Charge is a Pseudo Scalar

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Abstract

In terms of exterior differential forms A, and exterior differential form densities, J, the fundamental PDE's of Maxwell are represented in metric free form by the exterior differential system F-dA = 0 and J-dG = 0. Transformation properties of exterior diffential forms, based upon topological differences between differential forms and differential form densities, support the conclusion of E. J. Post that charge is a psuedo scalar under P and T transformations.

1. FUNDAMENTAL ASSUMPTIONS:

The use of differential forms should not be viewed as just another formalism of fancy. The Cartan technique goes well beyond the methods of tensor calculus (which is restricted to diffeomorphisms) and admits the study of topological evolution with respect to continuous maps that do not preserve topology. For example, the PDE's of Maxwell electrodynamics, unconstrained by metric, connections, or constitutive maps, can be formulated as the exterior differential system constructed on two topological constraints.

The first constraint is the postulate of potentials,

$$F - dA = 0, \tag{1.1}$$

where A is a 1-form of Action, with twice differentiable coefficients (potentials proportional to momenta) which induce a 2-form, F, of electromagnetic intensities

(**E** and **B** , related to forces). The closure of the exterior differential system, $dF = 0,$ generates the Maxwell-Faraday relations. The second topological constraint is the postulate of conserved charge currents,

$$J - dG = 0, \tag{1.2}$$

where G is an N-2 form *density* of field excitations (**D** and **H** , related to sources), and J is the N-1 form of charge-current densities. The partial differential equations equivalent to the exterior differential system are precisely the Maxwell-Ampere equations, and the closure $dJ = 0$ yields the charge conservation law. For details and other references, see [1]

It is to be noted that exterior differential forms are of two species: differential (pair) forms, ω, and differential (impair) form densities, Θ. The standard example of an exterior pair form will be taken as the electromagnetic 1-form of Action per unit charge, A. The standard example of an exterior impair form will be taken as the N-1 form density of electromagnetic charge current, J. The symmetry properties of the two species of exterior differential forms with respect to diffeomorphisms are different. Based on the assumptions given above that define the electromagnetic system from a topological point of view, the discrete symmetry properties of the various fields contained in the Maxwell system are to be examined in that which follows. The major result is that in order to be consistent with the topological viewpoint, the concept of charge must be interpreted as a pseudo-scalar with respect to P and T transformations in agreement with Post[2], and in disagreement with the "standard model" where charge is assumed to be a scalar [3] [4].

1.0.1. Exterior differential forms (Pair forms)

A differential 1-form, A, on a variety of independent variables, ξ^μ, is composed of coefficient functions $A_\sigma(\xi^\mu)$ and differentials $d\xi^\sigma$

$$A = \sum_\sigma A_\sigma(\xi^\mu) d\xi^\sigma = A_\sigma d\xi^\sigma \tag{1.3}$$

With respect to diffeomorphisms of the independent variables, this exterior differential 1-form is an invariant object. Relative to the differentiable, invertible map,

$$\varphi \ : \ \xi^{\mu} \Rightarrow \overline{\xi}^{\lambda} = \varphi^{\lambda}(\xi^{\mu}) \tag{1.4}$$

$$d\varphi \ : \ d\xi^{\mu} \Rightarrow d\overline{\xi}^{\lambda} = \{\partial\varphi^{\lambda}(\xi^{\nu})/\partial\xi^{\mu}\}d\xi^{\mu} = J_{\mu}^{\lambda}(\xi^{\nu})d\xi^{\mu}. \tag{1.5}$$

The differentials $d\xi^{\sigma}$ are linearly mapped from an initial "state" to a final "state" by means of the Jacobian matrix $J_{\mu}^{\lambda}(\xi^{\nu})$ of the diffeomorphism between the initial set of independent variables, ξ^{μ}, and the final set of independent variables, $\overline{\xi}^{\lambda}$. This push forward transformation is the epitome of the behavior of a contravariant object (index up). The coefficient functions $A_{\sigma}(\xi^{\mu})$ (with index down) are presumed to behave as covariant tensors with respect to the diffeomorphism, and are pushed forward by means of the linear map defined by the inverse of the Jacobian matrix. Hence the exterior differential form is an invariant object with respect to diffeomorphic transformations of the independent variables:

$$A = A_{\mu}(\xi^{\nu})d\xi^{\mu} \Rightarrow A_{\sigma}(\xi^{\nu})\delta_{\mu}^{\sigma}d\xi^{\mu} \Rightarrow A_{\sigma}(\xi^{\nu})\left[J_{\mu}^{\sigma}(\xi^{\nu})\right]^{-1}\left[J_{\mu}^{\sigma}(\xi^{\nu})\right]d\xi^{\mu} \Rightarrow \overline{A}_{\sigma}(\overline{\xi}^{\mu})d\overline{\xi}^{\sigma}. \tag{1.6}$$

