Differential Forms and the Pullback

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Abstract

Notes on the differences between contra-variant and co-variant tensors versus Pair and Impair differential forms.

1. Introduction

The object of this note is to clarify the differences between exterior differential forms and exterior differential form densities. The concepts are related to certain features of tensors and tensor analysis, which are, however, classically limited to (or defined by) the equivalence class of diffeomorphisms. Recall that diffeomorphisms are special cases of homeomorphisms, hence all topological features are preserved by the admissable maps of tensor analysis. Exterior differential forms are well defined over a larger class of transformations which only require a map with C1 differentiability. Hence exterior differential forms can carry topological information and can be used to study topological evolution. In tensor analysis, there exist two important categories of tensors, known as covariant tensors and contravariant tensors. These sets are distinguished by their transformation properties with respect to diffeomorphisms. By classical convention:

- The functional components of Covariant vectors are pushed forward from the initial state to the final state by means of the linear transformations induced by the Inverse of the Jacobian map of the diffeomorphism between independent variables.

- The functional components of Contravariant vectors are pushed forward by means of the linear transformations induced by the Jacobian map of the diffeomorphism between independent variables.

In the theory of exterior differential forms, there are also two important categories of exterior differential forms. The first category is defined as "pair forms", or "even forms", or simply "exterior differential forms". The second category is defined as "impair forms", or "odd forms", or as "exterior differential form densities". The two categories of differential forms are (like their tensor counterparts) distinguished by their transformation behavior with respect to C1 differentiable maps. However the important transformation properties are defined in terms of the Pullback The functional forms of the two species of differential forms, as defined on the final state, are pulled back to the initial state by means of linear transformations induced by the Transpose (for pair forms)and the Adjoint (for impair forms) of the Jacobian matrix, both of which exist, even though the Inverse to the Jacobian matrix does not exist.

- The functional components of "pair" exterior differential forms are pulled back from the final state to the initial state by means of the linear transformations induced by the Transpose of the Jacobian map between independent variables. The map from the initial state of independent variables to the final state of independent variables need not be a diffeomorphism, but must be C1 differentiable. For diffeomorphisms the coefficients of the pair differential forms behave like covariant tensors.

- The functional components of "impair" exterior differential form (densities) are pulled back from the final state to the initial state by means of the linear transformations induced by the Adjoint of the Jacobian map between independent variables. The map from the initial state of independent variables to the final state of independent variables need not be a diffeomorphism, but must be a C1 differentiable map. For diffeomorphisms the coefficients of the impair differential forms behave like contravariant tensor densities. For unimodular diffeomorphisms, the coefficients behave like contravariant tensors.

The coefficients of the impair forms behave as contravariant tensor densities (delta densities in the language of Schouten - densities in the language of Brilluoin) and are sensitive to the sign of the determinant and magnitude of the determinant. (Some authors define such forms as "twisted forms" but still use the covariant notation for the coefficients which IMO confuses the issues.) A prefered notation is (using ordered indicies) is:

$$A = a \text{ "pair" } exterior\ differential\ 1 - form \tag{1.1}$$
$$A = A_k(x)dx^k \tag{1.2}$$

$$F = a \text{ "pair" } exterior\ differential\ 2 - form : F = F_{jk}(x)dx^j\hat{}dx^k \tag{1.3}$$

$$J = an \text{ " Im} pair\text{" } exterior\ differential\ N - 1\ form\ density \tag{1.4}$$
$$J = J^k(x)\{dx^1\hat{}dx^2...\widehat{dx}^k...dx^N\} \tag{1.5}$$

$$G = an \text{ " Im} pair\text{" } exterior\ differential\ N - 2\ form\ density \tag{1.6}$$
$$G = G^{jk}(x)\{dx^1\hat{}dx^2...\widehat{dx}^j...\widehat{dx}^k...dx^N\} \tag{1.7}$$

The hatted symbol $\widehat{dx}^k$ means that that term dx^k is left out of the factors that form the volume element.

