```
> restart:
```

Pullbacks of maps to Euclidean spaces.

R. M. Kiehn
Updated 12/03/2001

The transformation properties of Maxwell's Electrodynamics in terms of D E B and H are carried out in detail for euclidean translations and euclidean rotations. Contrary to current dogma, in all cases proper transformations of the field components yield **invariant** formalisms for the PDE's that generate the Maxwell-Faraday and the Maxwell-Ampere equations. The relative motions do not change the PDE's but do modify the constitutive relations between E,B and D,H. Motions can induce currents that produce D and H field components, which are NOT associated with E and B field components.

When the components of the fields with respect to the independent variables are written with respect to the PullBack transformation schemes (E B are components of a pair form. D and H are components of a impair form) then the PDE's of Maxwell are invariant in format.

$$\mathrm{curl}(E, [x, y, z]) + \left(\frac{\partial}{\partial t} B\right) = 0.$$
$$\mathrm{div}(B, [x, y, z]) = 0$$
$$\mathrm{curl}(H, [x, y, z]) - \left(\frac{\partial}{\partial t} \mathrm{D}\right) = J$$
$$\mathrm{div}(\mathrm{D}, [x, y, z]) = \rho$$

PULLBACK Initial state Maxwell Equations with independent variables [r,theta,z,t] and fields E_PB,B_PB,D_PB,H_PB

$$\mathrm{curl}(E\_PB, [r, \theta, z]) + \left(\frac{\partial}{\partial t} B\_PB\right) = 0.$$
$$\mathrm{div}(B\_PB, [r, \theta, z]) = 0$$
$$\mathrm{curl}(H\_PB, [r, \theta, z]) - \left(\frac{\partial}{\partial t} D\_PB\right) = J\_PB$$
$$\mathrm{div}(D\_PB, [r, \theta, z]) = rho\_PB$$

TRANSLATIONS at Constant Velocity.

A: The initial state motion is assumed to be parallel to the direction of the current on the final state.
B: The initial state motion is assumed to be orthogonal to the direction of the current on the final state.

```
> restart;with(linalg):with(liesymm):with(difforms):defform(L=0,f=0,u=0,x=0,y=0,z=0,t=0,lambda=0,C=const,B=const,Phi=0,FF=0,phi=0,f1=0,f2=0,f3=0,JX=0,JY=0,JZ=0,Vx=const,Vy=const,Vz=const,AX=0,AY=0,AZ=0,A4=0,ax=const,DX=0,DY=0,DZ=0,HX=0,HY=0,HZ=0,j1=0,j2=0,j3=0,j4=0,rho=0,k=const,omega=const,JT=0,a=const,b=const,mu=const,epsilon=const,e=const,n=const,Omega=const,c=const,X=0,Y=0,Z=0,T=0,r=0,alpha=0,beta=0,theta=0,ZR=const):
```

```
Warning, the protected names norm and trace have been redefined and unprotected

Warning, the protected name close has been redefined and unprotected

Warning, the names &^, d and wdegree have been redefined
```

The following is the procedure for computing EM fields and currents given the potentials on the variety [x,y,z,t], with the constitutive assumption D = epsilon E, B = mu H. The procedure also evalutes the forms A, F, G, J on the final state.

USEFUL OUTPUT FUNCTIONS ARE
AF=vector potential
SP = scalar potential
EF = E field intensity
BF = B field intensity
DF = D field excitation
HF = H field excitation
JD = current density
CD = charge density
SPC = spin current
SPD = spin density
TFC = Torsion flux
HEL = Helicity
TF = Torsion field 4 components
SP = Spin field density 4 components
P1 = First Poincare invariant density
P2 = Second Poincare invariant
A1form = pair 1-form of potentials
F2form = pair 2-form of E, B field intensities
G2form = impair 2-form of D, H excitation densities
J3form = impair 3 form of charge current densities

```
> JCM:=proc(Ax,Ay,Az,phi)\
     local
A,A1,A2,A3,A4,BFC,EF1,EF2,EF3,JAC,JDC,ExBC,JTOT,Jcurl,Jt,Jh,Jd,Jdt,TFCa,SFCa:
     global
A1form,AF,SP,BF,EF,TF,HEL,P1,P2,DF,HF,CD,JA,JD,SPD,SF,SFD,ExB,G2form,F2form,J3f
orm,JTT,TFC,SFC:
     A1:=Ax:A2:=Ay:A3:=Az;A4:=phi:
A:=[A1,A2,A3]:AF:=factor(simplify(A)):SP:=simplify(phi):
A1form:=A1*d(x)+A2*d(y)+A3*d(z)-A4*d(t):
     EF1:=evalm(-grad(phi,[x,y,z])):
     EF2:=-[diff(A1,t),diff(A2,t),diff(A3,t)];
EF3:=[factor(EF1[1]+EF2[1]),factor(EF1[2]+EF2[2]),factor(EF1[3]+EF2[3])];
     EF:=[EF3[1],EF3[2],EF3[3]];
     BFC:=(curl([A1,A2,A3],[x,y,z])):
     BF:=[factor(BFC[1]),factor(BFC[2]),factor(BFC[3])];
HEL:=factor(innerprod(AF,BF));TFCa:=evalm(crossprod(EF,AF)+BF*phi);
     TF:=[factor(TFCa[1]),factor(TFCa[2]),factor(TFCa[3]),HEL];

P2:=-2*factor(innerprod(EF,BF));TFC:=[factor(TFCa[1]),factor(TFCa[2]),factor(TF
Ca[3])];
```

```
     HF:=[factor(BFC[1]/mu),factor(BFC[2]/mu),factor(BFC[3]/mu)];
DF:=[factor(epsilon*EF3[1]),factor(epsilon*EF3[2]),factor(epsilon*EF3[3])];
CD:=factor(diverge([DF[1],DF[2],DF[3]],[x,y,z]));
Jcurl:=curl([HF[1],HF[2],HF[3]],[x,y,z]);
     Jh:=[factor(Jcurl[1]),factor(Jcurl[2]),factor(Jcurl[3])];
Jdt:=-[diff(DF[1],t),diff(DF[2],t),diff(DF[3],t)];
Jd:=[factor(Jdt[1]),factor(Jdt[2]),factor(Jdt[3])];JTOT:=Jh+Jd:JD:=[factor(JTOT
[1]),factor(JTOT[2]),factor(JTOT[3])];
JTT:=[factor(JD[1]),factor(JD[2]),factor(JD[3]),CD];
F2form:=evalm(innerprod(BF,[d(y)&^d(z),-d(x)&^d(z),d(x)&^d(y)])+innerprod(EF,[d
(x)&^d(t),d(y)&^d(t),d(z)&^d(t)]));
SPD:=factor(innerprod(A,DF));G2form:=(HF[1]*d(x)&^d(t)+HF[2]*d(y)&^d(t)+HF[3]*d
(z)&^d(t)-DF[1]*d(y)&^d(z)+DF[2]*d(x)&^d(z)-DF[3]*d(x)&^d(y));J3form:=innerprod
(JTT,[d(y)&^d(z)&^d(t),-d(x)&^d(z)&^d(t),d(x)&^d(y)&^d(t),-d(x)&^d(y)&^d(z)]);

SFCa:=evalm(crossprod(AF,HF)+DF*phi);SFC:=[factor(SFCa[1]),factor(SFCa[2]),fact
or(SFCa[3])];
     SFD:=[factor(SFC[1]),factor(SFC[2]),factor(SFC[3]),SPD];
     P1:=innerprod(BF,HF)-innerprod(DF,EF)-innerprod(AF,JD)+CD*phi;
ExBC:=crossprod(EF,BF);ExB:=[factor(ExBC[1]),factor(ExBC[2]),factor(ExBC[3])];
  end proc:
> MAP:=proc(X,Y,Z,T,x,y,z,t) global JAC,ADJAC,DET,TRJAC,Map:
  Map:=[x,y,z,t]:JAC:=simplify(jacobian(Map,[X,Y,Z,T])):DET:=factor(simplify(de
  t(JAC))):TRJAC:=simplify(transpose(JAC)):ADJAC:=simplify(adjoint(JAC)): end
  proc:
>
>
```

TRANSLATION PARALLEL TO CURRENT

Consider a final state of independent variables, [x,y,z,t],
and a map from an initial state of cartesian variables [X,Y,Z,T].
Assert the existence of an additional constraint that represents a kinematic translation at constant velocity along the z axiz. Hence d(Z)/d(T) = Vz = constant.
The electromagnetic Action 1-form A will be given in terms of abstract functions Ax,Ay,Az,Phi, on the final state, with arguments in terms of [x,y,z,t].
The pair 2-form F = dA and its coefficients on the final state will be evaluated to yield E and B, the field intensities.
The final state will be presumed to be a euclidean space with classic constitutive properties in the sense that D = epsilon E and B = mu H.
Hence, on the final state, given the potentials, the field intensities can be computed, the field excitations D, H, can be evaluated to give the impair 2-form G.
From this hypothesis (or constraint) the impair 3-form of 4 current density will be computed from the impair 2-form G, such that J=dG.
The results on the final state then will be pulled back by the combined actions of functional substitution for the independent variables and their differentials, into the differential forms for A, F, G and J on the initial state.
For 1-forms and N-1 form densities, the pullbacks are particularly simple:
The 1-forms pull back by means of the transpose of the Jacobian matrix of the mapping.
The N-1 form densities pull back via the adjoint of the Jacobian matrix of the mapping.
The matrix elements of the Jacobian matrix need not be global constants.

```
>
```

```
> b:=0:c:=0:
> rr:=(x^2+y^2)^(1/2):AAA:=[z^2*y*b,-z^2*x*b,1-k^2*rr^2,c*z*y*x+omega*(1-k^2*rr^2)/k]:
> JCM(AAA[1],AAA[2],AAA[3],AAA[4]):
```

The exterior differential forms as specified on the final state.

```
> A1form:=wcollect(factor(simplify(A1form)));F2form:=
  wcollect(factor(simplify(F2form)));G2form:=
  wcollect(factor(simplify(G2form)));J3form:=
  wcollect(factor(simplify(J3form)));
```

$$A1form := (1 - k^2 x^2 - k^2 y^2)\, \mathrm{d}(z) + \frac{\omega (-1 + k^2 x^2 + k^2 y^2)\, \mathrm{d}(t)}{k}$$

$$F2form := -2 k^2 y\, (\mathrm{d}(y) \mathbin{\&\wedge} \mathrm{d}(z)) - 2 k^2 x\, (\mathrm{d}(x) \mathbin{\&\wedge} \mathrm{d}(z)) + 2 \omega k x\, (\mathrm{d}(x) \mathbin{\&\wedge} \mathrm{d}(t)) + 2 \omega k y\, (\mathrm{d}(y) \mathbin{\&\wedge} \mathrm{d}(t))$$

$$G2form := -2 \frac{k^2 y\, (\mathrm{d}(x) \mathbin{\&\wedge} \mathrm{d}(t))}{\mu} + \frac{2 k^2 x\, (\mathrm{d}(y) \mathbin{\&\wedge} \mathrm{d}(t))}{\mu} - 2 \varepsilon \omega k x\, (\mathrm{d}(y) \mathbin{\&\wedge} \mathrm{d}(z)) + 2 \varepsilon \omega k y\, (\mathrm{d}(x) \mathbin{\&\wedge} \mathrm{d}(z))$$

$$J3form := 4 \frac{k^2 \mathbin{\&\wedge}(\mathrm{d}(x), \mathrm{d}(y), \mathrm{d}(t))}{\mu} - 4 k \varepsilon \omega \mathbin{\&\wedge}(\mathrm{d}(x), \mathrm{d}(y), \mathrm{d}(z))$$

