The Projective Frame

Consider a global 1-form $A = A_k dx^k$ prolonged to a projective space of 1 dimension higher, to yield the 1-form $\omega = A_k dx^k - ds$. Construct the k associated vectors $\mathbf{e}_k$ to the 1-form ω. These vectors satisfy the equations

$$i(\mathbf{e}_k)\omega = 0.$$

These vectors, and their adjoint (which is proportional to the components of the 1-form itself), form a global basis frame on the projective domain of $k+1$ variables. For example,

$$[\mathbb{F}] = \begin{bmatrix} 1 & 0 & \ldots & A_1 \\ 0 & 1 & \ldots & A_2 \\ \ldots & \ldots & \ldots & \ldots \\ A_1 & A_2 & \ldots & -1 \end{bmatrix}$$

has a non-zero determinant

$$\det[\mathbb{F}] = -\{1 + (A_1)^2 + (A_2)^2 + \ldots\}.$$

The non-zero determinant guarantees the existence of a global inverse matrix, $\mathbb{F}^{-1}$. Compute the exterior derivative of the matrix equation, $\mathbb{F} \circ \mathbb{F}^{-1} = \mathbb{I}$ to construct the (right) Cartan matrix of connection 1-forms, $\mathbb{C}_r$,

$$d\mathbb{F} = \mathbb{F} \circ \mathbb{C}_r = \mathbb{F} \circ \{d\mathbb{F} \circ \mathbb{F}^{-1}\} = \mathbb{F} \circ \{-d\mathbb{F}^{-1} \circ \mathbb{F}\}.$$

However, each column vector of the Projective Basis Frame so constructed can be multiplied by an arbitrary non-zero function, so the method produces a large class of globally invertible basis frames.

Partition the (arbitrary) basis frame $\mathbb{F}$ in terms of the *associated* (other words would be horizontal, interior, or transversal) vectors, $\mathbf{e}_k$, and the *adjoint* (other words would be vertical, exterior, or normal) field, $\mathbf{n}_p$,

$$\mathbb{F} = [\mathbf{e}_k, \mathbf{n}] = [\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3; \mathbf{n}].$$

The corresponding Cartan matrix has the partition,

$$d\mathbb{F} = d\begin{bmatrix} e_1^1 & e_2^1 & \ldots & n^1 \\ e_1^2 & e_2^2 & \ldots & n^2 \\ \ldots & \ldots & \ldots & \ldots \\ e_1^n & e_2^n & \ldots & n^n \end{bmatrix} = \mathbb{F} \circ \mathbb{C} = \mathbb{F} \circ \begin{bmatrix} \Gamma_1^1 & \Gamma_2^1 & \ldots & \gamma^1 \\ \Gamma_1^2 & \Gamma_2^2 & \ldots & \gamma^2 \\ \ldots & \ldots & \ldots & \ldots \\ h_1 & h_2 & \ldots & \Omega \end{bmatrix}$$

The Cartan connection matrix, $\mathbb{C}$, is a matrix of differential 1-forms which can be evaluated explicitly from the functions that make up the basis frame if they admit first partial derivatives.

The Structural Equations

Consider a perspective point o as the origin and consider a position vector $\mathbf{R}$ to an arbitrary point p in the space of k+1 dimensions. Expand the differential of the position vector in terms of the same basis frame $\mathbb{F}$ and a set of Pfaffian 1-forms according to the formulas:

$$d\mathbf{R} = \mathbb{I} \circ \left| \begin{matrix} d\mathbf{x} \\ d\mathbf{y} \end{matrix} \right\rangle = \mathbb{F} \circ \mathbb{F}^{-1} \circ \left| \begin{matrix} d\mathbf{x} \\ d\mathbf{y} \end{matrix} \right\rangle = \mathbb{F} \circ \left| \begin{matrix} \sigma \\ \omega \end{matrix} \right\rangle,$$

