Projective Frames and Cartan's Structural Equations

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1. Introduction

In this article, Cartan's concept of a projective basis frame of functions, **F**, a vector valued set of 1-forms ϖ^σ (often called a Vierbein) and a matrix valued set of 1-forms, ω_β^α, (often called a connection) will be defined and constructed on a manifold using the technique of matrices and linear algebra. The algebraic method depends upon the Whitney theorem that any manifold can be embedded in a euclidean space of suitably higher dimension (and not exceeding 2n+1). It is a remarkable feature of the method that the matrix of curvature 2-forms, the two vectors of torsion 2-forms, and the structural equations on the partitioned space can be created both differentially or algebraically. Hopefully, the non-tensor (but tensor equivalent) matrix method will enable the appreciation of the Cartan concepts by an audience of engineers and applied scientists, whose skills with tensor analysis may be somewhat limited. Moreover the algebraic method demonstrates some new features that are often overlooked by use of the tensor methods.

In Cartan's early papers it was apparent that the theory of Lie groups led to certain structural equations that had a basis in differential geometry. These structural equations (Maurer) were necessary conditions that the Lie system of differential equations could be integrated. The structural equations are often written in the form,

$$d\varpi^\sigma + \omega_\beta^\sigma \varpi^\beta = 0 \tag{1.1}$$

and

$$d\omega_\alpha^\sigma + \omega_\beta^\sigma \omega_\alpha^\beta = 0. \tag{1.2}$$

for basis frames that are orthonormal.

A more general representation would be of the form

$$d\varpi^{\sigma} + C^{\sigma}_{\beta}\varpi^{\beta} = 0 \tag{1.3}$$

and

$$dC^{\sigma}_{\alpha} + C^{\sigma}_{\beta}C^{\beta}_{\alpha} = 0. \tag{1.4}$$

where the Cartan matrix of connections is not necessarily related to an orthonormal basis frame. (The matrix ω^{β}_{α} is anti symmetric, while the matrix C^{β}_{α} is not antisymmetric.)

These equations represent, in essence, a closure condition on the elements of the Vierbein and the Connection, a condition that implies that the differentials of the each set are composed of algebraic combinations of the set.

Applying the method to Riemannian geometries, where the connection of Levi-Cevita can be uniquely constucted in terms of the differentials of the metric, it became apparent that for spaces with non-zero Gauss Curvature, the structural equations above could have non zero Right Hand Sides. To a physicist, it is as if a source term were added (out of the blue) to the second of the equations of structure above, to yield

$$dC^{\sigma}_{\alpha} + C^{\sigma}_{\beta}C^{\beta}_{\alpha} = \Theta^{\sigma}_{\beta}. \tag{1.5}$$

Where did the source terms come from? The now classic interpretation is that they (the matix of curvature 2-forms, Θ^{σ}_{α}) come from the curvature of the manifold. If the manifold was flat, then the curvature 2-forms vanish. So far so good. However in the early days the first of the equations of structure above (the Vierbein equations) remained with out "source terms" on the RHS. Then "out of the blue again" Cartan, about 1922, conjectured that systems should be investigated where

$$d\varpi^{\sigma} + C^{\sigma}_{\beta}\varpi^{\beta} = \Sigma^{\sigma}. \tag{1.6}$$

The only trouble is that although visual descriptions of curvature are relatively easy to construct. (the spherical surface has curvature), visual descriptions of the vector of Torsion 2-forms has not been so easy to come by. The work of Eisenhart and Cartan (1925 - 1927) focused attention on the importance of manifolds linearly connected by a transitive group, and extended the work of Finsler to consider connections that were not generated by differentiations of the metric. Where Riemannian spaces admit a unique connection, and can have curvature, their induced Levi-Cevita connections are torsion free. The connections of the Levi-Cevita type and the affine type are often distinguished by the symbol Γ, such that for these systems the structural equations are written in the form

$$d\varpi^{\sigma} + \Gamma^{\sigma}_{\beta}\varpi^{\beta} = \Sigma^{\sigma} \tag{1.7}$$

and

$$d\Gamma^{\sigma}_{\ \alpha} + \Gamma^{\sigma}_{\beta}\Gamma^{\beta}_{\ \alpha} = \Theta^{\sigma}_{\beta}. \tag{1.8}$$

The symmetric part of $\Gamma^{\sigma}_{\ \alpha}$ is identified with the Christoffel connection and the antisymmetric parts of $\Gamma^{\sigma}_{\ \alpha}$ are identified with the "Torsion tensor". For the Affine connections, an anti-symmetric part is required to produce Σ^{σ}.

