Curvature and torsion of implicit hypersurfaces and the origin of charge

R. M. Kiehn

Emeritus, Phys. Dept., Univ. Houston
Updated 08/25/2001
http://www.cartan.pair.com
rkiehn2352@aol.com

Abstract

A formal correspondence is established between the curvature theory of generalized implicit hypersurfaces, the classical theory of electromagnetism as expressed in terms of exterior differential systems, and thermodynamics. Starting with a generalized implicit surface whose normal field is represented by an exterior differential 1-form, it is possible to deduce the curvature invariants of the implicit surface and to construct a globally closed vector density in terms of the Jacobian properties of the normal field. When the closed vector density is assigned the role of an intrinsic charge current density, and the components of the normal field are assigned the roles of the electromagnetic potentials, the theory is formally equivalent to an exterior differential system that generates the PDE's of both the Maxwell Faraday equations and the Maxwell Ampere equations. The interaction energy density between the potentials and the induced closed charge current density is exactly the similarity curvature invariant of highest degree (N-1) for the implicit surface. Although developed without direct contact with M-brane theory, these ideas of generalized implicit surfaces should have application to the study of p-branes that can have multiple components and envelopes. The theory suggests that gravitational collapse of mass energy density should include terms that involve the interaction between charge-current densities and electromagnetic potentials.

1. Introduction

1.1. An overview

The origin of charge has long been a mystery to physical theory, perhaps even more illusive than the concept of inertial mass. A major objective of this article is to examine the conjecture that the charge-current density of electromagnetism may have its origins in the differential geometry and topology of curvature and torsion, in a sense similar to the idea that mass density and gravity have their origins in the concept of curvature. The curvatures of interest are not those generated by a symmetric metric, but instead are those similarity invariants associated with a generalized implicit hypersurface. The generalized implicit hypersurfaces considered may not admit a global foliation as their normal fields need not satisfy the Frobenius integrability conditions. Hence such generalized hypersurfaces can support topological torsion as well as curvature.

An arbitrary 1-form of Action, A_0,whose coefficient functions may be considered as a set of electromagnetic potentials, when suitably scaled, can also play the role of the normal field to a generalized implicit hypersurface. The closure of the exterior differential system, $F_0 - dA_0 = 0$, always generates a system of PDE's which contain the Maxwell-Faraday equations. When the 1-form of Action is rescaled by use of a Holder norm, λ, such that resulting 1-form $A = A_0/\lambda$, is homogeneous of degree zero in its coefficient functions, the curvature features of the implicit hypersurface are completely specified in terms of the similarity invariants of the Jacobian matrix constructed from the components of teh renormalized 1-from, A. The curvature similarity invariant of highest degree (N-1) is defined as the Adjoint curvature, and is equal to the trace of the Jacobian Adjoint matrix.

It is now well known that given any non-closed 1-form, A_0,of twice differentiable functions, it is possible to deduce a 2-form of field intensities, F_0, and to show that the Maxwell-Faraday equations are always satisfied. However, without additional assumptions, how to produce or define a globally closed charge-current density is another matter. An algorithm from implicit surface theory will be used herein to construct, from the intrinsic properties of the implicit surface, such a globally closed N-1 form charge current density. The techniques thereby demonstrate the connection between implicit surface theory and the theory of electromagnetism. Although the Jacobian matrix to be constructed is globally singular, it is always possible to construct algebraically the matrix of cofactors transposed, defined as the Jacobian Adjoint matrix. Remarkably, multiplication of the covariant components of A by this singular Adjoint matrix yields an N-1 form density, or current, J_s, which is globally closed. The result is universally valid if the coefficients of the 1-form, A, are homogeneous of degree zero in its component functions. The global closure implies that there exists an N-2 form G_s such that $J_s - dG_s = 0$. The conclusion is valid for any Holder norm of any signature, of arbitrary isotropic index p, and of homogeneity index n =1. The PDE's associated with this exterior differential system are known to contain the Maxwell-Ampere equations.

These two Maxwell exterior differential systems lead to another N-1 form density, previously defined [1] in the four dimensional case as Topological Spin, $A\hat{}G_s$. This N-1 form always satisfies the equation

$$d(A\hat{}G_s) = F\hat{}G_s - A\hat{}J_s, \tag{1.1}$$

and demonstrates that twice the difference between the magnetic and electric energy densities of the field is cohomologous with the interaction energy density, $A\hat{}J_s$, generated by interaction of the Potentials and the Charge-Current density. (It has been suggested that this formula can be put into correspondence with the principle of equivalence, where the Field energy density $F\hat{}G$ plays the role of the gavitational field and the interaction energy density, $A\hat{}J$, plays the role of inertial energy density [2]). If the Holder norm used to make the initial 1-form homogeneous of degree 1 in its component functions is specialized to be of euclidean signature, isotropic index p = 2, and homogeneity index n =1 (which defines the Gauss map), then the N-1 form J_s is uniquely determined. It is a major result of this article to show that interaction energy density, $A\hat{}J_s$, is then always proportional to Adjoint curvature of the implicit hypersurface constructed from the 1-form of Action Potentials. On a four dimensional variety, $A\hat{}J_s$ is cubic in the principle curvatures.

With respect to a process defined by the induced charge-current density, Cartan's magic formula of topological evolution demonstrates a formal correspondence to the first law of thermodynamics [3]. The internal energy density of the physical system described by the 1-form, A, evolving in the direction field of the closed charge-current density, J_s, is exactly the coefficient of the Interaction energy N-form, $A\hat{}J_s$. This coefficient is exactly equal to the Adjoint curvature of the implicit surface. Hence a correspondence is established between the curvature theory of implicit hypersurfaces, the charge-current density interaction, and the internal energy of a thermodynamic system.

The implicit hypersurfaces can be put into equivalence classes depending upon the Pfaff dimension or class of the generating 1-form. Examples indicate that, depending upon the Pfaff dimension, the charge current densities are proportional to the adjoint curvatures and/or the topological torsion induced by the generalized implicit hypersurface.

It is important to realize that the method to be discussed involves curvatures, torsion and energy densities, but does not depend explicitly upon a metric, gauge constraints, or the Einstein field equations. In section 2 some topological and thermodynamic features of electromagnetism will be discussed. In section 3, the theory of generalized implicit hypersurfaces will be developed. In section 4 a number of examples will be summarized for generalized implicit hypersurfaces in N=3 and N=4 dimensions, demonstrating the claim that an intrinsic current exists, and that the intrinsic charge-current interaction with the potentials is equal to the N-1 similarity invariant of the hypersurface. The Maple programs that generated the examples can be downloaded from the internet [4]

1.2. Some Topological and Geometrical Features of Electromagnetism

1.2.1. Charge counting, conductors and insulators

From experience it is known that a given electromagnetic charge-current density J is conserved: $dJ = 0$ (the 4 vector density has zero divergence). However, a more important result is the observation of global charge neutrality, which can be attributed to a topological idea. As $dJ = 0$, is a global statement, there exists an N-2 form, G, such that $J - dG = 0$. This exterior differential system [2] is equivalent to the Maxwell-Ampere system of partial differential equations. The integral of G over a closed cycle in domains where $dG = 0$ yields values whose ratios are rational (Gauss' law of counting charges). When the closed integration domain is a boundary, the net charge enclosed is zero, yielding charge neutrality. These topological aspects can be used to distinguish insulators from conductors. Three dimensional insulators can be separated in the presence of an external $\mathbf{E}$ field into two physical components with each component interior enclosed by a two dimensional boundary. The external field distorts the internal charge distribution to produce a dipole field. Each physical component remains charge neutral when the external field is removed. Similarly three dimensional conductors can be separated into two physical components, but the presence of a remnant exterior electromagnetic field between the components indicates that the closed two dimensional varieties of each component are cycles, not necessarily boundaries. The components are said to be charged.