The domain thereby supports the exterior differential system:

$$d|\mathbf{R}\rangle - \mathbb{F}\circ\left|\begin{matrix}\sigma\\ \omega\end{matrix}\right\rangle = 0$$

and

$$d[\mathbb{F}] - [\mathbb{F}]\circ\mathbb{C} = 0$$

By the Poincare lemma, it follows that

$$dd\mathbf{R} = d\mathbb{F}\wedge\left|\begin{matrix}\sigma\\ \omega\end{matrix}\right\rangle + \mathbb{F}\circ\left|\begin{matrix}d\sigma\\ d\omega\end{matrix}\right\rangle = \mathbb{F}\circ\{\mathbb{C}\wedge\left|\begin{matrix}\sigma\\ \omega\end{matrix}\right\rangle + \left|\begin{matrix}d\sigma\\ d\omega\end{matrix}\right\rangle\} = 0$$

and

$$d\mathbb{F} = d\mathbb{F}\wedge\mathbb{C} + \mathbb{F}\wedge d\mathbb{C} = \mathbb{F}\circ\{\mathbb{C}\wedge\mathbb{C} + d\mathbb{C}\} = 0.$$

As the Frame matrix and the Cartan matrix are partitioned relative to the tangent (or interior) vectors **e** and the normal (or exterior) vectors, **n**, the Poincare lemma breaks up into linearly independent factors, each of which must vanish. The results are:

$$dd\mathbf{R} = \mathbf{e}\{d|\sigma\rangle + [\Gamma]\wedge|\sigma\rangle - \omega\wedge|\gamma\rangle\} + \mathbf{n}\{d\omega + \Omega\wedge\omega + \langle\mathbf{h}|\wedge|\sigma\rangle\} = 0$$

$$dd\mathbf{e} = \mathbf{e}\{d[\Gamma] + [\Gamma]\wedge[\Gamma] + |\gamma\rangle\wedge\langle\mathbf{h}|\} + \mathbf{n}\{d\langle\mathbf{h}| + \Omega\wedge\langle\mathbf{h}| + \langle\mathbf{h}|\wedge[\Gamma]\} = 0$$

$$dd\mathbf{n} = \mathbf{e}\{d|\gamma\rangle + [\Gamma]\wedge|\gamma\rangle - \Omega\wedge|\gamma\rangle\} + \mathbf{n}\{d\Omega + \Omega\wedge\Omega + \langle\mathbf{h}|\wedge|\gamma\rangle\} = 0$$

By reasons of linear independence, each of the curly bracket factors must vanish, leading to the results on the interior domain (coefficients of **e**):

$$d|\sigma\rangle + [\Gamma]\wedge|\sigma\rangle = \omega\wedge|\gamma\rangle \equiv |\Sigma\rangle = \left|\begin{matrix}\omega\wedge\gamma^1\\ \omega\wedge\gamma^2\\ \omega\wedge\gamma^3\end{matrix}\right\rangle \text{ the interior torsion vector of dislocation 2-forms.}$$

$$d[\Gamma] + [\Gamma]\wedge[\Gamma] = -|\gamma\rangle\wedge\langle\mathbf{h}| \equiv [\Theta] = \begin{bmatrix}\gamma^1\wedge h_1 & \gamma^1\wedge h_2 & \gamma^1\wedge h_3\\ \gamma^2\wedge h_1 & \gamma^2\wedge h_2 & \gamma^2\wedge h_3\\ \gamma^3\wedge h_1 & \gamma^3\wedge h_2 & \gamma^3\wedge h_3\end{bmatrix} \text{the matrix of interior curvature 2-forms}$$

$$\{d|\gamma\rangle + [\Gamma]\wedge|\gamma\rangle = \Omega\wedge|\gamma\rangle \equiv |\Psi\rangle = \left|\begin{matrix}\Omega\wedge\gamma^1\\ \Omega\wedge\gamma^2\\ \Omega\wedge\gamma^3\end{matrix}\right\rangle \text{the exterior torsion vector of disclination 2-forms.}$$

The first two equations are precisely Cartan’s equations of structure (on an affine domain).