Note that the inverse mapping is required to formulate the specific format of the functions $\overline{A}_{\sigma}(\overline{\xi}^{\mu}) = A_{\sigma}(\xi^{\nu})\left[J_{\mu}^{\sigma}(\xi^{\nu})\right]^{-1}$. However, initial data and functional coefficient formats given on the initial state can be pushed forward in a well defined manner to a final state, relative to diffeomorphisms. It is also true that given data and functional form on the final state, the functional forms and data are well defined on the initial state. This operation defines the pullback, in distinction to the classic push forward:

$$A = A_{\mu}(\xi^{\nu})d\xi^{\mu} = \overline{A}_{\sigma}(\varphi^{\mu}(\xi^{\nu}))\left[J_{\mu}^{\sigma}(\xi^{\nu})\right]d\xi^{\mu} \Leftarrow \overline{A}_{\sigma}(\overline{\xi}^{\mu})d\overline{\xi}^{\sigma}. \tag{1.7}$$

It is apparent that the preimage (pullback) is functionally well defined as $A_{\mu}(\xi^{\nu}) = \overline{A}_{\sigma}(\varphi^{\mu}(\xi^{\nu}))\left[J_{\mu}^{\sigma}(\xi^{\nu})\right]$, and does not invoke the inverse mapping nor the inverse Jacobian. Note that this formula agrees exactly with the covariant tensor definition if the inverse Jacobian exists. However, this pullback property is valid even when the map is not a diffeomorphism. In this sense, exterior differential forms go beyond the concept of tensors. Exterior differential forms are well behaved in a retrodictive sense, with respect to differentiable maps without a local, much less a global, inverse; tensors are not. Differential p-forms can be constructed from products of 1-forms, so that the concepts of pullback apply to all p-forms.

The bottom line is that exterior differential forms are well defined behavior with respect to maps which are not diffeomorphisms, but only in a retrodictive (not predictive) pullback (not pushforward) sense. The data and functional formats must be given it terms of the functions and independent variables on the final state if the mappings of the independent variables are not diffeomorphisms. Such differentiable mappings admit topological evolution, where diffeomorphisms do not. Hence differential forms may be used to describe topological evolution.

1.0.2. Exterior Differential Form densities. (Impair forms)

In projective geometry, where the concept of length, such as that defined in terms of a metric quadratic form, is not required, the concept of a vector is still useful idea. However it is the directional properties of the vector (its "line of action") that are of importance. The idea is that vectors are defined to be equivalent if they are the same to within a multiplying factor. sometimes such vectors are called pseudo-vectors. In the diffeomorphic subset, exterior differential forms can be constructed from pseudo-vectors, where the multiplying factor is determinant of diffeomorphism. The restriction to the determinant factor is why such impair forms are called "densities". However, for non-diffeomorphic maps, the impair forms have pullback properties different from the pullback properties of the pair exterior forms.

To demonstrate these concepts consider the N-1 form impair form, or current density, J. This form is in a sense a projective dual of a pair 1-form, A. It can be constructed by using the top down process in terms of coefficient functions defined in terms of a contravariant tensor density (this is not the same as a pair N-1 form whose coefficients are covariant tensors. A convenient notation is given by the formula:

$$Current\_density\_impair\_N-1\_form : J = J^{\sigma}(\overline{\xi})d\overline{\xi}^{1}\hat{}d\overline{\xi}^{2}\hat{}...\widehat{d\overline{\xi}^{\sigma}}...d\overline{\xi}^{N}. \quad (1.8)$$

The order sequence of differentials forms a local differential volume element. The hatted symbol $\widehat{d\overline{\xi}^{\sigma}}$ means that factor is left out of the N-form,

$$Vol = d\overline{\xi}^{1}\hat{}d\overline{\xi}^{2}\hat{}......d\overline{\xi}^{N}, \quad (1.9)$$

and is replaced by the contravariant (to within a factor) coefficients, $J^{\sigma}(\overline{\xi})$, all defined on the final state. Direct substitution (of the differentiable mappings expressing the differentials on the final state as linear combinations of the differentials on the initial state) into the impair N-1 form leads to the impair N-1 form on the initial state.

