Differential Forms and the Pullback

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Abstract

Differential p-forms are not p-rank tensors. They are scalars (to within a factor) composed of p-rank covaraint coefficients and p-rank contravariant differentials. The differences between contra-variant and co-variant tensors which are well behaved only with respect to C1 invertible diffeomorphisms are compaired to Pair and Impair differential forms, which, in addition to diffeomorphisms, are well behaved with respect to C1 non-invertible transformations, in a retrodictive sense.

1. Introduction

1.0.1. Pair and Impair exterior differential forms.

Exterior differential p-forms should not be confused with p-tensors. All exterior differential forms (with respect to diffeomorphic transformations) are invariant scalars to within a factor. When the factor is not unity, such scalars have been defined classically as "pseudo-scalars" [1]. A particularly useful choice of the pseudo-scalar factor is related to the determinant of the Jacobian of the diffeomorphic transformation, $\Delta = \det([\mathbb{J}])$. Explicitly, if the factor is related to Δ^{-k}, then such a pseudo-scalar has been defined as a density [2] of weight, k.

Exterior differential forms are well behaved with respect to diffeomorphic mappings of the independent variables. This means that the functional form of the coefficients can be pushed forward from given initial data to predicted final data, or pulled back from given final data to retrodicted initial data. However, exterior differential forms are also well behaved with respect to a wider class of differentiable mappings. The larger class of differentiable mappings includes those mappings that are not reversible, as well as diffeomorphisms which are reversible. The mapping functions for such transformations are C1-differentiable, and produce a Jacobian matrix of partial derivatives, $[\mathbb{J}]$, but the Jacobian need not be invertible. Such non-invertible differentiable transformations will be called "C1/d" (C1 differentiable mod diffeomorphisms) transformations in order to distinguish them from invertible C1 "diffeomorphisms". The "C1/d" transformations can be used to describe topological evolution, while the subset of diffeomorphisms cannot.

Relative to maps of the independent variables, it is useful for applications to specialize the theory of exterior differential forms to two species: Pair and Impair differential forms.

- Pair exterior differential forms are scalar invariants with respect to diffeomorphic transformations of the independent variables. Pair exterior differential forms are well defined pull_back scalars with respect to C1/d transformations, where the pull_back linear mappings of the coefficients are determined by the *transpose* of the Jacobian matrix. When applied to the theory of electromagnetism, it will become evident that the Pair forms are related to concepts of Forces and field intensities, E and B.

- Impair exterior differential forms are pseudo-scalar invariants; that is, Impair forms are scalars to within a factor Δ^{-1}. Impair exterior differential forms are well defined pull_back pseudo scalars with respect to C1/d transformations, where the pull_back linear mappings are determined by the *adjoint* of the Jacobian matrix. When applied to the theory of electromagnetism, it will become evident that the Impair forms are related to concepts of Sources, and field quantities, D and H.

- Note that relative to the constraint of special or proper unimodular groups of transformations, where $\Delta^{-1} = +1$, the two species of differential forms are not distinguishable. It is also remarkable that, relative to C1/d transformations, the pull_back is well defined, but the push_forward is not. Unique prediction fails, but retrodiction is deterministic. An "arrow of time" is built into the logic of differential forms.

1.0.2. The Constraints of Tensor Analysis

In tensor analysis, there exist two important categories of tensors, known as covariant tensors and contravariant tensors. These sets are distinguished by their transformation properties with respect to invertible diffeomorphisms. By classical convention:

- The functional components of Covariant vectors are pushed forward from the initial state to the final state by means of the linear transformations induced by the Inverse of the Jacobian matrix of the diffeomorphism between independent variables. It is usually not emphasized that Covariant vectors pulled back by means of maps induced by the transpose of the Jacobian matrix.