1.2.2. Domains of support

It has long been respected that physical work is required to produce charge separation, and that such charge separation leads to a potential difference between the charged components that can be used to produce useful work. In fact, the conventional physics approach to understanding electromagnetism is to start with some given distribution of charge currents and compute by some set of rules the associated potentials. In this article, the opposite procedure is exploited. The starting point will be given in terms of a set of potentials (functions), which can be used to construct a 1-form of Action, A_0,on a variety of independent variables. For C2 functions, exterior differentiation generates the exterior differential system, $F0 - dA_0 = 0$. The closure of this system is always equivalent to the system of PDE's that are known as the Maxwell-Faraday equations. The Maxwell-Faraday exterior differential system indicates that the domain of support for the 2-form of field intensities is usually open, or compact with boundary. The only possible exceptions are the torus and the Klein bottle. However these exceptions fail if the 2-form is of rank 4.

In this article an algorithm is presented whereby the potentials will lead to a well defined set of charge-currents, a procedure which is opposite to the conventional methods. However, the new method exploited herein has geometric and topological significance. The 1-form of Action Potentials will be made homogeneous of degree zero by division by a suitable Holder norm, λ, leading to the expression, $A = A_0/\lambda$. The Jacobian matrix of A will be constructed, as well as its Adjoint (matrix of cofactors transposed). In this sense, the components of 1-form A can be interpreted as the normal field to an implicit hypersurface. The similarity invariants of the Jacobian matrix determine the important curvature features of the hypersurface. The Jacobian matrix so constructed will always be singular and often is of maximal rank, N-1. The similarity invariant of highest degree, N-1, is equal to the trace of the Adjoint Jacobian matrix, and is therefor defined herein as the Adjoint curvature. The adjoint curvature plays a dominant part in the discussion that follows.

Multiplication of the components of A by the Adjoint matrix permit the construction of a closed N-1 form density, which will play the role of a deduced electromagnetic charge current density, J_s. The notation (with a subscript s) is such as to remind the reader that this current density was created from the singular Adjoint matrix generated from the 1-form of Potentials. As this N-1 form is closed (has zero divergence globally) there exists a G_s such that $J_s - dG_s = 0$. The PDE's created by this exterior differential system are equivalent to the Maxwell-

Ampere equations. In a space of 4 dimensions the properties of the N-2=2 form G_s are not the same as the properties of the 2-form F_0. The domain of support for the 2-form F_0 is not compact without boundary, while the domain of support for G_s can be compact without boundary.

1.2.3. Interaction energy density

In addition, the interaction energy density, defined as the N-form density,

$$A\hat{}J_s = F\hat{}G_s - d(A\hat{}G_s), \tag{1.2}$$

will be computed. The term $A\hat{}G$ in electromagnetic systems has been previously defined as "Topological Spin" [1] for it has the physical dimensions of joule-s in electromagnetic systems. The term $F\hat{}G$ represents twice the difference between magnetic and electric energy density and changes sign from a plasma to an electrostatic state. The equation is a statement of the cohomology of the two forms of energy density. In regions where $A\hat{}G$ is closed, the closed 3 dimensional integrals of $A\hat{}G$ have values whose ratios are rational [3] and are therefore countable. It will be demonstrated below that this interaction density is precisely equal to the Adjoint curvature of the hypersurface whose normal field is generated by the 1-form, A. On a variety of four dimensions, this result implies that interaction energy between the 4 potentials and the deduced (or intrinsic) charge current density is related to a cubic polynomial of the hypersurface curvatures, while the Gaussian sectional curvature (and therefor mass energy density) is quadratic in the surface curvatures. When the Jacobian matrix is of maximal rank N-2, the interaction energy vanishes. Note that if the interaction energy density is zero, the charge current density need not be zero. A special case exists such that if J_s is proportional to the Topological Torsion 3 form, $A\hat{}dA$, then the interaction energy density vanishes due to orthogonality of its two components. This special case will be discussed further below.

1.2.4. Topological evolution and internal energy density

Given a 1-form of Action A and a closed charge current density J, it is possible to use Cartan's magic formula [4] of topological evolution to demonstrate a correspondence between the implicit surface theory and the first law of thermodynamics. For evolutionary processes in the direction of the charge current density, Cartan's magic formula becomes

$$L(J)A = i(J)dA + d(i(J)A) = W + dU = Q \tag{1.3}$$

Using electromagnetic notation, on a variety $\{x, y, z, t\}$ the (virtual) work 1-form becomes

$$W = i(J)dA = (\rho\mathbf{E} + \mathbf{J}\times\mathbf{B})_k dx^k + (\mathbf{J}\circ\mathbf{E})dt \tag{1.4}$$

which is recognized as the product of the Lorentz force density times the differential displacement plus the dissipative power density times the increment dt.

In certain cases the induced charge current density, J_s will be proportional to the Topological Torsion field, $A\hat{}dA = i(T)dx\hat{}dy\hat{}dz\hat{}dt$. (An example of this case is presented below). In such cases, it follows that the evolution of the implicit surface is given by the expression,

$$L_{(J)}A = L_{(T)}A = i(T)dA + 0 = (\Gamma)\ A = (\mathbf{E}\circ\mathbf{B})A = Q. \tag{1.5}$$

It follows that the heat 3 form and the Topological Torsion 3 form are proportional:

$$Q\hat{}dQ = (\mathbf{E}\circ\mathbf{B})^2 A\hat{}dA. \tag{1.6}$$

From classical thermodynamics, when the process produces a heat 1-form Q which does not admit an integrating factor, such a process is thermodynamically irreversible. If the coefficient (the second Poincare invariant which is related to the 4 form $F\hat{}F$) is not zero, then irreversible processes exist when the topological torsion of the implicit surface is not zero. In order for Q to admit an integrating factor, the Frobenius integrability condition must be satisfied, or $Q\hat{}dQ = 0$. But if the surface 1-form is of Pfaff dimension 4, then $A\hat{}dA \neq 0$, and $\mathbf{E}.\mathbf{B} \neq 0$. It follows $Q\hat{}dQ \neq 0$, and such irreversible processes are artifacts of 4 dimensions.

Similarly, evaluation of the internal energy density for a process defined by the dynamics of the charge-current density becomes

$$U = (i(J)A) = \mathbf{A}\circ\mathbf{J} - \rho\phi. \tag{1.7}$$

which in classical field theory is defined as the interaction energy density. From the discussion above it is apparent that internal energy density is equivalent to the coefficient of the N form $A\hat{}J$, and at the same time is equal to the Adjoint curvature of the implicit hypersurface. It appears that the charge current interaction energy density, the thermodynamic internal energy density, and the adjoint curvature of the implicit surface generated by the 1-form of potentials are equivalent concepts.