The fields in engineering format on the final state

```
> R:=[x,y,z,t];Vector_potential:=AF;Scalar_potential:=factor(SP);E_field:=EF;B_field:=BF;Poincare2:=factor(P2);D_field:=DF;H_field:=HF;rho_charge_density:=CD;J_current_density:=(simplify(JD));;Poincare1:=factor(P1);PoyntingVector=ExB;Torsion_flux:=factor(evalm(TFC));Helicity:=HEL;Spin_current:=factor((SFC));Spin_density:=factor(SPD);Lagrangian_field_energy_density:=factor(simplify(innerprod(HF,BF)-innerprod(DF,EF)));Interaction_energy_density:=factor(AF[1]*JD[1]+AF[2]*JD[2]+AF[3]*JD[3]-CD*SP);
```

$$R := [x, y, z, t]$$

$$Vector\_potential := [0, 0, 1 - k^2 x^2 - k^2 y^2]$$

$$Scalar\_potential := -\frac{\omega (-1 + k^2 x^2 + k^2 y^2)}{k}$$

$$E\_field := [2 \omega k x, 2 \omega k y, 0]$$

$$B\_field := [-2 k^2 y, 2 k^2 x, 0]$$

$$Poincare2 := 0$$

$$D\_field := [2 \varepsilon \omega k x, 2 \varepsilon \omega k y, 0]$$

$$H\_field := \left[-2 \frac{k^2 y}{\mu}, 2 \frac{k^2 x}{\mu}, 0\right]$$

$$rho\_charge\_density := 4 \varepsilon \omega k$$

$$J\_current\_density := \left[0, 0, 4 \frac{k^2}{\mu}\right]$$

$$Poincare1 := 4 \frac{(2 k^2 x^2 + 2 k^2 y^2 - 1)(k^2 - \omega^2 \varepsilon \mu)}{\mu}$$

$$PoyntingVector = [0, 0, 4 k^3 \omega (x^2 + y^2)]$$

$$Torsion\_flux := [0, 0, 0]$$

$$Helicity := 0$$

$$Spin\_current := \left[2 \frac{(-1 + k^2 x^2 + k^2 y^2)\, x\, (k^2 - \omega^2 \varepsilon \mu)}{\mu}, 2 \frac{(-1 + k^2 x^2 + k^2 y^2)\, y\, (k^2 - \omega^2 \varepsilon \mu)}{\mu}, 0\right]$$

$$Spin\_density := 0$$

$$Lagrangian\_field\_energy\_density := 4\,\frac{k^2\,(x^2+y^2)\,(k^2-\omega^2\,\varepsilon\,\mu)}{\mu}$$

$$Interaction\_energy\_density := -4\,\frac{(-1+k^2\,x^2+k^2\,y^2)\,(k^2-\omega^2\,\varepsilon\,\mu)}{\mu}$$

```
> R_final_variables:=R;
```

$$R\_final\_variables := [x, y, z, t]$$

Variables on Initial State:

```
> X:=X:Y:=Y:Z:=Z:T:=T:
>
```

Define the mapping functions here
TRANSLATION PARALLEL TO CURRENT

```
> x:=X:y:=Y:z:=Z-Vz*T:t:=T:
>
> R_initial_variables:=[X,Y,Z,T];
```

$$R\_initial\_variables := [X, Y, Z, T]$$

```
> MAP(X,Y,Z,T,x,y,z,t):Mapping_functions:=Map;
```

$$Mapping\_functions := [X, Y, Z - Vz\,T, T]$$

```
> Jacobian:=evalm(JAC):DET:=DET:Adjoint:=evalm(ADJAC):
```

The map represents a translation along the Z axis.

Evaluate the exterior forms on the initial state by functional substitution and pullback:

```
> A1form:=wcollect(factor(simplify(A1form)));F2form:=
  wcollect(factor(simplify(F2form)));G2form:=
  wcollect(factor(simplify(G2form)));J3form:=
  wcollect(factor(simplify(J3form)));
```

$$A1form := -\frac{(-1+k^2\,X^2+k^2\,Y^2)\,(-k\,Vz-\omega)\,\mathrm{d}(T)}{k} + (1-k^2\,X^2-k^2\,Y^2)\,\mathrm{d}(Z)$$

$$F2form := -2\,k\,(-k\,Y\,Vz-\omega\,Y)\,(\mathrm{d}(Y)\ \&\wedge\ \mathrm{d}(T)) - 2\,k^2\,Y\,(\mathrm{d}(Y)\ \&\wedge\ \mathrm{d}(Z)) - 2\,k^2\,X\,(\mathrm{d}(X)\ \&\wedge\ \mathrm{d}(Z)) - 2\,k\,(-k\,X\,Vz-\omega\,X)\,(\mathrm{d}(X)\ \&\wedge\ \mathrm{d}(T))$$

$$G2form := 2\,\frac{k\,(\varepsilon\,\omega\,X\,\mu\,Vz+k\,X)\,(\mathrm{d}(Y)\ \&\wedge\ \mathrm{d}(T))}{\mu} - 2\,k\,\varepsilon\,\omega\,X\,(\mathrm{d}(Y)\ \&\wedge\ \mathrm{d}(Z)) + 2\,k\,\varepsilon\,\omega\,Y\,(\mathrm{d}(X)\ \&\wedge\ \mathrm{d}(Z)) + \frac{2\,k\,(-k\,Y-\varepsilon\,\omega\,Y\,\mu\,Vz)\,(\mathrm{d}(X)\ \&\wedge\ \mathrm{d}(T))}{\mu}$$

$$J3form := -4\,\varepsilon\,\omega\,k\ \&\wedge(\mathrm{d}(X), \mathrm{d}(Y), \mathrm{d}(Z)) + \frac{4\,k\,(k+Vz\,\mu\,\omega\,\varepsilon)\ \&\wedge(\mathrm{d}(X), \mathrm{d}(Y), \mathrm{d}(T))}{\mu}$$

```
> Spin3form:=(A1form&^G2form);Torsion3form:=(A1form&^F2form);
```

$$Spin3form := 2\,\frac{(-1+k^2\,X^2+k^2\,Y^2)\,X\,(k^2-\omega^2\,\varepsilon\,\mu)\ \&\wedge(\mathrm{d}(T), \mathrm{d}(Y), \mathrm{d}(Z))}{\mu} - \frac{2\,(-1+k^2\,X^2+k^2\,Y^2)\,Y\,(k^2-\omega^2\,\varepsilon\,\mu)\ \&\wedge(\mathrm{d}(T), \mathrm{d}(X), \mathrm{d}(Z))}{\mu}$$

$$Torsion3form := 0$$

```
> SP;
```

$$-\frac{\omega\,(-1+k^2\,X^2+k^2\,Y^2)}{k}$$

Pullback field Components on the initial state:

```
> AF_PB:=innerprod(TRJAC,[AF[1],AF[2],AF[3],-SP]);VPotential_PB:=simplify([AF_P
  B[1],AF_PB[2],AF_PB[3]]);ScalarPot_PB:=simplify(-AF_PB[4]);EF_PB:=factor(simp
```

```
lify((evalm(-grad(ScalarPot_PB,[X,Y,Z])-diff(VPotential_PB,T)))));BF_PB:=factor(simplify(curl(VPotential_PB,[X,Y,Z])));D1:=-getcoeff(G2form&^d(X)&^d(T)):D2:=getcoeff(G2form&^d(Y)&^d(T)):D3:=-getcoeff(G2form&^d(Z)&^d(T)):H1:=getcoeff(G2form&^d(Y)&^d(Z)):H2:=getcoeff(G2form&^d(X)&^d(Z)):H3:=-getcoeff(G2form&^d(X)&^d(Y)):DF_PB:=[factor(simplify(D1)),factor(simplify(D2)),factor(simplify(D3))];HF_PB:=[factor(simplify(H1)),factor(simplify(H2)),factor(simplify(H3))];JTPB:=innerprod(ADJAC,JTT):JD_PB:=[JTPB[1],JTPB[2],JTPB[3]];JC_PB:=JTPB[4];Poincare2_PB:=Poincare2;Poincare1_PB:=factor(simplify(DET*Poincare1));DET:=DET;factor(simplify(innerprod(BF,HF)-innerprod(EF,DF))):M_F:=evalm(curl(EF_PB,[X,Y,Z])+[diff(BF_PB[1],T),diff(BF_PB[2],T),diff(BF_PB[3],T)]);M_A:=evalm(curl(HF_PB,[X,Y,Z])-[diff(DF_PB[1],T),diff(DF_PB[2],T),diff(DF_PB[3],T)]);M_FdivB:=diverge(BF_PB,[X,Y,Z]);M_AdivD:=diverge(DF_PB,[X,Y,Z]);
```

$$AF\_PB := \left[0, 0, 1 - k^2 X^2 - k^2 Y^2, \frac{(-1 + k^2 X^2 + k^2 Y^2)(k\, Vz + \omega)}{k}\right]$$

$$VPotential\_PB := [0, 0, 1 - k^2 X^2 - k^2 Y^2]$$

$$ScalarPot\_PB := -\frac{(-1 + k^2 X^2 + k^2 Y^2)(k\, Vz + \omega)}{k}$$

$$EF\_PB := [2\, k\, X\, (k\, Vz + \omega), 2\, k\, Y\, (k\, Vz + \omega), 0]$$

$$BF\_PB := [-2\, k^2\, Y, 2\, k^2\, X, 0]$$

$$DF\_PB := [2\, k\, \varepsilon\, \omega\, X, 2\, k\, \varepsilon\, \omega\, Y, 0]$$

$$HF\_PB := \left[-2\, \frac{k\, Y\, (k + Vz\, \mu\, \omega\, \varepsilon)}{\mu}, 2\, \frac{k\, X\, (k + Vz\, \mu\, \omega\, \varepsilon)}{\mu}, 0\right]$$

$$JD\_PB := \left[0, 0, 4\, \frac{k\, (k + Vz\, \mu\, \omega\, \varepsilon)}{\mu}\right]$$

$$JC\_PB := 4\, \varepsilon\, \omega\, k$$

$$Poincare2\_PB := 0$$

$$Poincare1\_PB := 4\, \frac{(2\, k^2 X^2 + 2\, k^2 Y^2 - 1)(k^2 - \omega^2\, \varepsilon\, \mu)}{\mu}$$

$$DET := 1$$

$$M\_F := [0, 0, 0]$$

$$M\_A := \left[0, 0, 4\, \frac{k\, (k + Vz\, \mu\, \omega\, \varepsilon)}{\mu}\right]$$

$$M\_FdivB := 0$$

$$M\_AdivD := 4\, \varepsilon\, \omega\, k$$

Note that the constitutive relations between D and E and B and H on the initial state are not the same as for the final state.
The D fields are equal to epsilon E times the DET of the transformation, there by converting a tensor into a tensor density.
Simlarly the H fields are B divided by mu times the DET of the transformation -- almost.
There appears another term in the H fields on the final state due to the translation Vz. Indeed, motion along the z axis adds to the existing Current density in the fixed frame a component that is proportional to the moving charge density. This motion induces a component to the H field that encircles the z axis, but DOES NOT affect the associated B fields.

