physical system. If topological evolutionary solutions exist, the solutions need not be unique. Envelopes (Huygen wavelets) and wakes (edges of regression) and multiple phases are observational examples. In short, geometrically based concepts and methods based on tensor analysis are not entirely suitable for studying the non-equilbrium states of a plasma. The diffeomorphic constraint used in the definition of tensors does not admit topological evolution and change. Fortunately, Cartan's method of exterior differential calculus goes beyond the limitations of tensor analysis and permits the study of continuous topological evolution. It turns out that differential forms, although not deterministic or predictive with respect to non-invertible C1 maps, are functionally well behaved with respect to such maps, but in a retrodictive, or pullback sense [RMK 1976 b]. Given data on the final state exterior differential forms, the functional form of the exterior differential forms on the initial state are well defined for C1 maps without inverse. Tensors do not enjoy these properties.

In the language of exterior differential systems [Bryant 1991] it becomes evident that classical electromagnetism is equivalent to a set of topological constraints on a variety of independent variables. Certain integral properties of an electromagnetic system are deformation invariants with respect to all continuous evolutionary processes that can be described (to within a factor) by a vector direction field. These integral deformation invariants (which are topological properties) lead to the fundamental topological conservation laws described in the physical literature as the conservation of charge and the conservation of flux. Recall the definitions used to describe processes of continuous topological evolution [RMK1991 b]:

> A continuous process is defined as a map from an initial state of topology $T_{initial}$ into a final state of perhaps different topology T_{final} such that the limit points of the initial state are permuted among the limit points of the final state [Lipschutz 1965] [Kuratowski 1948]. Note that the initial and final topologies are not necessarily the same.
>
> A topological deformation invariant is defined as an integral over a closed manifold, or cycle, z, such that the Lie derivative of the closed p dimensional integral $\int_{zpd} \omega$ with respect to a deformed vector field, ρV^k, vanishes, for any choice of the deformation parameter, ρ.

$$\text{Deformation Invariant}: \quad L_{(\rho V^k)} \int_{zpd} \omega = 0 \quad any\ \rho. \tag{1.1}$$

If the deformation parameter is only a function of time, then the direction field can be put into correspondence with a kinematic "velocity" field where the concept of time and arc length are functionally related. In general, the vector field, ρV^k, need not be the generator of a single parameter group.

Those properties of a phsyical object that remain the same under continuous deformation are recognized by their topological, not geometric, properties. F. Klein defined a euclidean geometric property as an invariant of evolutionary processes that was represented by a (rigid) rotation or translation. Size and shape are examples of geometric properties of a physical object. Now consider those properties that are invariant with respect to continuous deformations. These properties are topological properties. Although somewhat oversimplified, the definition of a topological property as an invariant of a continuous deformation will find practical application in that which follows. Continuous deformations change size and shape but preserve topological properties of physical objects. For example, the number of holes in a thin rubber sheet stays the same as the rubber sheet is deformed. The number of holes is a topological property and is a deformation invariant. Topological evolution includes those situations where the number of holes stays the same, or changes. Thermodynamic irreversibility occurs when the process causes the number of holes to change. Topological change is a necessary requirement for irreversibility.

The idea of a deformation invariant has been applied by Cartan to define Hamiltonian mechanics. Consider, in a connected region, a tube of trajectories generated as flow lines tangent some vector field, ρV^k. Consider a closed integration chain, $z1d$, (a cycle) that connects points on the tube of different trajectories, and encircles the tube. Construct the integral of the 1-form of Action (which encodes the physical system),

$$\int_{z1d} A = \int_{z1d} \{pdq - H(p,q,t)dt\}. \tag{1.2}$$

Cartan's theorem [Cartan 1958 (1922)] then states that those vector fields, ρV^k, that leave the closed integral of Action a deformation invariant (for any positive continuous function ρ) have a Hamiltonian generator.

$$\text{Hamiltonian Vector fields :} \quad L_{(\rho V^k)} \int_{z1d} \{pdq - H(p,q,t)dt\} \Rightarrow 0 \quad any\ \rho. \tag{1.3}$$

Different values of the continuous deformation parameter $\rho(p,q,t)$ will "slide" and "deform" the points that make up the integration cycle, but will maintain the cycle on the same "tube of trajectories". Cartan thereby defined the equivalence class of conservative Hamiltonian processes in a topological manner by requiring that processes be the subsets of singly parameterized vector fields that leave the closed integral of the 1-form of Action a continuous deformation invariant.

Note that for physical systems that can be defined by a 1-form of Action, A, the derived 2-form $F = dA$ is an integral deformation invariant with respect to *all* continuous processes that can be defined by a singly parameterized vector field. This concept is at the basis of the Helmholtz theorems in hydrodynamics, and the conservation of flux in classical electromagnetism. Herein, this topological constraint will be called the postulate of potentials. When written as the equation, $F - dA = 0$, the postulate of potentials is to be recognized as an exterior differential system constraining the topology of the independent variables. From Stokes theorem, the (2 dimensional) domain of finite support for F can not, in general, be compact without boundary, unless the Euler characteristic vanishes. There are two exceptional cases for two dimensional domains, the torus and the Klein-Bottle, but these situations require the additional topological constraint that $F\hat{}F = 2(\mathbf{E} \circ \mathbf{B})\Omega_4 \Rightarrow 0$. The fields in these exceptional cases must reside on these exceptional compact surfaces, which form topological coherent structures in the electromagnetic field. For an electromagnetic action, the exceptional compact cases can only exist if $\mathbf{E} \circ \mathbf{B} = 0$. The resulting statement is that there do not exist compact domains of support without boundary when $\mathbf{E} \circ \mathbf{B} \neq 0$, a statement that will be of interest to thermodynamics of irreversible systems, and of plasma jets. Note that an evolutionary process could start with $F\hat{}F \neq 0$, and possibly evolve to a state with $F\hat{}F = 0$. If such residue states are compact without boundary, then they must be either tori or Klein bottles.

The definition of an electromagnetic system of charges and currents will require a second topological constraint imposed upon the domain of independent variables. This second postulate will be called the postulate of conserved currents. The electromagnetic domain not only supports the 1-form A, but also supports an N-1 form density, J, which is exact. The equivalent differential system, $J - dG = 0$, requires that the (N-1 dimensional) domain of support for J cannot be compact without boundary. However, the closed integrals of J are deformation integral invariants for *any* continuous evolutionary process that can be defined in terms of a singly parameterized vector field.

In that which follows, the classical Maxwell system will be displayed in terms of the vector formalism of Sommerfeld and Stratton [Sommerfeld 1952] [Stratton 1941]. The key feature is to note that the fields of intensities ($\mathbf{E}$ and $\mathbf{B}$) are considered as separate and distinct from the fields of excitation ($\mathbf{D}$ and $\mathbf{H}$), a historical thermodynamic distinction (championed by Sommerfeld) that is often masked in modern exposes of electromagnetic theory. Indeed, it will be demonstrated explicitly that the classic formalism of electromagnetism is a consequence of a system of two fundamental topological constraints

$$\text{Postulate of Potentials:} \qquad F - dA = 0, \tag{1.4}$$

$$\text{Postulate of Charge Currents} \ : \quad J - dG = 0, \tag{1.5}$$

defined on a domain of four ordered independent variables, abstractly written as $\{x, y, z, t\}$. The theory requires the existence of four fundamental exterior differential forms, $\{A, F, G, J\}$, which can be used to construct the complete Pfaff sequence [Schouten 1949] of forms by the processes of exterior differentiation and exterior multiplication. On a domain of four independent variables, the complete Pfaff sequence contains three 3-forms: the classic 3-form of charge current density, J, and the (apparently novel to many researchers) 3-forms of Spin Current density, $A\hat{}G$, [RMK 1977] and Topological Torsion-Helicity, $A\hat{}F$ [RMK 1990].

As the charge current 3-form, J, is a deformation invariant by construction (J is both closed and exact), it is of interest to determine topological refinements or constraints for which the 3-forms of Topological Spin Current and Topological Torsion will define physical topological conservation laws in the form of deformation invariants. The 3-forms, $A\hat{}G$, and $A\hat{}F$, are not necessarily closed, nor exact. Their exterior derivatives (divergences) are not necessarily zero. The values of the 4-forms created by exterior differentiation of these 3-forms define the topological Poincare invariants. As these 4-forms are exact by construction, they are deformation invariants and thereby define topological properties. The 3-forms are not necessarily, in themselves, deformation invariants.

However, when the Poincare invariants vanish (zero divergence) the closed integral of the corresponding 3-form generates a topological quantity (Topological Spin or Topological Torsion respectively) which is also a deformation invariant. In such situations, the 3-forms are closed, but not necessarily exact. Hence their closed integrals become deRham period integrals [Flanders 1963] [deRham 1960], and have rational ratios. Such is the stuff of topological quantization, which is independent from scales.

The concepts of a Topological Spin Current and the Topological Torsion vector have been utilized hardly at all in applications of classical electromagnetic theory. These 3-form objects are artifacts of non-equilibrium electromagnetic systems. Topological Torsion and Topological Spin have zero values in equilibrium systems. Just as the vanishing of the 3-form of charge current, $J = 0$, defines the topological domain called herein the plasma vacuum, the vanishing of the two other 3-forms can be used to refine the fundamental topology of the Maxwell system.

2. The Domain of Classical Electromagnetism

2.1. The classical Maxwell-Faraday and the Maxwell-Ampere equations.

Using the notation and the language of Sommerfeld and Stratton, the classic definition of an electromagnetic system is a domain of space-time $\{x, y, z, t\}$ which supports both the Maxwell-Faraday equations,

$$curl\ \mathbf{E} + \partial\mathbf{B}/\partial t = 0, \qquad div\ \mathbf{B} = 0, \tag{2.1}$$

and the Maxwell-Ampere equations,

$$curl\ \mathbf{H} - \partial\mathbf{D}/\partial t = \mathbf{J}, \qquad div\ \mathbf{D} = \rho. \tag{2.2}$$

2.2. The conservation of charge current

In every case, the charge current density for the Maxwell system satisfies the conservation law,

$$div\,\mathbf{J} + \partial\rho/\partial t = 0. \tag{2.3}$$

The charge-current densities are subsumed to be zero $[\mathbf{J},\ \rho] = 0$ for the vacuum state. For the Lorentz vacuum state, the field excitations, $\mathbf{D}$ and $\mathbf{H}$, are linearly connected to the field intensities, $\mathbf{E}$ and $\mathbf{B}$, by means of the Lorentz (homogeneous and isotropic) constitutive relations:

$$\mathbf{D} = \varepsilon\mathbf{E}\,, \quad \mathbf{B} = \mu\mathbf{H}. \tag{2.4}$$

The two vacuum constraints imply that the solutions to the homogeneous Maxwell equations also satisfy the vector wave equation, typically of the form

$$grad\ div\ \mathbf{B} - curl\,curl\,\mathbf{B} - \varepsilon\mu\partial^2\mathbf{B}/\partial t^2 = 0. \tag{2.5}$$

The constant wave phase velocity, v_p,is taken to be

$$v_p^2 = 1/\varepsilon\mu \equiv c^2 \tag{2.6}$$

Similar results can be obtained for the solid state where the constitutive constraints can be more complex [RMK 1991 d], and for the plasma state where the charge-current densities are not zero. It should be emphasized that the Lorentz constitutive equations form a severe topological (and not necessary) constraint on the general Maxwell electromagnetic system.

2.3. The existence of potentials

It is further subsumed that the classic Maxwell electromagnetic system is constrained by the statement that the field intensities are deducible from a system of twice differentiable potentials, $[\mathbf{A}, \phi]$:

$$\mathbf{B} = curl\ \mathbf{A}, \qquad \mathbf{E} = -grad\ \phi - \partial\mathbf{A}/\partial t. \tag{2.7}$$

This constraint topologically implies that domains that support non-zero values for the covariant field intensities, $\mathbf{E}$ and $\mathbf{B}$, can *not* be compact domains without a boundary. It is this constraint that distinguishes classical electromagnetism from Yang Mills theories, which have propensity for topological manifolds that are compact without boundary. Two other classical 3-vector fields are of interest, the Poynting vector $\mathbf{E} \times \mathbf{H}$ representing the flux of electromagnetic radiative energy, and the field momentum flux, $\mathbf{D} \times \mathbf{B}$.

3. The domain of Topological Electromagnetism

3.1. The Fundamental Exterior Differential Systems.

The formulation of Maxwell theory given above is relative to a choice of independent variables $\{x, y, z, t\}$, and uses classical vector analysis developed in euclidean 3-space. The topological features of the formalism are not immediately evident. However, electromagnetism has a formulation in terms of Cartan's exterior differential forms [RMK 1969] [RMK 1991 c]. Exterior differential forms do not depend upon a choice of coordinates, do not depend upon the a choice of metric, and are independent of the symmetry constraints imposed by gauge groups and connections. In such a formulation the equations of an electromagnetic system become recognized as consequences of topological constraints on a domain of independent variables.

The use of differential forms should not be viewed as just another formalism of fancy. The technique goes beyond the methods of tensor calculus, and admits the study of topological evolution. Recall that if an exterior differential system is valid on a final variety of independent variables $\{x, y, z, t\}$, then it is also true on any initial variety of independent variables $\{\xi^1, \xi^2, \xi^3, \xi^4\}$ that can be mapped onto $\{x, y, z, t\}$. The map need only be differentiable, such that the Jacobian matrix elements of the mapping are well defined *functions*. The Jacobian matrix does not have to have an inverse, so that the exterior differential system is *not* restricted to the equivalence class of diffeomorphisms. However, the field intensities on the initial variety are functionally well defined by the pullback mechanism, which involves algebraic composition with components of the Jacobian matrix transpose, and the process of functional substitution. This independence from a choice of independent variables (or coordinates) for Maxwell's equations was reported early on by Van Dantzig [Vandantzig 1934]. It follows that the Maxwell differential system is well defined in a covariant manner for both Galilean transformations as well as Lorentz transformations, or any other diffeomorphism. (The singular solution sets to the equations do not enjoy this universal property). In addition, it should be noted that the ideas of the exterior differential system imply that the closure equations of the Maxwell-Faraday type form a nested set, with exactly the same format, independent of the choice of the *number* of independent geometric variables. In addition, every physical system (such as fluid) that supports a 1-form of Action, has its version of the Maxwell-Faraday induction equations.

3.1.1. The Maxwell-Faraday exterior differential system.

The Maxwell-Faraday equations are a consequence of the exterior differential system

$$F - dA = 0, \tag{3.1}$$

where A is a 1-form of Action, with twice differentiable coefficients (potentials proportional to momenta) which induce a 2-form, F,of electromagnetic intensities ($\mathbf{E}$ and $\mathbf{B}$, related to forces and objects of intensities). The exterior differential system is a topological constraint that in effect defines field intensities in terms of the potentials. On a four dimensional space-time of independent variables, (x, y, z, t) with a volume element $\Omega_4 = dx\hat{}dy\hat{}dz\hat{}dt$, the 1-form of Action (representing the postulate of potentials) can be written in the form

$$A = \Sigma_{k=1}^{3} A_k(x, y, z, t)dx^k - \phi(x, y, z, t)dt = \mathbf{A}\circ d\mathbf{r} - \phi dt. \tag{3.2}$$

Subject to the constraint of the exterior differential system, the 2-form of field intensities, F,becomes:

$$F = dA = \{\partial A_k/\partial x^j - \partial A_j/\partial x^k\}dx^j\hat{}dx^k \tag{3.3}$$

$$= F_{jk}dx^j\hat{}dx^k = +\mathbf{B}_z dx\hat{}dy... + \mathbf{E}_x dx\hat{}dt... \tag{3.4}$$

where in usual engineering notation,

$$\mathbf{E} = -\partial\mathbf{A}/\partial t - grad\phi, \qquad \mathbf{B} = curl\ \mathbf{A} \equiv \partial A_k/\partial x^j - \partial A_j/\partial x^k. \tag{3.5}$$

The closure of the exterior differential system, $dF = 0$,

$$dF = ddA = \{curl\ \mathbf{E} + \partial\mathbf{B}/\partial t\}_x dy\hat{}dz\hat{}dt - .. + .. - div\,\mathbf{B}dx\hat{}dy\hat{}dz\} \Rightarrow 0, \tag{3.6}$$

generates the Maxwell-Faraday partial differential equations.:

$$\{curl\ \mathbf{E} + \partial\mathbf{B}/\partial t = 0, \qquad div\,\mathbf{B} = 0\}. \tag{3.7}$$

The component functions ($\mathbf{E}$ and $\mathbf{B}$) of the 2-form, F, transform as covariant tensor of rank 2. The topological constraint that F is exact, implies that the domain of support for the field intensities cannot be compact without boundary, unless the Euler characteristic vanishes. These facts distinguish classical electromagnetism from Yang-Mills field theories (where the domain of support for F is presumed to be compact without boundary). Moreover, the fact that F is subsumed to be exact and C1 differentiable excludes the concept of magnetic monopoles from classical electromagnetic theory on topological grounds. The closed integral of the 2-form F over any closed 2-manifold, $z2$, is a deformation (topological) invariant of any evolutionary process that can be described by a singly parameterized vector field, for

$$L_{\mathbf{V}}(\int_{z2} F) = \int_{z2} \{i(V)dF + d(i(V)F)\} = \tag{3.8}$$

$$\int_{z2} \{0 + d(i(V)F)\} = \int_{z2} d(i(V)F) = 0 \tag{3.9}$$

The integral is then a deformation invariant, for the result is valid even if the 4-vector field is distorted by an arbitrary function, $\beta\{x, y, z, t\}$, such that $\mathbf{V} \Rightarrow \beta(x, y, z, t)\mathbf{V}$. The points that make up the closed integration chain, or cycle , $z2$, are distorted and deformed by the arbitrary function, β, yet the value of the integral remains the same. The notation $\int_{z2}$implies that the 2D integration chain, $z2$, is closed. It can be a cycle (which is part of a boundary) or a boundary.

It is remarkable that the Maxwell-Faraday differential system can also be used to encode hydrodynamic systems, both of the equilibrium and non-equilibrium variety.

3.1.2. The Maxwell Ampere exterior differential system

The Maxwell Ampere equations are a consequence of second exterior differential system,

$$J - dG = 0, \tag{3.10}$$

where G is an N-2 form *density* of field excitations ($\mathbf{D}$ and $\mathbf{H}$), related to sources or objects of quantity), and J is the N-1 form of charge-current densities. The partial differential equations equivalent to the exterior differential system are precisely the Maxwell-Ampere equations. This second postulate, on a four dimensional domain of independent variables, assumes the existence of a N-2 form density given by the expression[1],

$$G = G^{34}(x,y,z,t)dx\hat{}dy... + G^{12}(x,y,z,t)dz\hat{}dt... = -\mathbf{D}^{z}dx\hat{}dy... + \mathbf{H}^{z}dz\hat{}dt... \tag{3.11}$$

Exterior differentiation produces an N-1 form,

$$J = \mathbf{J}^{z}(x,y,z,t)dx\hat{}dy\hat{}dt... - \rho(x,y,z,t)dx\hat{}dy\hat{}dz. \tag{3.12}$$

Matching the coefficients of the exterior expression $dG = J$ leads to the Maxwell-Ampere equations,

$$curl\,\mathbf{H} - \partial\mathbf{D}/\partial t = \mathbf{J} \quad and \quad div\,\mathbf{D} = \rho. \tag{3.13}$$

The fact that J is exact leads to the charge conservation law, $dJ = ddG = 0$,or

$$\partial\mathbf{J}^{x}/\partial x + \partial\mathbf{J}^{y}/\partial y + \partial\mathbf{J}^{z}/\partial z + \partial\rho/\partial t = 0. \tag{3.14}$$

The exterior differential system is a topological constraint for by Stokes theorem the support for G can be compact without boundary only if the domain is without charge-currents. The closure of the exterior differential system, $dJ = 0$, generates the charge-current conservation law. The integral of J over a closed 3 dimensional domain is a relative integral invariant (a deformation invariant) of any process that can be described in terms of a singly parametrized vector field. The formal statement is given by Cartan's magic formula [Marsden 1994], which describes continuous topological evolution in terms of the action of the Lie

[1]Note the signs.

derivative, with respect to a vector field, acting on the exterior differential 3-form, J :

$$L_{\mathbf{V}}(\int_{z3} J) = \int_{z3} \{i(V)dJ + d(i(V)J)\} = \int_{z3} \{0 + d(i(V)J)\} = 0. \qquad (3.15)$$

The Lie derivative of the closed integral is equal to zero for any 4-vector field V,when $dJ = 0$. The integral is then a deformation invariant, for the result is valid even if the 4-vector field is distorted by an arbitrary function, $\beta\{x, y, z, t\}$, such that $\mathbf{V} \Rightarrow \beta(x, y, z, t)\mathbf{V}$.