1.2.5. Gauge constraints are not used.

It is to be noted that given an N-1 form, J_s, the N-2 form G_s that satisfies the exterior differential system $J_s - dG_s = 0$ is not unique. Any closed N-2 form, q, such that $dq = 0$, may be added to G_s without changing the result of the exterior differential system. The same can be said for the 2-form F_0; there are many closed 1-forms γ, such that $d\gamma = 0$, that can be added to the 1-form A_0 and yet the exterior differential system yields the same values for F_0. It is the closed integrals of the closed but not exact components of q that determine the quantized charges. It is the closed integrals of closed but not exact components of γ that determine the quantized flux quanta. These concepts of ambiguity and non-uniqueness are often parlayed into specific theories, called gauge theories, where the non-uniqueness is restricted in form to some equivalence class. These problems of specific gauge equivalence class are not pertinent herein, for the results are formulated to be valid without a specification of gauge.

Classic Maxwell theory, as written in terms of the fields $\mathbf{E}$ and $\mathbf{B}$ only, is often said to be a U(1) gauge theory. However, when written in the language of exterior differential forms, Maxwell theory is *not* a U(1) gauge theory. The solution fields are not limited to those that can be constructed from complex functions. The topological Maxwell formulation is well behaved with respect to the general linear group, and with respect to differentiable maps without inverse. It is the imposition of zero charge current density (no interaction energy density) and the Lorentz constitutive constraint that reduces the general Maxwell theory to a U(1) gauge group theory [5]. The usual constraints are such that the both $F\hat{}G$ and $A\hat{}J$ vanish simultaneously. The constitutive constraints typically imposed are of the form $\mathbf{B} = \mu\mathbf{H}$ and $\mathbf{D} = \epsilon\mathbf{E}$ with the understanding that $\epsilon\mu c^2 = 1$ and $\mathbf{B} \circ \mathbf{B} - \mathbf{E} \circ \mathbf{E}/c^2 = 0$. An alternate formulation by Bateman [6] and Whittaker leads to the constitutive constraint, $\mathbf{D} = \alpha\mathbf{B}$ and $\mathbf{H} = \beta\mathbf{E}$, with the understanding that $(\alpha - \beta)\ (\mathbf{E} \circ \mathbf{B}) = \mathbf{0}$, The first case can be extended to include birefringence and Faraday rotation, while the second case generalizes to include rotational acceleration (Sagnac) effects and Optical Activity. Both simple formulations lead to the result that $F\hat{}G = 0$, which implies that the N-1 form $A\hat{}G$ is closed. Hence (subject to the constraints) over closed N-1 manifolds which are cycles, the integrals of the Topological Spin N-1=3 form can have ratios which are rational. The closed integrals are evolutionary deformation invariants, and thereby carry topological information.

1.2.6. Dissipation

It is of some importance to note, in certain topological circumstances, that the Jacobian induced currents, J_s, occur without the presence of **E** or **B** fields, (when $dA_0 = F_0 = 0$) or for other situations where **E** or **B** fields are present (where $F_0 \neq 0$), but where the dissipation coefficient **J**.**E** is zero. The implicit hypersurface method thereby seems to offer an alternative, non-quantum mechanical, understanding of what otherwise would be called superconducting currents. In the first case, the **B** field is excluded (Meisner effect) from the superconducting region, and in the second case (Hall effect) a large **B** field is present along with a non-dissipative current.

If the adjoint curvature of the generalized hypersurface is zero, then either the charge current density is zero, or the charge current density resides in the hypersurface, and thereby is orthogonal to the surface normal. In the latter special case, the direction field of the charge current density is proportional to the Topological Torsion vector [2] generated by the 3-form $A\hat{}dA$. Examples below indicate that there is a correlation between non-zero adjoint curvature and/or topological torsion and the existence of a charge current density.

2. Implicit Hypersurfaces

The algebra of the processes to be described can be staggering, especially for older folks who do not see too well. Hence a Maple program [9] has been provided to make the computation of examples a bit faster.

2.1. The normal and tangent fields

The classic implicit surface is generated by assigning a constant value to a function, $\phi(x, y, z..)$. It is important to recall that an implicit surface, in contrast to a parametric surface, can consist of more than one disconnected components. The gradient field to the given function represents a normal field to the surface, and tangent vectors which reside on the surface are orthogonal to the normal field at all points. As the normal field for the classic implicit surface is a gradient field, its associated 1-form is exact. If this normal gradient field is rescaled by a factor such that is homogeneous of degree zero in its functional arguments, then the Jacobian matrix of the rescaled normal field can be used to generate the curvatures of the implicit surface.

This procedure can be extended to the study of generalized implicit surfaces whose normal field is not representable by an exact 1-form. The 1-form representing the normal field can have arbitrary Pfaff dimension. If the Pfaff dimension of the 1-form is greater than 2, then the implicit surface can support topological torsion, $A\hat{}dA \neq 0$. It is necessary that the Pfaff dimension be greater than 2 if the implicit surface admits an envelope. A N-1 tangent vector basis can be constructed algebraically from the 1-form that represents the normal field. In the language of exterior differential forms, the tangent vectors, $\mathbf{e}$, have been described as the associate vectors relative to the 1-form A_0, and satisfy the equation, $i(\mathbf{e})A_0 = 0$. The array of tangent vectors and the normal field can be used to form a ND basis at any point in the implicit surface.

In fact it follows that starting from an arbitrary 1-form, defined herein as the Action 1-form of Potentials, A_0, on a variety of independent variables $(x, y, z, ...)$, it is possible to develop the curvature properties of the generalized implicit surface algebraically after admitting only one differentiation process. The components of A_0 will play the role of the normal field.

2.2. The homogeneous Holder norm and similarity curvature invariants.

After division by a suitable function of the coefficient potentials, λ, the original 1-form of Action

$$A_0 = (U(x, y, z, ...)dx + V(x, y, z, ...)dy + W(x, y, z, ...)dz...), \tag{2.1}$$

can be made homogeneous of degree zero in terms of those coefficient functions that define the potentials. It is to be emphasized that the homogeneity condition is not on the arguments of the coefficients, but on the coefficient functions themselves. The scaling function of choice, λ, is a Holder norm and is defined in terms of the covariant coefficients of the 1-form:

$$\lambda = (aU^p + bV^p + cW^p + ...)^{n/p}. \tag{2.2}$$

The index n will be defined as the homogeneity index; the index p will be described herein as the isotropic index, and the constants $(a, b, c...)$ are constant scale factors whose signs determine the signature. By choosing the index n to be unity, $n = 1$, the 1-form, A, defined as

$$A = A_0/\lambda = (Udx + Vdy + Wdz...)/\lambda = A_k dx^k \tag{2.3}$$

becomes homogeneous of degree zero in its coefficients. That is if every coefficient function is increased by a factor β then the coefficient function A_k does not change. This homogeneous degree zero 1-form, A_0/λ, is used to define an implicit hypersurface in the variety, whose geometrical properties can be expressed classically in terms of the similarity invariants of the associated *singular* Jacobian dyadic (or matrix). Classically these similarity invariants are "symmetric" functions of the surface curvatures. Examples are given below.