Some details and examples are presented below.

2. Functional Substitution and the PullBack Examples

Cartan's theory of exterior differential forms is NOT just another notational system of fancy. The important fact, often ignored, is that a differential form is not necessarily a tensor. In short, a differential form is a mathematical object that is well behaved with respect to differentiable maps, $y^k \Rightarrow x^\mu = \phi^\mu(y^k)$, from an initial state $\{y^k\}$ of independent variables to a final state $\{x^\mu\}$ of independent variables. Such maps need not be homeomorphisms. Hence the topology of the final state need not be the same as the topology of the initial state.

A tensor is an object which is well behaved if and only if the differentiable map ϕ^{μ} from initial to final state has a differentiable inverse. Such maps are defined to be diffeomorphisms, a subset of homeomorphisms, and as such the topology of the final state and the initial state must be the same.

Differential forms can be used to understand and study topological evolution, while tensors cannot.

2.0.1. Contravariant and covariant tensors

A tensor is a restricted class of mathematical objects that transform in a multi-linear manner relative to the Jacobian matrix $J = \left[\partial\phi^{\mu}/\partial y^{k}\right]$ of the differentiable map, and its inverse. The motivation for the definitions is most transparent when one considers the matrix inner product of two vectors, $\langle A_k| \circ \left|V^k\right\rangle$. Suppose that the Identity matrix can be written in terms of two factors, $[I] = [J]^{-1} \circ [J]$. Then the equation,

$$\langle A_k| \circ \left|V^k\right\rangle \Rightarrow \langle A_k| \circ [J]^{-1} \circ [J] \circ \left|V^k\right\rangle = \left\langle \overline{A}_k\right| \circ \left|\overline{V}^k\right\rangle, \tag{2.1}$$

can be interpreted in terms of a matrix transformation, where

$$\left|V^k\right\rangle \Rightarrow \left|\overline{V}^k\right\rangle = [J] \circ \left|V^k\right\rangle \tag{2.2}$$

and

$$\langle A_k| \Rightarrow \left\langle \overline{A}_k\right| = \langle A_k| \circ [J]^{-1} \tag{2.3}$$

in such a way that the value of the matrix row-column vector product before the transformation and after the transformation has the same value. The column vector $\left|V^k\right\rangle$ is the epitome of a contra-variant vector. The row vector $\langle A_k|$ is the epitome of a co-variant vector. The matrix $[J]$ is the Jacobian matrix of the transformation from intitial to final state. The matrix transformation rules above give primitive meaning to the concepts of co-variance and contra-variance. These rules in tensor analysis are extended to multi-linear objects.

The classical definitions consider a tensor to be a mathematical object whose functional form given on the initial state is well defined on the final state. Note that the tensor operations are a "push forward" from the initial to the final state. The transformation rules are of two types are defined as:

$$The\ Contravariant\ Rule\ (push\ forward) \quad : \tag{2.4}$$

$$\overline{V}^k(y) \;\Rightarrow\; V^\mu(x) = \left[\partial x^\mu(y)/\partial y^k\right] \overline{V}^k(y) \tag{2.5}$$

$$The\ Covariant\ Rule\ (push\ forward) \quad : \tag{2.6}$$

$$\overline{A}_k(y) \;\Rightarrow\; A_\mu(x) = \overline{A}_k(y)\partial y^k(x)/\partial x^\mu \tag{2.7}$$

In both cases an inverse mapping ($x = \phi^{-1}(y)$) is required if the arguments of the functions created on the final state are to be expressed in terms of the independent variables, x, of the final state.