THE PULLED BACK FIELD COMPONENTS E,B satisfy the MAXWELL - FARADAY PDE's in terms of the independent variables [X,Y,Z,T] with the constraint that d(Z)/d(T) = Vz.
THE PULLED BACK FIELD COMPONENTS D,H also satisfy the MAXWELL - AMPERE PDE's in in terms of the independent variables [X,Y,Z,T] with the constraint that d(Z)/d(T) = Vz.

****

TRANSLATION ORTHOGONAL TO CURRENT

```
> restart;with(linalg):with(liesymm):with(difforms):defform(L=0,f=0,u=0,x=0,y=0
,z=0,t=0,lambda=0,C=const,B=const,Phi=0,FF=0,phi=0,f1=0,f2=0,f3=0,JX=0,JY=0,J
Z=0,Vx=const,Vy=const,Vz=const,AX=0,AY=0,AZ=0,A4=0,ax=const,DX=0,DY=0,DZ=0,HX
=0,HY=0,HZ=0,j1=0,j2=0,j3=0,j4=0,rho=0,k=const,omega=const,JT=0,a=const,b=con
st,mu=const,epsilon=const,e=const,n=const,Omega=const,c=const,X=0,Y=0,Z=0,T=0
,r=0,alpha=0,beta=0,theta=0,ZR=const):
Warning, the protected names norm and trace have been redefined and unprotected

Warning, the protected name close has been redefined and unprotected

Warning, the names &^, d and wdegree have been redefined
```

```
>
```

Consider a final state of independent variables, [x,y,z,t],
and a map from an initial state of cartesian variables [X,Y,Z,T].
Assert the existence of an additional constraint that represents a kinematic translation at constant velocity along the z axiz. Hence d(X)/d(T) = Vx = constant.
The electromagnetic Action 1-form A will be given in terms of abstract functions Ax,Ay,Az,Phi, on the final state, with arguments in terms of [x,y,z,t].
The pair 2-form F = dA and its coefficients on the final state will be evaluated to yield E and B, the field intensities.
The final state will be presumed to be a euclidean space with classic constitutive properties in the sense that D = epsilon E and B = mu H.
Hence, on the final state, given the potentials, the field intensities can be computed, the field excitations D, H, can be evaluated to give the impair 2-form G.
From this hypothesis (or constraint) the impair 3-form of 4 current density will be computed from the impair 2-form G, such that J=dG.
The results on the final state then will be pulled back by the combined actions of functional substitution for the independent variables and their differentials, into the differential forms for A, F, G and J on the initial state.
For 1-forms and N-1 form densities, the pullbacks are particularly simple:
The 1-forms pull back by means of the transpose of the Jacobian matrix of the mapping.
The N-1 form densities pull back via the adjoint of the Jacobian matrix of the mapping.
The matrix elements of the Jacobian matrix need not be global constants.

```
> JCM:=proc(Ax,Ay,Az,phi)\
     local
A,A1,A2,A3,A4,BFC,EF1,EF2,EF3,JAC,JDC,ExBC,JTOT,Jcurl,Jt,Jh,Jd,Jdt,TFCa,SFCa:
     global
A1form,AF,SP,BF,EF,TF,HEL,P1,P2,DF,HF,CD,JA,JD,SPD,SF,SFD,ExB,G2form,F2form,J3f
orm,JTT,TFC,SFC:
     A1:=Ax:A2:=Ay:A3:=Az;A4:=phi:
A:=[A1,A2,A3]:AF:=factor(simplify(A)):SP:=simplify(phi):
A1form:=A1*d(x)+A2*d(y)+A3*d(z)-A4*d(t):
     EF1:=evalm(-grad(phi,[x,y,z])):
     EF2:=-[diff(A1,t),diff(A2,t),diff(A3,t)];
EF3:=[factor(EF1[1]+EF2[1]),factor(EF1[2]+EF2[2]),factor(EF1[3]+EF2[3])];
     EF:=[EF3[1],EF3[2],EF3[3]];
     BFC:=(curl([A1,A2,A3],[x,y,z])):
```

```
     BF:=[factor(BFC[1]),factor(BFC[2]),factor(BFC[3])];
HEL:=factor(innerprod(AF,BF));TFCa:=evalm(crossprod(EF,AF)+BF*phi);
     TF:=[factor(TFCa[1]),factor(TFCa[2]),factor(TFCa[3]),HEL];

P2:=-2*factor(innerprod(EF,BF));TFC:=[factor(TFCa[1]),factor(TFCa[2]),factor(TF
Ca[3])];
     HF:=[factor(BFC[1]/mu),factor(BFC[2]/mu),factor(BFC[3]/mu)];
DF:=[factor(epsilon*EF3[1]),factor(epsilon*EF3[2]),factor(epsilon*EF3[3])];
CD:=factor(diverge([DF[1],DF[2],DF[3]],[x,y,z]));
Jcurl:=curl([HF[1],HF[2],HF[3]],[x,y,z]);
     Jh:=[factor(Jcurl[1]),factor(Jcurl[2]),factor(Jcurl[3])];
Jdt:=-[diff(DF[1],t),diff(DF[2],t),diff(DF[3],t)];
Jd:=[factor(Jdt[1]),factor(Jdt[2]),factor(Jdt[3])];JTOT:=Jh+Jd:JD:=[factor(JTOT
[1]),factor(JTOT[2]),factor(JTOT[3])];
JTT:=[factor(JD[1]),factor(JD[2]),factor(JD[3]),CD];
F2form:=evalm(innerprod(BF,[d(y)&^d(z),-d(x)&^d(z),d(x)&^d(y)])+innerprod(EF,[d
(x)&^d(t),d(y)&^d(t),d(z)&^d(t)]));
SPD:=factor(innerprod(A,DF));G2form:=(HF[1]*d(x)&^d(t)+HF[2]*d(y)&^d(t)+HF[3]*d
(z)&^d(t)-DF[1]*d(y)&^d(z)+DF[2]*d(x)&^d(z)-DF[3]*d(x)&^d(y));J3form:=innerprod
(JTT,[d(y)&^d(z)&^d(t),-d(x)&^d(z)&^d(t),d(x)&^d(y)&^d(t),-d(x)&^d(y)&^d(z)]);

SFCa:=evalm(crossprod(AF,HF)+DF*phi);SFC:=[factor(SFCa[1]),factor(SFCa[2]),fact
or(SFCa[3])];
     SFD:=[factor(SFC[1]),factor(SFC[2]),factor(SFC[3]),SPD];
     P1:=innerprod(BF,HF)-innerprod(DF,EF)-innerprod(AF,JD)+CD*phi;
ExBC:=crossprod(EF,BF);ExB:=[factor(ExBC[1]),factor(ExBC[2]),factor(ExBC[3])];
   end proc:
> MAP:=proc(X,Y,Z,T,x,y,z,t) global JAC,ADJAC,DET,TRJAC,Map:
  Map:=[x,y,z,t]:JAC:=simplify(jacobian(Map,[X,Y,Z,T])):DET:=factor(simplify(de
  t(JAC))):TRJAC:=simplify(transpose(JAC)):ADJAC:=simplify(adjoint(JAC)): end
  proc:
>
>
>
> b:=0:c:=0:
> rr:=(x^2+y^2)^(1/2):AAA:=[z^2*y*b,-z^2*x*b,1-k^2*rr^2,c*z*y*x+omega*(1-k^2*rr
  ^2)/k]:
> JCM(AAA[1],AAA[2],AAA[3],AAA[4]):
```

The exterior differential forms as specified on the final state.

```
> A1form:=wcollect(factor(simplify(A1form)));F2form:=
  wcollect(factor(simplify(F2form)));G2form:=
  wcollect(factor(simplify(G2form)));J3form:=
  wcollect(factor(simplify(J3form)));
```

$$A1form := (1 - k^2 x^2 - k^2 y^2)\, d(z) + \frac{\omega (-1 + k^2 x^2 + k^2 y^2)\, d(t)}{k}$$

$$F2form := -2 k^2 y\, (d(y) \&\wedge d(z)) - 2 k^2 x\, (d(x) \&\wedge d(z)) + 2 \omega k x\, (d(x) \&\wedge d(t)) + 2 \omega k y\, (d(y) \&\wedge d(t))$$

$G2form :=$

$$-2 \frac{k^2 y\, (d(x) \&\wedge d(t))}{\mu} + \frac{2 k^2 x\, (d(y) \&\wedge d(t))}{\mu} - 2 \varepsilon \omega k x\, (d(y) \&\wedge d(z)) + 2 \varepsilon \omega k y\, (d(x) \&\wedge d(z))$$

$$J3form := -4\,\varepsilon\,\omega\,k\,\&\wedge(\mathrm{d}(x),\mathrm{d}(y),\mathrm{d}(z)) + \frac{4\,k^2\,\&\wedge(\mathrm{d}(x),\mathrm{d}(y),\mathrm{d}(t))}{\mu}$$