The doubly covariant Jacobian dyadic of coefficients is defined as the matrix of functions

$$Jacobian(A) = [\partial A_m/\partial x^n] = [JAC(A)]_{mn} = [\mathbb{J}] \tag{2.4}$$

The determinant of the Jacobian matrix so constructed ($n = 1$, any $a, b, c..., p$)) is always zero, indicating the existence of at least one zero eigen value (curvature or reciprocal radius). Hence the Jacobian matrix so constructed is singular, and induces a singular metric on the variety via the pullback $[g] = [\mathbb{J}]^{Transpose} \circ [\mathbb{J}]$. The zero determinant result also implies the existence of a global N-1 dimensional variety which in effect defines the implicit (hyper) surface. It is a standard geometrical procedure to construct the symmetric similarity invariants of the Jacobian matrix by forming the Cayley-Hamilton characteristic polynomial. Note that the induced symmetric metric $[g]$ does not carry the complete story of the surface properties inherent in the Jacobian dyadic, for the Jacobian matrix is not necessarily symmetric. As pointed out by Brand [7], the anti-symmetric components of the Jacobian dyadic also have important invariance properties. These additional invariants are developed in terms of the 2-form $F = dA$.

2.3. A globally closed current from the adjoint matrix

Next construct the doubly contravariant matrix $\left[\widehat{\mathbb{J}}\right]$ equal to the adjoint (matrix of co-factors transposed) of the doubly covariant Jacobian matrix. This adjoint matrix exists algebraically, even though the inverse of the singular Jacobian matrix, and the inverse of the induced singular metric does not. Use the adjoint matrix to construct the contravariant vector current, $|\mathbf{J}_s\rangle$,

$$|\mathbf{J}_s\rangle = [ADJ(A)]^{mn} \circ |\mathbf{A}\rangle = [\widehat{\mathbb{J}}]^{nm} \circ |\mathbf{A}\rangle\,, \tag{2.5}$$

and the N-1 form density, J_s :

$$J_s = i(\mathbf{J}_s)dx\hat{}dy\hat{}dz... \tag{2.6}$$

Remarkably for any Holder norm with $n = 1$, arbitrary signature, arbitrary scale factors, and arbitrary exponent p, the N-1 form, J_s, is closed.

$$dJ_s = 0 \quad for \;\; n = 1 \tag{2.7}$$

As this closure result is global, it follows that $J_s - dG_s = 0,$. which is equivalent to the Maxwell-Ampere equations. The subscript s is used to distinguish the fact that J_s has been deduced from the singular Jacobian matrix, and does not explicitly depend upon the field intensities, $F_0 = dA_0$ and some arbitrary constitutive constraint between F and G. Note that given a J_s the corresponding G_s is not uniquely determined. The N-2 form density, G_s, may have closed and exact components as well as closed non-exact components, neither of which contribute to a specific Charge-Current.

It is to be noted the induced metric is singular and therefor cannot be used to define a raising tensor as an inverse metric. Yet a raising tensor field $[\widehat{\mathbb{J}}]^{nm}$ can be functionally well defined in terms of the Adjoint of the Jacobian matrix. This raising tensor field, unlike a non-singular metric inverse field, is not symmetric. Moreover the Adjoint method applies to 1-forms (and therefor hypersurfaces) that do not satisfy the Frobenius condition of unique integrability. Hence, topological torsion, defined as the 3-form, $A_0\hat{}dA_0$, need not be zero. It will be demonstrated below that when the Holder norm is specialized to the Gauss map, $a = b = c = ... = 1$, $p = 2$, $n = 1$, then the coefficient of the interaction N form density, $A\hat{}J_s$, is equal to the $(N-1)^{th}$ similarity invariant of the Jacobian field. For all implicit surfaces, simple or not, this similarity invariant is equal to the trace of the Jacobian Adjoint matrix and is equal to the sum of all possible products of degree N-1 of the eigen values of the Jacobian matrix. This similarity invariant will be defined as the Adjoint Curvature of the implicit surface. In 3 dimensions the Adjoint curvature of simple implicit surfaces is equal to the Gauss sectional curvature.

2.4. The adjoint curvature and the interaction energy

In summary, a well defined procedure has been implemented to deduce a consistent exterior differential system in Maxwell - Electromagnetic format, starting from a set of potentials that define the coefficients of a 1-form of Action, A_0. It follows that the exterior differential system $F - dA = 0$ is always equivalent to the system of PDE's known as the Maxwell-Faraday equations. The induced system described above, $J_s - dG_s = 0$, generates the system of PDE's known as the Maxwell-Ampere equations. Note that no constitutive or duality constraints have been subsumed. It is known that the two combined exterior differential systems lead to a N-1 form, previously defined as topological spin, $A\hat{}G$, and a third exterior differential system, $d(A\hat{}G) - F\hat{}G + A\hat{}J = 0$. The last term, defined as the interaction energy and equal to $A\hat{}J_s$, can be evaluated in terms of the curvature invariants of the implicit hypersurface generated by the 1-form of Action Potentials. It is remarkable that the term $A\hat{}J_s$ is equal to the volume element multiplied by the $(N-1)^{th}$ similarity invariant, defined as the $Trace[\widehat{\mathbb{J}}]$.

$$Interaction\_energy = A\hat{}J_s = Trace[\widehat{\mathbb{J}}]dx\hat{}dy\hat{}dz... \tag{2.8}$$

As the $(N-1)^{th}$ similarity invariant can be interpreted in terms of a polynomial cubic in the curvatures of the hypersurface in 4D, it would appear that concept of the interaction energy between the charge current density and the potentials can be related to an expression cubic in the curvatures of the associated hypersurface.

3. Examples

The following examples will display some of the features of the theory of generalized implicit hypersurfaces in 3 and 4 dimensions. The 3D examples can be of two physically interesting categories based on the coordinate sets $\{x, y, z\}$ and $\{x, y, t\}$ From an electromagnetic interpretation, the first category has the properties of a 3D plasma. The second category admits an **E** field as well as a **B** field. Each of these 3D categories can be viewed as special cases of the 4D category based on coordinate variables of the type $\{x, y, z, t\}$ In addition there is a topological refinement of the categories which depend upon the Pfaff dimension of the 1-form, A_0, used to model the normal direction field of the implicit hypersurface. Most classical developments of implicit simple surface theory study those cases where the Pfaff dimension of the 1-form,.A, is unity. Such spaces do not support topological torsion. A rotating spherical surface does not support torsion. An expanding spherical surface does not support torsion. However an expanding and rotating spherical surface does support topological torsion.

Make sure you are aware that symbolic math programs using Maple programs [9] have been provided for you to check the details and extend the examples presented below.