- The functional components of Contravariant vectors are pushed forward from the initial state to the final state by means of the linear transformations induced by the Jacobian matrix of the diffeomorphism between independent variables. It is usually not emphasized that Contravaraint are pulled back by means of maps induced by the Inverse of the Jacobian matrix.\

A prefered notation for the two species of differential forms (using ordered or collective indicies) is demonstrated by a Pair 1-form, A, and an Impair N-1 form, J.

$$A = a\,"pair"\ exterior\ differential\ 1-form \tag{1.1}$$

$$A = A_k(x)dx^k \tag{1.2}$$

$$F = a\,"pair"\ exterior\ differential\ 2-form : F = F_{jk}(x)dx^j\hat{}dx^k \tag{1.3}$$

$$J = an\,"\mathrm{Im}pair"\ exterior\ differential\ N-1\ form\ density \tag{1.4}$$

$$J = J^k(x)\{dx^1\hat{}dx^2...\widehat{dx}^k...dx^N\} \tag{1.5}$$

$$G = an \text{ ''} \mathrm{Im} pair\text{''} \; exterior \; differential \; N-2 \; form \; density \quad (1.6)$$

$$G = G^{jk}(x)\{dx^1\hat{}dx^2...\widehat{dx}^j...\widehat{dx}^k...dx^N\} \quad (1.7)$$

The hatted symbol $\widehat{dx}^k$ means that that term dx^k is left out of the factors that form the volume element. Note that the coefficients of the Impair forms is defined with an upper index which is a collective compliment of "left out" factors in the volume element.

Some details and examples are presented below.

2. Functional Substitution and the pull_back Examples

Cartan's theory of exterior differential forms is NOT just another notational system of fancy. The important fact, often ignored, is that a differential form is not necessarily a tensor. In short, a differential form is a mathematical object that is well behaved with respect to differentiable maps, $y^k \Rightarrow x^\mu = \phi^\mu(y^k)$, from an initial state $\{y^k\}$ of independent variables to a final state $\{x^\mu\}$ of independent variables. Such maps need not be homeomorphisms. Hence the topology of the final state need not be the same as the topology of the initial state.

A tensor is an object which is well behaved if and only if the differentiable map ϕ^μ from initial to final state has a differentiable inverse. Such maps are defined to be diffeomorphisms, a subset of homeomorphisms, and as such the topology of the final state and the initial state must be the same.

Differential forms can be used to understand and study topological evolution, while tensors cannot.

2.0.3. Contravariant and covariant tensors

A tensor is a restricted class of mathematical objects that transform in a multilinear manner relative to the Jacobian matrix $[\mathbb{J}^\mu_k] = [\partial\phi^\mu(y)/\partial y^k]$ of the differentiable map, and its inverse. The motivation for the definitions is most transparent when one considers the matrix inner product of two vectors, $\langle A_k(y)| \circ |V^k(y)\rangle$, on the initial state. Suppose that the Identity matrix can be written in terms of two factors, $[\mathbb{I}] = [\mathbb{J}]^{-1} \circ [\mathbb{J}]$. Then the equation,

$$\langle A_k| \circ |V^k\rangle \Rightarrow \langle A_k| \circ [\mathbb{J}^\mu_k]^{-1} \circ [\mathbb{J}^\mu_k] \circ |V^k\rangle = \langle \overline{A}_\mu(y)| \circ |\overline{V}^\mu(y)\rangle, \quad (2.1)$$

can be interpreted in terms of a matrix transformation, where

$$\left|\overline{V}^{\mu}(y)\right\rangle = [\mathbb{J}_{k}^{\mu}(y)] \circ \left|V^{k}(y)\right\rangle \tag{2.2}$$

and

$$\langle A_k(y)| \Rightarrow \left\langle \overline{A}_{\mu}(y)\right| = \langle A_k| \circ [\mathbb{J}_{k}^{\mu}(y)]^{-1} \tag{2.3}$$

in such a way that the value of the matrix row-column vector product before the transformation and after the transformation has the same value. Note that the vector with components indexed by μ on the final state has arguments in terms of the independent variables on the initial state. IN order to produce a vector array of functions on the final state with arguments in terms of the variables x on the final state requires knowledge of the inverse mapping that expresses y in terms of x. The column vector $|V^{\mu}(y)\rangle$ is the epitome of a contra-variant vector. The row vector $\langle A_{\mu}|$ is the epitome of a co-variant vector. The matrix $[\mathbb{J}_{k}^{\mu}(y)]$ is the Jacobian matrix of the transformation from intitial to final state. The matrix transformation rules above give primitive meaning to the concepts of co-variance and contra-variance. These rules in tensor analysis are extended to multi-linear objects.