3.2. The Torsion and Spin 3-forms

As mentioned above, the method of exterior differential forms goes beyond the domain of classical tensor analysis, for it admits of maps from initial to final state that are without inverse. (Tensor analysis and coordinate transformations require that the Jacobian map from initial to final state has an inverse - the method of exterior differential forms does not.) Hence the theory of electromagnetism expressed in the language of exterior differential forms can support topological evolution, at least with respect to continuous processes without Jacobian inverse. With respect to such non-invertible maps, both tensor fields and differential forms are not functionally well defined in a predictive sense [RMK 1976 b]. Given the functional forms of a tensor field on an initial state, it is impossible to predict uniquely the functional form of the tensor field on the final state unless the map between initial and final state is invertible. However differential forms are functionally well defined in a retrodictive sense, by means of the pullback. Covariant anti-symmetric tensor fields pull back retrodictively with respect to the transpose of the Jacobian matrix (of functions) and functional substitution, and contravariant tensor densities pullback retrodictively with respect to the adjoint of the Jacobian matrix, and functional substitution. The transpose and the adjoint of the Jacobian exist, even if the Jacobian inverse does not.

The exterior differential forms that make up the electromagnetic system consist of the primitive 1-form, A, and the primitive N-2 form density, G,their exterior derivatives, and their algebraic intersections defined by all possible exterior products. The complete Maxwell system of exterior differential forms (the Pfaff sequence for the Maxwell system) is given by the set:

$$\{A, F = dA, G,\ J = dG,\ A\hat{}F,\ A\hat{}G,\ A\hat{}J,\ F\hat{}F,\ G\hat{}G\}. \qquad (3.16)$$

These forms and their unions may be used to form a topological base on the domain of independent variables. The Cartan topology constructed on this system of forms has the useful feature that the exterior derivative may be interpreted as a limit point, or closure, operator in the sense of Kuratowski [Kuratowski 1948]. The exterior differential systems that define the Maxwell-Ampere and the Maxwell-Faraday equations above are essentially topological constraints of closure. Note that the complete Maxwell system of differential forms (which assumes the existence of A) also generates two other exterior differential systems.

$$d(A\hat{}G) - (F\hat{}G - A\hat{}J) = 0, \tag{3.17}$$

and

$$d(A\hat{}F) - F\hat{}F = 0. \tag{3.18}$$

The two objects, $A\hat{}G$ and $A\hat{}F$ are three forms, not usually found in discussions of classical electromagnetism. The closed components of the first 3-form (density) were called topological spin [RMK 1969] and the closed components of the second 3-form were called topological torsion (or helicity) [RMK 1990]. By direct evaluation of the exterior product, and on a domain of 4 independent variables, each 3-form will have 4 components that can be symbolized by the 4-vector arrays

$$\text{Topological Spin-Current} : \mathbf{S}_4 = [\mathbf{A}\times\mathbf{H}+\mathbf{D}\phi, \mathbf{A}\circ\mathbf{D}] \equiv [\mathbf{S},\sigma], \tag{3.19}$$

and

$$\text{Topological Torsion-vector} : \mathbf{T}_4 = -[\mathbf{E}\times\mathbf{A}+\mathbf{B}\phi, \mathbf{A}\circ\mathbf{B}] \equiv [\mathbf{T},h], \tag{3.20}$$

which are to be compared with the charge current 4-vector density:

$$\text{Charge-Current} : \mathbf{J}_4 = [\mathbf{J},\rho], \tag{3.21}$$

The 3-forms then can be defined by the equivalent contraction processes

$$\text{Topological Spin 3-form} \doteq A\hat{}G \tag{3.22}$$

$$= i(\mathbf{S}_4)dx\hat{}dy\hat{}dz\hat{}dt = \mathbf{S}^x dy\hat{}dz\hat{}dt..... - \sigma dx\hat{}dy\hat{}dz \tag{3.23}$$

and

$$\text{Topological Torsion-helicity 3-form} \doteq A\hat{}F \tag{3.24}$$
$$= \; i(\mathbf{T}_4)\Omega_4 = \; \mathbf{T}^x dy\hat{}dz\hat{}dt..... - hdx\hat{}dy\hat{}dz. \tag{3.25}$$

The vanishing of the first 3-form is a topological constraint on the domain that defines topologically transverse electric (TTE) waves: the vector potential, $\mathbf{A}$, is orthogonal to $\mathbf{D}$, in the sense that $\mathbf{A} \circ \mathbf{D} = 0$. The vanishing of the second 3-form is a topological constraint on the domain that defines topologically transverse magnetic (TTM) waves: the vector potential, $\mathbf{A}$, is orthogonal to $\mathbf{B}$, in the sense that $\mathbf{A} \circ \mathbf{B} = 0$. When both 3-forms vanish, the topological constraint on the domain defines topologically transverse (TTEM) waves. For classic real fields this double constraint would require that vector potential, $\mathbf{A}$, is collinear with the field momentum, $\mathbf{D} \times \mathbf{B}$, and in the direction of the wave vector, $\mathbf{k}$.

The geometric notion of transversality modes of electromagnetic waves is a well known concept experimentally, but the association of transversality to topological issues is novel herein. Example solutions in the appendix indicate that the concept of geometric transversality and topological transversality are not the same. Examples 1-4 are all TEM modes in a geometric sense, as the $\mathbf{E}$ and $\mathbf{B}$ fields are orthogonal to the direction of propagation. However, Example 1 is TTM, but not TTE. Example 2 is TTE but not TTM. Example 3 is TTEM. Example 4 is not TTM and is not TTE.

Note that if the 2-form F was not exact, such topological concepts of transversality would be without meaning, for the 3-forms of Topological Spin and Topological Torsion depend upon the existence of the 1-form of Action. The Topological Torsion vector $\mathbf{T}_4$ and the Topological Spin current $\mathbf{S}_4$ are associated vectors to the 1-form of Action in the sense that

$$i(\mathbf{T}_4)A = 0 \quad and \quad i(\mathbf{S}_4)A = 0 \tag{3.26}$$

It will be demonstrated below that these results imply that evolution in the direction of the Topological Torsion vector $\mathbf{T}_4$ or the Topological Spin current $\mathbf{S}_4$ does not change the internal energy of the physical system. Evolution in the direction of $\mathbf{T}_4$ or $\mathbf{S}_4$ is locally adiabatic.

The closed components of the 1-form of Action do not effect the components of the 1-form of intensities, $F = dA = d(A_c + A_0) = 0 + d(A_0)$. However, these "gauge" additions do influence the topological dimension of the 1-form of Action. For example, let A_0 be of Pfaff Topological dimension 2, representing an isolated system where $A_0\hat{}dA_0 = 0$. Then by addition of a closed component to the original action, $A = A_c + A_0$ could have a topological dimension of 3, as

$$A\hat{}dA = (A_c + A_0)\hat{}dA_0 = A_c\hat{}dA_0 \neq 0. \tag{3.27}$$

So the addition of a closed component to the 1-form of Action could change the system from an isolated system of Pfaff dimension 2 to a closed system of Pfaff dimension 3. The 4-form $dA\hat{}dA$ is not influenced by the (gauge) addition to the original 1-form of Action.

$$dA\hat{}dA = dA_0\hat{}dA_0. \tag{3.28}$$

In Example 11 in the Appendix, a 1-form representing a Bohm-Aharanov-Abrikosov singular "vortex" string, $\gamma = b(ydx - xdy)/(x^2 + y^2)$, is added to a 1/r potential for a point source. The bare m/r "Coulomb" potential, $A_0 = m/\sqrt{(x^2 + y^2 + z^2)}dt$ exhibits no Topological Torsion but does exhibit Topological Spin. The 1/r potential term implies that $dA_0 \neq 0$. Hence the 1-form of Action representing a bare "coulomb" potential, is not in equilibrium, but does represent a connected "isolated" topology of Pfaff dimension 2. The combined 1-form of Action,

$$A = b(ydx - xdy)/(x^2 + y^2) + m/\sqrt{(x^2 + y^2 + z^2)}dt \tag{3.29}$$

even though $d\gamma = 0$, is of Pfaff dimension 3, not 2. The addition of the BAA term changes the topology of the 1-form from a connected topology to a disconnected topology. The Topological Torsion 3-form $A\hat{}F$ depends on both b and m, and is zero if $b = 0$, or if $m = 0$, reducing the Pfaff dimension of the modified 1-form back to 2. If $b = 0$ and $m \neq 0$, the 3-form $A\hat{}G$ is not zero.

3.3. The Topological Poincare Invariants

The exterior derivatives of the 3-forms of Spin and Torsion produce two 4-forms, $F\hat{}G - A\hat{}J$ and $F\hat{}F$, whose integrals over closed 4 dimensional domains are deformation invariants for the Maxwell system. These topological objects are related to the conformal invariants of a Lorentz system as discovered by Poincare and Bateman [Bateman 1914]. In the format of independent variables $\{x, y, z, t\}$, the exterior derivative corresponds to the 4-divergence of the 4-component Spin and Torsion vectors, $\mathbf{S}_4$ and $\mathbf{T}_4$. The functions so created define the Poincare topological invariants of the Maxwell system:

$$\begin{aligned} Poincare\ 1 &= d(A\hat{}G) = F\hat{}G - A\hat{}J \quad (3.30)\\ &= \{div_3(\mathbf{A}\times\mathbf{H}+\mathbf{D}\phi)+\partial(\mathbf{A}\circ\mathbf{D})/\partial t\}\Omega_4 \quad (3.31)\\ &= \{(\mathbf{B}\circ\mathbf{H}-\mathbf{D}\circ\mathbf{E})-(\mathbf{A}\circ\mathbf{J}-\rho\phi)\}\Omega_4 \quad (3.32)\end{aligned}$$

$$\begin{aligned} Poincare\ 2 &= d(A\hat{}F) = F\hat{}F \quad (3.33)\\ &= -\{div_3(\mathbf{E}\times\mathbf{A}+\mathbf{B}\phi)+\partial(\mathbf{A}\circ\mathbf{B})/\partial t\}\Omega_4 \quad (3.34)\\ &= \{2\mathbf{E}\circ\mathbf{B}\}\Omega_4 \quad (3.35)\end{aligned}$$

For the plasma vacuum state, with $J = 0$, zero values of the Poincare invariants require that the magnetic energy density is equal to the electric energy density ($1/2\mathbf{B}\circ\mathbf{H} = 1/2\mathbf{D}\circ\mathbf{E}$), and, respectively, that the electric field is orthogonal to the magnetic field ($\mathbf{E}\circ\mathbf{B} = 0$). Note that these constraints often are used as elementary textbook definitions of what is meant by electromagnetic waves. When either Poincare invariant vanishes, the corresponding closed 3-dimensional integral becomes a topological quantity in the sense of a deRham period integral. For example, when the first Poincare invariant vanishes, the closed integral of the 3-form of spin becomes a deformation invariant with quantized values (rational ratios):

$$Define : \text{Topological Spin} = \int_{z3} A\hat{}G \quad (3.36)$$

$$\text{In domains where } d(A\hat{}G) = 0, \textit{then} \int_{z3} A\hat{}G \text{ is a deformation invariant} \quad (3.37)$$

$$\begin{aligned} L_{\beta\mathbf{V}}\int_{z3} A\hat{}G &= \int_{z3}\{i(\beta V)d(A\hat{}G)+d(i(\beta V)(A\hat{}G)\} \quad (3.38)\\ &= \int_{z3}\{0+d(i(\beta V)(A\hat{}G)\} = 0. \quad (3.39)\end{aligned}$$

Similarly, when the second Poincare invariant vanishes, the closed integral of the 3-form of Torsion-helicity becomes a deformation invariant with quantized values:

$$Define \quad : \quad \text{Topological Torsion-Helicity} = \int_{z3} A\hat{\ }F \tag{3.40}$$

$$\text{In domains where } d(A\hat{\ }F) = 0, \quad then \int_{z3} A\hat{\ }F \text{ is a deformation invariant} \tag{3.41}$$

$$L_{\beta\mathbf{V}}(\int_{z3} A\hat{\ }F) = \int_{z3} \{i(\beta V)d(A\hat{\ }F) + d(i(\beta V)(A\hat{\ }F)\} \tag{3.42}$$

$$= \int_{z3} \{0 + d(i(\beta V)(A\hat{\ }F)\} = 0. \tag{3.43}$$

The 3 dimensional close integration chains, z_3, are closed cycles, and are not necessarily boundaries. It is important to realize that these topological conservation laws are valid in a plasma as well as in the vacuum, subject to the conditions of zero values for the Poincare invariants. On the other hand, topological transitions require that the Poincare invariants are not zero.

4. Topological Thermodynamics

It is fundamental to appreciate the idea that a plasma is essentially a non-equilibrium thermodynamic system. Plasma dynamics are mostly (but not always) turbulent, dissipative, and irreversible. The topological perspective of a plasma emphasizes these features, for it is possible to use the Cartan concepts of exterior differential systems to express precisely, and without statistics, what is meant by irreversible processes and what are the topological (not geometrical) constraints that define open, closed, and equilibrium thermodynamic systems. The combination of topological thermodynamics and topological electrodynamics leads to the conclusion that the plasma state of electrodynamics is usually of Pfaff topological dimension 4 (while equilibrium systems are of Pfaff topological dimension 1, and isolated systems are of Pfaff topological dimension 2), which can contain topological defect structures that are of Pfaff topological dimension 3 as defect structures in the Pfaff dimension 4 domains. Evolutionary processes in the Pfaff topological dimension 4 state are thermodynamically irreversible, while evolutionary processes in the Pfaff topological dimension 3 state can be Hamiltonian and therefore reversible.

As the emphasis in this article is on plasmas, this section on topological thermodynamics will be abbreviated, but is presented here in order to demonstrate how the universal topological viewpoint couples thermodynamic thinking to the theory of plasmas.

4.1. Cartan's Magic formula and the First Law of Thermodynamics

The topological view of thermodynamics described herein is based on three axioms.

1. Thermodynamic physical systems can be encoded in terms of a 1-form of covariant Action Potentials, $A_k(x, y, z, t)$, on a 4 dimensional abstract variety of ordered independent variables, $\{x, y, z, t\}$. The variety supports a volume element $\Omega_4 = dx\hat{}dy\hat{}dz\hat{}dt$.

2. Thermodynamic processes are assumed to be encoded, to within a factor, $\rho(x, y, z, t)$, in terms of contravariant vector direction fields, $\mathbf{V}_4(x, y, z, t)$.

3. Continuous topological evolution of the thermodynamic system can be encoded in terms of Cartan's magic formula (see p. 122 in [Marsden 1994]). The Lie differential, when applied to a exterior differential 1-form of Action, $A = A_k dx^k$, is equivalent *abstractly* to the first law of thermodynamics.

$$\begin{aligned}
\text{Cartan's Magic Formula} \quad L_{(\rho\mathbf{V}_4)}A &= i(\rho\mathbf{V}_4)dA + d(i(\rho\mathbf{V}_4)A) && (4.1)\\
\text{First Law of Thermodynamics} \quad &: W + dU = Q, && (4.2)\\
\text{Inexact 1-form of Heat} \quad L_{(\rho\mathbf{V}_4)}A &= W + dU = Q && (4.3)\\
\text{Inexact 1-form of Work} \quad W &= i(\rho\mathbf{V}_4)dA, && (4.4)\\
\text{Internal Energy} \quad U &= i(\rho\mathbf{V}_4)A. && (4.5)
\end{aligned}$$

In effect, Cartan's methods can be used to formulate precise mathematical definitions for many thermodynamic concepts in terms of topological properties -without the use of statistics or metric constraints. Moreover, the method applies to non-equilibrium thermodynamical systems and irreversible processes, again without the use of statistics or metric constraints.

In order to make the equations more suggestive to the reader, the symbolism for the variety of independent variables has been chosen to be of the format $\{x, y, z, t\}$, but be aware that no constraints of metric or connection are imposed upon this variety. For instance, it is NOT assumed that the base variety is euclidean.

4.2. The Pfaff Topological Dimension

One of the most useful topological tools is that defined as the Pfaff topological dimension (or class) of a 1-form [Schouten 1949]. Recall that it is possible to define many topologies on the same set of elements. For any given exterior differential 1-form of functions, say $A = A_k\{x, y, z, t\}dx^k$, it is possible to construct the Pfaff sequence of terms of the set,

$$\text{Pfaff sequence } \{A, dA, A\hat{}dA, dA\hat{}dA\}. \tag{4.6}$$

These elements may be used to construct a Cartan Topology (for any 1-form [**?**]). In the Cartan topology, the exterior derivative acts as limit point generator. Hence the union of a form and its exterior derivative creates the topological closure of the form.

For any given 1-form, the Pfaff sequence will contain M successive non-zero terms equal to or less than N, the number of geometric dimensions of the base independent variables. The number M is defined as the "Pfaff topological dimension" or class of the given 1-form. The three important 1-forms of thermodynamics, A, W, and Q, can have different Pfaff dimensions. Suppose the 1-form of work is defined in terms of two functions as $W = PdV$. The Pfaff sequence consists of the terms $\{W, dW, 0, 0\}$; hence in this specific example, the Pfaff dimension of W is 2. From the first law, under the assumption that $W = PdV$,

$$Q = W + dU = PdV + dU, \tag{4.7}$$

$$dQ = dW = dP\hat{}dW, \tag{4.8}$$

$$Q\hat{}dQ = W\hat{}dW + dU\hat{}dW = 0 + dU\hat{}dP\hat{}dV \tag{4.9}$$

$$dQ\hat{}dQ = 0. \tag{4.10}$$

Hence, the Pfaff dimension of 2 for the work 1-form can be associated with a Pfaff dimension of 3 for the Heat 1-form, unless the Pressure is a function of the internal energy and the volume. In this latter case, the Pfaff dimension of Q and W are both 2.

In this article, attention will be focused on dissipative Turbulent systems with thermodynamic irreversible processes such that the Pfaff topological dimensions of A, W, and Q will be maximal and equal to 4. (The techniques can be extended to higher dimensional spaces.) These Turbulent systems of Pfaff dimension 4 are NOT topologically equivalent to Equilibrium systems (for which the topological dimension is 2, at most). Topological defects in the Turbulent state will be associated with sets of space time where the Pfaff topological dimensions are not maximal. It is remarkable that such topological defect sets can form attractors causing self organization and long lived states of Pfaff dimension 3, which are far from equilibrium, created by irreversible evolutionary processes in the domain of Pfaff dimension 4.

It may be true that the functional form of a 1-form, A, yields a Pfaff topological dimension equal to 4 globally over the domain $U \subset \{x, y, z, t\}$, except for sub regions where the Pfaff dimension of A is 3 or less. These sub regions represent topological defects in the almost global domain of Pfaff dimension 4. The important concept of Pfaff topological dimension also can be used to define equivalence classes of physical systems and processes.

The concept of Pfaff "topological dimension" was developed more than 110 years ago (see page 290 of Forsyth [Forsyth (1890) 1959]), and has been called the "class" of a differential 1-form in the mathematical literature. More recent mathematical developments can be found in [Schouten 1949]. The method and its properties have been little utilized in the applied world of physics and engineering. Of key importance is the fact that the non-zero existence of the 3-form $A\hat{}dA = A\hat{}F$ of *Topological Torsion*, and the 4-form of *Topological Parity*, $dA\hat{}dA = F\hat{}F$ implies that the Pfaff topological dimension of the region is 3 or 4, respectively. Either value is an indicator that the electromagnetic physical system is NOT in thermodynamic equilibrium.

4.3. Physical Systems: Equilibrium, Isolated, Closed and Open

Physical systems and processes are elements of topological categories determined by the Pfaff topological dimension (or class) of the 1-forms of Action, A, Work, W, and Heat, Q. For example, the Pfaff topological dimension of the exterior differential 1-form of Action, A, determines the various species of thermodynamic systems in terms of distinct topological categories. The are two topological thermodynamic categories that are determined by the closure (or differential ideal) of the 1-form of Action, $A \cup dA$, and the closure of the 3-form of topological torsion, $A\hat{}dA \cup dA\hat{}dA$. The first category is represented by a connected Cartan topology, while the second category is represented by a disconnected Cartan topology. The Cartan topology is discussed in detail in [Baldwin 1991].

4.3.1. Connected Topology $A\hat{}F = 0$

- Equilibrium physical systems are elements such that the Pfaff topological dimension is 1.

  Isolated physical systems are elements such that the Pfaff topological dimension is 2, or less.

- Isolated systems of Pfaff dimension 2 need not be in equilibrium, but do not exchange radiation or mass with the environment.

4.3.2. Disconnected Topology $A\hat{}F \neq 0$

- Closed physical systems are elements such that the Pfaff topological dimension is 3. Closed systems can exchange radiation, but not mass, with the environment.

- Open physical systems are such that the Pfaff topological dimension is 4. Open physical systems can exchange both radiation and mass with the environment.

$$Systems \quad : \quad \text{defined by the Pfaff dimension of } A \tag{4.11}$$
$$dA = 0 \quad \text{Equilibrium - Pfaff dimension 1} \tag{4.12}$$
$$A\hat{}dA = 0 \quad \text{Isolated - Pfaff dimension 2} \tag{4.13}$$
$$d(A\hat{}dA) = 0 \quad \text{Closed - Pfaff dimension 3} \tag{4.14}$$
$$dA\hat{}dA \neq 0. \quad \text{Open - Pfaff dimension 4.} \tag{4.15}$$

Note that these topological specifications as given above are determined entirely from the functional properties of the physical system encoded as a 1-form of Action, A. The system topological categories do not involve a process, which is encoded (to within a factor) by some vector direction field, $\mathbf{V}_4$. However, the process $\mathbf{V}_4$ does influence the topological properties of the work 1-form W and the Heat 1-form Q.