The fields in engineering format on the final state

```
> R:=[x,y,z,t];Vector_potential:=AF;Scalar_potential:=factor(SP);E_field:=EF;B_field:=BF;Poincare2:=factor(P2);D_field:=DF;H_field:=HF;rho_charge_density:=CD;J_current_density:=(simplify(JD));;Poincare1:=factor(P1);PoyntingVector=ExB;Torsion_flux:=factor(evalm(TFC));Helicity:=HEL;Spin_current:=factor((SFC));Spin_density:=factor(SPD);Lagrangian_field_energy_density:=factor(simplify(innerprod(HF,BF)-innerprod(DF,EF)));Interaction_energy_density:=factor(AF[1]*JD[1]+AF[2]*JD[2]+AF[3]*JD[3]-CD*SP);
```

$$R := [x, y, z, t]$$

$$Vector\_potential := [0, 0, 1 - k^2 x^2 - k^2 y^2]$$

$$Scalar\_potential := -\frac{\omega(-1 + k^2 x^2 + k^2 y^2)}{k}$$

$$E\_field := [2\,\omega\,k\,x, 2\,\omega\,k\,y, 0]$$

$$B\_field := [-2\,k^2\,y, 2\,k^2\,x, 0]$$

$$Poincare2 := 0$$

$$D\_field := [2\,\varepsilon\,\omega\,k\,x, 2\,\varepsilon\,\omega\,k\,y, 0]$$

$$H\_field := \left[-2\frac{k^2\,y}{\mu}, 2\frac{k^2\,x}{\mu}, 0\right]$$

$$rho\_charge\_density := 4\,\varepsilon\,\omega\,k$$

$$J\_current\_density := \left[0, 0, 4\frac{k^2}{\mu}\right]$$

$$Poincare1 := -4\frac{(2\,k^2 x^2 + 2\,k^2 y^2 - 1)(-k^2 + \omega^2\,\varepsilon\,\mu)}{\mu}$$

$$PoyntingVector = [0, 0, 4\,\omega\,k^3\,(x^2 + y^2)]$$

$$Torsion\_flux := [0, 0, 0]$$

$$Helicity := 0$$

$$Spin\_current := \left[-2\frac{(-1 + k^2 x^2 + k^2 y^2)\,x\,(-k^2 + \omega^2\,\varepsilon\,\mu)}{\mu}, -2\frac{(-1 + k^2 x^2 + k^2 y^2)\,y\,(-k^2 + \omega^2\,\varepsilon\,\mu)}{\mu}, 0\right]$$

$$Spin\_density := 0$$

$$Lagrangian\_field\_energy\_density := -4\frac{k^2\,(x^2 + y^2)(-k^2 + \omega^2\,\varepsilon\,\mu)}{\mu}$$

$$Interaction\_energy\_density := 4\frac{(-1 + k^2 x^2 + k^2 y^2)(-k^2 + \omega^2\,\varepsilon\,\mu)}{\mu}$$

```
> R_final_variables:=R;
```

$$R\_final\_variables := [x, y, z, t]$$

Variables on Initial State:

```
> X:=X:Y:=Y:Z:=Z:T:=T:
>
```

Define the mapping functions here

ORTHOGONAL TO CURRENT TRANSLATION

```
> x:=X-Vx*T:y:=Y:z:=Z:t:=T:
>
> R_initial_variables:=[X,Y,Z,T];
```

$$R\_initial\_variables := [X, Y, Z, T]$$

```
> MAP(X,Y,Z,T,x,y,z,t):Mapping_functions:=Map;Jacobian:=evalm(JAC):DET:=DET:Adjoint:=evalm(ADJAC):
```

$$Mapping\_functions := [X - Vx\, T, Y, Z, T]$$

The map represents a translation along the X axis.

Evaluate the exterior forms on the initial state by functional substitution and pullback:

```
> A1form:=wcollect(factor(simplify(A1form)));F2form:=
  wcollect(factor(simplify(F2form)));d(F2form);G2form:=
  wcollect(factor(simplify(G2form)));J3form:=
  wcollect(factor(simplify(J3form)));
```

$$A1form := (1 - k^2 X^2 + 2 k^2 X\, Vx\, T - k^2 Vx^2 T^2 - k^2 Y^2)\, d(Z) + \frac{\omega(-1 + k^2 X^2 - 2 k^2 X\, Vx\, T + k^2 Vx^2 T^2 + k^2 Y^2)\, d(T)}{k}$$

$$F2form := 2 k (k\, Vx\, T - k X)(d(X) \,\&\wedge\, d(Z)) + 2 k (k X\, Vx - k\, Vx^2 T)(d(T) \,\&\wedge\, d(Z)) - 2 k^2 Y (d(Y) \,\&\wedge\, d(Z)) + 2 k (-\omega\, Vx\, T + \omega X)(d(X) \,\&\wedge\, d(T)) + 2 \omega k Y (d(Y) \,\&\wedge\, d(T))$$

$$0$$

$$G2form := 2 k \varepsilon \omega Y (d(X) \,\&\wedge\, d(Z)) - 2 k \varepsilon \omega Y\, Vx (d(T) \,\&\wedge\, d(Z)) - \frac{2 k (\varepsilon \omega \mu X - \varepsilon \omega \mu\, Vx\, T)(d(Y) \,\&\wedge\, d(Z))}{\mu} - \frac{2 k^2 Y (d(X) \,\&\wedge\, d(T))}{\mu} - \frac{2 k (k\, Vx\, T - k X)(d(Y) \,\&\wedge\, d(T))}{\mu}$$

$$J3form := 4 \frac{k^2 \,\&\wedge(d(X), d(Y), d(T))}{\mu} - 4 \varepsilon \omega k \,\&\wedge(d(X), d(Y), d(Z)) + 4 k \varepsilon \omega\, Vx \,\&\wedge(d(T), d(Y), d(Z))$$

```
> factor(wcollect(d(A1form)-F2form));
```

$$0$$

```
> Spin3form:=(A1form&^G2form);Torsion3form:=(A1form&^F2form);
```

$$Spin3form := 2 \frac{(-1 + k^2 X^2 - 2 k^2 X\, Vx\, T + k^2 Vx^2 T^2 + k^2 Y^2)(X - Vx\, T)(-k^2 + \omega^2 \varepsilon \mu) \,\&\wedge(d(Z), d(Y), d(T))}{\mu} - \frac{2 (-1 + k^2 X^2 - 2 k^2 X\, Vx\, T + k^2 Vx^2 T^2 + k^2 Y^2)\, Y (-k^2 + \omega^2 \varepsilon \mu) \,\&\wedge(d(Z), d(X), d(T))}{\mu}$$

$$Torsion3form := 0$$

```
> Spin3form :=
  -2*(-1+k^2*X^2-2*k^2*X*Vx*T+k^2*Vx^2*T^2+k^2*Y^2)*(Vx*T-X)*(-k^2+omega^2*epsilon*mu)/mu*`&^`(d(Z),d(Y),d(T))-2*(-1+k^2*X^2-2*k^2*X*Vx*T+k^2*Vx^2*T^2+k^2*Y^2)*Y*(-k^2+omega^2*epsilon*mu)/mu*`&^`(d(Z),d(X),d(T));
```

$$Spin3form := -2 \frac{(-1 + k^2 X^2 - 2 k^2 X\, Vx\, T + k^2 Vx^2 T^2 + k^2 Y^2)(Vx\, T - X)(-k^2 + \omega^2 \varepsilon \mu) \,\&\wedge(d(Z), d(Y), d(T))}{\mu} - \frac{2 (-1 + k^2 X^2 - 2 k^2 X\, Vx\, T + k^2 Vx^2 T^2 + k^2 Y^2)\, Y (-k^2 + \omega^2 \varepsilon \mu) \,\&\wedge(d(Z), d(X), d(T))}{\mu}$$

```
>
```

Pullback field Components on the initial state:

```
> AF_PB:=innerprod(TRJAC,[AF[1],AF[2],AF[3],-SP]);VPotential_PB:=simplify([AF_PB[1],AF_PB[2],AF_PB[3]]);ScalarPot_PB:=simplify(-AF_PB[4]);EF_PB:=factor(simplify((evalm(-grad(ScalarPot_PB,[X,Y,Z])-diff(VPotential_PB,T)))));BF_PB:=factor(simplify(curl(VPotential_PB,[X,Y,Z])));D1:=-getcoeff(G2form&^d(X)&^d(T)):D2:=getcoeff(G2form&^d(Y)&^d(T)):D3:=-getcoeff(G2form&^d(Z)&^d(T)):H1:=getcoeff(G2form&^d(Y)&^d(Z)):H2:=getcoeff(G2form&^d(X)&^d(Z)):H3:=-getcoeff(G2form&^d(X)&^d(Y)):DF_PB:=[factor(simplify(D1)),factor(simplify(D2)),factor(simplify(D3))];HF_PB:=[factor(simplify(H1)),factor(simplify(H2)),factor(simplify(H3))]
```

```
];JTPB:=innerprod(ADJAC,JTT):JD_PB:=[JTPB[1],JTPB[2],JTPB[3]];JC_PB:=JTPB[4];Poincare2_PB:=Poincare2;Poincare1_PB:=factor(simplify(DET*Poincare1));DET:=DET;M_F:=evalm(curl(EF_PB,[X,Y,Z])+[diff(BF_PB[1],T),diff(BF_PB[2],T),diff(BF_PB[3],T)]);M_A:=evalm(curl(HF_PB,[X,Y,Z])-[diff(DF_PB[1],T),diff(DF_PB[2],T),diff(DF_PB[3],T)]);M_FdivB:=diverge(BF_PB,[X,Y,Z]);M_AdivD:=diverge(DF_PB,[X,Y,Z]);
```

$$AF\_PB := \left[0, 0, 1 - k^2 X^2 + 2 k^2 X\, Vx\, T - k^2 Vx^2 T^2 - k^2 Y^2, \frac{\omega (-1 + k^2 X^2 - 2 k^2 X\, Vx\, T + k^2 Vx^2 T^2 + k^2 Y^2)}{k}\right]$$

$$VPotential\_PB := [0, 0, 1 - k^2 X^2 + 2 k^2 X\, Vx\, T - k^2 Vx^2 T^2 - k^2 Y^2]$$

$$ScalarPot\_PB := -\frac{\omega (-1 + k^2 X^2 - 2 k^2 X\, Vx\, T + k^2 Vx^2 T^2 + k^2 Y^2)}{k}$$

$$EF\_PB := [2 \omega k (X - Vx\, T), 2 \omega k Y, -2 k^2 X\, Vx + 2 k^2 Vx^2 T]$$

$$BF\_PB := [-2 k^2 Y, 2 k^2 X - 2 k^2 Vx\, T, 0]$$

$$DF\_PB := [2 k \varepsilon \omega (X - Vx\, T), 2 k \varepsilon \omega Y, 0]$$

$$HF\_PB := \left[-2 \frac{k^2 Y}{\mu}, 2 \frac{k^2 (X - Vx\, T)}{\mu}, 2 k \varepsilon \omega Y\, Vx\right]$$

$$JD\_PB := \left[4\, Vx\, \varepsilon \omega k, 0, 4 \frac{k^2}{\mu}\right]$$

$$JC\_PB := 4 \varepsilon \omega k$$

$$Poincare2\_PB := 0$$

$$Poincare1\_PB := -4 \frac{(2 k^2 X^2 - 4 k^2 X\, Vx\, T + 2 k^2 Vx^2 T^2 + 2 k^2 Y^2 - 1)(-k^2 + \omega^2 \varepsilon \mu)}{\mu}$$

$$DET := 1$$

$$M\_F := [0, 0, 0]$$

$$M\_A := \left[4\, Vx\, \varepsilon \omega k, 0, 4 \frac{k^2}{\mu}\right]$$

$$M\_FdivB := 0$$

$$M\_AdivD := 4 \varepsilon \omega k$$

Note that the constitutive relations between D and E and B and H on the initial state are not the same as for the final state.
The D fields are equal to epsilon E times the DET of the transformation, there by converting a tensor into a tensor density.
Simlarly the H fields are B divided by mu times the DET of the transformation -- almost.
There appears another term in the H fields on the final state due to the translation Vz. Indeed, motion along the z axis adds to the existing Current density in the fixed frame a component that is proportional to the moving charge density. This motion induces a component to the H field that encircles the z axis, but DOES NOT affect the associated B fields.