The selection of a transitive group connecting points on manifolds, when combined with the idea of preserving parallelism, leads to **two** possible connections. Whereas the Levi-Cevita connection, generated from a symmetric metric, enjoys a symmetry (in the two lower indices) the parallelism generated by transitive groups yields connections which are not symmetric. Cartan introduced the concept of (+) parallelism, and (-) parallelism, for the two asymmetric connections, and the concept of (0) parallelism for the symmetric connection.

1.1. Some speculations

Just how this relates the two components of the General Linear group is not clear to me at this time. However it is known that the two components are not connected. To be a group the determinant must not be zero. But the determinant could be plus or minus. Only the plus determinant group is near the identity (which has a positive determinant). The negative determinant cases must somehow be related to reflections. Or note that as the product of two minus det matrices gives a plus det matrix, the maybe the minus determinant cases are spinors.

For a physicist, the Cartan concept of plus and minus parallelism immedately suggests the conjecture that the two connections are (somehow) associated with the concept of left and right handedness, as in the propagation of polarized light. Another conjecture, that is exhibited by certain examples described below, presumes that the two connections are related to the differences between particle propagation versus wave propagation.

2. The Cartan method of Projective Frame Fields

2.1. Construction of the Projective Connection and Vierbein

For purposes of demonstration and immediate interest, the method will be applied to euclidean R4 which has a base coordinate representation of space-time {x,y,z,t}. The projective basis frame will be partitioned into space like tangent (or interior) vector fields, $\mathbf{e}_k$, and a normal field (or exterior) vector field, $\mathbf{n}$. On such a 4 dimensional space there exist many basis frames for describing linear systems. The elements of a matrix for the general linear group has 16 elements (functions of x,y,z,t). Given some functional form for this matrix, the domain of [x,y,z,t] can be constrained such that only those regions where the determinant of the matrix is non-zero are to be considered. Such a constraint defines a Projective Frame Field, $[\mathbf{F}]$. (The compliment to the project Frame will be discussed later). However, the essential assumption is that the Projective Basis Frame of functions has a global inverse (modulo the singular points). From this constraint,

$$[\mathbf{F}] \circ \left[\mathbf{F}^{-1}\right] = [\mathbf{1}], \tag{2.1}$$

it is possible to differentiate and apply the Leibniz rule to obtain

$$d[\mathbf{F}] = [\mathbf{F}] \circ [\mathbf{C}] = [\mathbf{F}] \circ \{-\left[d\mathbf{F}^{-1}\right] \circ [\mathbf{F}]\}. \tag{2.2}$$

The important result is to note that the differentials of any vector of the basis frame $[\mathbf{F}]$ is a linear combination of the elements of the basis frame, with the coefficients of linearity given by the Cartan matrix, $[\mathbf{C}]$. This concept of differential closure for elements of the Projective frame $[\mathbf{F}]$ is the key to the Cartan developoment. The matrix of linear connections $[\mathbf{C}]$ is immediately computable from the formula (assuming that the projective frame is C1)

$$[\mathbf{C}] = \{-\left[d\mathbf{F}^{-1}\right] \circ [\mathbf{F}]\}. \tag{2.3}$$

The Cartan connection matrix is now a matrix of 1-forms, $[\mathbf{C}]$, constructed from the differentials of the functions that make up the Projective Frame field matrix inverse, and every element is functionally well defined. The important point is, again, that the differentials of any basis vector in the Cartan system is a linear combination of the basis vectors; i.e., the process of differentiation is closed.