$$\begin{aligned} J &= J^{\sigma}(\xi^{\nu}) d\xi^{1} \hat{} d\xi^{2} \hat{} ... \widehat{d\xi^{\sigma}} ... d\xi^{N} && (1.10) \\ &= J^{\sigma}(\varphi^{\mu}(\xi^{\nu})) \left[J^{\sigma}_{\mu}(\xi^{\nu}) \right]^{adjoint} d\xi^{1} \hat{} d\xi^{2} \hat{} ... \widehat{d\xi^{\sigma}} ... d\xi^{N} \Leftarrow J^{\sigma}(\bar{\xi}) d\bar{\xi}^{1} \hat{} d\bar{\xi}^{2} \hat{} ... \widehat{d\bar{\xi}^{\sigma}} ... d\bar{\xi}^{N} && (1.11) \end{aligned}$$

It is apparent that the pullback preimage components of the current density on the initial state are related to the final state components multiplied by the Adjoint of the Jacobian matrix. Recall that for a pair 1-form the pullback depended on the Transpose of the Jacobian matrix. For the impair forms the pullback is dependent upon the Adjoint of the Jacobian matrix.

For diffeomorphisms the coefficients of a contravariant vector push forward with respect to the linear transformations induced by the Jacobian matrix of the transformation. The components of a contravariant density pushforward by means of the Jacobian divided by the determinant of the mapping. The contravariant vector components pull back via the linear transformations of the Jacobian inverse. The contravariant vector density pulls back with respect to the linear mapping defined by the Jacobian matrix inverse divided by the determinant of the transformation. However, this matrix is precisely the adjoint of the Jacobian matrix (the matrix of cofactors transposed.

Hence it is apparent from the pull back formula above that the coefficient functions of the charge - current density impair form pull back as a contravariant tensor density. For diffeomorphisms, the mappings are sensitive to the determinant of the transformation.

1.1. Differential forms are invariant with respect to either P or T transformations in {x,y,z,t} in 4D.

A parity transformation in 4D will be defined as the map

$$\begin{aligned} P &: \{x, y, z, t\} \Rightarrow \{-x, -y, -z, +t\} && (1.12) \\ P\_Jacobian &= \begin{bmatrix} -1 & & & \\ & -1 & & \\ & & -1 & \\ & & & +1 \end{bmatrix} && (1.13) \\ \text{Determinant } (P\_Jacobian) &= -1 && (1.14) \end{aligned}$$

A time inversion transformation in 4D will be defined as the map

$$T \quad : \quad \{x,y,z,t\} \Rightarrow \{+x,+y,+z,-t\} \tag{1.15}$$

$$T\_Jacobian = \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & -1 \end{bmatrix} \tag{1.16}$$

$$\text{Determinant } (T\_Jacobian) = -1 \tag{1.17}$$

Let ω be a pair differential form: then

$$P(\omega) = +\omega \tag{1.18}$$

$$T(\omega) = +\omega \tag{1.19}$$

1.2. Differential form densities change sign with respect to P or T transformations in {x,y,z,t}.

Let Θ be an impair form, or differential form density: then

$$P(\Theta) = -\Theta \tag{1.20}$$

$$T(\Theta) = -\Theta \tag{1.21}$$

2. Electromagnetism

2.1. EM Differential forms

Define the differential forms on $\{x,y,z,t\}$

$$A = \Sigma_{k=1}^{3} A_k(x,y,z,t)dx^k - \phi(x,y,z,t)dt = \mathbf{A}\circ d\mathbf{r} - \phi dt. \tag{2.1}$$

$$F = dA = \{\partial A_k/\partial x^j - \partial A_j/\partial x^k\}dx^j\hat{}\,dx^k = F_{jk}dx^j\hat{}\,dx^k = \mathbf{B}_z dx\hat{}\,dy ... + \mathbf{E}_x dx\hat{}\,dt ... \tag{2.2}$$

$$\begin{aligned} \text{Topological Torsion} \quad (A\hat{}F) &= i(\mathbf{T}_4)dx\hat{}dy\hat{}dz\hat{}dt && (2.3) \\ &= \mathbf{T}^x dy\hat{}dz\hat{}dt..... - hdx\hat{}dy\hat{}dz && (2.4) \end{aligned}$$

$$Torsion - vector : \mathbf{T}_4 = [\mathbf{E} \times \mathbf{A} + \mathbf{B}\phi, \mathbf{A} \circ \mathbf{B}] \equiv [\mathbf{T}, h]. \tag{2.5}$$