3.1. Simple surfaces of one component in 3D.

For simplicity, consider those surfaces generated by fixed values assigned to functions of the form $\phi = f(x, y) - z$. Such surfaces are of a single component and do not support a non-zero 2-form The differential of the function ϕ generates the exact 1-form

$$A_0 = (\partial\phi/\partial x)dx + (\partial\phi/\partial y)dy + (\partial\phi/\partial z)dz = (\partial f/\partial x)dx + (\partial f/\partial y)dy - dz \quad (3.1)$$

The 1-form associated with such surfaces is of Pfaff dimension 1. Choose the Holder norm equivalent to the Gauss map

$$\lambda = \{(\partial f/\partial x)^2 + (\partial f/\partial y)^2 + 1\}^{1/2}. \quad (3.2)$$

and construct the homogenous of degree zero 1-form

$$A = A_0/\lambda. \quad (3.3)$$

The homogeneous 1-form, A, can be of Pfaff dimension 2. Form the Jacobian matrix, $[\partial A_m/\partial x^n] = [\mathbb{J}]$, of the covariant components of the 1-form, A,and construct the similarity invariants, and the induced current. For this simple surface it is assumed that ϕ is linear in z. The determinant of the Jacobian matrix vanishes, which implies that the Jacobian matrix is singular and has no inverse. The remaining similarity invariants are:

$$\begin{aligned} Mean\_Curvature &= -1/2\{(\partial^2 f/\partial x^2)(1 + (\partial f/\partial y)^2) + (\partial^2 f/\partial y^2)(1 + (\partial f/\partial x)^2) \\ &\quad -2(\partial f/\partial x)(\partial f/\partial y)\partial^2 f/\partial x\partial y\}/\lambda^3 \end{aligned} \quad (3.4)$$

and

$$\begin{aligned} Adjoint\_Gauss\_Curvature &= Trace[\widehat{\mathbb{J}}] \\ &= \{(\partial^2 f/\partial x^2)(\partial^2 f/\partial y^2) - (\partial^2 f/\partial x\partial y)^2\}/\lambda^4 \end{aligned} \quad (3.5)$$

The induced current is of the form, $[0, 0, J_z]$ where

$$J_z = \{(\partial^2 f/\partial x^2)(\partial^2 f/\partial y^2) - (\partial^2 f/\partial x\partial y)^2\}/\lambda^3. \quad (3.6)$$

It follows that the N-form $A\hat{\ }J$ becomes

$$
\begin{aligned}
A\hat{}J_s &= (Adjoint\_Gauss\_Curvature)\, dx\hat{}dy\hat{}dz \\
&= \{(\partial^2 f/\partial x^2)(\partial^2 f/\partial y^2) - (\partial^2 f/\partial x\partial y)^2\} dx\hat{}dy\hat{}dz/\lambda^4
\end{aligned} \tag{3.7}
$$

and the coefficient of the interaction is precisely equal to the Adjoint curvature, which is equivalent to the classic Gauss curvature of the implicit surface in the 3 dimensional variety.

For exact 1-forms, the 2-form $F_0 = dA_0$ vanishes. Hence the general formula

$$A_0\hat{}J_s = F_0\hat{}G_s - d(A_0\hat{}G_s) \tag{3.8}$$

becomes

$$A\hat{}J_s = -d(A_0\hat{}G_s)/\lambda. \tag{3.9}$$

This result is equivalent to the Chern statement of the Gauss-Bonnet theorem: the Gauss curvature is integrable [9].

The singular induced (pullback) metric is given by the expression,

$$[g] = \begin{bmatrix} (\partial^2 f/\partial x^2)^2 + (\partial^2 f/\partial x\partial y)^2 & (\partial^2 f/\partial x\partial y)(\partial^2 f/\partial x^2 + \partial^2 f/\partial y^2) & 0 \\ (\partial^2 f/\partial x\partial y)(\partial^2 f/\partial x^2 + \partial^2 \phi/\partial y^2) & (\partial^2 f/\partial y^2)^2 + (\partial^2 f/\partial x\partial y)^2 & 0 \\ 0 & 0 & 0 \end{bmatrix} /\lambda^2, \tag{3.10}$$

and can be used to construct a line element, $(ds)^2$ on the two dimensional subspace.

3.2. Classical Implicit Surfaces with more than one component in 3D.

The classic implicit surface is defined by fixed values assigned to non-linear functions of the form $\phi(x, y, z)$. The differential of the function ϕ generates the exact 1-form of Pfaff dimension 1:

$$A_0 = (\partial\phi/\partial x)dx + (\partial\phi/\partial y)dy + (\partial\phi/\partial z)dz. \tag{3.11}$$

As before, choose the Holder norm equivalent to the Gauss map

$$\lambda = \{(\partial\phi/\partial x)^2 + (\partial\phi/\partial y)^2 + (\partial\phi/\partial z)^2\}^{1/2}. \tag{3.12}$$

and construct the homogenous of degree zero 1-form (which can be of Pfaff dimension 2):

$$A = A_0/\lambda. \tag{3.13}$$

Create the Jacobian matrix, $[\partial A_m/\partial x^n]$, of the covariant components of the 1-form, A,and construct the similarity invariants, and the induced current. Note that the Jacobian matrix is symmetric. The determinant of the Jacobian vanishes indicating the matrix is singular and without inverse. Then the remaining similarity invariants are constructed from the trace of the Jacobian matrix and the trace of the Adjoint matrix. The curvature formulas are best computed via a symbolic math program such as Maple [9].

The mean curvature becomes

$$\begin{aligned} & Mean\ Curvature \qquad (3.14) \\ = \; & -\{2(\partial\phi/\partial x)(\partial\phi/\partial y)(\partial^2\phi/\partial x\partial y) - \partial^2\phi/\partial z^2\left((\partial\phi/\partial y)^2 + (\partial\phi/\partial x)^2\right) \\ & +2(\partial\phi/\partial y)(\partial\phi/\partial z)(\partial^2\phi/\partial z\partial y) - \partial^2\phi/\partial x^2\left((\partial\phi/\partial y)^2 + (\partial\phi/\partial z)^2\right) \\ & +2(\partial\phi/\partial z)(\partial\phi/\partial x)(\partial^2\phi/\partial x\partial z) - \partial^2\phi/\partial y^2\left((\partial\phi/\partial x)^2 + (\partial\phi/\partial z)^2\right)\}/3\lambda^3 \end{aligned}$$

and the Adjoint - Gauss curvature becomes

$$\begin{aligned} & Adjoint - Gauss\ Curvature \qquad (3.15) \\ = \; & -\{2(\partial\phi/\partial x)(\partial\phi/\partial y)(\partial^2\phi/\partial x\partial y)(\partial^2\phi/\partial z^2) - (\partial^2\phi/\partial y^2)(\partial^2\phi/\partial x^2)(\partial\phi/\partial z)^2 \\ & +2(\partial\phi/\partial y)(\partial\phi/\partial z)(\partial^2\phi/\partial z\partial y)(\partial^2\phi/\partial x^2) - (\partial^2\phi/\partial z^2)(\partial^2\phi/\partial y^2)(\partial\phi/\partial x)^2 \\ & +2(\partial\phi/\partial z)(\partial\phi/\partial x)(\partial^2\phi/\partial x\partial z)(\partial^2\phi/\partial y^2) - (\partial^2\phi/\partial x^2)(\partial^2\phi/\partial z^2)(\partial\phi/\partial y)^2 \\ & +(\partial\phi/\partial x)^2(\partial^2\phi/\partial y\partial z)^2 + (\partial\phi/\partial y)^2(\partial^2\phi/\partial z\partial x)^2 + (\partial\phi/\partial z)^2(\partial^2\phi/\partial x\partial y)^2 \\ & -2(\partial\phi/\partial x)(\partial\phi/\partial y)(\partial^2\phi/\partial z\partial x)(\partial^2\phi/\partial z\partial y) \\ & -2(\partial\phi/\partial y)(\partial\phi/\partial z)(\partial^2\phi/\partial x\partial y)(\partial^2\phi/\partial x\partial z) \\ & -2(\partial\phi/\partial z)(\partial\phi/\partial x)(\partial^2\phi/\partial x\partial y)(\partial^2\phi/\partial x\partial z)\}/\lambda^4 \end{aligned}$$