Cartan also assumed that in this space which supports a global Projective basis frame, there exists an origin and a position vector $\mathbf{R}$ to a point in the domain. Expand the differential position vector in terms of the basis frame to obtain

$$d\mathbf{R}(x,y...)=[\mathbf{1}]\circ\left|\begin{array}{c}dx\\dy\\dz\\dt\end{array}\right\rangle=[\mathbf{F}]\circ\left[\mathbf{F}^{-1}\right]\circ\left|\begin{array}{c}dx\\dy\\dz\\dt\end{array}\right\rangle=[\mathbf{F}]\circ\left|\begin{array}{c}\sigma^x\\\sigma^y\\\sigma^z\\\omega\end{array}\right\rangle. \tag{2.4}$$

The formula then describes the genesis of the Vierbein 1-forms:

$$\left|\varpi^\beta\right\rangle=\left[\mathbf{F}^{-1}\right]\circ\left|\begin{array}{c}dx\\dy\\dz\\dt\end{array}\right\rangle=\left|\begin{array}{c}\sigma^x\\\sigma^y\\\sigma^z\\\omega\end{array}\right\rangle \tag{2.5}$$

The mystery of the Vierbein 1-forms is resolved.

A physical mechanism for defining the "position" or the existence of an absolute origin is lacking in the Cartan assumption. Furthermore it is not clear that the "origin" is located in the space of interest, but might actually be a point in some higher dimensional space, in the sense of a projection. For example, consider the projective plane where the perspective point is not in the plane at all. (This case will be studied in detail below.)

The column vector of differential 1-forms, $\left|\sigma^k\right\rangle$, representing the differential position vector in terms of the basis frame, is well defined algebraically in terms of the matrix product. These Cartan assumptions lead to the exterior differential system on R4:

$$d\left|\mathbf{R}\right\rangle-[\mathbf{F}]\circ\left|\varpi\right\rangle=0 \tag{2.6}$$

$$d\left[\mathbf{F}\right]-[\mathbf{F}]\circ[\mathbf{C}]=0. \tag{2.7}$$

Without the knowledge that the domain is R4, and the fact that the basis frame is a projective basis on R4, it is tough to see how to proceed. Also the question remains, what determines the frame $[\mathbf{F}]$, and what determines the position vector $\mathbf{R}$? However, as R4 is euclidean, everything is well defined, and the Poincare lemma can be applied

Suppose that the Cartan differentials are exact, and all of the functions in the matrices are differentiable. Then take the exterior derivative of both equations to yield, (using the Poincare lemma which states that $dd\left|\mathbf{R}\right\rangle=0$, and $dd\left[\mathbf{F}\right]=0$):

$$[\mathbf{F}]\left\{d\left|\sigma\right\rangle + [\mathbf{C}]\,\hat{}\,\left|\sigma\right\rangle\right\} = 0 \tag{2.8}$$

$$[\mathbf{F}]\left\{d\,[\mathbf{C}] + [\mathbf{C}]\,\hat{}\,[\mathbf{C}]\right\} = 0. \tag{2.9}$$

As $[\mathbf{F}]$ is invertible (criteria of linear independence of the projective basis vectors), these equations imply that

$$\left\{d\left|\sigma\right\rangle + [\mathbf{C}]\,\hat{}\,\left|\sigma\right\rangle\right\} = 0 \tag{2.10}$$

$$\left\{d\,[\mathbf{C}] + [\mathbf{C}]\,\hat{}\,[\mathbf{C}]\right\} = 0. \tag{2.11}$$

which are Cartan's structural equations for a flat space without torsion. Note that the matrix dot $\circ$ product symbol has been replace by the wedge product symbol $\hat{}$ to remind one that the matrix elements are 1-forms, and the standard matrix product of matrix elements has to preserve the exterior product.

2.2. Algebraic partitions and the Structural Equations.

To exemplify the algebraic methods consider that the Projective basis frame can be written in partitioned form as:

$$[F] = \begin{bmatrix} \mathbf{e}_\alpha & \mathbf{e}_\beta & .. & \mathbf{n} \end{bmatrix} = \begin{bmatrix} \mathbf{e}_\alpha^1 & \mathbf{e}_\beta^\alpha & ... & \mathbf{n}^1 \\ \mathbf{e}_\alpha^2 & \mathbf{e}_\beta^\beta & ... & \mathbf{n}^2 \\ ... & ... & ... & ... \\ \mathbf{e}_\alpha^n & ... & ... & \mathbf{n}^n \end{bmatrix}, \tag{2.12}$$