4.4. Equilibrium vs. Non-Equilibrium Systems

The intuitive idea for an equilibrium system comes from the experimental recognition that the intensive variables of pressure and temperature become domain constants in an equilibrium state: $dP \Rightarrow 0,\ dT \Rightarrow 0$. A definition made herein is that the Pfaff topological dimension in the interior of a physical system which is in the equilibrium state is at most 1 [Bamberg 1992]. The Cartan topology generated by the elements of the Pfaff sequence for A is then a connected topology of one component, $\{A \neq 0,\ dA = 0,\ A\hat{}dA = 0,\ dA\hat{}dA = 0\}$. Although the Pfaff topological dimension of A is at most 2 in the isolated state, processes in the equilibrium state are such that the Work 1-form and the Heat 1-form must be of Pfaff dimension 1. For suppose $W = PdV$, then $dW = dP\hat{}dV \Rightarrow 0$ if the pressure is a domain constant. Similarly, suppose $Q = TdS$, then $dQ = dT\hat{}dS \Rightarrow 0$ if the temperature is a domain constant. Hence both W and Q are of Pfaff dimension 1 for this equilibrium example. If the Pfaff dimension of the 1-form of Action is 1, then $dA \Rightarrow 0$. It follows in this more stringent case that $W \Rightarrow 0$, hence the Pressure must vanish, and Heat 1-form is a perfect differential, $Q = d(U)$.

Of particular interest herein are those regions of base variables for open, non-equilibrium, Turbulent physical systems, formed by the closure[2] of the 3-forms $A\hat{}dA$, $W\hat{}dW$, and $Q\hat{}dQ$. For such regions, the Pfaff topological dimension of the 1-forms, A, W, and Q, are all initially of Pfaff topological dimension 4, save for defect regions that are of Pfaff dimension 3. For example, evolutionary dissipative irreversible processes in such open systems can describe evolution to regions of base variables where the Pfaff topological dimension of the 1-form of Action, A, changes from 4 to 3. Such processes describe topological change in the physical system. For a given 1-form of Action, A, those regions of Pfaff topological dimension 3, once created, form topological "defect structures" in the closure of the 3-form, $A\hat{}F$. The defect structures of the 1-form of Action, A, (of Pfaff dimension 3) can behave as long lived (excited) states of the initial physical system, but they are far from equilibrium and are not isolated, for they are not of Pfaff topological dimension equal to 2 or less. Such excited states (of odd topological dimension) can admit extremal processes of kinematic perfection, and can have a Hamiltonian generator for the kinematics represented as first order ordinary differential equations. The Hamiltonian evolution remains contained in the defect structure, unless topological fluctuations destroy the kinematic perfection.

Such concepts can be applied to a model of cosmology (where the stars are the defect structures), to turbulent plasmas and fluids (where wakes are the defect structures), and to a better understanding of the arrow of time. Although the defects in the Turbulent non-equilibrium regime are not necessarily equilibrium

[2] The closure of the p-form Σ is the union of Σ and $d\Sigma$, which Cartan has called a differential ideal.

structures, once formed and self organized as coherent topological structures of Pfaff dimension 3, they can evolve along extremal trajectories that are not dissipative, and may even have a Hamiltonian representation. These "stationary", if not long lived (excited) states of Pfaff dimension 3, indeed are states "far" from the equilibrium state, which requires a Pfaff dimension of 1. Note that the word "far" does not imply a "distance". The Pfaff dimension 3 and 4 sets are not even "connected" to the equilibrium states in a topological sense. The non-equilibrium states of a physical system that are "near-by" to the equilibrium state, are "connected" to the equilibrium state, and are of Pfaff dimension 2.

The descriptive words of self-organized states far from equilibrium are abstracted from the intuition and conjectures of I. Prigogine [Kondepudi 1998]. However, the topological theory presented herein presents for the first time a solid, formal, mathematical justification (with examples) for the Prigogine conjectures. Precise definitions of equilibrium and non-equilibrium systems, as well as reversible and irreversible processes can be made in terms of the topological features of Cartan's exterior calculus. Thermodynamic irreversibility and the arrow of time are well defined in a topological sense [RMK 2003], a technique that goes beyond (and without) statistical analysis. Thermodynamic irreversibility and the arrow of time requires that the evolutionary process produce topological change.

4.5. Multiple Components

One of the most remarkable properties of the Cartan topology [Baldwin 1991], generated by the elements of a Pfaff sequence, is due to the fact that when $A\hat{}dA = 0$, (Pfaff dimension 2 or less) the Cartan topology of the physical system is reducible to a single connected topological component. This single component need not be simply connected. On the other hand when $A\hat{}dA \neq 0$, (Pfaff dimension 3 or more) the the Cartan topology of the physical system is a disconnected topology of more than one topological component. The bottom line is that when the Pfaff dimension is 3 or greater (such that conditions of the Frobenius unique integrability theorem are not satisfied), solution uniqueness to the Pfaffian differential equation, $A = 0$, is lost. If there exist solutions, there is more than one. Such concepts lead to propagating discontinuities (signals), envelope solutions (Huygen wavelets), an edge of regression (the spinodal line of phase transitions) a lack of time reversal invariance, and the existence of irreducible affine torsion in the theory of connections. It is the opinion of this author that a dogmatic insistence that "useful" physical theories must produce a unique outcome from given set of initial conditions is a severe topological constraint which has hindered the understanding of irreversibility and non-equilibrium systems.

4.6. Processes

4.6.1. Continuous Processes

All continuous processes may be put into equivalence classes as determined by the vector fields, $\mathbf{V}$, that generate the evolution of the physical system, A. For example, for the 1-form, A, those vector fields that satisfy the transversal equation,

$$Associated : i(\rho\mathbf{V})A = 0 \tag{4.16}$$

are said to be elements of the "associated class" of vector fields relative to the form A. For such thermodynamic processes, the change of internal energy is locally zero.

Those vectors that satisfy the equations,

$$Extremal : i(\rho\mathbf{V})dA = 0 \tag{4.17}$$

are said to be elements of the extremal class of vector fields. For such processes, the virtual work vanishes, $W = 0$. It should be noted that the 2-form dA admits a unique extremal vector only on topological spaces of odd Pfaff topological dimension. Such topological spaces define a Contact manifold, which serves (in higher dimensional cases) as the 2n+1 dimensional state space of mechanics. If the Pfaff dimension of the 1-form of Action, A, is 4, then a unique extremal vector does not exist. Similarly, the non-zero 2-form, dA, of maximal rank, defines a symplectic manifold of even (2n+2) dimensions. However, on the symplectic manifold there does exists a unique vector direction field, the Topological Torsion vector (described below), but no extremal vector.

Vectors which are both extremal and associated are said to be elements of the characteristic class of vector fields [Klein 1962].

$$Characteristic : i(\rho\mathbf{V})A = 0 \quad and \quad i(\rho\mathbf{V})dA = 0 \tag{4.18}$$

Note that characteristic flow lines generated by $\mathbf{V}$ of the Characteristic class preserve the Cartan topology, for each form of the Cartan topological base is invariant with respect to the action of the Lie derivative relative to characteristic flows. Characteristics are often associated with wave phenomena, and propagating discontinuities. Note that extremal processes relative to A are characteristic processes relative to dA.

4.6.2. Reversible and Irreversible Processes

The Pfaff topological dimension of the exterior differential 1-form of Heat, Q, determines important topological categories of processes. From classical thermodynamics "The quantity of heat in a reversible process always has an integrating factor" [Goldenblatt 1962] [Morse 1964] . Hence, from the Frobenius unique integrability theorem, which requires $Q\hat{}dQ = 0$, all reversible processes are such that the Pfaff dimension of Q is less than or equal to 2. Irreversible processes are such that the Pfaff dimension of Q is greater than 2. A dissipative irreversible topologically *turbulent* process is defined when the Pfaff dimension of Q is 4.

$$Processes = \text{defined by the Pfaff dimension } Q \tag{4.19}$$
$$Q\hat{}dQ = 0 \qquad \text{Reversible - Pfaff dimension 2} \tag{4.20}$$
$$d(Q\hat{}dQ) \neq 0. \qquad \text{Turbulent - Pfaff dimension 4.} \tag{4.21}$$

Note that the Pfaff dimension of Q depends on both the choice of a process, $\mathbf{V}_4$, and the physical system, A, upon which it acts. As reversible thermodynamic processes are such that $Q\hat{}dQ = 0$, and irreversible thermodynamic processes are such that $Q\hat{}dQ \neq 0$, Cartan's formula of continuous topological evolution, $L_{(\rho\mathbf{V}_4)}A = Q$, can be used to determine if a given process, $\mathbf{V}_4$, acting on a physical system, A, is thermodynamically reversible or not:

$$\left[\begin{array}{l} \text{Reversible Processes } \rho\mathbf{V}_4 : \; L_{(\rho\mathbf{V}_4)}A\hat{}L_{(\rho\mathbf{V}_4)}dA = 0, \\ \text{Irreversible Processes } \rho\mathbf{V}_4 : L_{(\rho\mathbf{V}_4)}A\hat{}L_{(\rho\mathbf{V}_4)}dA \neq 0. \end{array}\right] \tag{4.22}$$

Remarkably, Cartan's magic formula can be used to describe the continuous dynamic possibilities of both reversible and irreversible processes, in equilibrium or non-equilibrium systems, even when the evolution induces topological change, transitions between excited states, and changes of phase, such as condensations.

It is important to note that the direction field, $\mathbf{V}_4$, need not be topologically constrained such that it is singularly parameterized. That is, the evolutionary processes described by Cartan's magic formula are not necessarily restricted to vector fields that satisfy the topological constraints of kinematic perfection, such as $dx^k - v^k dt = 0$. A discussion of kinematic topological fluctuations, where

$dx^k - v^k dt = \Delta x^k \neq 0$, and a system first order equations (as well as the kinematic fluctuations, $dv^k - a^k dt = \Delta v^k \neq 0$, and a system of second order equations) is described below.

4.6.3. Adiabatic Processes - Reversible and Irreversible

The topological formulation permits a precise definition to be made for both reversible and an irreversible adiabatic processes in terms of the topological properties of Q. On a geometrical space of N dimensions, a 1-form will admit N-1 vector fields such that $i(V_A)Q = 0$. Such processes V_A are defined as local adiabatic processes [Bamberg 1992]. The N-1 null vectors will form a distribution of adiabatic processes orthogonal to the 1-form Q. The distribution of adiabatic processes will not form a smooth hypersurface, unless the Pfaff dimension of Q is 2 or less. In other words the null curves (adiabats) form an smooth hypersurface in the equilibrium state. Note that all adiabatic processes are defined by vector direction fields, to within an arbitrary factor, $\rho(x, y, z, t)$. That is, if $i(V_A)Q = 0$, then it is also true that $i(\rho V_A)Q = 0$.

The differences between the inexact 1-forms of Work and Heat become obvious in terms of this topological format. Both 1-forms depend on the process and on the physical system. However, Work is always transversal to the process, as $i(\mathbf{V}_4)W = i(\mathbf{V}_4)i(\mathbf{V}_4)dA = 0$, but Heat is not always transversal, as $i(\mathbf{V}_4)Q = i(\mathbf{V}_4)dU \Rightarrow 0$, only for adiabatic processes. It is this fundamental difference between Heat, Q, and Work, W, that lead to the Carnot-like statements that it is possible to convert work into heat with 100% efficiency, but it is not possible to convert heat into work with 100% efficiency.

Local adiabatic direction fields, defined as null curves of Q, do not imply that the Pfaff dimension of Q must be 2. That is, it is not obvious that Q can be written in the form, $Q = TdS$, as is possible on the manifold of equilibrium states. From the Cartan formulation it is apparent that if Q is not zero, then

$$\begin{aligned} i(\mathbf{V}_A)L_{(\mathbf{V}_A)}A &= i(\mathbf{V}_A)i(\mathbf{V}_A)dA + i(\mathbf{V}_A)d(i(\mathbf{V}_A)A) \\ &= 0(by\ transversality) + i(\mathbf{V}_A)d(i(\mathbf{V}_A)A) = i(\mathbf{V}_A)Q \end{aligned} \tag{4.23}$$

If Q is not zero, a locally Adiabatic process requires that ,

$$\text{Adiabatic process } i(\mathbf{V}_A)Q = i(\mathbf{V}_A)d(i(\mathbf{V}_A)A)) \Rightarrow 0, \quad Q \neq 0 \tag{4.24}$$

$$\text{with a sufficient condition } = i(\mathbf{V}_A)A \Rightarrow 0. \tag{4.25}$$

The sufficient condition implies that the process $\mathbf{V}_A$ is an associated vector of the 1-form of Action. If the heat 1-form is zero, then the process is a reversible adiabatic process of a special type. A reversible process is defined such that the Pfaff dimension of Q is less than 3; or, $Q\hat{}dQ = 0$. Hence $i(\mathbf{V}_A)(Q\hat{}dQ) = 0$ for reversible processes. But

$$i(\mathbf{V}_A)(Q\hat{}dQ) = (i(\mathbf{V}_A)Q)\hat{}dQ - Q\hat{}i(\mathbf{V}_A)dQ, \tag{4.26}$$

which permits reversible and irreversible adiabatic processes to be well distinguished [3] when $Q \neq 0$:

$$\text{Reversible Adiabatic Process} = -Q\hat{}i(\mathbf{V}_A)dQ \Rightarrow 0,\ i(\mathbf{V}_A)Q \Rightarrow 0, \tag{4.27}$$
$$\text{Irreversible Adiabatic Process} = -Q\hat{}i(\mathbf{V}_A)dQ \neq 0, \quad i(\mathbf{V}_A)Q \Rightarrow 0 \tag{4.28}$$

It is certainly true that if $L_{(\mathbf{V})}A = Q = 0$, *identically*, then all such processes are adiabatic, and reversible. (In sub-section, 2.6, it will be demonstrated how these thermodynamic ideas can be associated with the tensor processes of covariant differentiation and parallel transport.)

4.6.4. Processes classified by topological constraints

Cartan has shown that all Hamiltonian processes (systems with a generator of ordinary differential equations), $\rho\mathbf{V}_H$, satisfy the following equations of topological constraint on the work 1-form, W :

$$\text{A Hamiltonian } \mathbf{V}_H \text{ is either } \mathbf{V}_E \text{ or } \mathbf{V}_B \tag{4.29}$$
$$\text{Extremal Hamiltonian } \mathbf{V}_E \tag{4.30}$$
$$W_E = i(\rho\mathbf{V}_E)dA = 0 \qquad \text{Pfaff dimension} = 0 \tag{4.31}$$
$$\text{Bernoulli-Casimir Hamiltonian } \mathbf{V}_B \tag{4.32}$$
$$W_B = i(\rho\mathbf{V}_B)dA = d\Theta \quad \text{Pfaff dimension} = 1 \tag{4.33}$$

A special case occurs when the Bernoulli function is equal to the negative of the internal energy, for then the heat 1-form produced by this special Hamiltonian process vanishes.

[3] It is apparent that i(V)Q= 0 defines an adiabatic process, but not necessarily a reversible adiabatic process. This topological point clears up certain misconceptions that appear in the literature.

For symplectic processes (which are not strictly Hamiltonian) the situation is a bit more intricate, but in all cases the Pfaff dimension of the Work 1-form is at most 1.

$$\begin{aligned} & \text{Helmholtz Symplectic} && (4.34)\\ W_S &= i(\rho \mathbf{V}_S)dA = d\Theta + \gamma \ \text{ Pfaff dimension} = 1 && (4.35)\\ dW_S &= 0 \text{ as } \gamma \text{ is closed but not exact.} && (4.36) \end{aligned}$$

The closed but not exact forms, γ, introduce non-uniqueness into the definition of the work 1-form. As $dQ = d(\Theta+\gamma+U) = 0$, for all three processes defined above, the are all reversible (see equation (**??**)), but each class of processes must satisfy an additional topological constraint if the process is to be locally adiabatic:

$$\begin{aligned} \text{Adiabatic process } i(\mathbf{V}_A)Q &= i(\mathbf{V}_A)d(i(\mathbf{V}_A)A)) \Rightarrow 0, \quad Q \neq 0 && (4.37)\\ \text{with a sufficient condition } &= i(\mathbf{V}_A)A \Rightarrow 0. && (4.38) \end{aligned}$$

If $d\Theta = 0$, then $\rho\mathbf{V}_E$ is a characteristic process relative to the 2-form F. If the work 1-form is of Pfaff topological dimension 0, then the process is an extremal process relative to A (see equation 4.18).

Extremal processes cannot exist on a non-singular symplectic domain, because a non-degenerate anti-symmetric matrix (the coefficients of the 2-form dA) does not have null eigenvectors on space of even dimensions . Although unique extremal stationary states do not exist on the domain of Pfaff dimension 4, there can exist evolutionary invariant Bernoulli-Casimir functions, Θ, that generate non-extremal, "stationary"states. Such Bernoulli processes can correspond to energy dissipative symplectic processes, but they, as well as all symplectic processes, are reversible in the thermodynamic sense described below. The mechanical energy need not be constant, but the Bernoulli-Casimir function(s), Θ,are evolutionary invariant(s), and may be used to describe non-unique stationary state(s).

The equations, above, that define several familiar categories of processes, are in effect constraints on the topological evolution of any physical system represented by an Work 1-form, A. The Pfaff dimension of the 1-form of virtual work, $W = i(\mathbf{V})dA$ is 1 or less for all three categories. The Extremal constraint of equation (4.30) that the Pfaff dimension of W can be used to generate the Euler equations of hydrodynamics for a incompressible fluid. The Bernoulli-Casimir constraint of equation (4.32) can be used to generate the equations for a barotropic compressible fluid. The Helmholtz constraint of equation (4.36) can be used to generate the equations for a Stokes flow. All such processes are thermodynamically reversible as $dQ = 0$. None of these constraints on the Work 1-form, W, above will generate the Navier-Stokes equations, which require that the topological dimension of the 1-form of virtual work must be greater than 2.

A crucial idea is the recognition that non-equilibrium processes must be on domains of Pfaff dimension which support Topological Torsion, $A\hat{}dA \neq 0$, with its attendant properties of non-uniqueness, envelopes, regressions, and projectivized tangent bundles. Such domains are of Pfaff dimension 3 or greater. Moreover, as described below, it would appear that thermodynamic irreversibility must support a non-zero Topological Parity 4-form, $dA\hat{}dA \neq 0$. Such domains are of Pfaff dimension 4 or greater.

Although there does not exist a unique gauge independent stationary state on the symplectic manifold of Pfaff dimension 4, remarkably there does exist a unique vector field on the symplectic domain, with components that are generated by the 3-form $A\hat{}dA$. This unique (to within a factor) vector field is defined as the Torsion Current, $\mathbf{T}_4$, and satisfies (on the 2n+2=4 dimensional manifold) the equation,

$$i(\mathbf{T}_4)dx\hat{}dy\hat{}dz\hat{}dt = A\hat{}dA \tag{4.39}$$

This (four component) vector field, $\mathbf{T}_4$, has a non-zero divergence almost everywhere, for if the divergence is zero, then the 4-form $dA\hat{}dA$ vanishes, and the domain is no longer a symplectic manifold! The Torsion vector, $\mathbf{T}_4$, can be used to generate a dynamical system that will decay to the stationary states ($div_4(\mathbf{T}_4) \Rightarrow 0$) starting from arbitrary initial conditions. These processes are irreversible in the thermodynamic sense. It is remarkable that this unique evolutionary vector field, $\mathbf{T}_4$, is completely determined (to within a factor) by the physical system itself; e.g., the components of the 1-form, A, determine the components of the Torsion vector. Recall that if $L(\mathbf{v})A\hat{}L(\mathbf{v})dA \neq 0$, then the process V acting on the physical system, A, is irreversible. This topological definition implies that the three categories (above) of symplectic, Hamiltonian-Bernoulli or Hamiltonian-extremal processes, $\mathbf{V} \subset \mathbf{S}$, are reversible ($as\ L_{(\mathbf{S})}dA{=}dQ = 0$). However, for evolution in the direction of the Torsion vector, $\mathbf{T}_4$, direct computation demonstrates that the fundamental equations lead to a conformal evolutionary process, a process which is thermodynamically irreversible:

$$L_{(\mathbf{T}_4)}A = \sigma A \quad and \quad i(\mathbf{T}_4)A = 0, \tag{4.40}$$

such that

$$L_{(\mathbf{T}_4)}A\hat{}L_{(\mathbf{T}_4)}dA = Q\hat{}dQ = \sigma^2 A\hat{}dA \neq 0. \tag{4.41}$$

4.6.5. Topological Torsion and Conformal Irreversible Processes.

For any physical system encoded by a 1-form of Action, A, on a variety of 4 independent base variables, it is possible to construct the 3-form of Topological Torsion, $A\hat{}dA$. This 3-form is equivalent to the contraction of the Volume 4-form, Ω_4, with the 4 component vector, $\mathbf{T}_4$,

$$A\hat{}dA = A\hat{}F = i(\mathbf{T}_4)\Omega_4. \tag{4.42}$$

In other words, the direction field, $\mathbf{T}_4$, is defined entirely from the functional structure of the 1-form of Action, A, used to encode the physical system. Evolutionary processes can be defined to within a factor, ρ, by $\mathbf{T}_4$. This topological torsion process defined by, $\rho\mathbf{T}_4$, has remarkable properties.