The induced current J_s may be computed by multiplying the components of A with the Adjoint matrix relative to the Jacobian matrix, and it may be shown that the interaction N form is precisely equal to the volume element multiplied by the trace of the Adjoint matrix.

$$A\hat{\ }J_s = (Adjoint\_Gauss\_Curvature)\, dx\hat{\ }dy\hat{\ }dz \tag{3.16}$$

3.3. Implicit surfaces of the Bateman type

A generalized implicit surface generated by a 1-form which is of Pfaff dimension 2 has a representation of the Bateman type. That is, $A_0 = \alpha(x, y, z)db(x, y, z)$. The procedures are the same as above. Choose a Holder norm in the form of a Gauss map such that

$$\lambda = \alpha\{(\partial\beta/\partial x)^2 + (\partial\beta/\partial y)^2 + (\partial\beta/\partial z)^2\}^{1/2}. \tag{3.17}$$

Compute the Jacobian matrix which now can have an anti-symmetric part

3.4. Non integrable 2-surfaces in 3D

The procedure is the same as above, except now the 1-form A_0 has arbitrary coefficients,

$$A_0 = U(x, y, z)dx + V(x, y, z)dy + W(x, y, z)dz = \tag{3.18}$$

As before, choose the Holder norm equivalent to the Gauss map

$$\lambda = \{(U)^2 + (V)^2 + (W)^2\}^{1/2}. \tag{3.19}$$

and construct the homogenous of degree zero 1-form

$$A = A_0/\lambda. \tag{3.20}$$

Form the Jacobian matrix, $[\partial A_m/\partial x^n]$, of the covariant components of the 1-form, A,and construct the similarity invariants, and the induced current. The results are the same for the interaction N-form $A\hat{}J$.:

$$A\hat{}J_s = (Adjoint\_Curvature)\, dx\hat{}dy\hat{}dz \tag{3.21}$$

It should be remarked that the same procedures are valid in dimension N. If Vol is the N dimensional differential volume element, then in N dimensions,

$$A\hat{}J_s = (Adjoint\_Curvature)\ Vol \tag{3.22}$$

The computations can be lengthy, and it is advised that the symbolic math program provided be used [9].

3.5. An interpretation in terms of Electromagnetism

3.5.1. The 3-D Variety is Spatial: (x,y,z)

In this example, the variety of independent variables $\{x, y, z\}$, is presumed to be independent of time. Given the 1-form $A_0 = Udx + Vdy + Wdz$, it is possible to construct the field intensities, $F_0 = dA_0$ on the presumption that the 1-form is the 1-form of Action potentials for an electromagnetic field. By construction there is no time dependence and no scalar potential on the 3 D spatial domain. The 2-form F of field intensities consists of 3 components related to the curl of the vector potential, $\mathbf{A}_0 = [U, V, W]$:

$$\mathbf{B}_0 = curl\, \mathbf{A}_0. \tag{3.23}$$

By computing the Jacobian matrix of the rescaled 1-form, A, which is homogeneous of degree 0, the above procedures lead to a divergence free current. This current is a globally closed N-1=2-form, and so is related to the exterior derivative of some N-2=1-form. In this example, the 1-form of field excitations is defined by the symbolism,

$$G = H^x dx + H^y dy + H^z dz = \mathbf{H} \circ \mathbf{dr}. \tag{3.24}$$

It follows that the induced current is of the form

$$\mathbf{J}_s = curl\, \mathbf{H}, \tag{3.25}$$

but although there is a current there is no analogue to a charge density distribution for the time independent 3 dimensional format.

The interaction N=3-form becomes

$$\begin{aligned} A\hat{}J_s &= (Adjoint\_Gauss\_Curvature)\, dx\hat{}dy\hat{}dz \\ &= (\mathbf{A} \circ \mathbf{J}_s) dx\hat{}dy\hat{}dz \end{aligned} \tag{3.26}$$

The topological Spin N–1=2 form becomes

$$A_0\hat{}G = i(\mathbf{A}_0 \times \mathbf{H}) dx\hat{}dy\hat{}dz \tag{3.27}$$

and has a divergence equal to the first "Poincare" invariant,

$$d(A_0\hat{}G) = \{(\mathbf{B}_0 \circ \mathbf{H}) - (\mathbf{A} \circ \mathbf{J}_s)\} dx\hat{}dy\hat{}dz. \tag{3.28}$$

(The symbol $i(\mathbf{A}_0 \times \mathbf{H})$ is used for interior product operator and should not be confused with the imaginary $\sqrt{-1}$). The result is remarkable for it implies that the closed integral of the magnetic energy density $(\mathbf{B}_0 \circ \mathbf{H})$ minus the Gaussian curvature times λ, (*or* $\mathbf{A}_0 \circ \mathbf{J}_s$) of the surface created by the non-exact 1-form, A_0, is an invariant of any steady flow process on the variety $\{x, y, z\}$.

Note that it is not apparent nor true that $\mathbf{B}_0$ is linearly related to $\mathbf{H}$, where curl $\mathbf{H}$ is the source of the Adjoint closed current. As the 1-form is not necessarily integrable, the helicity $\mathbf{A}_0 \circ \mathbf{B}_0$ is not necessarily zero. In hydrodynamics, the spatial parts of the Action 1-form can be related to the fluid dynamics velocity field, and the $\mathbf{B}_0$ field is the fluid vorticity. In the integrable case the two direction fields associated with $\mathbf{A}_0$ and $\mathbf{B}_0$ form a surface (the Lamb surface).

Note that for the non-exact hypersurfaces in 3 dimensions, the similarity invariant is no longer a perfect differential, but is modified by the presence of the enstrophy (sqaure of the vorticity or $\mathbf{B}_0$ field). This example and others are displayed in [9].