and the differential position vector expanded in terms of the partitioned frame as:

$$d\mathbf{R} = [F] \circ \left| \begin{matrix} \sigma^\alpha \\ \sigma^\beta \\ ... \\ \omega \end{matrix} \right\rangle = \begin{bmatrix} \mathbf{e}_\alpha & \mathbf{e}_\beta & .. & \mathbf{n} \end{bmatrix} \circ \left| \begin{matrix} \sigma^\alpha \\ \sigma^\beta \\ ... \\ \omega \end{matrix} \right\rangle = \mathbf{e}_\alpha\sigma^\alpha + \mathbf{e}_\beta\sigma^\beta + ... + \mathbf{n}\omega \tag{2.13}$$

(The method of constructing or defining the basis frame is deferred until the next section.) This explicit example considers only one normal or exterior basis vector, $\mathbf{n}$, but there could as many as k+1. The general and corresponding partition of the Cartan connection matrix $[C]$ of 1-forms becomes

$$d\,[F] = [F] \circ [C] = [F] \circ \begin{bmatrix} \Gamma_\alpha^\alpha & \Gamma_\beta^\alpha & ... & \gamma^\alpha \\ \Gamma_\alpha^\beta & \Gamma_\beta^\beta & ... & \gamma^\beta \\ ... & ... & ... & ... \\ h_\alpha & h_\beta & ... & \Omega \end{bmatrix} \tag{2.14}$$

The $k \times k$ submatrix of 1-forms represented by $[\Gamma]$ will be defined as the interior connection coefficients on the subspace. Note that these 1-forms range over the k interior variables, and the p exterior variables (or external parameters). There is still some ambiguity for the choice of the basis frame $[F]$ on the euclidean space, but a common choice is the orthonormal frame of engineering analysis. (Another constructive procedure will be developed below in terms of a projective geometry based in the existence of a global 1-form of Action.)

To proceed, presume that the basis set $[F]$ is given. Then rewrite the structural equations for the euclidean space in terms of the partition. The equations for the exterior differential system can be partitioned into two parts: the first part relating to the k interior vectors (of $k+p$ components)

$$d\mathbf{e}_j = \mathbf{e}_k \Gamma_j^k + \mathbf{n} h_j \tag{2.15}$$

and a second part relating to the p exterior vectors, (in the example that follows, p=1)

$$d\mathbf{n} = \mathbf{e}_k \gamma^k + \mathbf{n}\Omega. \tag{2.16}$$

The differential position vector can be partitioned as

$$d\mathbf{R} = \mathbf{e}_k \sigma^k + \mathbf{n}\omega. \tag{2.17}$$

The coefficients of each of the basis vectors are differential 1-forms with a range of $k+p$.

When the exterior space and the interior space are transversal in the sense that $\mathbf{e}_k \circ \mathbf{n} = 0$, the interpretation of the various coefficients becomes more transparent. The factor Ω represents the change of $\mathbf{n}$ in the direction of $\mathbf{n}$, while the factor γ^k represents the change of $\mathbf{n}$ in the direction of the $\mathbf{e}_k$. The factor h_j represents the change in the $\mathbf{e}_k$ in the direction of $\mathbf{n}$. Note that if the adjoint vector $\mathbf{n}$ is constrained to have no change in the direction of $\mathbf{n}$, then the factor Ω vanishes.

Assuming that the functions that make up the basis vectors and the position vector are C1 differentiable, then the exterior derivative of these three sets of equations must vanish. Most of the features of the theory can be developed by successive application of the exterior derivative to the above equations, followed by algebraic substitution of the closure relations defined by the exterior differential system. As the Frame matrix and the Cartan matrix are partitioned relative to the tangent (or interior, or associated, or horizontal) vectors $\mathbf{e}_k$ and the normal (or exterior, or vertical) vectors, $\mathbf{n}$, the Poincare lemma breaks up into linearly independent factors, each of which must vanish separately. The results are given by structural equations (sums over repeated up and down indices):