1. The topological torsion process, $\rho\mathbf{T}_4$, is a null vector for the 1-form of Action, and therefor defines a local adiabatic process:

$$\rho\mathbf{T}_4 \text{ is locally adiabatic: } i(\rho\mathbf{T}_4)A = i(\rho\mathbf{T}_4)Q = 0. \tag{4.43}$$

2. The topological torsion process, $\rho\mathbf{T}_4$, is a conformal vector for the 1-form of Action,

$$L_{(\rho\mathbf{T}_4)}A = i(\rho\mathbf{T}_4)dA = \Gamma A. \tag{4.44}$$

with a conformal factor $\Gamma = divergence(\rho\mathbf{T}_4)$.

3. The topological torsion process, $\rho\mathbf{T}_4$, creates a thermodynamically irreversible process, when the 4-divergence of $\rho\mathbf{T}_4$ is not zero:

$$Q\hat{}dQ = \Gamma^2 A\hat{}dA \neq 0. \tag{4.45}$$

4. The topological torsion vector, $\rho\mathbf{T}_4$, is a reversible adiabatic characteristic vector relative to A when the 4-divergence, Γ, of $\rho\mathbf{T}_4$ is equal to zero.

$$\begin{aligned} \text{When } \Gamma &= 0, && (4.46)\\ Q\hat{}dQ &= \Gamma^2 A\hat{}dA = 0, && (4.47)\\ i(\rho\mathbf{T}_4)Q &= 0 && (4.48)\\ i(\rho\mathbf{T}_4)A &= 0,\ i(\rho\mathbf{T}_4)dA = 0. && (4.49) \end{aligned}$$

As the divergence of $A\hat{}F$ is non-zero only if $dA\hat{}dA$ is not zero, it is apparent that such a domain is of Pfaff topological dimension 4, relative to A. Recall that a physical system of Pfaff dimension 4 corresponds to an open thermodynamic system. In this sense, thermodynamic irreversibility is an artifact of 4 dimensions. Recall that the 4-form $F\hat{}F$ is exact, and is equal to $d(A\hat{}F)$. Hence the 4 dimensional integral becomes a deformation invariant if the integral over the boundary of a domain satisfies the condition,

$$\int\int\int\int_M F\hat{}F = \text{is an Absolute Deformation invariant, if} \quad (4.50)$$

$$0 = \int\int\int_{boundary\ of\ M} (\rho V^k)(A\hat{}F) \quad (4.51)$$

Similarly for the 4-form $dQ\hat{}dQ$

$$\int\int\int\int_M dQ\hat{}dQ = \text{is an Absolute Deformation invariant, if} \quad (4.52)$$

$$0 = \int\int\int_{boundary\ of\ M} (\rho V^k)(Q\hat{}dQ) \quad (4.53)$$

The latter condition can be put into correspondence with a closed system where only radiation is passed through the boundary, but mass is not. If the absolute criteria is not satisfied, mass can be exchanged through the boundary with the environment.

5. Deformation Invariants and the Plasma State.

5.1. Special evolutionary processes. The plasma process

As described in a previous section, the fundamental equation of topological evolution is given by Cartan's magic formula, which acts as a propagator on the forms that make up the exterior differential system. The differential system now under consideration will be the electromagnetic refinement of the hydrodynamic system, $A, F = dA, \}$, and is constructed from the augmented collection of exterior differential forms $\{A, F = dA, G, J = dG\}$. An evolutionary process is defined herein as a map that can be described by a singly parameterized vector field. If the Action of the Lie derivative on the complete system of Maxwell exterior differential forms vanishes for a particular choice of process, then that process leaves the entire Maxwell system absolutely invariant. As a topology can be constructed in terms of an exterior differential system, and if a special process leaves that system of forms invariant, then the topology induced by the system of forms is invariant; the process must be a homeomorphism.

However, for a given Maxwell system, it is more likely that only some of the exterior differential forms that make up the Maxwell system are invariant relative to an arbitrary process; others are not. Of particular interest are those forms which are relative integral invariants of continuous deformations. The closed integral of an exterior differential form that defines a relative integral invariant is not only invariant with respect to a process represented by particular vector field, but also with respect to longitudinal deformations of that process obtained by multiplying the particular vector field by an arbitrary non-zero function. For vector fields which are singly parameterized, this concept of longitudinal deformation is equivalent to a reparameterization of the vector field.

The development that follows is guided by Cartan's pioneering work, in which he examined those specialized processes for mechanical systems that leave the closed integrals of the 1-form of Action, A, a deformation (relative integral) invariant. Cartan proved that such processes always have a Hamiltonian representation. As described above, there are two classes of such Hamiltonian processes, the extremal class and the Bernoulli class:

$$\text{Hamiltonian Processes} \tag{5.1}$$

$$L_{(\beta\mathbf{V})}\int_{z1} A = \int_{z1} i(\beta V)dA + di(\beta V)A \Rightarrow 0. \tag{5.2}$$

$$\mathit{Extremal}: \qquad i(\beta\mathbf{V})dA = 0, \tag{5.3}$$

$$\mathit{Bernoulli}: \qquad i(\beta\mathbf{V})dA = d\Theta. \tag{5.4}$$

The closed integration chain $z1$ is not necessarily a boundary.

An electromagnetic system has not only the primitive 1-form, A, but also the N-2 form, G, which can undergo evolutionary processes. For electromagnetic systems, a set of equations similar to those that define Hamiltonian processes can be used to define specialized processes that leave the closed integrals of the N-2 form, G, of field excitations, a deformation (relative integral) invariant. These special processes will be defined as Plasma processes. Such process do not create free charge, but they can cause a change in the number of charge pairs of opposite sign. The equations that must be satisfied are of the form:

$$\text{Plasma Processes} \tag{5.5}$$

$$L_{(\beta\mathbf{V})}\int_{z2} G = \int_{z2} i(\beta V)dG + di(\beta V)G \Rightarrow 0. \tag{5.6}$$

$$Extremal : i(\beta V)dG = 0, \tag{5.7}$$

$$Bernoulli : i(\beta V)dG = d\omega. \tag{5.8}$$

In the Extremal case,

$$i(\beta V)dG = \beta\{(\mathbf{J} - \rho\mathbf{V})^x dy\hat{}dz - ... + (\mathbf{J}\times\mathbf{V})^x dx\hat{}dt... \Rightarrow 0, \tag{5.9}$$

implies that the extremal Plasma process obeys the classic expressions:

$$\text{Extremal Plasma process } \mathbf{J}_E = \rho\mathbf{V}. \tag{5.10}$$

5.1.1. The Topological Hall effect

In the Bernoulli case of a Plasma process the integrand must be proportional to an exact 2-form, $d\omega$. There is one obvious candidate, the 2-form, F :

$$i(\beta V)dG = \beta\{(\mathbf{J} - \rho\mathbf{V})^x dy\hat{}dz - ... + (\mathbf{J}\times\mathbf{V})^x dx\hat{}dt... = \sigma_{Hall}F. \tag{5.11}$$

The conductivity coefficient σ_{Hall} in the expression must be a domain constant. Comparing the components of the equation of constraint yields the properties of a Bernoulli Plasma process:

$$\text{Bernoulli Plasma process,} \quad \mathbf{J}_B = \rho\mathbf{V} + (\sigma_{Hall}/\beta)\mathbf{B}, \tag{5.12}$$

$$\text{and,} \quad (\mathbf{J}_B \times \mathbf{V}) = (\sigma_{Hall}/\beta)\mathbf{E}. \tag{5.13}$$

$$(\sigma_{Hall}/\beta)(\mathbf{J}_B \circ \mathbf{E}) \Rightarrow 0 \tag{5.14}$$

$$(\sigma_{Hall}/\beta)(\mathbf{V} \circ \mathbf{E}) \Rightarrow 0 \tag{5.15}$$

$$(\sigma_{Hall}/\beta)(\mathbf{E} \circ \mathbf{B}) \Rightarrow 0. \tag{5.16}$$

Thus the Bernoulli plasma process leads to a Plasma current $\mathbf{J}_B$ which is orthogonal to the $\mathbf{E}$ field and whose magnitude is proportional to the $\mathbf{B}$ field. To quote Landau and Lifshitz [Landau 1960] "As we see, it (the Hall effect) gives rise to a current perpendicular to the electric field, whose magnitude is proportional to the magnetic field." The conclusion is that the Bernoulli Plasma process generates a Hall effect, and requires that the second Poincare coefficient must vanish. It follows that the Topological Hall effect exists in non-equilibrium systems where the 1-form A is of Pfaff topological 3. Bernoulli plasma processes, which include the Hall component, are not dissipative in the sense that $(\mathbf{J}_B \circ \mathbf{E}) = 0$. Such Plasma processes do not change the net charge within the closed integration domain. That is, charges can be produced only in equal and opposite pairs by a "Plasma process".

The appearance of a magnetic conductance, σ_{Hall}, as presented herein, is novel to and dependent upon, the topological perspective of electromagnetism and plasmas. The concept is deduced from the sole assumption that the Plasma current defines a process direction field that preserves the closed integrals of the 2-form, G. The derivation has been accomplished without the constraints of metric, connection, gauge symmetry, and without statistics or quantum hypotheses. The concept of a non-dissipative Hall current does not depend on scales, and can be found in macro and cosmological domains, as well as in the microworld of quantum theory.

This invariance principle that conserves charge pairs is to be compared to the Helmholtz theorem which checks on the validity of the deformation integral invariance of the 2-form F.

$$L_{\beta\mathbf{V}} \int_{z2} F = \int_{z2} i(\beta V)dF = 0 \tag{5.17}$$

The closed integral of Helmholtz is an intrinsic topological (deformation) invariant of an electromagnetic system, for the 2-form F is exact by construction (the postulate of potentials). The Helmholtz integral is a deformation invariant for all evolutionary processes that can be described by a singly parameterized vector field. (This statement is not true for Yang Mills fields). Hence in a plasma, for which the evolutionary processes are constrained such that $\mathbf{J} = \rho\mathbf{V}$, both the closed integrals of F and G are deformation invariants. In the sense, the plasma is a topological refinement of the complete Maxwell system.

In the subsections that follow, various topological categories of plasma processes will be examined. The ideal and semi-ideal plasma processes will obey the plasma master equation, and the non-ideal plasma processes will not. The electromagnetic flux is a local (absolute) invariant of all semi-ideal plasma processes. This statement is similar to the classification of hydrodynamic flows. Ideal and semi-ideal hydrodynamic flows satisfy the Helmholtz theorem, and the local conservation of vorticity.

5.2. The ideal Plasma process = an extremal Hamiltonian process.

Recall that an extremal Hamiltonian process V_E is defined by the constraint that the Work 1-form $W = i(\beta V_E)dA$ be of Pfaff topological dimension 0.

$$\text{Extremal Hamiltonian process:} \quad i(\beta V_E)dA = 0. \tag{5.18}$$

An Ideal Plasma process is defined such that the the direction field created by the Plasma current, J is equivalent to the extremal Hamiltonian process, βV_E. It follows that

$$\begin{aligned}
\text{Ideal Plasma } J &= \beta V_E && (5.19)\\
i(\beta V_H)dA &= \beta(\mathbf{E} + \mathbf{V}_E \times \mathbf{B})_k dx^k + \beta(\mathbf{V}_E \circ \mathbf{E})dt = 0, && (5.20)\\
i(J)dA &= (\rho\mathbf{E} + \mathbf{J} \times \mathbf{B})_k dx^k + (\mathbf{J} \circ \mathbf{E})dt && (5.21)\\
&= \rho(\mathbf{E} + \mathbf{V} \times \mathbf{B})_k dx^k + (\rho\{\mathbf{V} + (\sigma_{mag}/\beta)\mathbf{B}\} \circ \mathbf{E})dt = 0 && (5.22)
\end{aligned}$$

The term involving a possible magnetic conductance can be ignored, as $(\mathbf{B} \circ \mathbf{E}) \Rightarrow 0$, for both the Bernoulli Plasma process and the Extremal Plasma process. These constraints define the "ideal" plasma current as an evolutionary vector direction field that satisfies the charge invariance equation, and the equation for an extremal field such that

$$\text{Ideal Plasma}: W = i(\rho V)dA = \{(\rho\mathbf{E} + \mathbf{J} \times \mathbf{B})_k dx^k + (\mathbf{J} \circ \mathbf{E})dt\} \Rightarrow 0, \tag{5.23}$$

The 1-form W is the 1-form of virtual work defined in terms of the Lorentz force. The resulting equation demonstrates that the concept of a Lorentz force, $\rho\mathbf{E} + \mathbf{J} \times \mathbf{B}$,has a topological foundation. It is apparent that if the Lorentz force vanishes,

$$\{\rho\mathbf{E} + \mathbf{J} \times \mathbf{B}) \Rightarrow 0, \tag{5.24}$$

and the Plasma current density is NOT ohmic,

$$(\mathbf{J} \circ \mathbf{E}) = (\rho\mathbf{V} \circ \mathbf{E}) + (\sigma_{Hall}/\beta)(\mathbf{E} \circ \mathbf{B}) \Rightarrow 0, \tag{5.25}$$

then the closed integral of the Action 1-form is also a deformation topological invariant of the Plasma process. Such a set of constraints,

$$W = i(J)dA = 0, \tag{5.26}$$

topologically defines the "ideal" plasma state as a plasma process for which the 1-form of virtual Work vanishes. Such plasmas are said to be "force-free". The ideal plasma is thereby a restriction of arbitrary processes to that unique extremal process that leaves invariant both the closed integrals of flux and the closed integrals of charge. The ideal plasma can not exist on a domain of a 4 dimensional variety where the second Poincare invariant is not zero (Pfaff dimension 4), for then the concept of an extremal process (which requires an odd Pfaff dimension, such as Pfaff dimension 3) does not exist.

5.3. The Bernoulli-Casimir plasma process is a semi-ideal plasma process.

The topological constraint that the 1-form of virtual work vanishes is sufficient but not necessary for a plasma process to preserve the closed integrals of the Action 1-form. Evolutionary invariance of the closed integral of Action does not require that the plasma process be unique. The 1-form of virtual Work, W, need not be zero, but only closed: $dW \Rightarrow 0$. By analogy to hydrodynamics, if the virtual Work 1-form is exact,

$$W = d\Theta \tag{5.27}$$

then the Lorentz force is represented by a spatial gradient, $\rho\mathbf{E} + \mathbf{J} \times \mathbf{B} = \nabla\Theta$, and the Power $-\mathbf{J} \circ \mathbf{E} = \partial\Theta/\partial t$. The function $\Theta(x, y, z, t)$ is a Bernoulli-Casimir function, and acts as the generator of a symplectic Hamiltonian flow. The (non-unique) Bernoulli-Casimir function is an evolutionary invariant for each process path, but is not necessarily a constant over the domain:

$$L_{\rho\mathbf{V}}(\Theta) = i(\rho V)d\Theta = i(\rho V)i(\rho V)A = 0. \tag{5.28}$$

In the language of hydrodynamics, there is a Bernoulli constant for each streamline, but the constant is different for different streamlines. The Bernoulli-Casimir function is not the same as the Hamiltonian energy function, but is more closely related to the thermodynamic concept of enthalpy. The Bernoulli-Casimir function can be used to generate a "Hamiltonian process", but the process is not uniquely defined as an extremal process.

For such symplectic plasma processes, the gradient of the Bernoulli-Casimir function is transverse to the **B** field only when the second Poincare invariant vanishes.

$$\rho\,\mathbf{E}\circ\mathbf{B}=\nabla\Theta\circ\mathbf{B}. \tag{5.29}$$

The implication (again) is that the Pfaff dimension of the Action 1-form cannot be 4. Similar expressions were studied in conjunction with topological conservation in MHD by Hornig and Schindler [**?**].

$$\rho\,\mathbf{E}\circ\mathbf{V}=\nabla\Theta\circ\mathbf{V}. \tag{5.30}$$

If the Ohmic *assumption* is made for the non-ideal plasma process, and is if the form $\mathbf{J}=\rho\mathbf{V}=\sigma_{ohm}(\mathbf{E}+\mathbf{V}\times\mathbf{B})$, then the symplectic condition leads to a Bernoulli format of the type $\mathbf{J}=(1/\rho\sigma_{ohm})grad(\Theta)$.When it is subsumed that the Bernoulli-Casimir function is related to temperature, such that

$$\mathbf{J}=(1/\rho\sigma_{ohm})grad(kT), \tag{5.31}$$

it would appear that for plasma motion along the **B** field lines, there can exist a dynamo action to produce an **E** field collinear with the magnetic field.

It is suggested that the large temperature gradient that exists in a plasma envelope about a rotating star (with a **B** field like a neutron star) can induce a current flow and an **E** acceleration field along the polar magnetic field lines. Like in a Bernoulli process in a fluid, the mechanical energy is not conserved, but the enthalpy (the Bernoulli-Casimir) is a invariant along any trajectory, and that invariant can be different from trajectory to neighboring trajectory.

5.4. The Stokes plasma process is a semi-ideal process that obeys the Master equation.

The constraint that the virtual work 1-form, W, generated by a plasma process, $W=i(J=\rho V)dA$, be closed, does not require that it be exact. The constraint of closure yields two vector conditions:

$$dW=0\Rightarrow curl(\rho\mathbf{E}+\mathbf{J}\times\mathbf{B})=0\qquad and\qquad \nabla(\mathbf{J}\circ\mathbf{E})=\partial(\rho\mathbf{E}+\mathbf{J}\times\mathbf{B})/\partial t. \tag{5.32}$$

The first vector condition implies that

$$\nabla\rho \times (\mathbf{E}+\mathbf{V}\times\mathbf{B}) + \rho\, curl(\mathbf{E}) + \rho\ curl(\mathbf{V}\times\mathbf{B}) = 0. \tag{5.33}$$

By using the Maxwell-Faraday equation, this topological constraint becomes the plasma master equation:

$$-\partial\mathbf{B}/\partial t + curl(\mathbf{V}\times\mathbf{B}) = -\nabla\ln\rho\times(\mathbf{E}+\mathbf{V}\times\mathbf{B}). \tag{5.34}$$

All of these ideal and semi ideal plasma processes enjoy the property that the electromagnetic flux is conserved locally. That is

$$L_{(J=\rho\mathbf{V})}(dA) = L_{\rho\mathbf{V}}F = d(i(\rho V)F) = 0. \tag{5.35}$$

5.5. Frozen-in lines.

It is of some interest to examine the evolution of the differential forms that make up an electromagnetic system relative to Plasma processes. The method is to construct the Lie derivative with respect a plasma process, $\mathbf{J} = \rho\mathbf{V}$, of all forms that make up the electromagnetic Pfaff sequence.

For an arbitrary vector field $\mathbf{Z}$ whose tangents define a line in space time, the N-1 form

$$Z = i(\gamma\mathbf{Z})dx\hat{\ }dy\hat{\ }dz\hat{\ }dt = i(\gamma\mathbf{Z})\Omega_4 \tag{5.36}$$

can be tested for evolutionary invariance relative to any other vector V. Suppose the effect of the evolutionary process is conformal:

$$L_{(V)}Z = i(V)d\,Z + d(i(V)Z) = \Gamma(x,y,z,t)Z \tag{5.37}$$

This statement implies that the points that make up the tangent line of the vector field Z remain on the tangent line. The points may be permuted but they do not leave the line. Such is the concept of a frozen in field. The points that make up a line evolve into points of the same line. The evolution need not be uniformly continuous, especially where the points are folded. Yet even in such cases the points of a line are still points of the line, even though rearranged in order. If for a given V the evolution of the lines of Z is conformal, then there exists a parametrization of V such that the evolution is uniform and invariant. A parametrization function $\beta(x,y,z,t)$ can be found such that

$$L_{(\beta V)}Z = \beta L_{(V)}Z + (L_{(V)}\beta)Z = (\beta \cdot \Gamma\ + i(V)d\beta)Z \Rightarrow 0. \qquad (5.38)$$

For the electromagnetic system there are three N-1 forms, which may or may not be frozen into the evolutionary process. Consider the 3-form of current.

$$L_{(V)}J = i(V)d\,J + d(i(V)J) \qquad (5.39)$$

As $dJ = 0$,

$$L_{(V)}J = d\{i(V)i(J)\Omega_4\} \qquad (5.40)$$

It follows that if $i(V)i(J)\Omega_4\} = 0$, the field lines of J are frozen-in (with $\Gamma = 0$). So the plasma evolutionary process evolutionary, with $\mathbf{J} = \rho\mathbf{V}$, is an example of a process that "freezes-in" the lines of current. However, there are many other evolutionary processes for which the J lines are frozen in.