The classical definitions consider a tensor to be a mathematical object whose functional form given on the initial state is well defined on the final state. Note that the tensor operations are a "push_forward" from the initial to the final state. The transformation rules are of two types are defined as:

$$\begin{aligned}
The\ Contravariant\ Rule &: (pushforward) && (2.4)\\
\left|V^{k}(y)\right\rangle &\Rightarrow |V^{\mu}(y)\rangle = \left[\partial x^{\mu}(y)/\partial y^{k}\right]\left|V^{k}(y)\right\rangle && (2.5)\\
&\Rightarrow \left|\overline{V}^{\mu}(x)\right\rangle = |V^{\mu}(y(x))\rangle && (2.6)
\end{aligned}$$

$$\begin{aligned}
The\ Covariant\ Rule &: (push\ forward) && (2.7)\\
A_k(y) &\Rightarrow A_{\mu}(y) = A_k(y)\left[\partial y^{k}(x)/\partial x^{\mu}\right] && (2.8)
\end{aligned}$$

In both cases an inverse mapping ($x = \phi^{-1}(y)$) is required if the arguments of the functions created on the final state are to be expressed in terms of the independent variables, x, of the final state.

When the tensor rule for covariant transformations is multiplied on both sides by the Jacobian matrix, and using the constraint inherent in the tensor definitions that an inverse Jacobian exists, a more general formula for covariant transformations can be obtained. If the functions that make up the covariant object are given on the final state, then the functions on the initial state are well defined in terms of the independent variables on the initial state. This more general situation works when the map is differentiable, but no inverse map exists. The process is defined as the covariant pull_back. (see http://www22.pair.com/csdc/ed3/ed3fre1.htm)

$$\textit{The Covariant Rule (pull\_back)} : |A_k(y)\rangle = A_\mu(x(y)) \left[\partial x^\mu(y)/\partial y^k\right] \Leftarrow A_\mu(x) \tag{2.9}$$

2.0.4. Retrodiction vs Prediction

It has been shown [1] that with respect to maps without differentiable inverses, the classic rules of tensor transformations do not permit the unique prediction of the functional form on the final state of either contravariant or covariant tensors, given the functional form of these tensors on the initial state. Numeric values on the final state sometimes can be predicted, but the functional form (defining neighborhoods) on the final state cannot be uniquely predicted. It also has been shown relative to maps without differentiable inverses that a given the functional form of a contra-tensor on the final state does not permit a unique retrodiction of the functional form on the initial state.

However, there are two situations where unique well defined retrodiction is possible in a functional sense. This retrodiction process is defined as the "Pull-Back" The first situation involves covariant tensors on the final state. The second situation involves contravariant antisymmetric tensor densities on the final state. Given a mapping and its Jacobian, $[J_k^\mu(y)]$, the two forms of pull_back may be written as:

$$\textit{pull\_back Covariant Rule} : \overline{A}_k(y) \Leftarrow A_\mu(x(y))\,[J_k^\mu(y)] \tag{2.10}$$

$$\textit{pull\_back Contravariant density Rule} : \overline{C}^k(y) \Leftarrow \left[AdJ_\mu^k(y)\right] C^\mu(x(y)). \tag{2.11}$$

In the notation above, $\left[AdJ_\mu^k(y)\right]$, is the adjoint of the Jacobian matrix (matrix of cofactors transposed) and does not depend upon the determinant of the transformation. The adjoint matrix may be algebraically determined even when the determinant of the matrix is zero. Hence the pull_back is well defined, even if an inverse does not exist. The contravariant density, or current, $C^\mu(x)$, is like the charge current density of electromagnetism. It is suggested herein that these two rules should be used as foundations for transformations of objects which would be considered to be tensors, if the mappings are constrained to be diffeomorphisms.

2.0.5. Derivation of the pull_back transformation rules:

First consider differential forms whose coefficients would be covariant tensor fields, if the Jacobian matrix has an inverse. A typical representation is a 1-form written in terms of the variables on the final state as:

$$A = A_\mu(x^\nu)\, dx^\mu = \langle A_\mu(x^\nu)| \circ |dx^\mu\rangle \, . \tag{2.12}$$

Consider the differentiable non-linear map

$$\phi : \; y^k \Rightarrow x^\mu = \phi^\mu(y^j), \tag{2.13}$$

$$d\phi : \; dy^k \Rightarrow dx^\mu = \left[\partial\phi^\mu(y^j)/\partial y^k\right] \, dy^k = \left[J_k^\mu(y^j)\right] \circ \left| \, dy^k \right\rangle . \tag{2.14}$$

