1. PROJECTIVE CARTAN-FINSLER CONNECTIONS

1.0.1. R. M. Kiehn, 1990 Lecture notes

Cartan techniques should be applied to the engineering sciences as well as to mathematical physics. The projectivized (Finsler) spaces with their homogeneity of degree 1 properties are the perfect vehicle for making advances in our understanding of irreversible thermodynamics.

There are many ways to understand Fiber Bundles and the theory of Connections, but the utility of the theorems are most easily deduced from the moving Frame method — Cartan's Repere Mobile. By using the moving Frame method, the (matrix of) curvature two forms can be deduced algebraically, and its closure can be proved without the extra differentiation implied (or required) by the Bianchi equations. Moreover there appears to be a third equation of structure on projective domains, not usually stated in the literature of affine domains.

To demonstrate the power of the method of the Repere Mobile. start with the idea of a position vector, $\mathbf{R}$, from an an arbitrary origin o to the point of interest, p, in a domain of variables $\{x^k, y^s\}$. At (any) point p assume the existence of a basis Frame $\mathbb{F}$ of functions with non-zero determinant. The columns of the Frame form a linearly independent set of basis vectors, but for the present, these basis vectors are not constrained by any special group structure on the set. A simple explanation of a base and a fiber is accomplished by partitioning the frame matrix into a linear independent set of columns - which can be put into correspondence with tangent and normal fields. The existence of the inverse matrix, of course, guarantees that the differential of every basis vector is decomposable into a linear combination of the original elements of the basis set, independent of the partition. The matrix of coefficients of this linear expansion defines the (right) Cartan matrix of connection 1-forms, $\mathbb{C}_r$,

$$d\mathbb{F} = \mathbb{F} \circ \mathbb{C}_r = \mathbb{F} \circ \{\mathbb{F}^{-1} \circ d\mathbb{F}\} = \mathbb{F} \circ \{-d\mathbb{F}^{-1} \circ \mathbb{F}\}$$

over the domain of support for the basis frame (where $\mathbb{F}^{-1}$ exists). (An alternate development would use the left Cartan matrix representation, $d\mathbb{F} = \mathbb{C}_l \circ \mathbb{F}$). Without this technique, the formulation of Cartan's structure equations appear as if they were plucked out of thin air. By use of the Frame matrix technique it appears that it is the group structure of the Cartan matrix, not the Frame, which has the most physical impact.

It is convenient to partition the (arbitrary) basis frame $\mathbb{F}$ in terms of the *associated* (horizontal, interior, coordinate or transversal) vectors, $\mathbf{e}_k$, and the *adjoint* (normal, exterior, parametric or vertical) field, $\mathbf{n}_p$,

$$\mathbb{F} = [\mathbf{e}_k, \mathbf{n}] = [\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{n}].$$

The corresponding Cartan matrix has the partition,

$$d\mathbb{F} = d \begin{bmatrix} e_1^1 & e_2^1 & ... & n^1 \\ e_1^2 & e_2^2 & ... & n^2 \\ ... & ... & ... & ... \\ e_1^n & e_2^n & ... & n^n \end{bmatrix} = \mathbb{F} \circ \mathbb{C} = \mathbb{F} \circ \begin{bmatrix} \Gamma_1^1 & \Gamma_2^1 & ... & \gamma^1 \\ \Gamma_1^2 & \Gamma_2^2 & ... & \gamma^2 \\ ... & ... & ... & ... \\ h_1 & h_2 & ... & \Omega \end{bmatrix}$$

The Cartan matrix, $\mathbb{C}$, is a matrix of differential 1-forms which can be evaluated explicitly from the functions that make up the basis frame if they admit first partial derivatives. Moreover, the differential of the position vector can be expanded in terms of the same basis frame and a set of Pfaffian 1-forms:

$$d\mathbf{R} = \mathbb{I} \circ \left| \begin{matrix} d\mathbf{x} \\ d\mathbf{y} \end{matrix} \right\rangle = \mathbb{F} \circ \mathbb{F}^{-1} \circ \left| \begin{matrix} d\mathbf{x} \\ d\mathbf{y} \end{matrix} \right\rangle = \mathbb{F} \circ \left| \begin{matrix} \boldsymbol{\sigma} \\ \boldsymbol{\omega} \end{matrix} \right\rangle,$$

where the vector $\left| \begin{matrix} \boldsymbol{\sigma} \\ \boldsymbol{\omega} \end{matrix} \right\rangle$ is a vector of 1-forms that can be computed explicity and algebraically.