THE PULLED BACK FIELD COMPONENTS E,B satisfy the MAXWELL - FARADAY PDE's in terms of the independent variables [X,Y,Z,T] with the constraint that d(Z)/d(T) = Vz.
THE PULLED BACK FIELD COMPONENTS D,H also satisfy the MAXWELL - AMPERE PDE's in in terms of the independent variables [X,Y,Z,T] with the constraint that d(Z)/d(T) = Vz.

```
> restart;with(linalg):with(liesymm):with(difforms):defform(L=0,f=0,u=0,x=0,y=0,z=0,t=0,lambda=0,C=const,B=const,Phi=0,FF=0,phi=0,f1=0,f2=0,f3=0,JX=0,JY=0,JZ=0,Vx=const,Vy=const,Vz=const,AX=0,AY=0,AZ=0,A4=0,ax=const,DX=0,DY=0,DZ=0,HX=0,HY=0,HZ=0,j1=0,j2=0,j3=0,j4=0,rho=0,k=const,omega=const,JT=0,a=const,b=const,mu=const,epsilon=const,e=const,n=const,Omega=const,c=const,X=0,Y=0,Z=0,T=0
```

```
,r=0,alpha=0,beta=0,theta=0,ZR=const):
Warning, the protected names norm and trace have been redefined and unprotected

Warning, the protected name close has been redefined and unprotected

Warning, the names &^, d and wdegree have been redefined
```

>

Pullbacks of maps to Euclidean spaces. Rotations

R. M. Kiehn
Updated 12/03/2001

***

Consider a final state of independent variables, [x,y,z,t],
and a map from cartesian variables [r,theta,z,t] with the additional constraint that represents a kinematic rotation at constant angular velocity about the z axiz. Hence d(theta)/d(t) = Omega = constant. The electromagnetic Action 1-form A will be given in terms of abstract functions Ax,Ay,Az,Phi, on the final state with arguments in terms of [x,y,z,t]. The pair 2-form F = dA and its coefficients on the final state will be evaluated to yield E and B, the field intensities.
The final state will be presumed to be a euclidean space with constitutive properties in the sense that D = epsilon E and B = mu H. Hence, on the final state, given the potentials, the field intensities can be computed, the field excitations D, H, can be evaluated to give the impair 2-form G. From this hypothesis (or constraint) the impair 3-form of 4 current density will be computed from the impair 2-form G, such that J=dG.
The results on the final state then will be pulled back by the combined actions of functional substitution for the independent variables and their differentials. into the differential forms for A, F, G and J.
For 1-forms and N-1 form densities, the pullbacks are particularly simple:
The 1-forms pull back by means of the transpose of the Jacobian matrix of the mapping.
The N-1 form densities pull back via the adjoint of the Jacobian matrix of the mapping.
The matrix elements of the Jacobian matrix need not be global constants.

The following is the procedure for computing EM fields and currents given the potentials on the variety [x,y,z,t], with the constitutive assumption D = epsilon E, B = mu H. The procedure also evalutes the forms A, F, G, J on the final state.

USEFUL OUTPUT FUNCTIONS ARE
AF=vector potential
SP = scalar potential
EF = E field intensity
BF = B field intensity
DF = D field excitation
HF = H field excitation
JD = current density
CD = charge density
SPC = spin current
SPD = spin density
TFC = Torsion flux
HEL = Helicity

TF = Torsion field 4 components
SP = Spin field density 4 components
P1 = First Poincare invariant density
P2 = Second Poincare invariant
A1form = pair 1-form of potentials
F2form = pair 2-form of E, B field intensities
G2form = impair 2-form of D, H excitation densities
J3form = impair 3 form of charge current densities

```
> JCM:=proc(Ax,Ay,Az,phi)\
     local
A,A1,A2,A3,A4,BFC,EF1,EF2,EF3,JAC,JDC,ExBC,JTOT,Jcurl,Jt,Jh,Jd,Jdt,TFCa,SFCa:
     global
A1form,AF,SP,BF,EF,TF,HEL,P1,P2,DF,HF,CD,JA,JD,SPD,SF,SFD,ExB,G2form,F2form,J3f
orm,JTT,TFC,SFC:
     A1:=Ax:A2:=Ay:A3:=Az;A4:=phi:
A:=[A1,A2,A3]:AF:=factor(simplify(A)):SP:=simplify(phi):
A1form:=A1*d(x)+A2*d(y)+A3*d(z)-A4*d(t):
     EF1:=evalm(-grad(phi,[x,y,z])):
     EF2:=-[diff(A1,t),diff(A2,t),diff(A3,t)];
EF3:=[factor(EF1[1]+EF2[1]),factor(EF1[2]+EF2[2]),factor(EF1[3]+EF2[3])];
     EF:=[EF3[1],EF3[2],EF3[3]];
     BFC:=(curl([A1,A2,A3],[x,y,z])):
     BF:=[factor(BFC[1]),factor(BFC[2]),factor(BFC[3])];
HEL:=factor(innerprod(AF,BF));TFCa:=evalm(crossprod(EF,AF)+BF*phi);
     TF:=[factor(TFCa[1]),factor(TFCa[2]),factor(TFCa[3]),HEL];

P2:=-2*factor(innerprod(EF,BF));TFC:=[factor(TFCa[1]),factor(TFCa[2]),factor(TF
Ca[3])];
     HF:=[factor(BFC[1]/mu),factor(BFC[2]/mu),factor(BFC[3]/mu)];
DF:=[factor(epsilon*EF3[1]),factor(epsilon*EF3[2]),factor(epsilon*EF3[3])];
CD:=factor(diverge([DF[1],DF[2],DF[3]],[x,y,z]));
Jcurl:=curl([HF[1],HF[2],HF[3]],[x,y,z]);
     Jh:=[factor(Jcurl[1]),factor(Jcurl[2]),factor(Jcurl[3])];
Jdt:=-[diff(DF[1],t),diff(DF[2],t),diff(DF[3],t)];
Jd:=[factor(Jdt[1]),factor(Jdt[2]),factor(Jdt[3])];JTOT:=Jh+Jd:JD:=[factor(JTOT
[1]),factor(JTOT[2]),factor(JTOT[3])];
JTT:=[factor(JD[1]),factor(JD[2]),factor(JD[3]),CD];
F2form:=evalm(innerprod(BF,[d(y)&^d(z),-d(x)&^d(z),d(x)&^d(y)])+innerprod(EF,[d
(x)&^d(t),d(y)&^d(t),d(z)&^d(t)]));
SPD:=factor(innerprod(A,DF));G2form:=(HF[1]*d(x)&^d(t)+HF[2]*d(y)&^d(t)+HF[3]*d
(z)&^d(t)-DF[1]*d(y)&^d(z)+DF[2]*d(x)&^d(z)-DF[3]*d(x)&^d(y));J3form:=innerprod
(JTT,[d(y)&^d(z)&^d(t),-d(x)&^d(z)&^d(t),d(x)&^d(y)&^d(t),-d(x)&^d(y)&^d(z)]);

SFCa:=evalm(crossprod(AF,HF)+DF*phi);SFC:=[factor(SFCa[1]),factor(SFCa[2]),fact
or(SFCa[3])];
     SFD:=[factor(SFC[1]),factor(SFC[2]),factor(SFC[3]),SPD];
     P1:=innerprod(BF,HF)-innerprod(DF,EF)-innerprod(AF,JD)+CD*phi;
ExBC:=crossprod(EF,BF);ExB:=[factor(ExBC[1]),factor(ExBC[2]),factor(ExBC[3])];
   end proc:
> MAP:=proc(X,Y,Z,T,x,y,z,t) global JAC,ADJAC,DET,TRJAC,Map:
```

```
Map:=[x,y,z,t]:JAC:=simplify(jacobian(Map,[X,Y,Z,T])):DET:=factor(simplify(det(JAC))):TRJAC:=simplify(transpose(JAC)):ADJAC:=simplify(adjoint(JAC)): end proc:
>
>
>
> b:=0:c:=0:
> rr:=(x^2+y^2)^(1/2):AAA:=[z^2*y*b,-z^2*x*b,1-k^2*rr^2,c*z*y*x+omega*(1-k^2*rr^2)/k]:
> JCM(AAA[1],AAA[2],AAA[3],AAA[4]):
```

The exterior differential forms as specified on the final state.

```
> A1form:=wcollect(factor(simplify(A1form)));F2form:=wcollect(factor(simplify(F2form)));G2form:=wcollect(factor(simplify(G2form)));J3form:=wcollect(factor(simplify(J3form)));
```

$$A1form := (1 - k^2 x^2 - k^2 y^2)\,\mathrm{d}(z) + \frac{\omega\,(-1 + k^2 x^2 + k^2 y^2)\,\mathrm{d}(t)}{k}$$

$$F2form := -2\,k^2\,y\,(\mathrm{d}(y)\ \&\wedge\ \mathrm{d}(z)) - 2\,k^2\,x\,(\mathrm{d}(x)\ \&\wedge\ \mathrm{d}(z)) + 2\,\omega\,k\,x\,(\mathrm{d}(x)\ \&\wedge\ \mathrm{d}(t)) + 2\,\omega\,k\,y\,(\mathrm{d}(y)\ \&\wedge\ \mathrm{d}(t))$$

$G2form :=$

$$-2\,\frac{k^2\,y\,(\mathrm{d}(x)\ \&\wedge\ \mathrm{d}(t))}{\mu} + \frac{2\,k^2\,x\,(\mathrm{d}(y)\ \&\wedge\ \mathrm{d}(t))}{\mu} - 2\,\varepsilon\,\omega\,k\,x\,(\mathrm{d}(y)\ \&\wedge\ \mathrm{d}(z)) + 2\,\varepsilon\,\omega\,k\,y\,(\mathrm{d}(x)\ \&\wedge\ \mathrm{d}(z))$$

$$J3form := -4\,\varepsilon\,\omega\,k\ \&\wedge(\mathrm{d}(x), \mathrm{d}(y), \mathrm{d}(z)) + \frac{4\,k^2\ \&\wedge(\mathrm{d}(x), \mathrm{d}(y), \mathrm{d}(t))}{\mu}$$