3.5.2. The 3D Variety is 2+1 time dependent (x,y,t)

In this example, the variety of independent variables $\{x, y, t\}$, is presumed to consist of two spatial variables and time. Given the 1-form $A_0 = Udx + VdY - \phi dt$, it is possible to construct the field intensities, $F_0 = dA_0$ on the presumption that the 1-form is the 1-form of Action potentials for an electromagnetic field. Each of the component functions can be functions of $\{x, y, t\}$ and $\phi(x, y, t)$ will play the role of the scalar potential. The 2-form of field intensities, F consists of one magnetic component, orthogonal to the xy plane (which means that it is in the t direction), and two electric components. In engineering format:

$$\mathbf{B}_t = \partial V/\partial x - \partial U/\partial y \qquad \mathbf{E}_x = -\partial U/\partial t - \partial \phi/\partial x \qquad \mathbf{E}_y = -\partial V/\partial t - \partial \phi/\partial y \tag{3.29}$$

By computing the Jacobian matrix of the rescaled 1-form which is homogeneous of degree 0, the above procedures lead to a divergence free current. This divergence free current is a globally closed N-1=2-form, and so is related to the exterior derivative of some N-2=1-form. In this example, the 1-form of field excitations is defined in terms of the symbols,

$$G_s = D^y dx - D^x dy + H^t dt. \tag{3.30}$$

It follows that, in engineering notation, the induced close current has the classic format,

$$\mathbf{J}_s = [\partial H^t/\partial y + \partial D^x/\partial t, \partial H^t/\partial x + \partial D^y/\partial t, +\partial D^y/\partial x + \partial D^x/\partial x] \tag{3.31}$$

$$\begin{aligned} &\simeq [curl\,\mathbf{H}^t + \partial\,\mathbf{D}/\partial t] \\ \rho &= div\mathbf{D}. \end{aligned} \tag{3.32}$$

In contrast to the previous example, it is now apparent that this time dependent system can have a non-zero charge distribution as well as a current. Note that the format for G_s is the canonical form of the Heisenberg system.

The interaction N=3-form becomes as before,

$$\begin{aligned} A\hat{}J_s &= (Adjoint\_Gauss\_Curvature)\,dx\hat{}dy\hat{}dz \\ &= (\mathbf{A}\circ\mathbf{J}_s - \rho\phi)dx\hat{}dy\hat{}dz. \end{aligned} \tag{3.33}$$

The topological Spin N–1=2 form becomes in component form,

$$A_0\hat{}G_s \Rightarrow [\mathbf{A}_0 H^t + \mathbf{D}\phi, \mathbf{A}\circ\mathbf{D}], \tag{3.34}$$

and has a divergence equal to the first "Poincare" invariant,

$$d(A_0\hat{}G_s) = \{(B_t H^t - \mathbf{D}\circ\mathbf{E}) - (\mathbf{A}_0\circ\mathbf{J}_s - \rho\phi_0)\}dx\hat{}dy\hat{}dz. \tag{3.35}$$

The result is remarkable for it implies that the closed integral of the Lagrangian energy density $(B_t H^t - \mathbf{D}.\mathbf{E})$ minus the Gaussian curvature times λ, $(or\ \mathbf{A}_0\circ\mathbf{J}_s - \rho\phi_0)$ of the surface created by the non-exact 1-form, A_0, is an invariant of any flow process on the variety $\{x, y, t\}$. For details see [9].

3.6. Four Dimension Hypersurfaces

In four dimensions, the analysis above continues to be true, with the fundamental result that the interaction density is related to the Adjoint curvature of the hypersurface defined by the 1-form of Action with coefficients that are homogeneous of degree zero. However, in 4 dimensions, it is possible to distinguish between the mean curvature, $Mean$, the Gauss sectional curvature, *Gauss*, and the Adjoint curvature, $Kubic$. The set $\{H, M, K\}$ are known as the similarity invariants of the Jacobian matrix. The Mean curvature, $Mean$, is proportional to the sum of the eigen values. The Sectional Gauss curvature *Gauss* is related to the sum of the three paired products of the eigenvalues of the Jacobian matrix For the 1-form which has been made homogeneous of degree zero in its component functions, the

Adjoint curvature $Kubic$ is the unique product of the three non-zero eigen values of the Jacobian matrix. . Again the interaction energy density N=4 form, $A\hat{}J_s$ has a coefficient exactly equal to the Adjoint similarity invariant, if the Holder norm is equivalent to the Gauss map with isotropic index p=2 and homogeneity index n = 1.

This observation yields a remarkable difference between mass energy density (related to Gaussian sectional second order curvature) and interaction energy density between the charge current density and the potentials (related to the Adjoint third order curvature). There are three situations of interest: Case 1, the charge-current density is zero, so that $J_s = 0$ and $A\hat{}J_s = 0$; Case 2, the charge-current density is not zero, so that $J_s \neq 0$ but $A\hat{}J_s = 0$ because of orthogonality; Case 3, the charge current density is not zero and not orthogonal to 1-form of Action, such that $A\hat{}J_s \neq 0$, and $J_s \neq 0$. On a space of N = 4 dimensions there are 3 direction fields that are orthogonal to the one form of potentials. If the 1-form A is of Pfaff dimension 4, then there is a unique vector direction field V such that $i(V)dA = \Gamma A$, and yet V is orthogonal to A. This direction field is determined by the topological torsion vector $A\hat{}dA$,with Γ proportional to the coefficient of the 4 form of topological parity, $dA\hat{}dA$. When $\Gamma \Rightarrow 0$, the Pfaff dimension is three, and the direction field generated by the topological torsion 3-form, $A\hat{}dA$, becomes a characteristic vector field of the 1-form A. Characteristic vector fields are homeomorphisms that preserve topology.

Examples indicate that Case 1, $J_s = 0$, is satisfied if the scalar potential is zero (any vector potential) or if the vector potential is zero (any scalar potential), for then the charge current density does not exist and the Adjoint curvature vanishes. The case which admits only a Scalar potential yields a system with **E** fields and no **B** fields. A time independent Vector potential without a scalar potential yields the opposite situation with **B** fields and no **E** fields. The Time dependent vector field case admits both **B** fields and **E** fields.

Examples indicate that Case 3 is usually satisfied if the 1-form of potentials has both a scalar and vector components which are both time and space dependent. A special situation occurs if the potentials are not explicitly time dependent, for then the spatial current density is zero but the charge density is not zero. The Adjoint curvature is not zero, and the interaction energy density does not vanish. This example gives credence to the suggestion that the origin of charge density is cubic curvature. The time independent case can support non zero topological Torsion and non-zero topological Parity.

In the examples presented below, it appears that if a charge current density exists, and it is not related to an irreversible process, then it is related to the third order curvature invariants of the implicit surface defined by the 1-form of Potentials. The algebra is a bit formidable. Hence a Maple program is presented [9] for which the reader can verify the computations, and modify the program to check his own hierarchy of examples. Only the results of the computations are described here. Only the isotropic Holder norm equivalent to the Gauss map, where p = 2, n =1 and with euclidean signature, is utilized in the examples presented below unless specified otherwise. The utility of the other Holder norms, that leace the coefficients of the 1-form homogeneous of degree zero have not been studied.

3.6.1. 4D Example 1, Time dependent Scalar potential only

When applied to a 1-form that consists of a single (time and spatially dependent) scalar potential,

$$A_0 = -\phi(x, y, z, t)\, dt, \tag{3.36}$$

it follows that the implicit surface is flat. All of the curvature similarity invariants vanish. First, renormalize A_0 by dividing through by a Holder norm, λ, such as to make the new 1-form homogeneous of degree 0. Then construct the Jacobian matrix of the components of $A = A_0/\lambda$. It follows for the example that all of the curvature similarity invariant vanish.

$$\begin{aligned} Mean &= 0, && (3.37) \\ Gauss &= 0 && (3.38) \\ Kubic &= 0. && (3.39) \end{aligned}$$

There is zero induced charge current density; $J_s = 0$. There can be an electric field, **E**, but no magnetic field, **B**. The interaction energy as well as the Adjoint curvature are zero; $A\hat{}J_s = 0$. The Pfaff dimension of the 1-form is at most 2. The hypersurface may be considered to be a 3-plane orthogonal to the time coordinate. If the scalar potential is independent from time, the problem is related to classical electrostatics. The implicit surface is flat, without bending or tension. The Coulomb potential falls into this class of examples.