Substitute these formulas into the expression for the differential 1-form expressed in terms of the independent variables on the final state:

$$\begin{aligned} A &= \langle A_\mu(x^\nu)| \circ |dx^\mu\rangle = \left\langle A_\mu(\phi^\nu(y^j))\right| \circ \left[J_k^\mu(y^j)\right] \circ \left| \, dy^k \right\rangle & (2.15)\\ &= \left\langle \overline{A}_k(y^j)\right| \circ \left| \, dy^k \right\rangle . & (2.16)\end{aligned}$$

The coefficents $\left\langle \overline{A}_k(y^j)\right|$ are well defined functions on the initial state, with arguments in terms of the initial state variables. Now if the map from initial to final state is such that the Jacobian is an invertible matrix, then the coefficient variables

$$\langle A_\mu(x^\nu)| \circ \left[J_k^\mu(y^j)\right] = \left\langle A_\mu(\phi^\nu(y^j))\right| \circ \left[J_k^\mu(y^j)\right] = \left\langle \overline{A}_k(y^j)\right| \tag{2.17}$$

is equivalent to the transformation rules of a covariant tensor field in classic tensor analysis:

$$\textit{pull\_back Rule} : A_\mu \partial x^\mu/\partial y^k = \overline{A}_k \tag{2.18}$$

Note: The pull_back coefficients are not tensor equivalents when the Jacobian matrix is not invertible. However, the pull_back is always well defined in terms of the language which uses the Jacobian matrix transpose. The transpose always exists even though the inverse does not.

2.0.6. Example 1: pull_back of a 1-form

Consider the example map from 3 to 3 dimensions, for which an **inverse Jacobian does not exist.**

$$\phi : \{X, Y, Z\} \Rightarrow \{x, y, z\} = \{XY, Y^3, X\} \tag{2.19}$$

$$d\phi : \{dX, dY, dZ\} \Rightarrow \{dx, dy, dz\} = \{Y dX + X dY, 3Y^2 dY, dX\} \tag{2.20}$$

$$\begin{aligned} Jacobian\ [J_k^\mu(X)] &= \left[\partial\phi^\mu(x^j)/\partial X^k\right] = \begin{bmatrix} Y & X & 0 \\ 0 & 3Y^2 & 0 \\ 1 & 0 & 0 \end{bmatrix} && (2.21) \\ \det\,[J_k^\mu(X)] &= 0 && (2.22) \end{aligned}$$

$$\begin{aligned} Given\ on\ final\ state\ \ A &= y\,dx - xdy + dz && (2.23) \\ Substitute\ \phi\ and\ d\phi &= Y^3(Y dX + X dY) - XY3Y^2 dY + dX \\ pull\_back\ to\ initial\ state &= (Y^4+1)dX - (2Y^3X)dY + 0dZ && (2.24) \end{aligned}$$

$$\begin{aligned} Coefficients\ on\ final\ state\ \ \langle A_\mu| &= [y,\ -x\,,+1] && (2.25) \\ pull\_back\ to\ initial\ state\ \left\langle \overline{A}_k\right| &= [(Y^4+1),\ -(2Y^3X), 0] && (2.26) \\ &= \langle A_\mu| \circ [J_k^\mu] && (2.27) \\ &= \langle y,\ -x\,,+1| \circ \begin{bmatrix} Y & X & 0 \\ 0 & 3Y^2 & 0 \\ 1 & 0 & 0 \end{bmatrix} && (2.28) \\ &= \left\langle yY+1\,, yX - x3Y^2,\ 0\right| && (2.29) \\ &= \left\langle (Y^4+1),\ -(2Y^3X),\ 0\right| && (2.30) \end{aligned}$$

The example demonstrates the fact that the coefficients of the differential forms do not behave as tensors with respect to the non-invertible, but differentiable map. Yet, everything is well defined, functionally, with respect to the pull_back operation.