The formulas created by the topological constraints given above are valid on any set of independent variables, but expressions on 4 dimensions of space time for "frozen-in" lines are not quite the same as those that appear in the engineering literature based on euclidean 3-space [Kochine 1964]. Either the time-like component of the 4-vector W must vanish, or the process V must be explicitly time-independent for the general formulas to be in precise agreement with the engineering expressions. [RMK 1975 a]

It is important to note that in space time the "frozen-in" lines must be related to 3-forms, and not to the two form components **E** and **B**. These latter objects can produce "frozen-in" lines only on the exceptional 2-surfaces, the torus and the Klein bottle, (and then only when $\mathbf{E}\circ\mathbf{B}$ =0). The 3-form of Torsion has spatial components that are dominated by the **B** field (in the limit $\mathbf{E} \to 0$), such that "frozen-in" lines of Torsion might have the appearance of "frozen-in" lines of **B**. The 3-form of Spin has lines that can be dominated by **D**. However the explicit formulas for the 3-forms of Torsion and Spin are not dependent upon a choice of constitutive relations that act as geometrical constraints on the 2-forms of F and G.

5.6. Evolution of the lines of topological torsion with respect to plasma currents.

Consider the evolution of the lines of topological torsion

$$L_{(J=\rho V)}A\hat{\ }F = i(\rho V)d(A\hat{\ }F) + d(i(\rho V)A\hat{\ }F) \quad (5.41)$$

$$= i(\rho V)d(A\hat{\ }F) + d\{(i(\rho V)A)\hat{\ }F - A\hat{\ }i(\rho V)F\} \quad (5.42)$$

First consider those systems where the second Poincare invariant vanishes, $F\hat{\ }F = 0$. The lines in space time which are tangent to the 3-form $A\hat{\ }F$ then have zero divergence. The lines can only start and stop on boundary points, or they are closed on themselves. The Torsion lines can be either parallel to the plasma current or they can be orthogonal to the plasma current. As the electromagnetic current is exact, any three dimensional domain of support for a finite plasma current cannot be compact without a boundary. If the lines of plasma current start and stop on boundary points, then the lines of torsion can form closed loops that link the current lines. It is the concept of linkages that is of interest to the theory of magnetic knots.

Consider that plasma process such that the evolution is in the direction of the Torsion lines. As in this situation,

$$(i(J)A\hat{\ }F) = (i(\rho V)A\hat{\ }F) \Rightarrow (i(\gamma \mathbf{T}_4)A\hat{\ }F) \quad (5.43)$$

$$= \gamma(i(\mathbf{T}_4)(i(\mathbf{T}_4)\Omega_4 = 0, \quad (5.44)$$

the 3-form of Torsion is a local invariant whenever the second Poincare invariant vanishes; $\mathbf{E} \circ \mathbf{B} \Rightarrow 0$. In other words, $F\hat{\ }F \neq 0$ is a local necessary condition for topological change. It is also a remarkable fact that any evolution in the direction of the Torsion vector leaves the Action 1-form conformally invariant, in the sense that:

$$L_{(\gamma \mathbf{T}_4)}A = i(\gamma \mathbf{T}_4)dA + di(\gamma \mathbf{T}_4)A = \gamma(\mathbf{E} \circ \mathbf{B})A + 0. \quad (5.45)$$

The torsion vector on a domain of 4 variables is transverse to the 1-form of Action, as $A\hat{\ }(A\hat{\ }F) = 0$. Evolution in the direction of the Torsion vector in not Hamiltonian, unless the second Poincare invariant vanishes. If the second Poincare invariant is not zero, evolution in the direction of the Torsion vector is thermodynamically irreversible.

5.7. Evolution of the lines of Topological Spin Current with respect to plasma currents.

Consider the evolution of the lines of Spin current

$$L_{(\rho V)}A\hat{}G = i(\rho V)d(A\hat{}G) + d(i(\rho V)A\hat{}G) \tag{5.46}$$
$$= i(\rho V)d(A\hat{}G) + d\{(i(\rho V)A)\hat{}G - A\hat{}i(\rho V)G\} \tag{5.47}$$

First consider those systems where the first Poincare invariant vanishes, $F\hat{}G - A\hat{}J = 0$. The lines in space time which are tangent to the 3-form $A\hat{}G$ then have zero divergence. The lines can only start and stop on boundary points, or they are closed on themselves. The Spin lines are either parallel to the plasma current or they are orthogonal to the plasma current. As the electromagnetic current is exact, any three dimensional domain of support for a finite plasma current cannot be compact without a boundary. If the lines of plasma current do not stop or start on boundary points (current loops), then the Spin lines which terminate on boundary points can be linked by the current loops.

The concept of the spin vector depends on the existence of G, but not on the concept of $J = dG$. That is, the Spin vector can be associated with separated domains of charges, which can be compact domains without boundary. These domains of separated charge are complements of those regions which, as they support finite charge current densities, are regions that can not be compact without boundary.

5.8. Applications to Electromagnetism and Plasmas

All of the development of previous sections will carry over to the electromagnetic system, which also subsumes the postulate of potentials. The topological torsion 3-form, $A\hat{}dA$, induces the torsion current

$$\mathbf{T}_4 = -[(\mathbf{E}\times\mathbf{A} + \mathbf{B}\phi); \mathbf{A}\circ\mathbf{B}] \equiv [\mathbf{T}, h]. \tag{5.48}$$

If $div_4\mathbf{T} = 2\ \mathbf{E}\circ\mathbf{B} \neq 0$, the electromagnetic 1-form, A, defines a domain of Pfaff dimension 4. Such domains cannot support topologically transverse magnetic waves ($as\ \ A\hat{}F \neq 0$). Evolutionary processes (including plasma currents) that are proportional to the Torsion current are thermodynamically irreversible, if $\sigma = \mathbf{E}\circ\mathbf{B} \neq 0$. However, the conformal properties of evolution in the direction of the Torsion current lead to extraordinary properties when the plasma current is in the direction of the Torsion vector. From the thermodynamic arguments presented above, based on the postulate of potentials for an arbitrary system, but using the notation of an electromagnetic system, it follows that

$$L_{(\mathbf{T}_4)}A = \sigma A = (\mathbf{E} \circ \mathbf{B})A \tag{5.49}$$

and

$$L_{(\mathbf{T}_4)}(A\hat{}F) = 2\sigma A\hat{}F = 2(\mathbf{E} \circ \mathbf{B})\ A\hat{}F. \tag{5.50}$$

It follows that motion along the direction of the torsion vector freezes-in the lines of the torsion vector in space time, but the process is irreversible unless the second Poincare invariant is zero. The time evolution of the deformable coherent structure is recognizable even though it thermodynamically decays!

Recall that the definition of a plasma current, J, is equivalent to an evolutionary process such that

$$\text{Definition of a plasma Current } J: \qquad L_{(J)}G = 0. \tag{5.51}$$

Consider a plasma current which is also in the direction of the Torsion vector. Then

$$L_{(J)}A\hat{}G = (L_{(J)}A)\hat{}G + A\hat{}L_{(J)}G \tag{5.52}$$

$$= (L_{(\gamma\mathbf{T}_4)}A)\hat{}G + A\hat{}L_{(\gamma\mathbf{T}_4)}G = \gamma \cdot (\mathbf{E} \circ \mathbf{B})\ A\hat{}G + 0 \tag{5.53}$$

For plasma motions in the direction of the (possibly dissipative) torsion vector, both the "lines"of the Spin vector are "frozen in" and the lines of the Torsion vector are "frozen in". Such "frozen in"objects can be used to give a topological definition of deformable coherent structures in a plasma. Moreover, as the evolutionary process causes the frozen in structures to deform and decay, it is conceivable that evolution could proceed to form stationary (but not stagnant) states (where $\mathbf{E} \circ \mathbf{B} \Rightarrow 0$), such that the frozen in field line structures become local deformation invariants, or topological defects.

In conclusion, electromagnetic coherent structures are evolutionary deformable (and perhaps decaying) domains of Pfaff dimension 4, which form stationary states of topological defects (including the null state) in regions of Pfaff dimension 3, where $\mathbf{E} \circ \mathbf{B} = 0$. (Note that all semi-ideal plasma current processes are reversible in a thermodynamic sense.)

6. Appendix: Examples

These examples of electromagnetic physical systems with topological torsion and topological spin have been formulated in terms of a Maple program for accuracy in solving problems with a high level of algebraic complexity. When the expressions become too complex for convenient display in the text, the reference [Maple EM 2003] is cited.

6.1. Electromagnetic Waves in the Vacuum with Spin and Torsion

As the formalism of the Topological Spin current $\mathbf{S}_4$ and the Topological Torsion 4-vector $\mathbf{T}_4$ may be unfamiliar to many readers, it is useful to compare four classes of unusual vacuum wave solutions with the usual wave solutions. The first solutions exemplify the meaning of Topological Transversality and the statements made above about TTEM modes not radiating. The "unusual waves" have their vector potential, $\mathbf{A}$, orthogonal to the wave vector, $\mathbf{k}$, describing the direction of the wave front. So in each case the solutions are geometrically transverse, but not necessarily topologically transverse. In each unusual example, the current density (if the phase velocity condition, $(\omega/k)^2 - 1/(\varepsilon\mu) = 0$, is not satisfied) is in the direction of the vector potential and is, therefore, also orthogonal to the wave vector. The usual wave solutions have their vector potential parallel to the wave vector. The four unusual cases belong to equivalence classes defined by the constraints

$$\begin{aligned}
\text{Equilibrium Pfaff dimension 2} &: & (6.1)\\
\text{No Torsion, No Spin } (A\hat{}F &= 0,\ A\hat{}G = 0) & (6.2)\\
\text{Non-Equilibrium Pfaff dimension 3} &: & (6.3)\\
\text{TTM - No Torsion, finite Spin } (A\hat{}F &= 0,\ A\hat{}G \neq 0,\ d(A\hat{}G) = 0) & (6.4)\\
\text{TTE - Finite torsion, No Spin } (A\hat{}F &\neq 0,\ A\hat{}G = 0, d(A\hat{}F) = 0) & (6.5)\\
\text{Finite torsion, Finite Spin } (d(A\hat{}F) &= 0,\ d(A\hat{}G) = 0) & (6.6)\\
\text{Non-Equilibrium Pfaff dimension 4} &: & (6.7)\\
(d(A\hat{}F) &\neq 0,\ d(A\hat{}G) \neq 0). & (6.8)
\end{aligned}$$

In each case, each component of the fields satisfies the wave equation, subject to the phase velocity relation, $(\omega/k)^2 - 1/(\varepsilon\mu) = 0$. The current density, J, is proportional to the vector potential, A, (in a fashion reminiscent to the London conjecture) multiplied by the same phase velocity relation. The examples do not generate any charge current distributions when the phase velocity equation is satisfied (the phase velocity equals the speed of light as determined by the constitutive equations).

In each example given below, the 1-form of Action is specified and the field intensities are computed. Then the Spin Current and the Torsion vector are evaluated. A satisfactory vacuum solution will satisfy the Lorentz-Lorenz vacuum conditions of zero charge current densities, which may be satisfield subject to a phase velocity "dispersion" relation of the type ($\varepsilon\mu c^2 = 1$). The Poynting vector is computed, and the Poincare invariants are evaluated.

The simplest "eikonal" phase function is defined by the formula $\Theta = (\pm kz \mp \omega t)$ representing outbound waves, or $\Theta = (\pm kz \pm \omega t)$ for inbound waves. It is important to realize that there are two methods for determining if a physical object is coming at you or going away. The first method is based upon the phase velocities of waves (and the signs in the phase function) while the second method depends upon a kinematic (particle) distinction which is related to the group velocity concept. Examples below will demonstrate the differences, which often appear in the theory of wave guides;

Examples of the four classes of these simple (but unusual) wave types correspond to:

6.1.1. Example 1. Real Linear Polarization: $A\hat{}G \neq 0,\ d(A\hat{}G) \neq 0,\ A\hat{}F = 0$

Consider the Potentials for and outbound wave:

$$A = [\cos(kz - \omega t), \cos(kz - \omega t), 0, 0] \tag{6.9}$$

and their induced fields (assuming $\mathbf{D} = \varepsilon\mathbf{E},\ \mathbf{B} = \mu\mathbf{H}$):

$$\mathbf{E} = [-\sin(kz - \omega t), -\sin(kz - \omega t), 0]\omega \tag{6.10}$$

$$\mathbf{B} = [\sin(kz - \omega t), -\sin(kz - \omega t), 0]k \tag{6.11}$$

$$\text{Topological Torsion } \mathbf{T}_4 = [0, 0, 0, 0]. \tag{6.12}$$

$$Poincare\ II = 2\mathbf{E} \circ \mathbf{B} = 0 \tag{6.13}$$

$$\mathbf{J}_4 = [-\cos(kz-\omega t), -\cos(kz-\omega t), 0, 0](-k^2+\varepsilon\mu\omega^2)/\mu \tag{6.14}$$

$$\rho = 0 \tag{6.15}$$

$$\text{Topological spin } \mathbf{S}_4 = [0, 0, -k/\mu, -\varepsilon\omega]\, 2\cos(kz-\omega t)\sin(kz-\omega t). \tag{6.16}$$

$$Poincare\ I = 2\{(\sin(kz-\omega t))^2 - (\cos(kz-\omega t))^2\}(-k^2+\varepsilon\mu\omega^2)/\mu \tag{6.17}$$

$$\mathbf{E}\times\mathbf{H} = [0, 0, 1](2\,\omega k/\mu)(\sin(kz-\omega t))^2) \tag{6.18}$$

$$\begin{aligned}
(\mathbf{J}\circ\mathbf{E}) &= 2\{\cos(kz-\omega t)\sin(kz-\omega t)\}(k^2-\varepsilon\mu\omega^2)/\mu && (6.19)\\
(\mathbf{E}\circ\mathbf{B}) &= 0, && (6.20)\\
\textbf{TTM } (\mathbf{A}\circ\mathbf{B}) &= 0, && (6.21)\\
\textit{not } \textbf{TTE as } (\mathbf{A}\circ\mathbf{D}) &= -2\omega\epsilon\{\cos(kz-\omega t)\sin(kz-\omega t)\} && (6.22)
\end{aligned}$$

This class of potentials generates a set of complex field intensities and excitations, and a current density proportional to the vector potential. If the dispersion relation $(k^2 - \varepsilon\mu\omega^2) = 0$ is satisfied, then the solutions are acceptable vacuum solutions, with a vanishing charge current density. The Torsion vector vanishes identically, independent from the dispersion condition, but the Spin vector does not. The first Poincare invariant vanishes subject to the constraint of the dispersion relation. The second Poincare invariant vanishes identically. The solution corresponds to a linear state of polarization at 45° with respect to the x-axis, with the electric and the magnetic fields in phase. There is a non-zero Poynting vector along the z axis, which is orthogonal to the vector potential. Note that the radiated power has a time average which is zero. If the charge current density is not zero (due to a fluctuation in the dispersion relation) the charge current vector is orthogonal to the Spin current vector.

If the assumed potentials are changed so as to represent an inbound wave $(kz - \omega t) \Rightarrow (kz + \omega t)$, then $\mathbf{E}$ field changes sign, the $\mathbf{B}$ field remains the same, the current density remains the same, the 4th component of $\mathbf{S}_4$ changes sign, and $(\mathbf{A}\circ\mathbf{D})$ changes sign.

6.1.2. Example 2. Real Circular Polarization: $A\hat{}G = 0, A\hat{}F \neq 0$

Consider the Potentials for an outbound wave:

$$A = [\cos(kz - \omega t), \sin(kz - \omega t), 0, 0] \tag{6.23}$$

and their induced fields (assuming $\mathbf{D} = \varepsilon\mathbf{E}$, $\mathbf{B} = \mu\mathbf{H}$):

$$\mathbf{E} = [-\sin(kz - \omega t), \cos(kz - \omega t), 0]\omega$$

$$\mathbf{B} = [-\cos(kz - \omega t), -\sin(kz - \omega t), 0]k$$

$$\mathbf{J}_4 = [-\cos(kz - \omega t), +\sin(kz - \omega t), 0, 0](-k^2 + \varepsilon\mu\omega^2)/\mu$$

$$\rho = 0 \tag{6.24}$$

$$\mathbf{S}_4 = [0, 0, 0, 0] \, .$$

$$Poincare \ I = 0 \tag{6.25}$$

$$\mathbf{T}_4 = [0, 0, \omega, k].$$

$$Poincare \quad II = 0 \tag{6.26}$$

$$\mathbf{E} \times \mathbf{H} \; = [0, 0, 1](\omega k/\mu)$$

$$\begin{aligned} (\mathbf{J} \circ \mathbf{E}) &= 0 \\ (\mathbf{E} \circ \mathbf{B}) &= 0, \\ \textit{not}\ \text{TTM}\ \ (\mathbf{A} \circ \mathbf{B}) &= k, \\ \text{TTE}\ \ (\mathbf{A} \circ \mathbf{D}) &= 0 \end{aligned}$$

This class of potentials generates a set of complex field intensities and excitations, and a current density proportional to the vector potential. If the dispersion relation $(k^2 - \varepsilon\mu\omega^2) = 0$ is satisfied, then the solutions are acceptable vacuum solutions, with a vanishing charge current density. The Spin vector vanishes identically, but the Torsion vector does not. In fact, the torsion vector is constant. The solution corresponds to a circular state of polarization with the constant magnetic and electric amplitudes rotating about the z axis. The Poynting vector is not zero and is a constant, time independent, vector. This wave solution is geometrically transverse (TEM), yet it produces power as it is not topologically transverse (TTEM). If the dispersion relation is not precisely satisfied, the current vector is orthogonal to the Torsion vector and parallel to the vector potential. Both Poincare invariants vanish identically. The soliton like solution should be compared to the wave guide solution of example 5 below, which is also TEM, but does not radiate. The example solution demonstrates the correspondence betwen Topological Torsion and circular polarization.

6.1.3. Example 3. Complex Linear Polarization: $A\hat{\ }G = 0, A\hat{\ }F = 0$

Consider the Potentials for an outbound wave:

$$A = [\cos(kz - \omega t), \sqrt{-1}\cos(kz - \omega t), 0, 0] \tag{6.27}$$

and their induced fields (assuming $\mathbf{D} = \varepsilon\mathbf{E}$, $\mathbf{B} = \mu\mathbf{H}$):

$$\mathbf{E} = [-\sin(kz - \omega t), -\sqrt{-1}\sin(kz - \omega t), 0]\omega \tag{6.28}$$

$$\mathbf{B} = [+\sqrt{-1}\sin(kz - \omega t), -\sin(kz - \omega t), 0]k \tag{6.29}$$

$$\text{Topological Torsion } \mathbf{T}_4 = [0, 0, 0, 0]. \tag{6.30}$$

$$Poincare\ II = 2\mathbf{E} \circ \mathbf{B} = 0 \tag{6.31}$$

$$\mathbf{J}_4 = [-\cos(kz - \omega t), -\sqrt{-1}\cos(kz - \omega t), 0, 0](-k^2 + \varepsilon\mu\omega^2)/\mu \tag{6.32}$$

$$\rho = 0 \tag{6.33}$$

$$\text{Topological spin } \mathbf{S}_4 = [0, 0, 0, 0]\,. \tag{6.34}$$

$$Poincare\ I = 0 \tag{6.35}$$

$$\mathbf{E}\times\mathbf{H} \;= [0,0,0]. \tag{6.36}$$

$$\begin{aligned}
(\mathbf{J}\circ\mathbf{E}) &= 0 && (6.37)\\
(\mathbf{E}\circ\mathbf{B}) &= 0, && (6.38)\\
\textbf{TTM as } (\mathbf{A}\circ\mathbf{B}) &= 0, && (6.39)\\
\textbf{TTE } \;(\mathbf{A}\circ\mathbf{D}) &= 0 && (6.40)
\end{aligned}$$

This class of potentials generates a set of complex field intensities and excitations, and a current density proportional to the vector potential. The fields are said to be complex linearly polarized because the complex **B** field is a complex scalar multiple of the complex **E** field. If the dispersion relation $(k^2 - \varepsilon\mu\omega^2) = 0$ is satisfied, then the solutions are acceptable vacuum solutions, with a vanishing charge current density. Note that both the Torsion vector and the Spin vector vanish identically. The complex square of both the electric and the magnetic field vectors vanish. Both Poincare invariants vanish independent from the dispersion constraint. Although the fields are propagating, there is no momentum flux and the Poynting vector is zero. The **E** and **B** fields are (complex) collinear. This example is perhaps the simplest member of the class of Bateman-Whittaker complex solutions described in Example 10, below.