$$dd\mathbf{R} = \mathbf{e}_k\{d\left|\boldsymbol{\sigma}^k\right\rangle + [\mathbf{\Gamma}^k_m]\hat{}\,|\boldsymbol{\sigma}^m\rangle + \left|\boldsymbol{\gamma}^k\hat{}\omega\right\rangle\} + \mathbf{n}\{d\omega + \Omega\hat{}\omega + \langle\mathbf{h}_m|\,\hat{}\,|\boldsymbol{\sigma}^m\rangle\} = 0 \quad (2.18)$$

$$dd\mathbf{e}_j = \mathbf{e}_k\{d[\mathbf{\Gamma}^k_j] + [\mathbf{\Gamma}^k_m]\hat{}[\mathbf{\Gamma}^m_j] + \left|\boldsymbol{\gamma}^k\right\rangle\,\hat{}\,\langle\mathbf{h}_j|\} + \mathbf{n}\{d\,\langle\mathbf{h}_j| + \Omega\hat{}\,\langle\mathbf{h}_j| + \langle\mathbf{h}_m|\,\hat{}[\mathbf{\Gamma}^m_j]\} = 0 \quad (2.19)$$

$$dd\mathbf{n} = \mathbf{e}_k\{d\left|\boldsymbol{\gamma}^k\right\rangle + [\mathbf{\Gamma}^k_m]\hat{}\,|\boldsymbol{\gamma}^m\rangle + \left|\boldsymbol{\gamma}^k\hat{}\Omega\right\rangle\} + \mathbf{n}\{d\Omega + \Omega\hat{}\Omega + \langle\mathbf{h}_m|\,\hat{}\,|\boldsymbol{\gamma}^m\rangle\} = 0. \quad (2.20)$$

Each of the bracket {} factors must vanish, for by hypothesis the basis vectors are linearly independent (and non-zero). The outcome (due to the partition) is six structural equations, three related to the interior domain of the partition and three related to the exterior domain of the partition. Each bracket factor is composed of 2-forms.

Consider the three interior structural equations rewritten as

$$d\left|\boldsymbol{\sigma}^k\right\rangle + [\mathbf{\Gamma}^k_m]\hat{}\,|\boldsymbol{\sigma}^m\rangle = \omega\hat{}\left|\boldsymbol{\gamma}^k\right\rangle \equiv \left|\mathbf{\Sigma}^k\right\rangle \quad (2.21)$$

$$d[\mathbf{\Gamma}^k_j] + [\mathbf{\Gamma}^k_m]\hat{}[\mathbf{\Gamma}^m_j] = -\left|\boldsymbol{\gamma}^k\right\rangle\,\hat{}\,\langle\mathbf{h}_j| \equiv \left[\mathbf{\Theta}^k_j\right] \quad (2.22)$$

$$d\left|\boldsymbol{\gamma}^k\right\rangle + [\mathbf{\Gamma}^k_m]\hat{}\,|\boldsymbol{\gamma}^m\rangle = \Omega\hat{}\left|\boldsymbol{\gamma}^k\right\rangle \equiv \left|\mathbf{\Psi}^k\right\rangle \quad (2.23)$$

On the left hand side of each structural equation, given the interior connection coefficients, $[\mathbf{\Gamma}^k_m]$, differential processes are used to construct the vector or matrix arrays of 2-forms. The vector array $\left|\mathbf{\Sigma}^k\right\rangle$ is defined as the translation *affine* torsion 2-forms. These objects have been used to analyze the concept of dislocation defects in crystals. The matrix array $\left[\mathbf{\Theta}^k_j\right]$ is defined as the curvature 2-forms.

These concepts appear in the classical literature of Cartan (and others). What is new from the construction presented herein, is that these arrays of 2-forms also can be computed algebraically, with out the need for another differentiation. Moreover, the third and new interior structural equation yields another vector array of rotational *expansion-twist* torsion 2-forms, $\left|\mathbf{\Psi}^k\right\rangle$. This latter array can represent the concept of disclinations in liquid crystals. Each array of torsion 2-forms depends upon distinct and different 1-forms, ω and Ω. If the 1-form ω vanishes (as it does for all parametrically described surfaces) then there does not exist any two forms of affine torsion. If the 1-form Ω vanishes then there does not exist any two forms of expansion-twist torsion.