When the tensor rule for covariant transformations is multiplied on both sides by the Jacobian matrix, and using the constraint inherent in the tensor definitions that an inverse Jacobian exists, a more general formula for covariant transformations can be obtained. If the functions that make up the covariant object are given on the final state, then the functions on the initial state are well defined in terms of the independent variables on the initial state. This more general situation works when the map is differentiable, but no inverse map exists. The process is defined as the covariant pullback. (see http://www22.pair.com/csdc/ed3/ed3fre1.htm)

$$The\ Covariant\ Rule\ (pullback) : \overline{A}_k(y) = A_\mu(x(y)) \left[\partial x^\mu(y)/\partial y^k\right] \Leftarrow A_\mu(x) \tag{2.8}$$

2.0.2. Retrodiction vs Prediction

It has been shown [1] that with respect to maps without differentiable inverses, the classic rules of tensor transformations do not permit the unique prediction of the functional form on the final state of either contravariant or covariant tensors, given the functional form of these tensors on the initial state. Numeric values on the final state sometimes can be predicted, but the functional form (defining neighborhoods) on the final state cannot be uniquely predicted. It also has been shown relative to maps without differentiable inverses that a given the functional form of a contra-tensor on the final state does not permit a unique retrodiction of the functional form on the initial state.

However, there are two situations where unique well defined retrodiction is possible in a functional sense. This retrodiction process is defined as the "Pull-Back" The first situation involves covariant tensors on the final state. The second situation involves contravariant antisymmetric tensor densities on the final state. Given a mapping and its Jacobian, $[J_k^\mu(y)]$, the two forms of pull back may be written as:

$$PullBack\ Covariant\ Rule : \overline{A}_k(y) \Leftarrow A_\mu(x(y))\,[J_k^\mu(y)] \tag{2.9}$$

$$PullBack\ Contravariant\ density\ Rule : \overline{C}^k(y) \Leftarrow \left[AdJ_\mu^k(y)\right] C^\mu(x(y)). \tag{2.10}$$

In the notation above, $\left[AdJ_\mu^k(y)\right]$, is the adjoint of the Jacobian matrix (matrix of cofactors transposed) and does not depend upon the determinant of the transformation. The adjoint matrix may be algebraically determined even when the determinant of the matrix is zero. Hence the pull back is well defined, even if an inverse does not exist. The contravariant density, or current, $C^\mu(x)$, is like the charge current density of electromagnetism. It is suggested herein that these two rules should be used as foundations for transformations of objects which would be considered to be tensors, if the mappings are constrained to be diffeomorphisms.

2.0.3. Derivation of the Pullback transformation rules:

First consider differential forms whose coefficients would be covariant tensor fields, if the Jacobian matrix has an inverse. A typical representation is a 1-form written in terms of the variables on the final state as:

$$A = A_\mu(x^\nu)\,dx^\mu = \langle A_\mu(x^\nu)| \circ |dx^\mu\rangle . \tag{2.11}$$

Consider the differentiable non-linear map

$$\phi :\ y^k \Rightarrow x^\mu = \phi^\mu(y^j), \tag{2.12}$$

$$d\phi :\ dy^k \Rightarrow dx^\mu = \left[\partial\phi^\mu(y^j)/\partial y^k\right]\, dy^k = \left[J_k^\mu(y^j)\right] \circ \left| dy^k \right\rangle . \tag{2.13}$$

Substitute these formulas into the expression for the differential 1-form expressed in terms of the independent variables on the final state:

$$\begin{aligned} A &= \langle A_\mu(x^\nu)| \circ |dx^\mu\rangle = \left\langle A_\mu(\phi^\nu(y^j))\right| \circ \left[J_k^\mu(y^j)\right] \circ \left| dy^k \right\rangle && (2.14)\\ &= \left\langle \overline{A}_k(y^j)\right| \circ \left| dy^k \right\rangle . && (2.15)\end{aligned}$$

The coefficents $\left\langle \overline{A}_k(y^j)\right|$ are well defined functions on the initial state, with arguments in terms of the initial state variables. Now if the map from initial to final state is such that the Jacobian is an invertible matrix, then the coefficient variables

$$\langle A_\mu(x^\nu)| \circ \left[J_k^\mu(y^j)\right] = \left\langle A_\mu(\phi^\nu(y^j))\right| \circ \left[J_k^\mu(y^j)\right] = \left\langle \overline{A}_k(y^j)\right| \tag{2.16}$$

is equivalent to the transformation rules of a covariant tensor field in classic tensor analysis:

$$PullBack\ Rule : A_\mu \partial x^\mu / \partial y^k = \overline{A}_k \tag{2.17}$$

Note: The pullback coefficients are not tensor equivalents when the Jacobian matrix is not invertible. However, the PullBack is always well defined in terms of the language which uses the Jacobian matrix transpose. The transpose always exists even though the inverse does not.