6.1.4. Example 4. Complex Circular Polarization: $A\hat{}G \neq 0, A\hat{}F \neq 0$

Consider the complex Potentials for an outbound wave:

$$A = [\cos(kz-\omega t), \sqrt{-1}\sin(kz-\omega t), 0, 0] \tag{6.41}$$

and their induced fields (assuming $\mathbf{D} = \varepsilon\mathbf{E}$, $\mathbf{B} = \mu\mathbf{H}$):

$$\mathbf{E} = [-\sin(kz-\omega t), \sqrt{-1}\cos(kz-\omega t), 0]\omega \tag{6.42}$$

$$\mathbf{B} = [-\sqrt{-1}\cos(kz-\omega t), -\sin(kz-\omega t), 0]k \tag{6.43}$$

$$\text{Topological Torsion } \mathbf{T}_4 = [0,0,\omega,k]. \tag{6.44}$$

$$\mathit{Poincare\ II} = 2\mathbf{E}\circ\mathbf{B} = 0 \tag{6.45}$$

$$\mathbf{J}_4 = [-\cos(kz-\omega t), -\sqrt{-1}\sin(kz-\omega t), 0](k^2 - \varepsilon\mu\omega^2)/\mu \tag{6.46}$$

$$\rho = 0 \tag{6.47}$$

$$\text{Topological Spin } \mathbf{S}_4 = -[0, 0, k/\mu, \varepsilon\omega]\, 2\cos(kz-\omega t)\sin(kz-\omega t). \tag{6.48}$$

$$\textit{Poincare } \; I = 2(k^2 - \varepsilon\mu\omega^2)\{(\cos(kz-\omega t))^2 - (\sin(kz-\omega t))^2\} \tag{6.49}$$

$$\mathbf{E}\times\mathbf{H} \; = [0, 0, -\omega k/\mu]\{(\cos(kz-\omega t))^2 - (\sin(kz-\omega t))^2\}. \tag{6.50}$$

$$(\mathbf{J}\circ\mathbf{E}) = 2(k^2-\varepsilon\mu\omega^2)\cos(kz-\omega t)\sin(kz-\omega t)/\mu \tag{6.51}$$

$$(\mathbf{E}\circ\mathbf{B}) = 0, \tag{6.52}$$

$$\textit{not } \textbf{TTM as } (\mathbf{A}\circ\mathbf{B}) = +\sqrt{-1}k, \tag{6.53}$$

$$\textit{not } \textbf{TTE } (\mathbf{A}\circ\mathbf{D}) = -2\varepsilon\omega\cos(kz-\omega t)\sin(kz-\omega t). \tag{6.54}$$

Subject to the phase velocity constraint $(k^2 - \varepsilon\mu\omega^2) \Rightarrow 0$, the solution is an admissable vacuum solution, for then both the charge current density and the first Poincare invariant vanish.

6.1.5. Example 5. Waveguide TM modes

The next example is composed from a kinematic form, $(dz - v_g dt)$, multiplied by the arbitrary wave function, $f(x,y)\cos(kz-\omega t)$. If $v_g > 0$, then as t increases, the "distance" gets larger (outbound to the observer) . If $v_g < 0$, then as t increases, the distance gets smaller (inbound to the observer). Consider the Potentials ($\mathbf{A} = [A_1, A_2, A_3]$, $\varphi = -A_4$)

$$A = [0, 0, 1, -v_g]f(x,y)\cos(kz-\omega t), \tag{6.55}$$

and their induced fields (note that a "group" velocity v_g is used in the definition of the kinematic form, not the phase velocity, $v_p = \omega/k$):

$$\mathbf{E} = [-v_g\partial f/\partial x, -v_g\partial f/\partial y, f(x,y)\tan(kz-\omega t)(-v_g+v_p)k]\cos(kz-\omega t) \quad (6.56)$$

$$\mathbf{B} = [\partial f(x,y)/\partial y\cos(kz-\omega t), -\partial f(x,y)/\partial x\cos(kz-\omega t), 0] \quad (6.57)$$

$$\begin{aligned}
\mathbf{J}_4 &= [k\partial f/\partial x\sin(kz-\omega t)(\varepsilon\mu v_g v_p - 1), && (6.58)\\
&\quad -k\partial f/\partial y\sin(kz-\omega t)(\varepsilon\mu v_g v_p - 1), && (6.59)\\
&\quad -(\nabla^2 f + \alpha f)\cos(kz-\omega t), && (6.60)\\
&\quad -v_g\varepsilon\mu(\nabla^2 f + \beta f)\cos(kz-\omega t)]/\mu && (6.61)
\end{aligned}$$

$$\alpha = k^2\varepsilon\mu v_p(v_p - v_g), \qquad \beta = k^2(v_p - v_g)/v_g, \quad (6.62)$$

$$\nabla^2 f = \partial f^2(x,y)/\partial x^2 + \partial f^2(x,y)/\partial y^2 \quad (6.63)$$

$$\rho = -v_g\varepsilon\mu(\nabla^2 f + \beta f)\cos(kz-\omega t)/\mu \quad (6.64)$$

$$\text{Topological Torsion } \ \mathbf{T}_4 = [0,0,0,0]. \quad (6.65)$$

$$Poincare\ II = 2\mathbf{E}\circ\mathbf{B} = 0 \quad (6.66)$$

$$\begin{aligned}
\mathbf{S}_4 &= [-(v_g/v_p - 1)f\partial f/\partial x\cos(kz-\omega t)^2, && (6.67)\\
&\quad -(v_g/v_p - 1)f\partial f/\partial x\cos(kz-\omega t)^2, && (6.68)\\
&\quad -k(v_g/v_p - 1)f^2\sin(kz-\omega t)\cos(kz-\omega t), && (6.69)\\
&\quad -\mu k(v_g - v_p)f^2\sin(kz-\omega t)\cos(kz-\omega t)]/\mu && (6.70)
\end{aligned}$$

$$Poincare\ \ I = \text{complicated function [Maple EM 2003]} \quad (6.71)$$

$$\begin{aligned}
\mathbf{E}\times\mathbf{H} &= [(v_p/v_g - 1)f\partial f/\partial x\sin(kz-\omega t),\\
&\quad (v_p/v_g - 1)f\partial f/\partial y\sin(kz-\omega t),\\
&\quad ((\partial f/\partial x)^2 + (\partial f/\partial y)^2)\cos(kz-\omega t)](v_g/\mu)\cos(kz-\omega t)
\end{aligned}$$

$$(\mathbf{J}\circ\mathbf{E}) = \ \beta/k\{f\nabla^2 f\} + \gamma\{(\partial f/\partial x)^2 + (\partial f/\partial y)^2\} + \delta\{f^2\} \text{ [Maple EM 2003]} \tag{6.72}$$

$$\begin{aligned}(\mathbf{E}\circ\mathbf{B}) &= 0, & (6.73)\\ \textbf{TTM as } (\mathbf{A}\circ\mathbf{B}) &= 0, & (6.74)\\ \textit{not } \textbf{TTE } (\mathbf{A}\circ\mathbf{D}) &= -(v_g - v_p)k(f^2\cos(kz-\omega t)\sin(kz-\omega t). & (6.75)\end{aligned}$$

$$(\mathbf{B}\circ\mathbf{H} - \mathbf{D}\circ\mathbf{E}) = -(\{\varepsilon\mu(\omega/k)^2 - 1\}/\mu)\cos(kz-\omega t)^2\{(\nabla f)^2 \ + f(\nabla^2 f)\}$$

Note that in this solution, the fourth component of the Action is scaled by the "group velocity", v_g, not the "speed of light" which is defined as $c = \sqrt{1/\xi\mu}$. The phase velocity, $v_p = \omega/k$, differs from the group velocity, v_g, and the vacuum speed, c. Again, two constraint conditions (dispersion relations) are required for the solution to be a vacuum solution without charge currents. One of the constraint conditions demands that the product of the group and the phase velocity equals the square of the speed of light as determined from the constitutive properties:

$$v_p \cdot v_g = 1/\varepsilon\mu = c^2. \tag{6.76}$$

The second constraint required for the vacuum state $(\mathbf{J} = 0, \rho = 0)$ implies the functions $f(x,y)$ satisfies the Helmholtz equation with parameter, λ,

$$\nabla^2 f + \lambda^2 f = 0 \tag{6.77}$$

$$\lambda^2 = k^2(v_p/v_g - 1). \tag{6.78}$$

Such charge current free TM modes are also TTM modes; the Torsion vector is identically zero, but the Spin vector is not. Further note that the **E** field has a longitudinal component when the group velocity and the phase velocity are not the same. For the TTM mode magnetic mode constraint is satisfied, $\mathbf{A}\circ\mathbf{B} = 0,$ but the TTE mode constraint is not, $\mathbf{A}\circ\mathbf{D} \neq 0$. The second Poincare invariant vanishes, $\mathbf{E}\circ\mathbf{B} = 0,$but for this solution, the first Poincare invariant does not vanish. Not only is the Spin vector not zero, but also its divergence is not zero. The energy flow is in the direction of the wave vector, $\mathbf{k}$, but not in the direction of the field momentum, $\mathbf{D}\times\mathbf{B},$ and the energy propagates with the group velocity v_g.

6.1.6. Example 6. Waveguide TEM modes that do not radiate.

By a slight modification of the preceeding example, consider the Potentials when the group velocity and the phase velocity are the same. Then the potentials become

$$A = [0, 0, f(x, y), -(\omega/k)f(x, y)]\cos(kz - \omega t) \tag{6.79}$$

and their induced fields (using $\mathbf{D} = \varepsilon\mathbf{E}$, $\mathbf{B} = \mu\mathbf{H}$) are:

$$\mathbf{E} = [-(\omega/k)\partial f/\partial x, -(\omega/k)\partial f/\partial y, 0]\cos(kz - \omega t) \tag{6.80}$$

$$\mathbf{B} = [\partial f/\partial y, -\partial f/\partial x, 0]\cos(kz - \omega t) \tag{6.81}$$

$$\text{Topological Torsion } \mathbf{T}_4 = [0, 0, 0, 0]. \tag{6.82}$$

$$Poincare\ II = 2\mathbf{E} \circ \mathbf{B} = 0 \tag{6.83}$$

$$\begin{aligned}
\mathbf{J}_4 &= [J_x, J_y, J_z, \rho]/\mu, &(6.84)\\
J_x &= \partial f/\partial x \sin(kz - \omega t)(k^2 - \varepsilon\mu\omega^2)/k, &(6.85)\\
J_y &= -\partial f/\partial y \sin(kz - \omega t)(k^2 - \varepsilon\mu\omega^2)/k, &(6.86)\\
J_z &= -\cos(kz - \omega t)\{\partial^2 f/\partial x^2 + \partial^2 f/\partial y^2\}, &(6.87)\\
\rho &= -\cos(kz - \omega t)(\varepsilon\mu\omega/k)\{\partial^2 f/\partial x^2 + \partial^2 f/\partial y^2\} &(6.88)
\end{aligned}$$

$$\begin{aligned}
\text{Topological Spin } \mathbf{S}_4 &= -[f\partial f/\partial x, f\partial f/\partial y, 0, 0]\beta &(6.89)\\
\beta &= (1 - \varepsilon\mu(\omega/k)^2)(\cos(kz - \omega t))^2/\mu. &(6.90)
\end{aligned}$$

$$\begin{aligned}
Poincare\ \ I &= (1 - \varepsilon\mu(\omega/k)^2)(\cos(kz - \omega t))^2 &(6.91)\\
&\quad \{f\partial^2 f/\partial x^2 + f\partial^2 f/\partial y^2 + (\partial f/\partial x)^2 + (\partial f/\partial y)^2)\}/\mu &(6.92)
\end{aligned}$$

$$\begin{aligned}
\mathbf{E} \times \mathbf{H} &= [0, 0, \omega/(k\mu)]\gamma &(6.93)\\
\gamma &= \{(\cos(kz - \omega t))^2((\partial f/\partial x)^2 + (\partial f/\partial y)^2)\}. &(6.94)
\end{aligned}$$

$$\begin{aligned}
(\mathbf{J} \circ \mathbf{E}) &= -(1-\varepsilon\mu(\omega/k)^2)\cos(kz-\omega t)\sin(kz-\omega t) && (6.95)\\
& \quad \{(\partial f/\partial x)^2 + (\partial f/\partial y)^2\}/\mu && (6.96)\\
(\mathbf{E} \circ \mathbf{B}) &= 0, && (6.97)\\
\textbf{TTM as } (\mathbf{A} \circ \mathbf{B}) &= 0, && (6.98)\\
\textbf{TTE } (\mathbf{A} \circ \mathbf{D}) &= 0. && (6.99)
\end{aligned}$$

Note that the vector potential, $\mathbf{A}$, is parallel to both the wave vector, $\mathbf{k}$, and the field momentum, $\mathbf{D} \times \mathbf{B}$. The Torsion vector and the second Poincare invariant are identically zero. The solution produces transverse currents and spin densities unless a dispersion relation, $\varepsilon\mu(\omega/k)^2 = 1$, is satisfied. Subject to the dispersion constraints, this classic solution has both a zero Torsion vector and a zero Spin vector. Both $\mathbf{A} \circ \mathbf{D} = 0$ and $\mathbf{A} \circ \mathbf{B} = 0$. The wave front normal field is in the spatial direction of the potential, by construction. The candidate solution subject to the dispersion relation is both topologically transverse TTEM and geometrically transverse, TEM .

However, even if the dispersion relations are satisfied, the geometric TEM solution produces finite charge current densities, unless the function $f(x, y)$ is a solution of the two dimensional Laplace equation, $\nabla^2 f = 0$. This further constraint implies that the TEM solution produces no radiated power in the charge free state, for $\mathbf{E} \times \mathbf{H} \Rightarrow 0$ as $\nabla^2 f \Rightarrow 0$. In the previous example, the constraint that the system be TTEM is relaxed, and radiated power is achieved in a TTM mode.

6.1.7. Example 7. An irreversible Type 1 vacuum solution for which $\mathbf{E} \circ \mathbf{B} > 0$

Consider the potentials representing a "rotation" in the x-y plane, and a "translation" in the z-t plane:

$$\mathbf{A} = [+y, -x, ct]/\lambda^2 , \quad \phi = cz/\lambda^2, \quad where\ \lambda = -c^2t^2 + x^2 + y^2 + z^2. \quad (6.100)$$

The induced fields are:

$$\mathbf{E} = [-2(cty - xz), +2(ctx + yz), -(c^2t^2 + x^2 + y^2 - z^2)]2c/\lambda^3$$

$$\mathbf{B} = [-2(cty + xz), +2(ctx - yz), +(c^2t^2 + x^2 + y^2 - z^2)]2/\lambda^3.$$

$$\text{Topological Torsion } \mathbf{T}_4 = 2c[x, y, z, t]/\lambda^4. \tag{6.101}$$

$$Poincare\ II = 2\mathbf{E} \circ \mathbf{B} = -8c/\lambda^4 \tag{6.102}$$

$$\mathbf{J}_4 = [J_x, J_y, J_z, \rho](\varepsilon\mu c^2 - 1)/\mu\lambda^4 \tag{6.103}$$

$$J_x = [4y\lambda + 6ct(yct - zx), \tag{6.104}$$

$$J_y = 4x\lambda + 6ct(xct + zy), \tag{6.105}$$

$$J_z = 8ct(2x^2 + 2y^2 - z^2 + c^2t^2), \tag{6.106}$$

$$\rho = 0 \tag{6.107}$$

$$\text{Topological Spin } \mathbf{S}_4 = -[S_x, S_y, S_z, \sigma]/\mu\lambda^5 \text{ [Maple EM 2003]} \tag{6.108}$$

$$Poincare\ I = -4(1 - \varepsilon\mu c^2)\chi/\mu\lambda^6) \tag{6.109}$$

$$\chi = \chi(x, y, z, t) \tag{6.110}$$

$$\mathbf{E} \times \mathbf{H} = [x, y, 2z\varsigma)]16c^2t\eta/(\mu\lambda^6) \tag{6.111}$$

$$\eta = (x^2 + y^2 - z^2 + c^2t^2) \tag{6.112}$$

$$\varsigma = (x^2 + y^2)/\eta. \tag{6.113}$$

$$(\mathbf{J} \circ \mathbf{E}) = -16(1 - \varepsilon\mu(\omega/k)^2)(\gamma/\mu\lambda^7) \tag{6.114}$$

$$\gamma = \gamma(x, y, z, t) \tag{6.115}$$

$$(\mathbf{E} \circ \mathbf{B}) = -8c/\lambda^4 \neq 0, \tag{6.116}$$

$$\text{not } \mathbf{TTM}\ (\mathbf{A} \circ \mathbf{B}) = 2ct/\lambda^4 \neq 0, \tag{6.117}$$

$$\text{not } \mathbf{TTE}\ (\mathbf{A} \circ \mathbf{D}) = -2\varepsilon c^2t(+c^2t^2 + 3x^2 + 3y^2 - z^2 \neq 0. \tag{6.118}$$

Subject to the dispersion relation, $\varepsilon\mu c^2 = 1$, and the Lorentz constitutive conditions, these time dependent wave functions satisfy the homogeneous Maxwell equations without charge currents, and are therefore acceptable vacuum solutions. The Poynting vector does not vanish, but first Poincare invariant vanishes, as does $(\mathbf{J} \circ \mathbf{E})$.

$$J_{+t} = dG = [0, 0, 0, 0] \tag{6.119}$$

The extensive algebra involved in these and other computations in this article were checked with a Maple symbolic mathematics program [Maple EM 2003] . It is to be noted that when the substitution $t \Rightarrow -t$ is made in the functional forms for the potentials, the modified potentials fail to satisfy the vacuum Lorentz conditions for zero charge-currents. The algebraic results for the charge current density are somewhat complicated, but the bottom line is that

$$J_{-t} = dG \neq [0, 0, 0, 0]. \tag{6.120}$$

It appears that the valid vacuum solution presented above is not time-reversal invariant in a directional sense where by dt is presumed to increase, but the translation in the z-t plane is reversed.

The Spin current density for this non-transverse vacuum wave example is evaluated as:

$$\begin{aligned} Spin \ &: \ \mathbf{S}_4 = [x(3\lambda^2 - 4y^2 - 4x^2), y(3\lambda^2 - 4y^2 - 4x^2), z(\lambda^2 - 4y^2 - 4x^2), \\ &\quad t(\lambda^2 - 4y^2 - 4x^2)](2/\mu)/\lambda^{10}, \end{aligned} \tag{6.121}$$

and has zero divergence, subject to the condition $\varepsilon\mu c^2 = 1$. The solution has magnetic helicity as $\mathbf{A} \circ \mathbf{B} \neq 0$ and is radiative in the sense that the Poynting vector, $\mathbf{E} \times \mathbf{H} \neq 0$.

Both the Spin current and the Torsion vector are non-zero, which implies that this solution represents waves which are neither TTM nor TTE. They are not transverse waves in any sense. However, the first Poincare invariant vanishes, implying that the Spin integral is a deformation invariant, and is conserved. The second Poincare invariant is not zero, which implies that the Torsion-Helicity integral is not a topological invariant. These solutions are not simple transverse waves for both $\mathbf{A} \circ \mathbf{B} \neq 0$, and $\mathbf{A} \circ \mathbf{D} \neq 0$. Note that the physical units of the second Poincare invariant are that of an energy density multiplied by an impedance (ohms). As the second Poincare invariant is not zero, it is impossible to find a compact without boundary two surface that contains non-zero lines of magnetic field. That is, a closed 2-torus of magnetic field lines does not exist.