One physical objective of this article is to associate the 2-forms $\left|\mathbf{\Sigma}^k\right\rangle$ with shears of (affine) translation of parallel planes, and the 2-forms $\left|\mathbf{\Psi}^k\right\rangle$ with shears of expansion twists. A second physical objective is to associate the structural equation

$$d[\mathbf{\Gamma}_j^k] + [\mathbf{\Gamma}_m^k]\hat{}[\mathbf{\Gamma}_j^m] = -\left|\boldsymbol{\gamma}^k\right\rangle \hat{} \langle\mathbf{h}_j| \qquad (2.24)$$

with an extension of the Einstein field equations that would be valid on non-Riemannian spaces.

$$[\mathbf{G}_j^k] \Longleftarrow d[\mathbf{\Gamma}_j^k] + [\mathbf{\Gamma}_m^k]\hat{}[\mathbf{\Gamma}_j^m] = -\left|\boldsymbol{\gamma}^k\right\rangle \hat{} \langle\mathbf{h}_j| \Rightarrow \left[\mathbf{T}_j^k\right]. \qquad (2.25)$$

An important feature of the Einstein Ansatz is that the divergence of the Ricci tensor for a metric space is zero, hence the divergence of the stress energy tensor must be zero, which is pleasing on physical grounds. What is remarkable herein, is that when the Cartan matrix of connection 1-forms is antisymmetric, (the case of an orthogonal frame field) the exterior derivative of the matrix of curvature 2-forms, $\left[\mathbf{\Theta}_j^k\right]$, vanishes (see below), which implies that both the differential construction of the curvature 2-forms and the algebraic construction of the curvature 2-forms are closed. Locally there exists a set of 1-forms (the potentials) that generate the curvature 2-forms. The topology of the subspace will be dictated by the cohomology of the curvature 2-forms. From the deRham theorems, this closed integrals are quantized (have values whose ratios are rational). This result does not depend upon an interior metric, but does depend upon the group structure of the connection.

In the formula above, on the left is the construction based upon interior geometry of the "connection" and differential processes, and on the right is the stress energy tensor computed algebraically from the exterior features of the embedded system.

The three exterior structural equations are

$$d\omega + \Omega\hat{}\omega = -\langle \mathbf{h}_m | \hat{} | \boldsymbol{\sigma}^m \rangle \equiv L \tag{2.26}$$

$$d\langle \mathbf{h}_j | + \Omega\hat{}\langle \mathbf{h}_j | = -\langle \mathbf{h}_m | \hat{}[\mathbf{\Gamma}_j^m] \equiv \langle \mathbf{J} | \tag{2.27}$$

$$d\Omega + \Omega\hat{}\Omega = -\langle \mathbf{h}_m | \hat{} | \boldsymbol{\gamma}^m \rangle \equiv S \tag{2.28}$$

The physical significance of these structural equations has yet to be determined. However, it is important to recognize that the 2-form S is always exact.

The method developed above indicates that there exist a number of exact 2-form structures. Recall that each exact 2-form is an evolutionary deformable integral invariant, that can be used to establish a topological conservation law. These exact 2-forms can be read off from the structural equations:

$$[\mathbf{\Gamma}_m^k]\hat{}\,|\boldsymbol{\sigma}^m\rangle - \omega\hat{}\,\left|\boldsymbol{\gamma}^k\right\rangle = -d\left|\boldsymbol{\sigma}^k\right\rangle \tag{2.29}$$

$$[\mathbf{\Gamma}_m^k]\hat{}[\mathbf{\Gamma}_j^m] + \left|\boldsymbol{\gamma}^k\right\rangle\,\hat{}\,\langle \mathbf{h}_j | = -d[\mathbf{\Gamma}_j^k] \tag{2.30}$$

$$[\mathbf{\Gamma}_m^k]\hat{}\,|\boldsymbol{\gamma}^m\rangle - \Omega\hat{}\,\left|\boldsymbol{\gamma}^k\right\rangle = -d\left|\boldsymbol{\gamma}^k\right\rangle \tag{2.31}$$

$$\Omega\hat{}\omega + \langle \mathbf{h}_m | \hat{} | \boldsymbol{\sigma}^m \rangle = -d\omega \tag{2.32}$$

$$\Omega\hat{}\langle \mathbf{h}_j | + \langle \mathbf{h}_m | \hat{}[\mathbf{\Gamma}_j^m] = -d\langle \mathbf{h}_j | \tag{2.33}$$

$$\Omega\hat{}\Omega + \langle \mathbf{h}_m | \hat{} | \boldsymbol{\gamma}^m \rangle = -d\Omega \tag{2.34}$$

It is to be observed that the structural equations above represent exterior differential systems that define topological properties of the domain. The objects on the RHS need not be exact, but can have closed components in the sense of deRham that define cohomological properties of the partitions.