The fields in engineering format on the final state

```
> R:=[x,y,z,t];Vector_potential:=AF;Scalar_potential:=factor(SP);E_field:=EF;B_field:=BF;Poincare2:=factor(P2);D_field:=DF;H_field:=HF;rho_charge_density:=CD;J_current_density:=(simplify(JD));;Poincare1:=factor(P1);PoyntingVector=ExB;Torsion_flux:=factor(evalm(TFC));Helicity:=HEL;Spin_current:=factor((SFC));Spin_density:=factor(SPD);Lagrangian_field_energy_density:=factor(simplify(innerprod(HF,BF)-innerprod(DF,EF)));Interaction_energy_density:=factor(AF[1]*JD[1]+AF[2]*JD[2]+AF[3]*JD[3]-CD*SP);
```

$$R := [x, y, z, t]$$

$$Vector\_potential := [0, 0, 1 - k^2 x^2 - k^2 y^2]$$

$$Scalar\_potential := -\frac{\omega\,(-1 + k^2 x^2 + k^2 y^2)}{k}$$

$$E\_field := [2\,\omega\,k\,x, 2\,\omega\,k\,y, 0]$$

$$B\_field := [-2\,k^2\,y, 2\,k^2\,x, 0]$$

$$Poincare2 := 0$$

$$D\_field := [2\,\varepsilon\,\omega\,k\,x, 2\,\varepsilon\,\omega\,k\,y, 0]$$

$$H\_field := \left[-2\,\frac{k^2\,y}{\mu}, 2\,\frac{k^2\,x}{\mu}, 0\right]$$

$$rho\_charge\_density := 4\,\varepsilon\,\omega\,k$$

$$J\_current\_density := \left[0, 0, 4\,\frac{k^2}{\mu}\right]$$

$$Poincare1 := -4\,\frac{(2\,k^2\,x^2 + 2\,k^2\,y^2 - 1)\,(-k^2 + \omega^2\,\varepsilon\,\mu)}{\mu}$$

$$PoyntingVector = [0, 0, 4\,\omega\,k^3\,(x^2 + y^2)]$$

$$Torsion\_flux := [0, 0, 0]$$

$$Helicity := 0$$

$$Spin\_current := \left[-2\,\frac{(-1+k^2 x^2+k^2 y^2)\,x\,(-k^2+\omega^2\,\varepsilon\,\mu)}{\mu},\, -2\,\frac{(-1+k^2 x^2+k^2 y^2)\,y\,(-k^2+\omega^2\,\varepsilon\,\mu)}{\mu},\, 0\right]$$

$$Spin\_density := 0$$

$$Lagrangian\_field\_energy\_density := -4\,\frac{k^2\,(x^2+y^2)\,(-k^2+\omega^2\,\varepsilon\,\mu)}{\mu}$$

$$Interaction\_energy\_density := 4\,\frac{(-1+k^2 x^2+k^2 y^2)\,(-k^2+\omega^2\,\varepsilon\,\mu)}{\mu}$$

```
> R_final_variables:=R;
```

$$R\_final\_variables := [x, y, z, t]$$

Variables on Initial State:

```
> X:=r:Y:=theta:Z:=z:T:=t:
>
```

Define the mapping functions here

ROTATION ABOUT z axis

```
> x:=r*cos(theta-Omega*t):y:=r*sin(theta-Omega*t):z:=z:t:=T:
>
> R_initial_variables:=[X,Y,Z,T];
```

$$R\_initial\_variables := [r, \theta, z, t]$$

```
> MAP(X,Y,Z,T,x,y,z,t):Mapping_fucntions:=Map;Jacobian:=evalm(JAC):DET:=DET:Adjoint:=evalm(ADJAC);
```

$$Mapping\_fucntions := [r\cos(-\theta+\Omega\,t),\, -r\sin(-\theta+\Omega\,t),\, z,\, t]$$

$$Adjoint := \begin{bmatrix} r\cos(-\theta+\Omega\,t) & -r\sin(-\theta+\Omega\,t) & 0 & 0 \\ \sin(-\theta+\Omega\,t) & \cos(-\theta+\Omega\,t) & 0 & r\,\Omega \\ 0 & 0 & r & 0 \\ 0 & 0 & 0 & r \end{bmatrix}$$

The mapping represents a rotation about the z axis.

Evaluate the exterior forms on the initial state by functional substitution and pullback:

```
> A1form:=wcollect(factor(simplify(A1form)));F2form:=
  wcollect(factor(simplify(F2form)));G2form:=
  wcollect(factor(simplify(G2form)));J3form:=
  wcollect(factor(simplify(J3form)));
```

$$A1form := -(k\,r-1)\,(k\,r+1)\,\mathrm{d}(z) + \frac{(k\,r-1)\,(k\,r+1)\,\omega\,\mathrm{d}(t)}{k}$$

$$F2form := -2\,k^2\,r\,(\mathrm{d}(r)\ \&\wedge\ \mathrm{d}(z)) + 2\,k\,r\,\omega\,(\mathrm{d}(r)\ \&\wedge\ \mathrm{d}(t))$$

$$G2form := 2\,k\,r^2\,\varepsilon\,\omega\,\Omega\,(\mathrm{d}(t)\ \&\wedge\ \mathrm{d}(z)) - 2\,k\,r^2\,\varepsilon\,\omega\,(\mathrm{d}(\theta)\ \&\wedge\ \mathrm{d}(z)) + \frac{2\,k^2\,r^2\,(\mathrm{d}(\theta)\ \&\wedge\ \mathrm{d}(t))}{\mu}$$

$$J3form := 4\,\frac{k^2\,r\ \&\wedge(\mathrm{d}(r), \mathrm{d}(\theta), \mathrm{d}(t))}{\mu} - 4\,k\,r\,\varepsilon\,\omega\ \&\wedge(\mathrm{d}(r), \mathrm{d}(\theta), \mathrm{d}(z)) + 4\,k\,r\,\varepsilon\,\omega\ \&\wedge(\mathrm{d}(r), \mathrm{d}(t), \mathrm{d}(z))\,\Omega$$

```
> Spin3form:=(A1form&^G2form);Torsion3form:=(A1form&^F2form);
```

$$Spin3form := 2\,\frac{(k\,r-1)\,(k\,r+1)\,r^2\,(-k^2+\omega^2\,\varepsilon\,\mu)\ \&\wedge(\mathrm{d}(z), \mathrm{d}(\theta), \mathrm{d}(t))}{\mu}$$

$$Torsion3form := 0$$

```
> AF_PB:=innerprod(TRJAC,[AF[1],AF[2],AF[3],-SP]);VPotential_PB:=simplify([AF_PB[1],AF_PB[2],AF_PB[3]]);ScalarPot_PB:=simplify(-AF_PB[4]);EF_PB:=factor(simplify((evalm(-grad(ScalarPot_PB,[X,Y,Z])-diff(VPotential_PB,T)))));BF_PB:=fact
```

```
or(simplify(curl(VPotential_PB,[X,Y,Z])));D1:=-getcoeff(G2form&^d(X)&^d(T)):D2:=getcoeff(G2form&^d(Y)&^d(T)):D3:=-getcoeff(G2form&^d(Z)&^d(T)):H1:=getcoeff(G2form&^d(Y)&^d(Z)):H2:=getcoeff(G2form&^d(X)&^d(Z)):H3:=-getcoeff(G2form&^d(X)&^d(Y)):DF_PB:=[factor(simplify(D1)),factor(simplify(D2)),factor(simplify(D3))];HF_PB:=[factor(simplify(H1)),factor(simplify(H2)),factor(simplify(H3))];JTPB:=innerprod(ADJAC,JTT):JD_PB:=[JTPB[1],JTPB[2],JTPB[3]];JC_PB:=JTPB[4];Poincare2_PB:=Poincare2;Poincare1_PB:=factor(simplify(DET*Poincare1));DET:=DET;M_F:=evalm(curl(EF_PB,[X,Y,Z])+[diff(BF_PB[1],T),diff(BF_PB[2],T),diff(BF_PB[3],T)]);M_A:=evalm(curl(HF_PB,[X,Y,Z])-[diff(DF_PB[1],T),diff(DF_PB[2],T),diff(DF_PB[3],T)]);M_FdivB:=diverge(BF_PB,[X,Y,Z]);M_AdivD:=diverge(DF_PB,[X,Y,Z]);
```

$AF\_PB :=$

$$\left[0,\ 0,\ 1-k^2 r^2 \cos(-\theta+\Omega\, t)^2 - k^2 r^2 \sin(-\theta+\Omega\, t)^2,\ \frac{\omega\,(-1+k^2 r^2 \cos(-\theta+\Omega\, t)^2 + k^2 r^2 \sin(-\theta+\Omega\, t)^2)}{k}\right]$$

$$VPotential\_PB := [0,\ 0,\ 1-k^2 r^2]$$

$$ScalarPot\_PB := -\frac{\omega\,(-1+k^2 r^2)}{k}$$

$$EF\_PB := [2\,k\,r\,\omega,\ 0,\ 0]$$

$$BF\_PB := [0,\ 2\,k^2\,r,\ 0]$$

$$DF\_PB := [2\,k\,r^2\,\varepsilon\,\omega,\ 0,\ 0]$$

$$HF\_PB := \left[0,\ 2\,\frac{k^2 r^2}{\mu},\ -2\,k\,r^2\,\varepsilon\,\omega\,\Omega\right]$$

$$JD\_PB := \left[0,\ 4\,r\,\Omega\,\varepsilon\,\omega\,k,\ 4\,\frac{k^2 r}{\mu}\right]$$

$$JC\_PB := 4\,k\,r\,\varepsilon\,\omega$$

$$Poincare2\_PB := 0$$

$$Poincare1\_PB := -4\,\frac{r\,(2\,k^2 r^2-1)\,(-k^2+\omega^2\,\varepsilon\,\mu)}{\mu}$$

$$DET := r$$

$$M\_F := [0,\ 0,\ 0]$$

$$M\_A := \left[0,\ 4\,r\,\Omega\,\varepsilon\,\omega\,k,\ 4\,\frac{k^2 r}{\mu}\right]$$

$$M\_FdivB := 0$$

$$M\_AdivD := 4\,k\,r\,\varepsilon\,\omega$$

>

>

Note that the constitutive relations between D and E and B and H on the initial state are not the same as for the final state.
The D fields are equal to epsilon E times the DET of the transformation, there by converting a tensor into a tensor density.
Simlarly the H fields are B divided by mu times the DET of the transformation -- almost.
There appears another term in the H fields on the final state due to the rotation Omega. Indeed, motion of the charge density about the z axis appears to create a contribution to the current density that encircles the z axis. Such a current density induces a component of H along the z axis and related to the rotation rate.
This rotational motion of the charge density influences the H field, but DOES NOT affect the associated B fields.