By the Poincare lemma, it follows that

$$dd\mathbf{R} = d\mathbb{F}\hat{} \left| \begin{matrix} \boldsymbol{\sigma} \\ \boldsymbol{\omega} \end{matrix} \right\rangle + \mathbb{F} \circ \left| \begin{matrix} d\boldsymbol{\sigma} \\ d\boldsymbol{\omega} \end{matrix} \right\rangle = \mathbb{F} \circ \{\mathbb{C}\hat{} \left| \begin{matrix} \boldsymbol{\sigma} \\ \boldsymbol{\omega} \end{matrix} \right\rangle + \left| \begin{matrix} d\boldsymbol{\sigma} \\ d\boldsymbol{\omega} \end{matrix} \right\rangle\} = 0$$

and

$$d\mathbb{F} = d\mathbb{F}\hat{}\mathbb{C} + \mathbb{F}\hat{}d\mathbb{C} = \mathbb{F} \circ \{\mathbb{C}\hat{}\mathbb{C} + d\mathbb{C}\} = 0.$$

As the Frame matrix and the Cartan matrix are partitioned relative to the tangent (or interior) vectors $\mathbf{e}$ and the normal (or exterior) vectors, $\mathbf{n}$, the Poincare lemma breaks up into linearly independent factors, each of which must vanish. The results are:

$$dd\mathbf{R} = \mathbf{e}\{d\,|\boldsymbol{\sigma}\rangle + [\boldsymbol{\Gamma}]\hat{}\,|\boldsymbol{\sigma}\rangle - \omega\hat{}\,|\gamma\rangle\} + \mathbf{n}\{d\omega + \Omega\hat{}\omega + \langle\mathbf{h}|\,\hat{}\,|\boldsymbol{\sigma}\rangle\} = 0$$

$$dd\mathbf{e} = \mathbf{e}\{d[\boldsymbol{\Gamma}] + [\boldsymbol{\Gamma}]\hat{}[\boldsymbol{\Gamma}] + |\gamma\rangle\;\hat{}\;\langle\mathbf{h}|\} + \mathbf{n}\{d\,\langle\mathbf{h}| + \Omega\hat{}\,\langle\mathbf{h}| + \langle\mathbf{h}|\,\hat{}[\boldsymbol{\Gamma}]\} = 0$$

$$dd\mathbf{n} = \mathbf{e}\{d\,|\boldsymbol{\gamma}\rangle + [\mathbf{\Gamma}]\hat{}\,|\boldsymbol{\gamma}\rangle - \Omega\hat{}\,|\boldsymbol{\gamma}\rangle\} + \mathbf{n}\{d\Omega + \Omega\hat{}\Omega + \langle\mathbf{h}|\,\hat{}\,|\boldsymbol{\gamma}\rangle\} = 0$$

By reasons of linear independence, each of the curly bracket factors must vanish, leading to the results on the interior domain (coefficients of $\mathbf{e}$):

$$d\,|\boldsymbol{\sigma}\rangle + [\mathbf{\Gamma}]\hat{}\,|\boldsymbol{\sigma}\rangle = \omega\hat{}\,|\boldsymbol{\gamma}\rangle \equiv |\mathbf{\Sigma}\rangle = \left|\begin{array}{c} \omega\hat{}\gamma^1 \\ \omega\hat{}\gamma^2 \\ \omega\hat{}\gamma^3 \end{array}\right\rangle \text{ the interior torsion vector of dislocation 2-forms.}$$

$$d[\mathbf{\Gamma}] + [\mathbf{\Gamma}]\hat{}[\mathbf{\Gamma}] = -\,|\boldsymbol{\gamma}\rangle\ \hat{}\ \langle\mathbf{h}| \equiv [\mathbf{\Theta}] = \begin{bmatrix} \gamma^1\hat{}h_1 & \gamma^1\hat{}h_2 & \gamma^1\hat{}h_3 \\ \gamma^2\hat{}h_1 & \gamma^2\hat{}h_2 & \gamma^2\hat{}h_3 \\ \gamma^3\hat{}h_1 & \gamma^3\hat{}h_2 & \gamma^3\hat{}h_3 \end{bmatrix} \text{ the matrix of interior curvature}$$

$$\{d\,|\boldsymbol{\gamma}\rangle + [\mathbf{\Gamma}]\hat{}\,|\boldsymbol{\gamma}\rangle = \Omega\hat{}\,|\boldsymbol{\gamma}\rangle \equiv |\mathbf{\Psi}\rangle = \left|\begin{array}{c} \Omega\hat{}\gamma^1 \\ \Omega\hat{}\gamma^2 \\ \Omega\hat{}\gamma^3 \end{array}\right\rangle \text{ the exterior torsion vector of disclination 2-forms}$$

The first two equations are precisely Cartan's equations of structure (on an affine domain).

The last equation appears to be a new equation of structure valid on a projective domain, when $\Omega \neq 0$.

$|\mathbf{\Psi}\rangle$ physically seems to represent a different kind of "torsion" which I am trying to put into correspondence with disclination defects. Recall that Kondo has developed the theory of dislocation defects based on $|\mathbf{\Sigma}\rangle$.