2.0.4. Example 1: Pullback of a 1-form

Consider the example map from 3 to 3 dimensions, for which an inverse Jacobian does not exist.

$$\phi : \{X, Y, Z\} \Rightarrow \{x, y, z\} = \{XY, Y^3, X\} \tag{2.18}$$

$$d\phi : \{dX, dY, dZ\} \Rightarrow \{dx, dy, dz\} = \{YdX + XdY, 3Y^2dY, dX\} \tag{2.19}$$

$$Jacobian\ [J_k^\mu(X)] = \left[\partial\phi^\mu(x^j)/\partial X^k\right] = \begin{bmatrix} Y & X & 0 \\ 0 & 3Y^2 & 0 \\ 1 & 0 & 0 \end{bmatrix} \tag{2.20}$$

$$\det [J_k^\mu(X)] = 0 \tag{2.21}$$

$$Given\ on\ final\ state\ A = y\,dx - xdy + dz \tag{2.22}$$

$$Substitute\ \phi\ and\ d\phi = Y^3(YdX + XdY) - XY3Y^2dY + dX$$

$$PullBack\ to\ initial\ state = (Y^4 + 1)dX - (2Y^3X)dY + 0dZ \tag{2.23}$$

$$
\begin{aligned}
Coefficients\ on\ final\ state\ \ \langle A_\mu| &= [y,\ -x\ ,+1] && (2.24)\\
PullBack\ \ to\ initial\ state\ \left\langle \overline{A}_k \right| &= [(Y^4+1),\ -(2Y^3X),0] && (2.25)\\
&= \langle A_\mu| \circ [J^\mu_k] && (2.26)\\
&= \langle y,\ -x\ ,+1| \circ \begin{bmatrix} Y & X & 0 \\ 0 & 3Y^2 & 0 \\ 1 & 0 & 0 \end{bmatrix} && (2.27)\\
&= \left\langle yY+1\ ,yX - x3Y^2,\ 0 \right| && (2.28)\\
&= \left\langle (Y^4+1),\ -(2Y^3X),\ 0 \right| && (2.29)
\end{aligned}
$$

The example demonstrates the fact that the coefficients of the differential forms do not behave as tensors with respect to the non-invertible, but differentiable map. Yet, everything is well defined, functionally, with respect to the PullBack operation.

2.0.5. Example 2: Pullback of a 2-form

For the same map given in the first example, consider the 2-form below

$$
\begin{aligned}
& && Given\ on\ final\ state\\
F &= && F_{xy}dx\hat{}dy + F_{yz}dy\hat{}dz + F_{zx}dz\hat{}dx && (2.30)\\
& && Substitute\\
&= && F_{xy}(3Y^3dX\hat{}dY) - F_{yz}(3Y^2)dX\hat{}dY + F_{zx}(XdX\hat{}dY) && (2.31)\\
& && PullBack\ \ to\ initial\ state\\
&= && \{F_{xy}(3Y^3) - F_{yz}(3Y^2) + F_{zx}(X)\}dX\hat{}dY && (2.32)
\end{aligned}
$$

$$
\begin{aligned}
& && Coefficients\ on\ final\ state\\
\langle F_{\mu\nu}| &= && \langle F_{xy},\ F_{yz}\ ,F_{zx}| && (2.33)\\
& && PullBack\ \ to\ \ initial\ \ state\\
\left\langle \overline{F}_{XY} \right| &= && \left\langle \{-F_{yz}(3Y^2) + F_{zx}(X) + F_{xy}(3Y^3)\},0,0 \right| && (2.34)
\end{aligned}
$$