3. The Bianchi identities and other conserved 3-forms

An application of the exterior derivative to the equations of the set of section 3 leads to constraints on systems of 3-forms as further necessary conditions. Exterior differentiation of each of the brackets yields the system of 3-form equations:

$$d\,|\mathbf{\Sigma}\rangle + [\mathbf{\Gamma}]\hat{}\,|\mathbf{\Sigma}\rangle = [\mathbf{\Theta}]\,\hat{}\,|\boldsymbol{\sigma}\rangle \tag{3.1}$$

$$d\,[\mathbf{\Theta}] + [\mathbf{\Gamma}]\hat{}\,[\mathbf{\Theta}] = [\mathbf{\Theta}]\,\hat{}\,[\mathbf{\Gamma}] \tag{3.2}$$

$$d\,|\mathbf{\Psi}\rangle + [\mathbf{\Gamma}]\hat{}\,|\mathbf{\Psi}\rangle = [\mathbf{\Theta}]\,\hat{}\,|\gamma\rangle \tag{3.3}$$

with similar expressions for the exterior curvature components of the structural equations (the parts that depend upon Ω,not Γ). The 3-form equations lead to exact 3-form structures and topological deformation invariants in terms of the integrals of

$$[\mathbf{\Gamma}]\hat{}\,|\mathbf{\Sigma}\rangle - [\mathbf{\Theta}]\,\hat{}\,|\boldsymbol{\sigma}\rangle = -d\,|\mathbf{\Sigma}\rangle \tag{3.4}$$

$$[\mathbf{\Gamma}]\hat{}\,[\mathbf{\Theta}] - [\mathbf{\Theta}]\,\hat{}\,[\mathbf{\Gamma}] = -d\,[\mathbf{\Theta}] \tag{3.5}$$

$$[\mathbf{\Gamma}]\hat{}\,|\mathbf{\Psi}\rangle - [\mathbf{\Theta}]\,\hat{}\,|\gamma\rangle = -d\,|\mathbf{\Psi}\rangle \tag{3.6}$$

The equation (4.2) involving $d\,[\mathbf{\Theta}]$ is the equivalent to the Bianchi identities in classical tensor analysis.

The process of repetitive exterior differentiation of these expressions can proceed until the totality of necessary compatibility conditions on a system of N+1 forms is generated. From then on, the process of exterior differentiation leads to no new information.

4. Some Examples: Parametric Surfaces:

Parametric surfaces may be viewed as a map from N-1 parameters into a space (often euclidean) of dimension N. The position vector $\mathbf{R}(u,v)$ to the surface has N components with each component a function of N-1 parameters. The partial derivatives of the position vector form a set of N-1 linearly independent (tangent) vectors $\mathbf{e}_k$ of dimension N. There exists a unique adjoint vector, $\mathbf{N}$, algebraically constructed with components proportional to the (N-1) by (N-1) determinants of the "tangent" vector components. The N contravariant column vectors form a basis frame, $[F]$. No concept of distance (metric) has been established. Each column vector of the basis frame could be multiplied by an arbitrary function. In particular, the adjoint vector, $\mathbf{N}$, could be scaled by an arbitrary function of the parameters, $\mathbf{n} \to \mathbf{N}/\rho(u,v)$. In such a case the determinant of the basis frame becomes a complicated algebraic expression in the components of the tangent vectors. For a Monge surface, the Frame matrix is computed from the position vector, $\mathbf{R}(u,v) = [u, v, Z(u,v)], as$

$$[F] = \begin{bmatrix} 1 & 0 & \partial Z(u,v)/\partial u/\rho(u,v) \\ 0 & 1 & \partial Z(u,v)/\partial v/\rho(u,v) \\ (\partial Z(u,v)/\partial u & \partial Z(u,v)/\partial v & 1/\rho(u,v) \end{bmatrix}, \tag{4.1}$$