The last equation appears to be a new equation of structure valid on a projective domain, **when** $\Omega \neq 0$.

The vector of torsion-like 2-forms $|\mathbf{\Psi}\rangle$ physically seems to represent a different kind of ”torsion”, a torsion which I am trying to put into correspondence with disclination defects and circulation phenomena in fluids. Recall that Kondo has developed the theory of dislocation defects based on vector of affine torsion 2-forms $|\mathbf{\Sigma}\rangle$.

There are also three equations of structure on the exterior domain (coefficients of **n**) which are given by the constructions:

$$d\omega + \Omega\wedge\omega = -\langle\mathbf{h}|\wedge|\sigma\rangle$$

$$d\langle\mathbf{h}| + \Omega\wedge\langle\mathbf{h}| = -\langle\mathbf{h}|\wedge[\Gamma]$$

$$d\Omega + \Omega\hat{}\Omega = \theta = -\langle \mathbf{h}|\hat{}|\gamma\rangle \text{ the exterior curvature 2-forms}$$

Algebraic and Differential Methods

A remarkable result (to this author) of this constructive procedure is the fact that the matrix of interior curvature 2-forms, $[\Theta]$, can be constructed in two ways. The classical method utilizes differential processes

$$[\Theta] = \{d[\Gamma]+[\Gamma]\hat{}[\Gamma]\},$$

while the second method is purely algebraic

$$[\Theta] = \{-|\gamma\rangle\hat{}\langle\mathbf{h}|\}.$$

The degree of partial derivatives contained in the algebraic (exterior) expression for the interior curvature $\{-|\gamma\rangle\hat{}\langle\mathbf{h}|\}$ is one less than the classic expression built on the interior connection coefficients, $\{d[\Gamma]+[\Gamma]\hat{}[\Gamma]\}$.

Exterior differentiation of the matrix of interior curvature 2-forms constructed in an algebraic manner gives a simple proof of the theorem that the matrix of curvature 2-forms is closed:

$$d[\Theta] = -d|\gamma\rangle\hat{}\langle\mathbf{h}| = (-|d\gamma\rangle\hat{}\langle\mathbf{h}|) + (|\gamma\rangle\hat{}\langle d\mathbf{h}|) =$$

$$([\Gamma]\hat{}|\gamma\rangle\hat{}\langle\mathbf{h}|) - (\Omega\hat{}|\gamma\rangle\hat{}\langle\mathbf{h}|) - (|\gamma\rangle\hat{}\Omega\hat{}\langle\mathbf{h}|) - (|\gamma\rangle\hat{}\langle\mathbf{h}|\hat{}[\Gamma]) = 0$$

It is important to note that due to the partition, the exterior curvature is a closed (in this example a scalar valued) 2-form $\theta = -\langle\mathbf{h}|\hat{}|\gamma\rangle$ with

$$d\theta = -\langle d\mathbf{h}|\hat{}|\gamma\rangle + \langle\mathbf{h}|\hat{}|d\gamma\rangle = +\Omega\hat{}\langle\mathbf{h}|\hat{}|\gamma\rangle + \langle\mathbf{h}|\hat{}[\Gamma]\hat{}|\gamma\rangle - \langle\mathbf{h}|\hat{}[\Gamma]\hat{}|\gamma\rangle + \langle\mathbf{h}|\hat{}\Omega\hat{}|\gamma\rangle = 0.$$

Remarks

The concept that a single (global) 1-form dictates much of the topology and geometry of the domain of support is of interest to physics, for the usual starting point of many physical theories is the assumption that the physical system can be represented in terms of a suitable 1-form of Action. Without further assumptions, save for C2 differentiability, the above constructive method permits evaluation of the important geometrical and topological defect structures induced by the physical constraints of such a 1-form of Action.