2.0.6. Example 3. Pullback of a contravariant tensor delta density and the adjoint Jacobian

Now consider the volume element 3-form

$$Vol3 := dx\hat{}dy\hat{}dz = (\det[J])dX\hat{}dY\hat{}dZ \tag{2.35}$$

and the Current N-1 form delta-density,

$$C = i(C^{\mu})Vol3 = C^{x}dy\hat{}dz - C^{y}dz\hat{}dx + C^{z}dx\hat{}dy. \tag{2.36}$$

The adjoint matrix to the Jacobian (for which no inverse exists) is

$$[AdJ] = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -3Y^2 & X & 3Y^3 \end{bmatrix}. \tag{2.37}$$

Substitution yields

$$\begin{aligned} C &= -C^{x}(3Y^2)dX\hat{}dY + C^{y}(XdX\hat{}dY) + C^{z}3Y^3dX\hat{}dY & (2.38)\\ &= \{-(3Y^2)C^{x} + (X)C^{y} + 3Y^3C^{z}\}dX\hat{}dY & (2.39)\\ &= \overline{C}^{Z}dX\hat{}dY, & (2.40)\end{aligned}$$

from which it is apparent that

$$\left| \begin{array}{c} \overline{C}^{X} \\ \overline{C}^{Y} \\ \overline{C}^{Z} \end{array} \right\rangle \Leftarrow [AdJ] \circ \left| \begin{array}{c} C^{x} \\ C^{y} \\ C^{z} \end{array} \right\rangle. \tag{2.41}$$

Note that these results are valid for the case where the inverse Jacobian does not exist.

For those cases where the inverse matrix does exist, it is apparent that the Current coefficients do not transform as a contravariant tensor, but instead transform as a contravariant tensor delta density: Multiplication of both sides of the preceding equation by the Jacobian matrix yields:

$$[J_k^{\mu}] \circ \left|\overline{C}^{k}(Y^j)\right\rangle = J_k^{\mu}\overline{C}^{k} \Rightarrow \Delta \cdot C^{\mu}(x) = (\det[J_k^{\mu}]) \cdot C^{\mu}(x). \tag{2.42}$$

If the coefficients transformed as a contravariant tensor, then the preceeding formula would have been written as

$$J_k^{\mu}\overline{C}^{k} \Rightarrow C^{\mu}(x). \tag{2.43}$$

Tensor delta densities are sensitive to the sign and magnitude of the determinant of the mapping.

3. Summary: Pair forms and Impair forms

The bottom line is that retrodiction and functional substitution leads to two well defined species of differential form objects: Pair (even) forms with antisymmetric coefficients, which pull back relative to the Transpose of the Jacobian of the mapping, and Impair (odd) forms with antisymmetric coefficients objects that pull back via Adjoint of the Jacobian matrix. These latter objects are sensitive to orientations, where the former are not. Both objects are not dependent upon an invertible differential mapping, hence can be used to study continuous topological evolution. However, when the maps from an initial state of independent variables to a final state of independent variables are unimodular det =1 diffeomorphisms, then the coefficients of the pair forms behave as covariant tensors, and the coefficients of the impair forms behave as contravariant tensors. It the diffeomorphisms are not such that det of the Jacobian matrix is 1, then the coefficients transform as contravariant tensor densities.

- The 2-form F of electromagnetic field intensities is a pair differential form. The integrals of exterior differential forms are invariants of diffeomorphisms which may change the sign of the N-volume (a change of orientation). Such integrals are scalars.
- The N-2 form G is an impair 2 form density in a space of 4 dimensions. The integrals of impair differential form densities change sign with respect to diffeomorphisms that change the N-volume orientation. Such integrals are pseudoscalars and can lead to enantiomorphic pairs.

4. References

R. M. Kiehn, Retrodictive Determinism, Int. J. Engng. Sci 1976 **14** pp. 249-754