Curvature and torsion of implicit hypersurfaces and the origin of charge-currents

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Abstract

A formal correspondence is established between the curvature theory of generalized implicit hypersurfaces, the classical theory of electromagnetism as expressed in terms of exterior differential systems, and thermodynamics. Starting with a generalized implicit surface whose normal field is represented by an exterior differential 1-form, it is possible to deduce the curvature invariants of the implicit surface and to construct a globally closed vector density in terms of the Jacobian properties of the normal field. When the closed vector density is assigned the role of an intrinsic charge current density, and the components of the normal field are assigned the roles of the electromagnetic potentials, the theory is formally equivalent to an exterior differential system that generates the PDE's of both the Maxwell Faraday equations and the Maxwell Ampere equations. The interaction energy density between the potentials and the induced closed charge current density is exactly the similarity curvature invariant of highest degree (N-1) for the implicit surface. Evolution of the 1-form of Action with respect to the direction field defined by the charge-current permits the interaction energy density to be identified with the internal energy of thermodynamics. These ideas of generalized implicit surfaces should have application to the study of p-branes that can have multiple components and envelopes. The theory suggests that gravitational collapse of mass energy density should include terms that involve the interaction between charge-current densities and electromagnetic potentials. Such interaction energies are cubic in curvatures, and are to be compared with metric-mass energies which are quadratic in curvatures.

1. Introduction

1.1. The implicit surface generated by electromagnetic potentials, A

The origin of charge has long been a mystery to physical theory, perhaps even more elusive than the concept of inertial mass. A major objective of this article is to examine the conjecture that the charge-current density of electromagnetism may have part, if not all, of its origins in the differential geometry and topology of curvature and torsion. The concept is in a sense similar to the idea that mass density and gravity have their origins in the concept of metric curvature. The curvatures of interest herein are not, however, those generated by a symmetric metric, but instead are those curvatures related to the similarity invariants of a generalized implicit hypersurface. The generalized implicit hypersurfaces considered herein may not admit a global foliation as their normal fields need not satisfy the Frobenius integrability conditions. Hence such generalized hypersurfaces can support topological torsion as well as curvature. Although other (for example, hydrodynamic) interpretations of the results to be presented are possible, the electromagnetic nomenclature will be used for purposes of more rapid comprehension.

An arbitrary 1-form of Action, A_0, whose coefficient functions may be considered as a set of electromagnetic potentials, can also play the role of the normal direction field to a generalized implicit hypersurface. The closure of the exterior differential system, $F_0 - dA_0 = 0$, always generates a system of PDE's which contain the Maxwell-Faraday equations, thereby establishing the first half of Maxwell theory. The 1-form of Action can be rescaled by use of a Holder norm, λ, such that the resulting 1-form $A = A_0/\lambda$, is homogeneous of degree zero in its coefficient functions. The curvature features of the implicit hypersurface are completely specified in terms of the similarity invariants of the Jacobian matrix, $[\mathbb{J}] = \left[\partial A_k/\partial x^k\right]$.

It is important to realize that the method to be discussed involves curvatures, torsion and energy densities, but does not depend explicitly upon a metric, gauge constraints, or the Einstein field equations.

1.2. The induced charge-current density J_s

The remaining half of electromagnetic theory is the Maxwell-Ampere equations, which depend upon the existence of a globally closed charge-current density. Although the Jacobian matrix described above is globally singular, $\det[\mathbb{J}] = 0$, it is always possible to construct algebraically the matrix of cofactors transposed, or $[\mathbb{J}]^{adjoint}$. If the components of the 1-form, A, do not form a null eigen vector, it is remarkable that multiplication of these components by $[\mathbb{J}]^{adjoint}$ yields an N-1 form density, or current, J_s, which is globally closed, and therefor can be utilized to play the role of a charge-current density. The global closure implies that there exists an N-2 form G_s such that $J_s - dG_s = 0$. The PDE's associated with this exterior differential system are known to contain the Maxwell-Ampere equations. Note that the two exterior differential systems establish topological constraints on the variety. For example, the domain of support for the 2-form F is not compact without boundary, while the domain of support for G_s can be compact without boundary. The N-2 form, G_s , (like the 1-form of Action) is not uniquely determined by the exterior differential system, as it may contain closed, or closed and exact, components that do not contribute to the charge current density, $J_s = dG_s$.