There are also three equations of structure on the exterior domain (coefficients of $\mathbf{n}$) which are given by the constructions:

$$d\omega + \Omega\hat{}\omega = -\,\langle\mathbf{h}|\,\hat{}\,|\boldsymbol{\sigma}\rangle$$

$$d\,\langle\mathbf{h}| + \Omega\hat{}\,\langle\mathbf{h}| = -\,\langle\mathbf{h}|\,\hat{}[\mathbf{\Gamma}]$$

$$d\Omega + \Omega\hat{}\Omega = \theta = -\,\langle\mathbf{h}|\,\hat{}\,|\boldsymbol{\gamma}\rangle \text{ the exterior curvature 2-forms}$$

1.1. COMMENTS

A remarkable result (to me) of this construction is the fact that the matrix of interior curvature 2-forms, $[\boldsymbol{\Theta}]$, can be constructed in two ways. The classical method utilizes differential processes $\{d[\boldsymbol{\Gamma}] + [\boldsymbol{\Gamma}]\hat{}[\boldsymbol{\Gamma}]\}$, while the second method is purely algebraic $\{- |\boldsymbol{\gamma}\rangle \;\hat{}\; \langle \mathbf{h}|\}$. The order of partial derivatives contained in the algebraic (exterior) expression for the interior curvature $\{- |\boldsymbol{\gamma}\rangle \;\hat{}\; \langle \mathbf{h}|\}$ is one less than the classic expression built on the connection coefficients, $\{d[\boldsymbol{\Gamma}] + [\boldsymbol{\Gamma}]\hat{}[\boldsymbol{\Gamma}]\}$.

Exterior differentiation of the matrix of interior curvature 2-forms yields:

$$d[\boldsymbol{\Theta}] = -d\,|\boldsymbol{\gamma}\rangle \;\hat{}\; \langle \mathbf{h}| = (-\,|d\boldsymbol{\gamma}\rangle \;\hat{}\; \langle \mathbf{h}|) + (|\boldsymbol{\gamma}\rangle \;\hat{}\; \langle d\mathbf{h}|) =$$

$$([\boldsymbol{\Gamma}]\hat{}\,|\boldsymbol{\gamma}\rangle \;\hat{}\; \langle \mathbf{h}|) - (\Omega\hat{}\,|\boldsymbol{\gamma}\rangle \;\hat{}\; \langle \mathbf{h}|) - (|\boldsymbol{\gamma}\rangle \;\hat{}\,\Omega\hat{}\,\langle \mathbf{h}|) - (|\boldsymbol{\gamma}\rangle \;\hat{}\,\langle \mathbf{h}|\,\hat{}[\boldsymbol{\Gamma}]) = 0$$

The fundamental result is that the matrix of 2-forms that forms the interior curvature matrix is closed!

It is important to note that due to the partition, the exterior curvature is a closed (in this example a scalar valued) 2-form $\theta = -\,\langle \mathbf{h}|\,\hat{}\,|\boldsymbol{\gamma}\rangle$ with

$$d\theta = -\,\langle d\mathbf{h}|\,\hat{}\,|\boldsymbol{\gamma}\rangle + \langle \mathbf{h}|\,\hat{}\,|d\boldsymbol{\gamma}\rangle = +\Omega\hat{}\,\langle \mathbf{h}|\,\hat{}\,|\boldsymbol{\gamma}\rangle + \langle \mathbf{h}|\,\hat{}[\boldsymbol{\Gamma}]\hat{}\,|\boldsymbol{\gamma}\rangle - \langle \mathbf{h}|\,\hat{}[\boldsymbol{\Gamma}]\hat{}\,|\boldsymbol{\gamma}\rangle + \langle \mathbf{h}|\,\hat{}\Omega\hat{}\,|\boldsymbol{\gamma}\rangle = 0.$$

Both the exterior and the interior curvature 2-forms can be matrix valued depending upon the partition of the Frame. Each curvature matrix exhibits a set of similarity invariants deduced from the coefficients of the Cayley-Hamilton characteristic polynomial. It would appear therefore that their are two species of Chern characteristic classes that can be constructed from the Cayley-Hamilton polynomial similarity invariants.

If (in the example) the projective Cartan matrix is constrained to be euclidean, then $\Omega = 1$, and both $\mathbf{h} = 0$,and $\boldsymbol{\gamma} = 0$. Hence both the interior and the exterior curvature vanish. Indeed, then both types of torsion 2-forms vanish.

On the otherhand, if the Cartan matrix is anti-symmetric (as it must be for an orthonormal frame matrix) then $\Omega = 0$, and $\boldsymbol{\gamma} = -\mathbf{h}$. Hence, the exterior curvature vanishes, and $|\boldsymbol{\Psi}\rangle = 0$, but the domain could support interior curvature and dislocation torsion 2-forms, $|\boldsymbol{\Sigma}\rangle \neq 0$.

If the Cartan matrix is left affine, then $\mathbf{h} = 0, \Omega = 1$. The interior and exterior domains are flat, but the structure could admit both forms of torsion 2-forms.