The determinant of the Frame matrix is

$$\det[F] = \{1 + (\partial Z(u,v)/\partial u)^2 + (\partial Z(u,v)/\partial v)^2\}/\rho(u,v)$$

and is never zero for real functions, and $\rho(u,v) > 0$. Hence the Cartan matrix exists globally. The 1-forms that make up the differential of the position vector are given by the expression: $\begin{bmatrix} 1 & 0 & \partial Z(u,v)/\partial u/\rho(u,v) \\ 0 & 1 & \partial Z(u,v)/\partial v/\rho(u,v) \\ (\partial Z(u,v)/\partial u & \partial Z(u,v)/\partial v & 1/\rho(u,v) \end{bmatrix}$

$$\left[F^{-1}\right] \circ \left| \begin{matrix} dx \\ dy \\ \ldots \\ dz \end{matrix} \right\rangle \Rightarrow \left| \begin{matrix} \sigma^\alpha \\ \sigma^\beta \\ \ldots \\ \omega \end{matrix} \right\rangle = \left| \begin{matrix} du \\ dv \\ \ldots \\ 0 \end{matrix} \right\rangle.$$

Note that the 1-form, $\omega \Rightarrow 0$, vanishes identically for any parametrized surface (Monge or otherwise). Hence, the affine torsion 2-forms,

$$-\omega\hat{}\left|\boldsymbol{\gamma}^k\right\rangle \equiv \left|\boldsymbol{\Sigma}^k\right\rangle = |\mathbf{0}\rangle$$

vanish for all parametric surfaces. Dislocation defects do NOT admit a description in terms of parametric surfaces.

Ruled surfaces are generated by "straight" lines, and are special cases of parametric surfaces. Therefore, ruled surfaces can have rotational twisted torsion, but not translational affine torsion.

Consider the one form ϖ constructed from the components of the adjoint field: $\varpi = \{N_x dx + N_y dy + N_z dz\}/\rho \equiv B_u(u,v)du + B_v(u,v)dv$. As the pullback form only involves two variables, there always exists an integrating factor such that the components of the adjoint field are proportional to a gradient field. (This result is not true in higher dimensions.)

If the scaling function for the adjoint field is chosen such that

$$\rho(u,v) = [+(N1)^2 + (N2)^2 + ...]^{1/2}$$

then the parametrized surfaces the expansion-twist 1-form vanishes, $\Omega(u,v,du,dv) \Rightarrow 0$. It follows that

$$d\left|\boldsymbol{\gamma}^k\right\rangle + [\boldsymbol{\Gamma}^k_m]\hat{}\,\left|\boldsymbol{\gamma}^m\right\rangle = -\Omega\hat{}\,\left|\boldsymbol{\gamma}^k\right\rangle \equiv \left|\boldsymbol{\Psi}^k\right\rangle \Rightarrow 0.$$

The Torsion 2-forms of the second type, $\left|\boldsymbol{\Psi}^k\right\rangle$, vanish for parametric representations which normalize the surface adjoint field with the quadratic norm. If the surface vector is everywhere quadratically normalizable (without expansion), then parametric methods cannot describe disclination defects.

For the Holder norm, $\rho(u,v) = [+(N1)^4 + (N2)^4 + ...]^{1/4}$, $\Omega(u,v,du,dv)$ is not zero, and the second type of Torsion 2-forms, $-\Omega\hat{}\,\left|\boldsymbol{\gamma}^k\right\rangle \equiv \left|\boldsymbol{\Psi}^k\right\rangle$, can exist.

5. OTHER EXAMPLES TO FOLLOW

5.1. References

Leon Brillouin, "Tensors in Mechanics and Elasticity", Academic Press, NY, 1964 p.93

See
http://www.uh.edu/~rkiehn/pdf/parametric.pdf
for examples of the Repere Mobile used for parametric surfaces.

See
http://www.uh.edu/~rkiehn/pdf/implinor.pdf
for examples of the Repere Mobile used for implicit surfaces.