These two Maxwell exterior differential systems, $F - dA = 0$, and $J_s - dG_s = 0$ can be used to deduce two additional differential systems that augment, but do not change, the PDE's of the classical Maxwell theory. These augmentations depend upon the existence of two 3-forms, previously defined as Topological Torsion, $A\hat{}F$, and Topological Spin, $A\hat{}G_s$ [1]. Constraints of equilibrium and uniqueness (which are not invoked herein) will cause these 3-forms to be null. Exterior differentiation of $A\hat{}G_s$ leads to the equation:

$$d(A\hat{}G_s) = F\hat{}G_s - A\hat{}J_s, \tag{1.1}$$

and demonstrates that twice the difference between the magnetic and electric energy densities of the field, $F\hat{}G_s$, is cohomologous with the interaction energy density, $A\hat{}J_s$. A major feature of this article is to present the idea that the interaction energy density, $A\hat{}J_s$, is proportional to the hypersurface curvature similarity invariant of degree (N-1). This similarity invariant is defined as the Adjoint curvature and is equal to the trace of the Jacobian Adjoint matrix, $[\mathbb{J}]^{adjoint}$. Note that closed but not exact gauge contributions to A and to G can influence the value of the competing terms, $F\hat{}G_s$ and $A\hat{}J_s$.

1.3. Charge counting, conductors and insulators when dG=0.

From experience it is known that a given electromagnetic charge-current density J is conserved: $dJ = 0$ (the 4 vector density has zero divergence). However, a more important result is the observation of global charge neutrality, which can be attributed to a topological idea. As $dJ = 0$, is a global statement, there exists an N-2 form, G, such that $J - dG = 0$. This exterior differential system, $J - dG = 0$, is equivalent to the Maxwell-Ampere system of partial differential equations. The integral of G over a closed cycle in domains where $dG = 0$ yields values whose ratios are rational (Gauss' law of counting charges). When the closed integration domain is a boundary, the net charge enclosed is zero, yielding charge neutrality. These topological aspects can be used to distinguish insulators from conductors. Three dimensional insulators in contact can be separated in the presence of an external **E** field into two physical components with each component interior enclosed by a two dimensional boundary. The external field distorts the internal charge distribution to produce a dipole field. Each physical component remains charge neutral when the external field is removed. Similarly, three dimensional conductors in contact can be separated into two physical components, but the presence of a remnant exterior electromagnetic field between the components indicates that the closed two dimensional varieties of each component are cycles, not necessarily boundaries. The components are said to be charged.

1.4. The interaction energy density, $A\hat{}J_s$

The interaction energy density, $A\hat{}J_s$, satisfies the cohomological constraint 1.1 and can be evaluated, given A and G_s. The Lagrangian field energy density term, $F\hat{}G$, in electromagnetic format is equal to twice the difference of the magnetic and electric energy density of the electromagnetic field. The term has different signs depending on whether the system is dominated by a plasma or electrostatic state. In regions where $A\hat{}G$ is closed (has zero divergence), the closed 3 dimensional integrals of $A\hat{}G$ have values whose ratios are rational and are therefore countable (quantized in units of h). It is demonstrated in the appendix that this interaction density is precisely equal to the Adjoint curvature of the hypersurface whose normal direction field is generated by the 1-form, A. On a variety of four dimensions, this result implies that interaction energy, $A\hat{}J_s$ is a cubic function of the hypersurface curvatures, while the Gaussian sectional curvature (and therefor mass energy density) is quadratic in the surface curvatures. When the Jacobian matrix is of maximal rank N-2, the interaction energy vanishes. Note that if the interaction energy density is zero, the charge current density need not be zero. A special situation exists when J_s is proportional to the Topological Torsion 3 form, $A\hat{}dA$, for then the interaction energy density vanishes due to orthogonality of its two factors. An example of this special case is given below, where the Hopf map is used to formulate an implicit surface 1-form. Maple programs are available for computing the features of generalized implicit hypersurfaces, demonstrating the claim that an intrinsic charge-current exists, and proving that the intrinsic charge-current interaction with the potentials is equal to the Adjoint curvature of the implicit hypersurface [4].