THE PULLED BACK FIELD COMPONENTS E,B satisfy the MAXWELL - FARADAY PDE's in terms of the independent variables [r,theta,z,t] with the constraint that d(theta)/d(t) = Omega.
THE PULLED BACK FIELD COMPONENTS D,H also satisfy the MAXWELL - AMPERE PDE's in in terms of the independent variables [r,theta,z,t] with the constraint that d(theta)/d(t) = Omega.

```
> restart;with(linalg):with(liesymm):with(difforms):defform(L=0,f=0,u=0,x=0,y=0,z=0,t=0,lambda=0,C=const,B=const,Phi=0,FF=0,phi=0,f1=0,f2=0,f3=0,JX=0,JY=0,JZ=0,Vx=const,Vy=const,Vz=const,AX=0,AY=0,AZ=0,A4=0,ax=const,DX=0,DY=0,DZ=0,HX=0,HY=0,HZ=0,j1=0,j2=0,j3=0,j4=0,rho=0,k=const,omega=const,JT=0,a=const,b=const,mu=const,epsilon=const,e=const,n=const,Omega=const,c=const,X=0,Y=0,Z=0,T=0,r=0,alpha=0,beta=0,theta=0,ZR=const,zz=0,tt=0):
Warning, the protected names norm and trace have been redefined and unprotected

Warning, the protected name close has been redefined and unprotected

Warning, the names &^, d and wdegree have been redefined
>
```

Pullbacks of maps to Euclidean spaces. Rotations + Translations

R. M. Kiehn
Updated 12/03/2001

***

Consider a final state of independent variables, [x,y,z,t],
and a map from cartesian variables [r,theta,z,t] with the additional constraint that represents a kinematic rotation at constant angular velocity about the z axiz. Hence d(theta)/d(t) = Omega = constant. The electromagnetic Action 1-form A will be given in terms of abstract functions Ax,Ay,Az,Phi, on the final state with arguments in terms of [x,y,z,t]. The pair 2-form F = dA and its coefficients on the final state will be evaluated to yield E and B, the field intensities.
The final state will be presumed to be a euclidean space with constitutive properties in the sense that D = epsilon E and B = mu H. Hence, on the final state, given the potentials, the field intensities can be computed, the field excitations D, H, can be evaluated to give the impair 2-form G. From this hypothesis (or constraint) the impair 3-form of 4 current density will be computed from the impair 2-form G, such that J=dG.
The results on the final state then will be pulled back by the combined actions of functional substitution for the independent variables and their differentials. into the differential forms for A, F, G and J.
For 1-forms and N-1 form densities, the pullbacks are particularly simple:
The 1-forms pull back by means of the transpose of the Jacobian matrix of the mapping.
The N-1 form densities pull back via the adjoint of the Jacobian matrix of the mapping.
The matrix elements of the Jacobian matrix need not be global constants.

The following is the procedure for computing EM fields and currents given the potentials on the variety [x,y,z,t], with the constitutive assumption D = epsilon E, B = mu H. The procedure also evalutes the forms A, F, G, J on the final state.

USEFUL OUTPUT FUNCTIONS ARE

AF=vector potential
SP = scalar potential
EF = E field intensity
BF = B field intensity
DF = D field excitation
HF = H field excitation
JD = current density
CD = charge density
SPC = spin current
SPD = spin density
TFC = Torsion flux
HEL = Helicity
TF = Torsion field 4 components
SP = Spin field density 4 components
P1 = First Poincare invariant density
P2 = Second Poincare invariant
A1form = pair 1-form of potentials
F2form = pair 2-form of E, B field intensities
G2form = impair 2-form of D, H excitation densities
J3form = impair 3 form of charge current densities

```
> JCM:=proc(Ax,Ay,Az,phi)\
     local
A,A1,A2,A3,A4,BFC,EF1,EF2,EF3,JAC,JDC,ExBC,JTOT,Jcurl,Jt,Jh,Jd,Jdt,TFCa,SFCa:
     global
A1form,AF,SP,BF,EF,TF,HEL,P1,P2,DF,HF,CD,JA,JD,SPD,SF,SFD,ExB,G2form,F2form,J3f
orm,JTT,TFC,SFC:
     A1:=Ax:A2:=Ay:A3:=Az;A4:=phi:
A:=[A1,A2,A3]:AF:=factor(simplify(A)):SP:=simplify(phi):
A1form:=A1*d(x)+A2*d(y)+A3*d(z)-A4*d(t):
     EF1:=evalm(-grad(phi,[x,y,z])):
     EF2:=-[diff(A1,t),diff(A2,t),diff(A3,t)];
EF3:=[factor(EF1[1]+EF2[1]),factor(EF1[2]+EF2[2]),factor(EF1[3]+EF2[3])];
     EF:=[EF3[1],EF3[2],EF3[3]];
     BFC:=(curl([A1,A2,A3],[x,y,z])):
     BF:=[factor(BFC[1]),factor(BFC[2]),factor(BFC[3])];
HEL:=factor(innerprod(AF,BF));TFCa:=evalm(crossprod(EF,AF)+BF*phi);
     TF:=[factor(TFCa[1]),factor(TFCa[2]),factor(TFCa[3]),HEL];

P2:=-2*factor(innerprod(EF,BF));TFC:=[factor(TFCa[1]),factor(TFCa[2]),factor(TF
Ca[3])];
     HF:=[factor(BFC[1]/mu),factor(BFC[2]/mu),factor(BFC[3]/mu)];
DF:=[factor(epsilon*EF3[1]),factor(epsilon*EF3[2]),factor(epsilon*EF3[3])];
CD:=factor(diverge([DF[1],DF[2],DF[3]],[x,y,z]));
Jcurl:=curl([HF[1],HF[2],HF[3]],[x,y,z]);
     Jh:=[factor(Jcurl[1]),factor(Jcurl[2]),factor(Jcurl[3])];
Jdt:=-[diff(DF[1],t),diff(DF[2],t),diff(DF[3],t)];
Jd:=[factor(Jdt[1]),factor(Jdt[2]),factor(Jdt[3])];JTOT:=Jh+Jd:JD:=[factor(JTOT
[1]),factor(JTOT[2]),factor(JTOT[3])];
JTT:=[factor(JD[1]),factor(JD[2]),factor(JD[3]),CD];
F2form:=evalm(innerprod(BF,[d(y)&^d(z),-d(x)&^d(z),d(x)&^d(y)])+innerprod(EF,[d
```

```
(x)&^d(t),d(y)&^d(t),d(z)&^d(t)]));
SPD:=factor(innerprod(A,DF));G2form:=(HF[1]*d(x)&^d(t)+HF[2]*d(y)&^d(t)+HF[3]*d
(z)&^d(t)-DF[1]*d(y)&^d(z)+DF[2]*d(x)&^d(z)-DF[3]*d(x)&^d(y));J3form:=innerprod
(JTT,[d(y)&^d(z)&^d(t),-d(x)&^d(z)&^d(t),d(x)&^d(y)&^d(t),-d(x)&^d(y)&^d(z)]);

SFCa:=evalm(crossprod(AF,HF)+DF*phi);SFC:=[factor(SFCa[1]),factor(SFCa[2]),fact
or(SFCa[3])];
    SFD:=[factor(SFC[1]),factor(SFC[2]),factor(SFC[3]),SPD];
    P1:=innerprod(BF,HF)-innerprod(DF,EF)-innerprod(AF,JD)+CD*phi;
ExBC:=crossprod(EF,BF);ExB:=[factor(ExBC[1]),factor(ExBC[2]),factor(ExBC[3])];
  end proc:
> MAP:=proc(X,Y,Z,T,x,y,z,t) global JAC,ADJAC,DET,TRJAC,Map:
  Map:=[x,y,z,t]:JAC:=simplify(jacobian(Map,[X,Y,Z,T])):DET:=factor(simplify(de
  t(JAC))):TRJAC:=simplify(transpose(JAC)):ADJAC:=simplify(adjoint(JAC)): end
  proc:
>
>
>
> b:=0:c:=0:
> rr:=(x^2+y^2)^(1/2):AAA:=[z^2*y*b,-z^2*x*b,1-k^2*rr^2,c*z*y*x+omega*(1-k^2*rr
  ^2)/k]:
> JCM(AAA[1],AAA[2],AAA[3],AAA[4]):
```

The exterior differential forms as specified on the final state.

```
> A1form:=wcollect(factor(simplify(A1form)));F2form:=
  wcollect(factor(simplify(F2form)));G2form:=
  wcollect(factor(simplify(G2form)));J3form:=
  wcollect(factor(simplify(J3form)));
```

$$A1form := (1 - k^2 x^2 - k^2 y^2)\, d(z) + \frac{\omega(-1 + k^2 x^2 + k^2 y^2)\, d(t)}{k}$$

$$F2form := -2 k^2 y\,(d(y) \&\wedge d(z)) - 2 k^2 x\,(d(x) \&\wedge d(z)) + 2\,\omega\, k\, x\,(d(x) \&\wedge d(t)) + 2\,\omega\, k\, y\,(d(y) \&\wedge d(t))$$

$G2form :=$

$$-2\frac{k^2 y\,(d(x) \&\wedge d(t))}{\mu} + \frac{2 k^2 x\,(d(y) \&\wedge d(t))}{\mu} - 2\,\varepsilon\,\omega\, k\, x\,(d(y) \&\wedge d(z)) + 2\,\varepsilon\,\omega\, k\, y\,(d(x) \&\wedge d(z))$$

$$J3form := -4\,\varepsilon\,\omega\, k \,\&\wedge(d(x), d(y), d(z)) + \frac{4 k^2 \&\wedge(d(x), d(y), d(t))}{\mu}$$