2.0.7. Example 2: pull_back of a 2-form

For the same map given in the first example, consider the 2-form below

$$\begin{aligned} & & Given\ on\ final\ state \\ F &=& F_{xy}dx\hat{}dy + F_{yz}dy\hat{}dz + F_{zx}dz\hat{}dx && (2.31) \\ & & Substitute \\ &=& F_{xy}(3Y^3dX\hat{}dY) - F_{yz}(3Y^2)dX\hat{}dY + F_{zx}(XdX\hat{}dY) && (2.32) \\ & & pull\_back\ to\ initial\ state \\ &=& \{F_{xy}(3Y^3) - F_{yz}(3Y^2) + F_{zx}(X)\}dX\hat{}dY && (2.33) \end{aligned}$$

$$\begin{aligned} & & Coefficients\ on\ final\ state \\ \langle F_{\mu\nu}| &=& \langle F_{xy},\ F_{yz}\ ,F_{zx}| && (2.34) \\ & & pull\_back\ to\ initial\ state \\ \left\langle \overline{F}_{XY}\right| &=& \left\langle \{-F_{yz}(3Y^2) + F_{zx}(X) + F_{xy}(3Y^3)\}, 0, 0\right| && (2.35) \end{aligned}$$

2.0.8. Example 3. pull_back of a contravariant tensor delta density and the adjoint Jacobian

Now consider the volume element 3-form

$$Vol3 := dx\hat{}dy\hat{}dz = (\det[J])dX\hat{}dY\hat{}dZ \qquad (2.36)$$

and the Current N-1 form delta-density,

$$C = i(C^\mu)Vol3 = C^x dy\hat{}dz - C^y dz\hat{}dx + C^z dx\hat{}dy. \qquad (2.37)$$

The adjoint matrix to the Jacobian (for which no inverse exists) is

$$[AdJ] = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -3Y^2 & X & 3Y^3 \end{bmatrix}. \qquad (2.38)$$

Substitution yields

$$\begin{aligned} C &=& -C^x(3Y^2)dX\hat{}dY + C^y(XdX\hat{}dY) + C^z 3Y^3 dX\hat{}dY && (2.39) \\ &=& \{-(3Y^2)C^x + (X)C^y + 3Y^3C^z\}dX\hat{}dY && (2.40) \\ &=& \overline{C}^Z dX\hat{}dY, && (2.41) \end{aligned}$$

from which it is apparent that

$$\left|\begin{array}{c}\overline{C}^X\\ \overline{C}^Y\\ \overline{C}^Z\end{array}\right\rangle \Leftarrow [AdJ]\circ\left|\begin{array}{c}C^x\\ C^y\\ C^z\end{array}\right\rangle. \tag{2.42}$$

Note that these results are valid for the case where the inverse Jacobian does not exist.

For those cases where the inverse matrix does exist, it is apparent that the Current coefficients do not transform as a contravariant tensor, but instead transform as a contravariant tensor delta density: Multiplication of both sides of the preceding equation by the Jacobian matrix yields:

$$[J_k^\mu]\circ|\overline{C}^k(Y^j)\rangle = J_k^\mu\overline{C}^k \Rightarrow \Delta\cdot C^\mu(x) = (\det[J_k^\mu])\cdot C^\mu(x). \tag{2.43}$$

If the coefficients transformed as a contravariant tensor, then the preceeding formula would have been written as

$$J_k^\mu\overline{C}^k \Rightarrow C^\mu(x). \tag{2.44}$$

Tensor delta densities are sensitive to the sign and magnitude of the determinant of the mapping.

3. Summary: Pair forms and Impair forms

The bottom line is that retrodiction and functional substitution leads to two well defined species of differential form objects: Pair (even) forms with antisymmetric coefficients, which pull_back relative to the Transpose of the Jacobian of the mapping, and Impair (odd) forms with antisymmetric coefficients objects that pull_back via Adjoint of the Jacobian matrix. These latter objects are sensitive to orientations, where the former are not. Both objects are not dependent upon an invertible differential mapping, hence can be used to study continuous topological evolution. However, when the maps from an initial state of independent variables to a final state of independent variables are unimodular det =1 diffeomorphisms, then the coefficients of the pair forms behave as covariant tensors, and the coefficients of the impair forms behave as contravariant tensors. It the diffeomorphisms are not such that det of the Jacobian matrix is 1, then the coefficients transform as contravariant tensor densities.

- The 2-form F of electromagnetic field intensities is a pair differential form. The integrals of exterior differential forms are invariants of diffeomorphisms which may change the sign of the N-volume (a change of orientation). Such integrals are scalars.
- The N-2 form G is an impair 2 form density in a space of 4 dimensions. The integrals of impair differential form densities change sign with respect to diffeomorphisms that change the N-volume orientation. Such integrals are pseudoscalars and can lead to enantiomorphic pairs.

4. References

R. M. Kiehn, Retrodictive Determinism, Int. J. Engng. Sci 1976 **14** pp. 249-754