2.1.1. EM Differential form densities

$$G = G^{34}(x, y, z, t)dx\hat{}dy... + G^{12}(x, y, z, t)dz\hat{}dt... = \mathbf{D}^z dx\hat{}dy...\mathbf{H}^z dz\hat{}dt... \tag{2.6}$$

$$J = \mathbf{J}^z(x, y, z, t)dx\hat{}dy\hat{}dt... - \rho(x, y, z, t)dx\hat{}dy\hat{}dz. \tag{2.7}$$

$$Charge \ - Current \ density : \mathbf{J}_4 = [\mathbf{J}, \rho], \tag{2.8}$$

$$\begin{aligned} \text{Topological Spin} \quad (A\hat{}G) &= i(\mathbf{S}_4)dx\hat{}dy\hat{}dz\hat{}dt && (2.9) \\ &= \mathbf{S}^x dy\hat{}dz\hat{}dt..... - \sigma dx\hat{}dy\hat{}dz && (2.10) \end{aligned}$$

$$Spin - Current \ density : \mathbf{S}_4 = [\mathbf{A} \times \mathbf{H} + \mathbf{D}\phi, \mathbf{A} \circ \mathbf{D}] \equiv [\mathbf{S}, \sigma], \tag{2.11}$$

3. PT transformations in 4D

$$\begin{aligned} P &: \ x \Rightarrow -x, \ y \Rightarrow -y, \ z \Rightarrow -z, && (3.1) \\ \Delta &= dx\hat{}dy\hat{}dz\hat{}dt \Rightarrow -1 && (3.2) \end{aligned}$$

$$\begin{aligned} T &: \ t \Rightarrow -t, && (3.3) \\ \Delta &= dx\hat{}dy\hat{}dz\hat{}dt \Rightarrow -1 && (3.4) \end{aligned}$$

3.1. 1. Discrete Symmetry behavior of the 2-form F

$$\begin{bmatrix} & \mathbf{E} & \mathbf{B} \\ P: & -\mathbf{E} & +\mathbf{B} \\ T: & -\mathbf{E} & +\mathbf{B} \end{bmatrix} \tag{3.5}$$

This table is in agreement with Post, and NOT with the "standard" model of Sakurai and Henley $\{P(E) \Rightarrow -E,\ P(B) \Rightarrow +B,\quad T(E) \Rightarrow +E,\ T(B) \Rightarrow -B\}$

3.2. 2. Discrete Symmetry behavior of the 1-form A

$$\begin{bmatrix} & \mathbf{A} & \phi \\ P: & -\mathbf{A} & +\phi \\ T: & +\mathbf{A} & -\phi \end{bmatrix} \tag{3.6}$$

This table is in agreement with Post, and NOT with the "standard" model of Sakurai and Henley which claims that $\{P(A) \Rightarrow -A,\ P(\phi) \Rightarrow +\phi,\quad T(A) \Rightarrow -A,\ T(\phi) \Rightarrow +\phi\}$

3.3. 3 Discrete Symmetry behavior of the N-2=2 form (density) G

$$\begin{bmatrix} & \mathbf{D} & \mathbf{H} \\ P: & -\mathbf{D} & +\mathbf{H} \\ T: & -\mathbf{D} & +\mathbf{H} \end{bmatrix} \tag{3.7}$$

This table is in agreement with Post, and NOT with the "standard" model of Sakurai and Henley which claims that $\{P(D) \Rightarrow -D,\ P(H) \Rightarrow +H,\quad T(D) \Rightarrow +D,\ T(H) \Rightarrow -H\}$

3.4. 4. Discrete Symmetry behavior of the 3-form of Charge-Current (density) J

$$\begin{bmatrix} & \mathbf{J} & \rho \\ P: & -\mathbf{J} & +\rho \\ T: & +\mathbf{J} & -\rho \end{bmatrix} \tag{3.8}$$

This table is in agreement with Post, and NOT with the "standard" model of Sakurai and Henley $\{P(J) \Rightarrow -J,\ P(\rho) \Rightarrow +\rho,\quad T(J) \Rightarrow -J,\ T(\rho) \Rightarrow +\rho\}$

3.5. 5. Discrete Symmetry behavior for Charge defined as $Q = \int\int\int \rho dx\hat{}dy\hat{}dz$

$$\begin{bmatrix} & Q \\ P: & -Q \\ T: & -Q \end{bmatrix} \tag{3.9}$$

This table is in agreement with Post, and NOT with the "standard" model of Sakurai and Henley. To be consistent, Sakurai and Henley would have to say $\{P(Q) \Rightarrow -Q, \quad T(Q) \Rightarrow +Q.\}$. BUT in their text [3] and [4], they claim that Q is a scalar under both P and T, which is INCONSISTENT with their other claims.