The fields in engineering format on the final state

```
> R:=[x,y,z,t];Vector_potential:=AF;Scalar_potential:=factor(SP);E_field:=EF;B_
  field:=BF;Poincare2:=factor(P2);D_field:=DF;H_field:=HF;rho_charge_density:=C
  D;J_current_density:=(simplify(JD));;Poincare1:=factor(P1);PoyntingVector=ExB
  ;Torsion_flux:=factor(evalm(TFC));Helicity:=HEL;Spin_current:=factor((SFC));S
  pin_density:=factor(SPD);Lagrangian_field_energy_density:=factor(simplify(inn
  erprod(HF,BF)-innerprod(DF,EF)));Interaction_energy_density:=factor(AF[1]*JD[
  1]+AF[2]*JD[2]+AF[3]*JD[3]-CD*SP);
```

$$R := [x, y, z, t]$$

$$Vector\_potential := [0, 0, 1 - k^2 x^2 - k^2 y^2]$$

$$Scalar\_potential := -\frac{\omega(-1 + k^2 x^2 + k^2 y^2)}{k}$$

$$E\_field := [2\,\omega\, k\, x, 2\,\omega\, k\, y, 0]$$

$$B\_field := [-2 k^2 y, 2 k^2 x, 0]$$

$$Poincare2 := 0$$

$$D\_field := [2\,\varepsilon\,\omega\,k\,x,\ 2\,\varepsilon\,\omega\,k\,y,\ 0]$$

$$H\_field := \left[-2\,\frac{k^2\,y}{\mu},\ 2\,\frac{k^2\,x}{\mu},\ 0\right]$$

$$rho\_charge\_density := 4\,\varepsilon\,\omega\,k$$

$$J\_current\_density := \left[0,\ 0,\ 4\,\frac{k^2}{\mu}\right]$$

$$Poincare1 := -4\,\frac{(2\,k^2\,x^2 + 2\,k^2\,y^2 - 1)\,(-k^2 + \omega^2\,\varepsilon\,\mu)}{\mu}$$

$$PoyntingVector = [0,\ 0,\ 4\,\omega\,k^3\,(x^2 + y^2)]$$

$$Torsion\_flux := [0,\ 0,\ 0]$$

$$Helicity := 0$$

$$Spin\_current := \left[-2\,\frac{(-1 + k^2\,x^2 + k^2\,y^2)\,x\,(-k^2 + \omega^2\,\varepsilon\,\mu)}{\mu},\ -2\,\frac{(-1 + k^2\,x^2 + k^2\,y^2)\,y\,(-k^2 + \omega^2\,\varepsilon\,\mu)}{\mu},\ 0\right]$$

$$Spin\_density := 0$$

$$Lagrangian\_field\_energy\_density := -4\,\frac{k^2\,(x^2 + y^2)\,(-k^2 + \omega^2\,\varepsilon\,\mu)}{\mu}$$

$$Interaction\_energy\_density := 4\,\frac{(-1 + k^2\,x^2 + k^2\,y^2)\,(-k^2 + \omega^2\,\varepsilon\,\mu)}{\mu}$$

```
> R_final_variables:=R;
```

$$R\_final\_variables := [x,\ y,\ z,\ t]$$

Variables on Initial State:

```
> X:=r:Y:=theta:Z:=zz:T:=tt:
>
```

Define the mapping functions here

ROTATION ABOUT z axis plus translation along z axis

```
> x:=r*cos(theta-Omega*T);y:=r*sin(theta-Omega*T);z:=Z-Vz*tt;t:=T;
```

$$x := r\cos(-\theta + \Omega\,tt)$$

$$y := -r\sin(-\theta + \Omega\,tt)$$

$$z := zz - Vz\,tt$$

$$t := tt$$

```
>
> R_initial_variables:=[X,Y,Z,T];
```

$$R\_initial\_variables := [r,\ \theta,\ zz,\ tt]$$

```
> MAP(X,Y,Z,T,x,y,z,t):Mapping_fucntions:=Map;Jacobian:=evalm(JAC);DET:=DET:Adj
  oint:=evalm(ADJAC):
```

$$Mapping\_fucntions := [r\cos(-\theta + \Omega\,tt),\ -r\sin(-\theta + \Omega\,tt),\ zz - Vz\,tt,\ tt]$$

$$Jacobian := \begin{bmatrix} \cos(-\theta + \Omega\,tt) & r\sin(-\theta + \Omega\,tt) & 0 & -r\sin(-\theta + \Omega\,tt)\,\Omega \\ -\sin(-\theta + \Omega\,tt) & r\cos(-\theta + \Omega\,tt) & 0 & -r\cos(-\theta + \Omega\,tt)\,\Omega \\ 0 & 0 & 1 & -Vz \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

The mapping represents a rotation about the z axis.

Evaluate the exterior forms on the initial state by functional substitution and pullback:

```
> A1form:=wcollect(factor(simplify(A1form)));F2form:=
  wcollect(factor(simplify(F2form)));G2form:=
  wcollect(factor(simplify(G2form)));J3form:=
  wcollect(factor(simplify(J3form)));
```

$$A1form := -(k\,r-1)\,(k\,r+1)\,d(zz) - \frac{(k\,r-1)\,(k\,r+1)\,(-k\,Vz-\omega)\,d(tt)}{k}$$

$$F2form := -2\,k^2\,r\,(d(r)\ \&\wedge\ d(zz)) + 2\,k\,r\,(k\,Vz+\omega)\,(d(r)\ \&\wedge\ d(tt))$$

$$G2form := 2\,\frac{k\,r^2\,(k+\varepsilon\,\omega\,\mu\,Vz)\,(d(\theta)\ \&\wedge\ d(tt))}{\mu} - 2\,k\,r^2\,\varepsilon\,\omega\,(d(\theta)\ \&\wedge\ d(zz)) + 2\,k\,r^2\,\varepsilon\,\omega\,\Omega\,(d(tt)\ \&\wedge\ d(zz))$$

$$J3form := -4\,k\,r\,\varepsilon\,\omega\ \&\wedge(d(r), d(\theta), d(zz)) + \frac{4\,k\,r\,(k+\varepsilon\,\omega\,\mu\,Vz)\ \&\wedge(d(r), d(\theta), d(tt))}{\mu}$$
$$+ 4\,k\,r\,\varepsilon\,\omega\ \&\wedge(d(r), d(tt), d(zz))\,\Omega$$

```
> Spin3form:=(A1form&^G2form);Torsion3form:=(A1form&^F2form);
```

$$Spin3form := 2\,\frac{(k\,r-1)\,(k\,r+1)\,r^2\,(-k^2+\omega^2\,\varepsilon\,\mu)\ \&\wedge(d(zz), d(\theta), d(tt))}{\mu}$$

$$Torsion3form := 0$$

```
>
```

Pullback field Components on initial state:

```
> AF_PB:=innerprod(TRJAC,[AF[1],AF[2],AF[3],-SP]);VPotential_PB:=simplify([AF_PB[1],AF_PB[2],AF_PB[3]]);ScalarPot_PB:=simplify(-AF_PB[4]);EF_PB:=factor(simplify((evalm(-grad(ScalarPot_PB,[X,Y,Z])-diff(VPotential_PB,T)))));BF_PB:=factor(simplify(curl(VPotential_PB,[X,Y,Z])));D1:=-getcoeff(G2form&^d(X)&^d(T)):D2:=getcoeff(G2form&^d(Y)&^d(T)):D3:=-getcoeff(G2form&^d(Z)&^d(T)):H1:=getcoeff(G2form&^d(Y)&^d(Z)):H2:=getcoeff(G2form&^d(X)&^d(Z)):H3:=-getcoeff(G2form&^d(X)&^d(Y)):DF_PB:=[factor(simplify(D1)),factor(simplify(D2)),factor(simplify(D3))];HF_PB:=[factor(simplify(H1)),factor(simplify(H2)),factor(simplify(H3))];JTT;JTPB:=innerprod(ADJAC,JTT):JD_PB:=[JTPB[1],JTPB[2],JTPB[3]];JC_PB:=JTPB[4];Poincare2_PB:=Poincare2;Poincare1_PB:=factor(simplify(DET*Poincare1));DET:=DET;M_F:=evalm(curl(EF_PB,[X,Y,Z])+[diff(BF_PB[1],T),diff(BF_PB[2],T),diff(BF_PB[3],T)]);M_A:=evalm(curl(HF_PB,[X,Y,Z])-[diff(DF_PB[1],T),diff(DF_PB[2],T),diff(DF_PB[3],T)]);M_FdivB:=diverge(BF_PB,[X,Y,Z]);M_AdivD:=diverge(DF_PB,[X,Y,Z]);
```

$$AF\_PB := \left[0, 0, 1-k^2\,r^2\cos(-\theta+\Omega\,tt)^2 - k^2\,r^2\sin(-\theta+\Omega\,tt)^2,\right.$$
$$\left.\frac{(-1+k^2\,r^2\cos(-\theta+\Omega\,tt)^2+k^2\,r^2\sin(-\theta+\Omega\,tt)^2)\,(k\,Vz+\omega)}{k}\right]$$

$$VPotential\_PB := [0, 0, 1-k^2\,r^2]$$

$$ScalarPot\_PB := -\frac{-k\,Vz-\omega+k^3\,r^2\,Vz+k^2\,r^2\,\omega}{k}$$

$$EF\_PB := [2\,k\,r\,(k\,Vz+\omega), 0, 0]$$

$$BF\_PB := [0, 2\,k^2\,r, 0]$$

$$DF\_PB := [2\,k\,r^2\,\varepsilon\,\omega, 0, 0]$$

$$HF\_PB := \left[0, 2\,\frac{k\,r^2\,(k+\varepsilon\,\omega\,\mu\,Vz)}{\mu}, -2\,k\,r^2\,\varepsilon\,\omega\,\Omega\right]$$

$$\left[0, 0, 4\,\frac{k^2}{\mu}, 4\,\varepsilon\,\omega\,k\right]$$

$$JD\_PB := \left[0, 4\,r\,\Omega\,\varepsilon\,\omega\,k, 4\,\frac{k\,r\,(k+\varepsilon\,\omega\,\mu\,Vz)}{\mu}\right]$$

$$JC\_PB := 4\,r\,\varepsilon\,\omega\,k$$

$$Poincare2\_PB := 0$$

$$Poincare1\_PB := -4\,\frac{r\,(2\,k^2\,r^2-1)\,(-k^2+\omega^2\,\varepsilon\,\mu)}{\mu}$$

$$DET := r$$

$$M\_F := [0, 0, 0]$$

$$M\_A := \left[0,\ 4\,r\,\Omega\,\varepsilon\,\omega\,k,\ 4\,\frac{k\,r\,(k+\varepsilon\,\omega\,\mu\,Vz)}{\mu}\right]$$

$$M\_FdivB := 0$$

$$M\_AdivD := 4\,r\,\varepsilon\,\omega\,k$$

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Note that the constitutive relations between D and E and B and H on the initial state are not the same as for the final state.
The D fields are equal to epsilon E times the DET of the transformation, there by converting a tensor into a tensor density.
Simlarly the H fields are B divided by mu times the DET of the transformation -- almost.
There appears another term in the H fields on the final state due to the rotation Omega. Indeed, motion of the charge density about the z axis appears to create a contribution to the current density that encircles the z axis. Such a current density induces a component of H along the z axis and related to the rotation rate.
This rotational motion of the charge density influences the H field, but DOES NOT affect the associated B fields.

THE PULLED BACK FIELD COMPONENTS E,B satisfy the MAXWELL - FARADAY PDE's in terms of the independent variables [r,theta,z,t] with the constraint that d(theta)/d(t) = Omega.
THE PULLED BACK FIELD COMPONENTS D,H also satisfy the MAXWELL - AMPERE PDE's in in terms of the independent variables [r,theta,z,t] with the constraint that d(theta)/d(t) = Omega.

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