If the Jacobian matrix is used to define a class of right Cartan connection coefficients, the such spaces are without structure. All of the similarity invariants of the Frame matrix are zero. Such is the case for the euclidean (or Lorentzian) equivalence class of spaces.

3.6.2. 4D Example 2, Time dependent or time independent Vector potentials only

In this example,the 1-form is presumed to be of the form

$$A_0 = A_x(x,y,z,t)dx + A_y(x,y,z,t)dy + A_z(x,y,z,t)dz, \tag{3.40}$$

The time dependent potentials admit both a magnetic field **B** and an electric field **E**, and the time independent potentials only admit a **B** field.. The invariant Adjoint Jacobian technique indicates that a charge current density is not induced, $J_s = 0$, and the interaction energy is identically zero, $A\hat{}J_s = 0$. The time dependent potentials are of Pfaff dimension 4, and can support non-zero Helicity and non-zero Parity without inducing a charge current density. The time independent potentials are Pfaff dimension 3, hence the topological parity, $F\hat{}F$, is zero and the closed integrals of topological torsion have rational ratios. The Gaussian curvature and the mean curvature are not necessarily zero, although the Adjoint curvature is always zero; $A\hat{}J_s = 0$. Hence the 3D hypersurface has degenerated into a 2 dimensional surface in 4 dimensions.

$$Mean \neq 0, \tag{3.41}$$
$$Gauss \neq 0 \tag{3.42}$$
$$Kubic = 0. \tag{3.43}$$

The equivalence class of spaces generated by the Jacobian as a frame matrix are not flat, and have structure. However such spaces can be reduced to two dimensional hypersurfaces.

3.6.3. 4D Example 3, Vector and Scalar Potentials without explicit time dependence.

In this example,the 1-form is presumed to be of the form

$$A_0 = A_x(x,y,z)dx + A_y(x,y,z)dy + A_z(x,y,z)dz - \phi(x,y,z)dt. \tag{3.44}$$

The results of the formalism indicate that there is no spatial current density, but there can exist a charge density. The adjoint curvature is not necessarily zero, $A\hat{}J_s \neq 0$.

$$
\begin{aligned}
Mean &\neq 0, && (3.45)\\
Gauss &\neq 0 && (3.46)\\
Kubic &\neq 0. && (3.47)
\end{aligned}
$$

The Pfaff dimension can be as high as 4, with the 1-form supporting both a Helicity 3-form and a Parity 4 form. The similarity invariants indicate that the equivalence class of such spaces is related to 3 dimensional hypersurfaces.

3.6.4. 4D Example 4, Bateman - Whittaker solutions

In this example,the 1-form is presumed to be of the form

$$A_0 = \alpha(x, y, z, t)d\beta(x, y, z, t). \tag{3.48}$$

The results of the formalism indicate that there can be both a current density and a charge density, $J_s \neq 0$. The Pfaff dimension is 2 or less. The Helicity 3 form and the Parity 4-form vanish. The **E** field and **B** field are always orthogonal. However, if the function β is independent from time (but α remains explicitly time dependent) then the Adjoint curvature and the induced charge current density also vanish. The 3D hypersurface reduces to a 2D hypersurface that supports mean and Gauss curvature, but not cubic curvature invariants, $A\hat{}J_s = 0$.

$$
\begin{aligned}
Mean &\neq 0, && (3.49)\\
Gauss &\neq 0 && (3.50)\\
A\hat{}J_s\ \ Kubic &= 0 && (3.51)\\
Top\_Torsion &\neq 0 && (3.52)\\
J_s &\neq 0 && (3.53)
\end{aligned}
$$

This result corresponds to what are called time harmonic solutions in the engineering literature, and gives yet more credence to the idea that charge current densities can be related to cubic curvatures.

3.6.5. 4D Example 5, A Hopf map solution.

In this example,the Hopf 1-form is presumed to be of the form

$$A_0 = b(ydx. - xdy) + a(tdz - zdt). \tag{3.54}$$

The 1-form of Potentials depends on the coefficients a and b which are presumed to take on values ± 1. There are two cases corresponding to left and right handed "polarizations": $a = b$ or $a = -b$. (There actually are 6 cases to consider, by cyclically permuting the variables, and these can be combined to represent spinor solutions.[2]) What is remarkable for this solution, is that both the mean curvature and the Adjoint curvature of the implicit hypersurface in 4D vanish, for any choice of a or b. The Gauss curvature is non-zero, positive real and is equal to the square of the radius of a 4D euclidean sphere. The cubic interaction energy density is zero.

$$
\begin{aligned}
Mean &= 0, && (3.55)\\
Gauss &> 0 && (3.56)\\
A\hat{}J_s\ Kubic &= 0 && (3.57)\\
Top\_Torsion &\neq 0 && (3.58)\\
J_s &\neq 0 && (3.59)
\end{aligned}
$$

This situation occurs when the three curvatures of the implicit 3-surface are $\{0, +i\omega, -i\omega)$.. The Hopf surface is a 3D imaginary *minimal* two dimensional hyper surface in 4D and has two non-zero imaginary curvatures! Strangely enough the charge-current density is not zero, but it is proportional to the topological Torsion vector that generates the 3 form $A\hat{}F$. The topological Parity 4 form is not zero, and depends on the sign of the coefficients a and b. In other words the 'handedness' of the different 1-forms determines the orientation of the normal field with respect to the implicit surface. This set of circumstances corresponds to the Case 3 situation described above where the charge current interaction density is zero, but the charge current density is not zero. The spatial scalar product of A and J is balance by $\rho\phi$. It is known that a process described by a vector proportional to the topological torsion vector in a domain where the topological parity $(4ba)$ is non-zero is thermodynamically irreversible [10].

This example demonstrates that in special cases the charge-current density is not proportional to the adjoint curvature of the implicit minimal surface. However, the case corresponds to the special situation where the interaction energy, alias the internal energy relative to the given process, is zero, and yet the process is non-zero, but thermodynamically irreversible.

4. Summary and applications to p-brane theories

Every Pfaffian 1-form whose coefficients are functionally homogeneous of degree zero can be used to describe the normal field to an implicit surface. The curvature similarity invariants can be computed from the Jacobian matrix of the homogeneous 1 form. For those p-branes which are 3 dimensional implicit surfaces in 4 dimensions, it is possible to deduce an electromagnetic interpretation with an intrinsic charge current density. The interaction energy density of this charge current density and the potentials that define the implicit surface is exactly the cubic curvature similarity invariant of the implicit hypersurface. As the curvature radii get smaller and smaller, the electromagnetic interaction energy being proportional to the cube of the curvatures could conceivably prevent if not impede gravitational collapse. Certainly such terms should be included in the dynamics of collapsing mass systems. This effect, like the Bohm-Aharanov effect, does not depend explicitly upon the field strengths, **E** and **B**. Such a possibility appears to have been neglected in metric based curvature theories.

5. Acknowledgments

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