However, as the first Poincare invariant is zero it is possible to construct a deformation invariant in terms of the deRham period integral over a closed 3 dimensional submanifold

$$Spin = \int_{z3} \{S_x dy\hat{}dz\hat{}dt - S_y dx\hat{}dz\hat{}dt + S_z dx\hat{}dy\hat{}dt - \sigma dx\hat{}dy\hat{}dz\}. \quad (6.122)$$

6.1.8. Example 8. An irreversible vacuum solution of type 2 for which $\mathbf{E} \circ \mathbf{B} < 0$

Consider the potentials (complimentary to example 7)

$$\mathbf{A} = [+ct, -z, +y]/\lambda^4\ , \quad \phi = cx/\lambda^4, \quad where\ \lambda^2 = -c^2t^2 + x^2 + y^2 + z^2 \quad (6.123)$$

and their induced fields:

$$\mathbf{E} = [+(-c^2t^2 + x^2 - y^2 - z^2), +2(ctz + yx), -2(cty - zx)]2c/\lambda^6$$

$$\mathbf{B} = [+(-c^2t^2 + x^2 - y^2 - z^2), +2(-ctz + yx), +2(cty + zx)]2/\lambda^6.$$

As in the previous example above, these fields satisfy the Maxwell-Faraday equations, and the associated excitations satisfy the Maxwell-Ampere equations without producing a charge current 4-vector. However, it follows by direct computation that the second Poincare invariant, and the Torsion 4-vector are of opposite signs to the values computed for the previous example:

$$\mathbf{E} \circ \mathbf{B} = +4c/\lambda^8 \quad and \quad \mathbf{A} \circ \mathbf{B} = +2ct/\lambda^8\ .$$

6.1.9. Example 9. A classical photon? Superposition of the two complementary examples of type 1 and type 2.

When the potentials of examples type 1 and type 2 above are combined by addition or subtraction, the resulting wave is topologically transverse magnetic, but not topological transverse electric. Not only does the second Poincare invariant vanish under superposition, but so also does the Torsion 4 vector. Conversely, the examples above show that there can exist topologically transverse magnetic waves which can be decomposed into two non-transverse waves. Under the assumption that $(\varepsilon\mu c^2 - 1) \Rightarrow 0$, the superposed solutions satisfy the vacuum condition of zero-charge current density, but the Spin 4 vector does not vanish. Such waves are not topologically transverse electric. However, the first Poincare invariant vanishes, which implies that the Spin remains a conserved topological quantity for the superposition. The spin current density for the combined examples is given by the formula:

$$\begin{aligned}
\text{Topological Spin } \mathbf{S}_4 &= [S_x, S_y, S_z, \sigma]\, 4c/\lambda^5 && (6.124)\\
S_x &= 2cx(y+ct)^2 && (6.125)\\
S_y &= y(y+ct)(x^2 - y^2 + z^2 - 2cty - c^2t^2) && (6.126)\\
S_z &= -2cz(y+ct)^2 && (6.127)\\
\sigma &= -(y+ct)(x^2 + y^2 + z^2 + 2cty + c^2t^2) && (6.128)\\
\lambda &= -c^2t^2 + x^2 + y^2 + z^2 && (6.129)
\end{aligned}$$

while the Torsion current is a zero vector

$$\mathbf{T}_4 = [0, 0, 0, 0].$$

In addition, for the superposed example, the spatial components of the Poynting vector are equal to the Spin current density vector multiplied by γ, such that

$$\mathbf{E} \times \mathbf{H} = \gamma\ \mathbf{S}, \quad \textit{with}\ \gamma = -(x^2 + y^2 + z^2 + 2cty + c^2t^2)/2c(y+ct)\lambda^2.$$

These results seem to give classical credence to the Planck assumption that vacuum state of Maxwell's electrodynamics supports quantized angular momentum, (the conserved spin integral) and that the energy flux must come in multiples of the spin quanta. In other words, these combined irreversible solutions of examples type 1 and type 2 have the appearance of the photon

6.1.10. Example 10a. A Plasma Accretion disc from Hedge Hog B field solutions.

When the **B** field is radial in direction, and either inbound or outbound, the field is called a "hedgehog" field. Many such structures can be generated by multiplying a closed, but not exact, 1-form of the type,

$$
\begin{aligned}
A_{closed} &= [y, -x, 0]/(ax^2 + by^2) && (6.130)\\
dA_{closed} &= 0, && (6.131)
\end{aligned}
$$

by the function $\Gamma = m\sqrt{-1}/\sqrt{ax^2 + by^2 + cz^2}$. The solution to be presented below generates a family of different topological singularities depending on the non-isotropic coefficients (a, b, c). For simplicity of display the isotropy coefficients will be chosen to give z axis symmetry; i.e., $a = b$. The potentials are then of the format:

$$
\begin{aligned}
\mathbf{A} &= \Gamma(x, y, z)[y, -x, 0]/(ax^2 + ay^2)\ , && (6.132)\\
with\quad \Gamma &= -(z\,m)\sqrt{-1}/\sqrt{ax^2 + ay^2 + cz^2} && (6.133)\\
\phi &= 0,\quad and\quad x^2 + y^2 + z^2 = r^2. && (6.134)
\end{aligned}
$$

Note that the Γ term (like the Dirac Hedge Hog solution in the next example) is imaginary. These potentials induce the field intensities and charge current densities below:

$$\mathbf{A} = \sqrt{-1}zm[y, -x, 0]/\{\sqrt{(ax^2 + ay^2 + cz^2)}(ax^2 + ay^2)\},\quad \phi = 0 \qquad (6.135)$$

$$\mathbf{B} = \sqrt{-1}am[x, y, z]/(ax^2 + ay^2 + cz^2)^{3/2} \qquad (6.136)$$

$$\mathbf{E} = [0, 0, 0] \qquad (6.137)$$

$$
\begin{aligned}
\text{Topological Torsion} &= \text{A\^{}F},\quad \text{Poincare II} = d(A\hat{\ }F) && (6.138)\\
\mathbf{T4} &= [0, 0, 0, 0]. && (6.139)\\
\text{Poincare II} &= 2(\mathbf{E} \circ \mathbf{B}) = 0. && (6.140)
\end{aligned}
$$

$$
\begin{aligned}
\mathbf{J} &= \sqrt{-1}(3azm(c - a)/\mu)[-y, +x, 0]/(ax^2 + ay^2 + cz^2)^{5/2} && (6.141)\\
&= \sqrt{-1}3(c - a)(ax^2 + ay^2)/\mu(ax^2 + ay^2 + cz^2)^2 \cdot \mathbf{A}, && (6.142)\\
\rho &= 0. && (6.143)
\end{aligned}
$$

$$\text{Topological Spin} = \text{A\^{}G}, \quad \text{Poincare I} = d(A\hat{\ }G) \tag{6.144}$$

$$\mathbf{S4} = (azm^2/\mu)[zx, zy, -(ax^2+ay^2), 0]/\{(ax^2+ay^2)(ax^2+ay^2+cz^2)^2\}. \tag{6.145}$$

$$\text{Poincare I} = (\mathbf{B}\circ\mathbf{H} - \mathbf{D}\circ\mathbf{E}) - (\mathbf{A}\circ\mathbf{J} - \rho\phi) \tag{6.146}$$

$$= (am^2/\mu)(ax^2+ay^2+4az^2-3cz^2)/(ax^2+ay^2+cz^2)^3. \tag{6.147}$$

$$\mathbf{A}\circ\mathbf{B} = 0 \quad \text{(TTM mode if zero)} \tag{6.148}$$

$$\mathbf{A}\circ\mathbf{D} = 0 \quad \text{(TTE mode if zero)} \tag{6.149}$$

The Lorentz force and the dissipative terms become:

$$\text{Lorentz 4-force} = \rho\mathbf{E} + \mathbf{J}\times\mathbf{B} \tag{6.150}$$

$$= (3(c-a)a^2m^2/\mu)[z^2x, z^2y, -z(x^2+y^2)]/(ax^2+ay^2+cz^2)^4 \tag{6.151}$$

$$= 3a(c-a)(ax^2+ay^2)\cdot\mathbf{S4}. \tag{6.152}$$

$$\text{Poynting Vector } \mathbf{E}\times\mathbf{H} = [0,0,0]. \tag{6.153}$$

$$\text{Dissipative power } \mathbf{J}\circ\mathbf{E} = 0. \tag{6.154}$$

$$\text{Interaction Energy } A\hat{\ }J = 3(c-a)(z^2m^2a/\mu)/(ax^2+ay^2+cz^2)^3. \tag{6.155}$$

The potentials, the charge current and the **B** field are sensitive to the rotation direction, which is governed by the parameter, m. The topological spin and the Lorentz force are not sensitive to the circulation sense, m. For the isotropic case, $a = c$, there is no induced current density and no Lorentz force, yet the Topological Spin and the interaction energy density are non-zero. In cases, the dissipative power, the Poynting vector vanish.

For oblate

The singularities for the potentials are the z axis and the origin, for positive anisotropy coefficients. If c is negative and a is positive, then the singular set for the denominator $ax^2+ay^2+cz^2 \Rightarrow 0$ generates a cone. The cone centered on the z axis, with its vertex at the origin. The cone is oblate when the ratio $|a/c|$ is very small, and prolate when $|a/c|$ is very large. The prolate cone is remindful of jets, and the oblate cone is remindful of flat spirals. The region between the cone and the z axis generates imaginary values for the **A** and **B** fields, but not for the Topological Spin, or the Lorentz force, or the interaction energy.

The **B** field is similar to the famous Dirac Hedgehog (see the next example) field often associated with "magnetic monopoles". The magnetic field is radial. Although its divergence is zero everywhere, the lines of the field do not close on themselves. The **B** field starts and stops on points of a topological boundary. Note that the rotational orientation of the flux circulation integral (whether m is positive or negative) determines whether or not the **B** field is pointing inbound or outbound relative to the origin. The topological (radial) orientation is related to the rotational sense in this example. It is also true that the induced current **J** (linear in m) depends upon the rotational orientation, and it is proportional to the vector potential, but the field is opposite in a rotational sense. However the Lorentz force, the Topological Torsion and the Interaction Energy do not depend upon the sense of the rotational circulation, as they are quadratic in m.

The Hedgehog **B** field in this example has zero divergence everywhere except at the zeros of the denominator, which herein is interpreted as a topological obstruction, or defect, to be excised from consideration.

In this example, the non-zero plasma current density, **J** , has a sense of "circulation" about the z axis, and is proportional to the vector potential, reminding one of a London current, $\mathbf{J} = \lambda \cdot \mathbf{A}$. The "London" parameter , λ, due to the circulation is

$$\lambda = (3(c-a)/\mu) \cdot (ax^2 + ay^2)/(ax^2 + ay^2 + cz^2)^2. \tag{6.156}$$

The London formula depends strongly upon the anisotropy, which is confirmed in superconductivity experiments. Indeed the hedgehog example in an anisotropic situation generates a non-dissipative current density with some of the attributes of a super current. Also note that the spatial components of the Topological Spin are proportional to the direction field of the spatial components of the Lorentz force. The proportionality factor is exactly equal to the (same!) London parameter, λ.

The formula for the Lorentz force demonstrates that the system of circulating currents, is directed radially away (centrifugally) from the rotational axis, and yet is such that the plasma is attracted to the $z = 0, xy$ plane. Independent of the sense of rotation, the Lorentz force is divergent in the radial plane and convergent in the direction of the z axis, towards the $z = 0$ plane. The conjecture is that this electromagnetic field for the rotating plasma would have the tendency to form an accretion disk of plasma in the presence of a central gravitational field.

It is apparent that the helicity density and the second Poincare invariant are zero. In fact, the 3-form of topological torsion vanishes identically. Although the 3-form of Topological Torsion vanishes identically, the 3-form of Topological Spin is not zero. It is also true that the divergence of the 3-form of spin is not zero, for the first Poincare invariant is not zero.

If the system is spherical, then the deformation parameter vanishes, $b^2 = 0$, the Lorentz force vanishes, the first Poincare invariant vanishes, the charge current 3-form vanishes, but the Topological Spin does not vanish. There can exist closed domains such the the closed integrals of Topological spin are rational to within a common factor (topological quantization).

6.1.11. Example 10b. The Dirac Hedgehog

The famous Dirac hedgehog solution also generates a radial **B** field solution inversely proportional to the cube of a length. The Dirac potentials, and therefor the **B** field intensity, are pure Imaginary, yet the Topological Spin and the first Topological Poincare invariant are real:

$$\mathbf{A} = \Gamma(x,y,z)[y,-x,0] \ , \tag{6.157}$$

$$with \quad \Gamma = \sqrt{-1}(m/2)/\{r(z-r)\} \tag{6.158}$$

$$\phi = 0, \quad and \ \ x^2+y^2+z^2 = r^2. \tag{6.159}$$

These potentials induce the field intensities and charge current densities below:

$$\mathbf{A} = \sqrt{-1}m/2[y,-x,0]/\{r(z-r)\}, \quad \phi = 0 \tag{6.160}$$

$$\mathbf{B} = \sqrt{-1}m[x,y,z]/(r^2 - b^2z^2)^{3/2} \tag{6.161}$$

$$\mathbf{E} = [0,0,0] \tag{6.162}$$

$$\mathbf{D} = \epsilon\mathbf{E} \qquad \mathbf{H} = \mathbf{B}/\mu \tag{6.163}$$

$$\mathbf{J} = [0,0,0], \tag{6.164}$$

$$\rho = 0. \tag{6.165}$$

In addition, the potentials induce the 3-forms of Topological Torsion and Topological Spin, and their divergences which define the first and second Topological Poincare invariants:

$$\text{Topological Spin} = \text{A}\hat{}\text{G}, \quad \text{Poincare I} = d(A\hat{}G) \quad (6.166)$$

$$\mathbf{S4} = (zm^2/4\mu)[-zx, -zy, (x^2+y^2), 0]/\{(z-r)(r^4)\}. \quad (6.167)$$

$$\text{Poincare I} = (\mathbf{B}\circ\mathbf{H} - \mathbf{D}\circ\mathbf{E}) - (\mathbf{A}\circ\mathbf{J} - \rho\phi) \quad (6.168)$$

$$= -m^2/4(r)^4 \quad (6.169)$$

$$\text{Topological Torsion} = \text{A}\hat{}\text{F}, \quad \text{Poincare II} = d(A\hat{}F) \quad (6.170)$$

$$\mathbf{T4} = [0,0,0,0]. \quad (6.171)$$

$$\text{Poincare II} = 2(\mathbf{E}\circ\mathbf{B}) = 0. \quad (6.172)$$

$$\mathbf{A}\circ\mathbf{B} = 0 \quad \text{(TTM mode if zero)} \quad (6.173)$$

$$\mathbf{A}\circ\mathbf{D} = 0 \quad \text{(TTE mode if zero)} \quad (6.174)$$

The Lorentz force and the dissipative terms become:

$$\text{Lorentz 4-force} = \rho\mathbf{E} + \mathbf{J}\times\mathbf{B} = [0,0,0]$$

$$\text{Poynting Vector } \mathbf{E}\times\mathbf{H} = [0,0,0]. \quad (6.175)$$

$$\text{Dissipative power } \mathbf{J}\circ\mathbf{E} = 0. \quad (6.176)$$

$$\text{Interaction Energy } A\hat{}J = 0 \quad (6.177)$$

6.1.12. Example 11. Coulomb plus Abrikosov vortex singularity.

Consider the Potentials for the combined vortex singularity and Coulomb 1/r potential:

$$A = [by/(x^2+y^2), -bx/(x^2+y^2), 0, -(1/4\pi\epsilon)m/r], \quad (6.178)$$

$$r = \sqrt{x^2+y^2+z^2}. \quad (6.179)$$

The induced fields (assuming $\mathbf{D} = \varepsilon\mathbf{E}$, $\mathbf{B} = \mu\mathbf{H}$) are:

$$\mathbf{E} = (1/4\pi\epsilon)m[x,y,z]/r^3 \quad (6.180)$$

$$\mathbf{B} = [0, 0, 0] \tag{6.181}$$

$$\text{Topological Torsion } \mathbf{T}_4 = (1/4\pi\epsilon)mb[-zx, -zy, (x^2 + y^2), 0]/(r^3(x^2 + y^2)). \tag{6.182}$$

$$Poincare2 = 0 \tag{6.183}$$

$$\mathbf{J}_4 = [0, 0, 0, 0] \tag{6.184}$$

$$\rho = 0 \tag{6.185}$$

$$\text{Topological spin } \mathbf{S}_4 = (1/4\pi\epsilon)^2\epsilon m^2/r^4[x, y, z, 0]\,. \tag{6.186}$$

$$Poincare1 = -(1/4\pi\epsilon)^2\epsilon m^2/r^4 \tag{6.187}$$

$$\mathbf{E} \times \mathbf{H} \ = [0, 0, 0]. \tag{6.188}$$

$$(\mathbf{J} \circ \mathbf{E}) = 0 \tag{6.189}$$

$$(\mathbf{E} \circ \mathbf{B}) = 0, \tag{6.190}$$

$$(\mathbf{A} \circ \mathbf{B}) = 0, \tag{6.191}$$

$$(\mathbf{A} \circ \mathbf{D}) = 0 \tag{6.192}$$

This example demonstrates how the addition of a closed but not exact contribution to a 1-form of Action can influence the topological features of an electromagnetic system. The Topological Torsion vanishes without the "rotation" or angular momentum term, with strength determined by the coefficient b. Note that the Topological Spin term does not vanish. The combined system is not in thermodynamic equilibrium.

6.1.13. Example 12. Bateman-Whittaker solutions.

In the modern language of differential forms, Bateman [Bateman 1914] (and Whittaker) determined that if two *complex* functions $\alpha(x, y, z, t)$ and $\beta(x, y, z, t)$ are used to define the 1-form of Action,

$$A = \alpha d\beta - \beta d\alpha \Rightarrow \mathbf{A} = \alpha\nabla\beta - \beta\nabla\alpha, \quad \phi = -(\alpha\partial\beta/\partial t - \beta\partial\alpha/\partial t) \qquad (6.193)$$

then the derived 2-form $F = 2d\alpha\hat{}d\beta$ generates the complex field intensities,

$$\mathbf{E} = (\partial\alpha/\partial t)\nabla\beta - (\partial\beta/\partial t)\nabla\alpha \;\; and \;\; \mathbf{B} = \nabla\alpha \times \nabla\beta,$$

which of course satisfy the Maxwell-Faraday equations. If in addition, the functions α and β satisfy the complex Bateman constraints:

$$\nabla\alpha \times \nabla\beta = \pm(i/c)[(\partial\alpha/\partial t)\nabla\beta - (\partial\beta/\partial t)\nabla\alpha],$$

then the complex field excitations computed from the Lorentz vacuum constitutive constraints will satisfy the Maxwell-Ampere equations for the vacuum, without charge currents. It is apparent immediately that the second Poincare invariant is identically zero for such solutions. It is also apparent immediately that the Torsion vector is identically zero. What is not immediately apparent is that first Poincare invariant and the Spin vector vanish identically as well. In fact, the constrained complex solutions of the Bateman type are examples of topologically transverse (TTEM) waves. The Bateman solutions do not radiate!

As an explicit example, consider

$$\alpha = (x \pm iy)/(z - r), \quad \beta = (r - ct), \quad r = \sqrt{x^2 + y^2 + z^2}).$$

These functions satisfy the Bateman conditions (and, it should be mentioned, the Eikonal equation subject to the dispersion relation $\varepsilon\mu c^2 = 1$). The $\mathbf{E}$ and the $\mathbf{B}$ fields are complex (and complicated algebraically)

$$\begin{aligned}\mathbf{B} &= [yx + \sqrt{-1}(z^2 + y^2 - rz), -(z^2 + x^2 - rz) - \sqrt{-1}xy, \\ &\quad (r^2 + z^2 - 2rz)/(r - z)\,)(y - \sqrt{-1}x)]2/(r(z - r)^2)\end{aligned}$$

$$\begin{aligned}\mathbf{E} &= [-\sqrt{-1}yx + (y^2 + z^2 - rz), \sqrt{-1}(x^2 + z^2 - rz) - xy, \\ &\quad (z - r\,)(x + \sqrt{-1}y)]2c/(r(z - r)^2)\end{aligned}$$

$$\mathbf{S}_4 = [0, 0, 0, 0].$$

$$\mathbf{T}_4 = [0, 0, 0, 0].$$

$$\mathbf{E} \times \mathbf{H} = [0, 0, 0], \quad \mathbf{D} \times \mathbf{B} = [0, 0, 0], \quad \mathbf{E} \circ \mathbf{E} = 0, \quad \mathbf{B} \circ \mathbf{B} = 0$$

$$(\mathbf{B} \circ \mathbf{H} - \mathbf{D} \circ \mathbf{E}) = 0 \qquad (\mathbf{E} \circ \mathbf{B}) = 0$$

The functions α and β that satisfy the Bateman condition may be used to construct an arbitrary function, $F(\alpha, \beta)$, and remarkably enough, the arbitrary function $F(\alpha, \beta)$ satisfies the Eikonal equation,

$$(\nabla F)^2 - \varepsilon\mu(\partial F/\partial t)^2 = 0. \tag{6.194}$$

From experience with Eikonal solutions and wave equations, it might be thought that Eikonal solutions are sufficient. However, the Bateman conditions are necessary, for both the candidate solutions

$$\alpha = (x \pm iy)/(z - ct), \quad \beta = (r - ct), \quad r = \sqrt{x^2 + y^2 + z^2)}. \tag{6.195}$$

satisfy the Eikonal equation, but not the Bateman conditions. They do not generate TTEM modes in the vacuum. For arbitrary functions the algebra can become quite complex. A Maple symbolic mathematics program for computing the various terms is available (see references below)

6.2. Spinor plasma solutions from the Hopf Map.

Consider the map from R4(X,Y,Z,S) to R3(u,v,w) given by the formulas

$$\mathbf{H}1 = [u1, v1, w1] = [2(XZ + YS), 2(XS - YZ), (X^2 + Y^2) - (Z^2 + S^2)] \tag{6.196}$$

These formulas define the format of a Hopf map. The 3 component Hopf vector $\mathbf{H}1$ is real and has the property that

$$\mathbf{H}1 \cdot \mathbf{H}1 = (u1)^2 + (v1)^2 + (w1)^2 = (X^2 + Y^2 + Z^2 + S^2)^2. \tag{6.197}$$

Hence a real (and imaginary) 4 dimensional sphere maps to a real 3 dimensional sphere. If the functions $[u1, v1, w1]$ are defined as $[x/ct, y/ct, z/ct]$, then the 4D sphere $(X^2+Y^2+Z^2+S^2)^2 = 1 \Rightarrow (x/ct)^2, (y/ct)^2, (z/ct)^2 = 1$, implies that the Hopf map formulas are equivalent to the 4D light cone. The Hopf map can also be represented in terms of complex functions by a map from C2 to R3, as given by the formulas

$$\mathbf{H}1 = [u1, v1, w1] = [\alpha \cdot \beta^* + \beta \cdot \alpha^*, i(\alpha \cdot \beta^* - \beta \cdot \alpha^*), \alpha \cdot \alpha^* - \beta \cdot \beta^*. \qquad (6.198)$$

The variables α and β also can be viewed as two distinct complex variables defining ordered pairs of the four variables $[X, Y, Z, S]$. The classic format given above for $\mathbf{H}1$ can be obtained from the expansion, $\alpha = X + iY$, $\beta = Z + iS$. Other selections for the ordered pairs of (X, Y, Z, S) (along with permutations of the 3 vector components) give distinctly different Hopf vectors. For example, the ordered pairs, $\alpha = X + iZ$, $\beta = Y + iS$, give

$$\mathbf{H}2 = [2(YX - SZ), X^2 + Z^2 - Y^2 - S^2, -2(ZY + SX)] \qquad (6.199)$$

which is another Hopf vector, a map from R4 to R3, but with the property that $\mathbf{H}2$ is orthogonal to $\mathbf{H}1$:

$$\mathbf{H}2 \cdot \mathbf{H}1 = 0. \qquad (6.200)$$

Similarly, a third linearly independent orthogonal Hopf vector $\mathbf{H}3$ can be found

$$\mathbf{H}3 = [X^2 + Y^2 - Z^2 - S^2, -2(YX + SZ), 2(-ZX + SY)] \qquad (6.201)$$

such that

$$\mathbf{H}2 \cdot \mathbf{H}1 = \mathbf{H}3 \cdot \mathbf{H}2 = \mathbf{H}2 \cdot \mathbf{H}3 = 0. \qquad (6.202)$$

$$\mathbf{H}1 \cdot \mathbf{H}1 = \mathbf{H}2 \cdot \mathbf{H}2 = \mathbf{H}3 \cdot \mathbf{H}3 = (X^2 + Y^2 + Z^2 + S^2)^2. \qquad (6.203)$$

The three linearly independent Hopf vectors can be used as a basis of R3 excluding those points where the quartic form vanishes. The mapping functions (u, v, w) of the Hopf vector can be differentiated with respect to (X, Y, Z, S) to produce a set of three exact 1- forms whose coefficients form 3 independent 4 component vectors on R4. A 4th linearly independent vector can be created algebraically and forms the "adjoint" field for the given Hopf vector. This direction field can be used to construct a non-integrable 1-form, A, of Pfaff dimension 4. These three exact 1-forms and the non-integrable 1-form can be used as a basis frame for the space. The exterior differentials of the basis frame produce the usual Cartan connection which is not affine-torsion free in its subspaces. By this mechanism the differential structure of R4 as induced by the Hopf map is determined.