3.6. 6. Discrete Symmetry behavior of $A\hat{}F$ (not a density)

$$\begin{bmatrix} & (\mathbf{E}\times\mathbf{A}+\mathbf{B}\phi) & \mathbf{A}\circ\mathbf{B} \\ P: & +(\mathbf{E}\times\mathbf{A}+\mathbf{B}\phi) & -\mathbf{A}\circ\mathbf{B} \\ T: & -(\mathbf{E}\times\mathbf{A}+\mathbf{B}\phi) & +\mathbf{A}\circ\mathbf{B} \end{bmatrix} \tag{3.10}$$

The table is NOT the consistent with the charge current density table of this subsection. But that is what would be expected as $A\hat{}F$ is formed from the product of a 1-form and a 2-form, creating a 3-form and not a 3-form density..

3.7. 7. Discrete Symmetry behavior of $A\hat{}G$ (a density)

$$\begin{bmatrix} & (\mathbf{A}\times\mathbf{H}+\mathbf{D}\phi) & \mathbf{A}\circ\mathbf{D} \\ P: & -(\mathbf{A}\times\mathbf{H}+\mathbf{D}\phi) & +\mathbf{A}\circ\mathbf{D} \\ T: & +(\mathbf{A}\times\mathbf{H}+\mathbf{D}\phi) & -\mathbf{A}\circ\mathbf{D} \end{bmatrix} \tag{3.11}$$

This table has the same properties as the charge current table in this section, which is pleasing as it is a product of a differential form and a differential form density. $A\hat{}G$ can be distinguished from $A\hat{}F$ under P and T.

3.8. 8. The Hall Impedance is impair.

If the Hall impedance Z_{hall} is defined in terms of the ratio of the 1-form A to the 2-form density G, then the physical dimension of the Hall impedance is h/e^2 and it is impair. Alternately, if the Hall Impedance is defined in terms of the ratio of the 3-form $A\hat{}F$ to the 3-form density, $A\hat{}G$, [5] then the physical dimension of the Hall impedance is h/e^2 and it is impair.

$$\begin{bmatrix} & \mathbf{Z}_{Hall} \\ P: & -\mathbf{Z}_{Hall} \\ T: & -\mathbf{Z}_{Hall} \end{bmatrix} \tag{3.12}$$

3.9. 9. The Free space Impedance is pair.

If the Free Space impedance is defined as the ratio of $\sqrt{\frac{\mu}{\varepsilon}}$, and as both ε and μ are impair, it follows that $Z_{freespace} = \sqrt{\frac{\mu}{\varepsilon}}$ and is pair.

$$\begin{bmatrix} & \mathbf{Z}_{freespace} \\ P: & +\mathbf{Z}_{freespace} \\ T: & +\mathbf{Z}_{freespace} \end{bmatrix} \tag{3.13}$$

3.10. 10. The fine structure constant is impair.

To within a factor of 2 the ratio of the Hall impedance to the Free Space impedance is equal to the fine structure constant. Hence it follows that the fine structure is impair.

$$\begin{bmatrix} & \boldsymbol{\alpha} \\ P: & -\boldsymbol{\alpha} \\ T: & -\boldsymbol{\alpha} \end{bmatrix} \tag{3.14}$$

So starting from the classical electron radius, which is pair, the electron Compton wavelength is impair, and the electron Bohr orbit is pair.

4. Conclusion: Post is correct.

Post's concept of charge as a pseudo scalar [2] requires that differential form densities behave differently than do differential forms under P or T transformations in 4D. These ideas are in agreement with the fact that tensor densities involve the determinant of the transformation (not the magnitude of the determinant). The results given above are NOT in agreement with the standard Sakurai - Henley model [3], but have a credence level built on topological ideas and diffeomorphic invariance, and not upon geometrical constraints of metric and connection.

5. References

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