For **H**1, the 4 independent 1 forms are given by the expressions (where $\Lambda(X,Y,Z,S)$ is an arbitrary scaling function):

$$d(u1) = 2Zd(X) + 2Sd(Y) + 2Xd(Z) + 2Yd(S) \tag{6.204}$$

$$d(u2) = 2Sd(X) - 2Zd(Y) - 2Yd(Z) + 2Xd(S) \tag{6.205}$$

$$d(u3) = 2Xd(X) + 2Yd(Y) - 2Zd(Z) - Sd(S) \tag{6.206}$$

$$A = \{-Yd(X) + Xd(Y) - Sd(Z) + Zd(S)\}/\Lambda \tag{6.207}$$

A Frame Matrix can be generated by the coefficients of the 4 independent 1-forms, such that $Det\,F = (Z^2+S^2+Y^2+X^2)^2/\Lambda$. It is some interest to examine the properties of the adjoint 1-form, A,defined hereafter as the Hopf 1-form. For $\Lambda = 1$, it follows that the Hopf 1-form is of Pfaff dimension 4. It is also of interest to consider factors Λ that are of the format of the Holder norm, where n and p are integers, and (a,b,k,m) are arbitrary constants.

$$\Lambda = (aX^p + bY^p + kZ^p + mS^p)^{n/p} \tag{6.208}$$

The exponents n and p determine the homogeneity of the resulting 1-form, which is given below an ambiguous format (the plus of minus sign)

$$A_{\pm} = A(\pm)/\Lambda = \{\pm(Yd(X) - Xd(Y)) - Sd(Z) + Zd(S)\}/\Lambda.$$

For example, for $n = p = 2$, the scaling factor becomes related to the classic quadratic form. The scaled Hopf 1-form, A, is then homogeneous of degree zero. For arbitrary n and p, the 3-form of topological (Hopf) torsion becomes:

$$\text{Topological Torsion} = (A_{\pm})\hat{}d(A_{\pm}) \tag{6.209}$$

$$= i(\mathbf{T}_4)d(X)\hat{}d(Y)\hat{}d(Z)\hat{}d(S), \tag{6.210}$$

where the topological torsion 4 vector is equal to:.

$$\mathbf{T}_4 = \pm[X, Y, Z, S]/\Lambda. \tag{6.211}$$

The Torsion vector, $\mathbf{T}_4$, for the Hopf map is proportional to the position vector from the four dimensional origin and represents an expansion or a contraction process. The factor Λ depends upon the integers n and p as well as the constants (a,b,k,m).

The Topological Parity 4-form, whose coefficient is the 4 divergence of the Torsion vector, $\mathbf{T}_4$, becomes

$$\text{Topological Parity} = d(A_\pm)\hat{}(d(A_\pm) \tag{6.212}$$

$$= -4(\pm\Lambda)^{(-2n/p)})(n-2)d(X)\hat{}d(Y)\hat{}d(Z)\hat{}d(S)() \tag{6.213}$$

It is most remarkable that for n=2, any p and any (a,b,k,m), the topological parity vanishes; the scaled Hopf 1-form is of Pfaff dimension 3, not 4. In such cases the ratios of the integrals of the topological torsion 3 form over various closed manifolds are rational, and the closed integrals of the 3-form are topological deformation invariants. (coherent structures).

Also note that if the scaling factor is restricted to values such that $n = 4,\ p = 2,\ a = b = k = m = 1,$ then the Frame matrix is unimodular, and the scaled Hopf 1-form is homogeneous of degree -2, relative to the substitution $X \Rightarrow \gamma X$, *etc.* (A somewhat different definition of homogeneity relative to the volume element will be given below.) Emphasis is to be placed on those examples for which $n = 4\ or\ 2,\ p = 2,\ a = b = k = 1,\ \ m = \pm 1.$

6.2.1. Spinors as linear combinations of Hopf Maps

The 3D isotropic (null) complex position vector, $[z1, z2, z3]$ can be decomposed into a real and an imaginary part, such that both parts have the same magnitude and are orthogonal. In short, the Cartan Spinor, [16] can be represented as the complex sum of two Hopf vectors. The spinors come in two triples of the form

$$|\boldsymbol{\sigma}_{12}\rangle = |\mathbf{H}1\rangle + i\ |\mathbf{H}2\rangle \quad with \quad \langle\boldsymbol{\sigma}_{12}| \circ |\boldsymbol{\sigma}_{12}\rangle = 0 \tag{6.214}$$

$$|\boldsymbol{\sigma}_{23}\rangle = |\mathbf{H}2\rangle + i\ |\mathbf{H}3\rangle \quad with \quad \langle\boldsymbol{\sigma}_{23}| \circ |\boldsymbol{\sigma}_{23}\rangle = 0 \tag{6.215}$$

$$|\boldsymbol{\sigma}_{31}\rangle = |\mathbf{H}3\rangle + i\ |\mathbf{H}1\rangle \quad with \quad \langle\boldsymbol{\sigma}_{31}| \circ |\boldsymbol{\sigma}_{31}\rangle = 0 \tag{6.216}$$

These complex combinations of Hopf vectors can be used to generate solutions for which the topological torsion vanishes, and yet the topological spin is finite and quantized..

6.2.2. Electromagnetism of signature zero Hopf 1-forms

Guided by prior investigations, it is of interest to use the scaled Hopf 1-form as the generator of electromagnetic field intensities. The coefficients of the scaled Hopf 1-form can be put into correspondence with the classic vector and scalar potentials, $[\mathbf{A}, \phi]$ (using $S = CT$ where C is a constant). The Action for the first example is then of the format,

$$A_{\pm} = A_{0\pm}/\Lambda = \{\pm(+Yd(X) - Xd(Y)) - CTd(Z) + CZd(T)\}/\Lambda \tag{6.217}$$

When the number of minus signs in the quadratic form is zero (index or signature 0), and the exponents are $n = 4, p = 2$, define

$$\lambda_0 = (X^2 + Y^2 + Z^2 + C^2T^2), \quad and \;\; \Lambda = \lambda_0^2 \tag{6.218}$$

The coefficients of the scaled Hopf 1-form can be put into correspondence with the classic vector and scalar potentials, $[\mathbf{A}, \phi]$, using S = CT where C is a constant. The Action for the first example is then of the format,

$$A_{\pm} = A_{0\pm}/\lambda_0^2 = \{\pm(+Yd(X) - Xd(Y)) - CTd(Z) + CZd(T)\}/\lambda_0^2 \tag{6.219}$$

For this choice it is remarkable that the derived 2-form has coefficients ($\mathbf{E}$ and $\mathbf{B}$) that are proportional to the same Hopf Map. The adjoint 1-form generated from one Hopf map has a limit set which is another Hopf map. Using the minus ambiguity (parity) sign, leads to the classic result that $\mathbf{E}^2 = C^2\mathbf{B}^2$, but with the not-usual result that the $\mathbf{E}$ field is anti-parallel to the $\mathbf{B}$ field (If the positive ambiguity (parity) sign is used, the $\mathbf{E}$ and $\mathbf{B}$ fields are parallel.):

$$F = dA\,,\; \mathbf{B} = curl\mathbf{A},\; \mathbf{E} = -grad\phi - \partial\mathbf{A}/\partial T \tag{6.220}$$

$$\mathbf{B} = [2(CTY + XZ), -2(-YZ + CTX), (-X^2 - Y^2 + Z^2 + (CT)^2)](2/\lambda_0^3) \tag{6.221}$$

$$\mathbf{E} = [-2(CTY + XZ), 2(-YZ + CTX), -(-X^2 - Y^2 + Z^2 + (CT)^2)](2C/\lambda_0^3) \tag{6.222}$$

It is natural to ask if these $\mathbf{E}$ and $\mathbf{B}$ fields admit a Lorentz symmetry constitutive constraint such that vacuum state is charge current free. Recall that a constitutive constraint is a relationship between a 2-form, F, and a 2-form density G, such that the coefficients of $G(\mathbf{D},\mathbf{H})$ are related to the coefficients of $F(\mathbf{E},\mathbf{B})$. A Lorenz vacuum condition implies that the fields are solutions of the vector wave equation. The question becomes, "If is presumed that $\mathbf{D} = \varepsilon\mathbf{E}$ and $\mathbf{B} = \mu\mathbf{H}$, do the Maxwell Ampere equations generate a zero 3 form of charge current? ". Direct computation of the index zero Hopf 1-form indicates that $dG = J \neq 0$, unless $\varepsilon\mu C^2 + 1 = 0$. Hence the scaled Hopf Action, where the scaling is of signature zero, does **not** describe a charge current free vacuum, for real positive values of ε, μ, and C. On the other hand, if it is presumed that the domain is such that say μ,or ε, is negative, then the Hopf Map, scaled as above, does generate charge-current free wave solutions. Negative ε appears to hold in metals and the Earth's ionosphere. Recent announcements indicate constructions that yield negative μ. [Physics Today May 2000]. However, for situations where ε or μ are negative, the Hopf wave solutions imply that the Spin angular momentum $A\hat{}G$ is not a deformation invariant (hence Spin angular momentum of the field is not conserved.)

6.2.3. Electromagnetism of signature one Hopf 1-forms

When the number of minus signs in the quadratic form is one (index or signature 1), and the exponents are n = 4, p = 2, define

$$\lambda_1 = (x^2 + y^2 + z^2 - C^2t^2), \quad and \quad \Lambda = \lambda_1^2 \tag{6.223}$$

(lower case letters will be used for Index one Hopf 1-forms). For his choice, it is remarkable that the derived 2-form has coefficients ($\mathbf{E}$ and $\mathbf{B}$) that are proportional to different Hopf Maps. The Action 1-form is the same as above, but with a different denominator. This fact leads to the classic result that $\mathbf{E}^2 = C^2\mathbf{B}^2$, but now the $\mathbf{E}$ field is not collinear with the $\mathbf{B}$ field. Using the negative ambiguity (parity) sign leads to the fields:

$$F = dA\,,\ \mathbf{B} = curl\mathbf{A},\ \mathbf{E} = -grad\phi - \partial\mathbf{A}/\partial T \tag{6.224}$$

$$\mathbf{B} = [2(Cty + xz), -2(-yz + Ctx), (-x^2 - y^2 + z^2 - (Ct)^2)](2/\lambda_1^3) \tag{6.225}$$

$$\mathbf{E} = [2(Cty - xz), 2(-yz - Ctx), -(-x^2 - y^2 + z^2 + (Ct)^2)](2C/\lambda_1^3) \tag{6.226}$$

The Spin current density for this non-transverse wave example is evaluated as:

$$
\begin{aligned}
\text{Topological Spin } \mathbf{S}_4 &= [S_x, S_y, S_z, \sigma]\,(2/\mu)/\lambda_1^5 && (6.227)\\
S_x &= x(3\lambda_1^2 - 4y^2 - 4x^2) && (6.228)\\
S_y &= y(3\lambda_1^2 - 4y^2 - 4x^2) && (6.229)\\
S_z &= z(\lambda_1^2 - 4y^2 - 4x^2) && (6.230)\\
\sigma &= t(\lambda_1^2 - 4y^2 - 4x^2) && (6.231)\\
\lambda_1 &= -c^2t^2 + x^2 + y^2 + z^2 && (6.232)
\end{aligned}
$$

and has zero divergence. The Torsion current may be evaluated as

$$Torsion : \mathbf{T}_4 = -[x, y, z, t]2c/\lambda_1^4. \tag{6.233}$$

and has a non-zero divergence equal to the second Poincare invariant

$$Poincare\ 2 = -2\mathbf{E} \circ \mathbf{B} = +8c/\lambda_1^4. \tag{6.234}$$

The solution has magnetic helicity as $\mathbf{A} \circ \mathbf{B} \neq 0$ and is radiative in the sense that the Poynting vector, $\mathbf{E} \times \mathbf{H} \neq 0$.

Independent from any other constraints, it is possible to construct the 3-form of Topological Torsion, and its exterior differential defined as Topological Parity. The Topological parity can be either positive, zero, or negative. For the example Hopf 1-form given above (using the negative ambiguity sign), the Topological Torsion is represented to within a factor by a position vector $[-x, -y, -z, -t]$ inbound in 4 dimensions, and having a negative divergence or parity. If the positive sign of the ambiguity factor is changed, then the parity of the form changes sign. For example, for the 1-form, $A = A_{(1+)}/\lambda = \{+yd(x) - xd(y) - Ctd(z) + zCd(t)\}/\lambda_1^2$, the 4 -form of topological parity is positive, and the topological torsion is represented by an outbound position vector (to within a factor).

Similar to the investigation described above for the index zero Hopf vectors, it is natural to ask if these $\mathbf{E}$ and $\mathbf{B}$ fields admit a Lorentz symmetry constitutive constraint such that vacuum state is charge current free. Again, such a condition implies that the fields are solutions of the vector wave equation. Direct computation of the Maxwell Ampere equations indicates that $dG = J = 0$ if the phase velocity constraint vanishes, $\varepsilon\mu C^2 - 1 = 0$. Hence the scaled Hopf Action, where the scaling is of index one, **does** describe a charge current free vacuum, for real positive values of ε, μ, and C.

It is some interest to give the charge current solutions to show how the "phase factor " $(\varepsilon\mu C^2 - 1) \Rightarrow 0$ establishes the vacuum charge free conditions. The example results for the components of the current density are: (note $\rho = 0$):

$$J^x = -(yx^2 + yz^2 + 5yC^2t^2 - 6zCtx + y^3)(\varepsilon\mu C^2 - 1)4/\lambda_1^3 \tag{6.235}$$

$$J^y = (x^3 + xy^2 + xz^2 + 5xC^2t^2 + 6zCty)(\varepsilon\mu C^2 - 1)4/\lambda_1^3 \tag{6.236}$$

$$J^z = -(2x^2 + 2y^2 - z^2 + C^2t^2)(\varepsilon\mu C^2 - 1)8Ct/\lambda_1^3 \tag{6.237}$$

It is conjectured that fluctuations of the "perfect" vacuum phase relations, where $\varepsilon\mu C^2 - 1 \neq 0$, are associated with ZPF. Note that there are possible charge-current free (singular wave solutions) that are governed by curves in space time. There curve are generated by the intersection of the three surfaces created by setting each of the coefficients of the current density equal to zero. These solutions are valid for any phase velocity and could be a source of "needle" radiation.

The solution given above is not free of Topological Torsion, $A\hat{}F$, and there is a non-zero value of the second Poincare invariant, $\mathbf{E} \cdot \mathbf{B} \neq 0$. However, the Spin 3-form $A\hat{}G$ is also non-zero, but it has, subject to the phase constraint, a zero 4-divergence. (The first Poincare invariant is zero.) The divergence of the Spin 3-form, has 2 parts. The first part is twice the conventional Lagrange density of the fields, $(\mathbf{B} \cdot \mathbf{H} - \mathbf{D} \cdot \mathbf{E})$. The second part is the interaction between the potentials and the charge currents, $(\mathbf{A} \cdot \mathbf{J} - \rho\phi)$. When the divergence of the 3-form is zero, then the closed integrals of Topological Spin are deformation invariants, and have closed integrals with rational (quantized) ratios. That is, with regard to any singly parametrized vector field, V, describing an evolutionary process,

$$\begin{aligned} L_{(\beta V)} \int_{z3} (A\hat{}G) &= \int_{z3} i(\beta V) d(A\hat{}G) + \int_{z3} d(i(\beta V) A\hat{}G) \\ &= 0 + 0 \supset \textit{evolutionary invariance.} \end{aligned} \tag{6.238}$$

The function β is an arbitrary deformation parameter.

6.2.4. Twistors composed by superposing two index 1 Hopf 1-forms

By superposing (adding or subtracting) two different, index 1, Hopf 1-forms (which will be shown below to be equivalent to a Penrose twistor solution) it is possible to construct a vacuum (charge current free wave) solution to the Maxwell system, subject to the constraint that the phase speed satisfies the phase velocity equation, $(\varepsilon\mu C^2 - 1) = 0$.

As an example consider another Hopf 1-form of signature 1 formulated as

$$A = \{Ctd(x) + zd(y) - yd(z) - xCd(t)\}/\lambda_1^2 \tag{6.239}$$

Similar formulas for the field intensities can be determined as above. Note that the parity of the Hopf forms to be superposed can be the same or different. If the parity of the two superposed Hopf 1-forms are opposite, then without consideration of the phase constraint, the Topological Torsion of the "twistor" 1-form vanishes, $A\hat{}F = 0$. Yet the quantized topological spin3-form $A\hat{}G$ does not vanish, and moreover, subject to the phase constraint, the closed integrals of the Spin 3 form are conserved. This result implies that such a construction yields "quantized" values for the Spin integrals. These formulations can be compared with the Penrose twistor definitions in terms of differential forms [Penrose 1999]

6.2.5. Self dual solutions

It is possible to construct a two-form G (without using the Lorentz vacuum constitutive definitions) in terms of two arbitrary functions, α and β, from the dual relations:

$$G = i(*d\alpha)\hat{}i(*d\beta)Vol = i(*d\alpha)\hat{}i(*d\beta)dx\hat{}dy\hat{}dz\hat{}dt. \tag{6.240}$$

The functions α and β used in the dual construction are not required to be solutions of the Bateman condition. However, the resulting "self-dual" field excitations are **not** the same as those generated by the Bateman method, unless the functions also satisfy the Bateman conditions of complex collinearity. In the self dual formulas the * operator is the Hodge * operator with respect to the Lorentz metric modified by the impedance of free space. The resulting self-dual excitations constructed from the two arbitrary functions indeed satisfy the Maxwell-Ampere equations, in virtue of the Maxwell-Faraday equations and the dispersion relation. The construction yields:

$$\mathbf{H} = \sqrt{-1}/\mu c(\partial\alpha/\partial t)\nabla\beta - (\partial\beta/\partial t)\nabla\alpha \quad and \quad \mathbf{D} = -\sqrt{-1}\varepsilon/c\nabla\alpha \times \nabla\beta. \tag{6.241}$$

The self-dual construction, however, implies a chiral (non-Lorentz) constitutive relation of the type $\mathbf{D} = -[\gamma] \circ \mathbf{B}$ and $\mathbf{H} = [\gamma^{\dagger}] \circ \mathbf{E}$, and will not be considered further in this article.

6.2.6. References

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