1.5. Topological evolution, internal energy density, dissipation

Given a 1-form of Action A and a closed charge current density J_s, it is possible to use Cartan's magic formula [3] of topological evolution to demonstrate a correspondence between the implicit surface theory and the first law of thermodynamics. For evolutionary processes in the direction of the charge current density, Cartan's magic formula becomes

$$L_{(J_s)}A = i(J_s)dA + d(i(J_s)A) = W + dU = Q \tag{1.2}$$

Using electromagnetic notation, on a variety $\{x, y, z, t\}$ the (virtual) work 1-form becomes

$$W = i(J_s)dA = (\rho\mathbf{E} + \mathbf{J} \times \mathbf{B})_k dx^k + (\mathbf{J} \circ \mathbf{E})dt \tag{1.3}$$

which is recognized as the product of the Lorentz force density times the differential displacement plus the dissipative power density times the increment dt.

In certain cases the induced charge current density, J_s will have a component proportional to the Topological Torsion field, $A\hat{}dA = i(T)dx\hat{}dy\hat{}dz\hat{}dt$. An example of this case is presented below. If J_s is proportional to the Topological Torsion current, T, it follows that the evolution of the implicit surface is given by the expression,

$$L_{(J_s)}A = L_{(T)}A = i(T)dA + 0 = (\Gamma)\ A = (\mathbf{E} \circ \mathbf{B})A = Q. \tag{1.4}$$

It follows that the heat 3 form, $Q\hat{}dQ$, and the Topological Torsion 3 form, $A\hat{}F$, are proportional:

$$Q\hat{}dQ = (\mathbf{E} \circ \mathbf{B})^2 A\hat{}dA. \tag{1.5}$$

From classical thermodynamics, when a process produces a heat 1-form Q which does not admit an integrating factor, then such a process is thermodynamically irreversible. From Frobenius theory, an integrating factor does not exist if $Q\hat{}dQ \neq 0$. If the implicit surface 1-form is of Pfaff dimension 4, then $A\hat{}dA \neq 0$, and the topological parity 4 form, $dA\hat{}dA = 2(\mathbf{E} \circ \mathbf{B})dx\hat{}dy\hat{}dz\hat{}dt \neq 0$. So if the induced charge-current density has a component in the direction of the Topological Torsion field, then the associated process is thermodynamically irreversible. Such irreversible processes are artifacts of 4 dimensions.

Similarly, evaluation of the internal energy density for a process defined by the dynamics of the charge-current density becomes $U = (i(J_s)A) = \mathbf{A} \circ \mathbf{J} - \rho\phi$, which is identical to the coefficient of the interaction energy density. The dissipative irreversible component of the evolutionary process, which is proportional to the Topological Torsion current, does not contribute to the internal energy, as $i(T)A = 0$. Hence a correspondence has been established between the curvature theory of implicit surfaces, the charge-current density interaction, and the internal energy of a thermodynamic system. In the reversible situations, where $(\mathbf{E} \circ \mathbf{B}) = 0$, the implicit hypersurface method thereby seems to offer an alternative, non-quantum mechanical, understanding of what otherwise would be called superconducting currents. It is possible to have charge currents without dissipation. In one case, (the Meisner effect) the **B** field is excluded from the superconducting region, and in another case (the Hall effect) a large **B** field is present along with a non-dissipative but transverse current.

1.6. Four Dimensional Hypersurfaces

In this example, the Hopf map is used to deduce a 1-form of Pfaff dimension 4:

$$A_0 = b(ydx - xdy) + a(tdz - zdt). \tag{1.6}$$

This 1-form of Potentials depends on the coefficients a and b which are presumed to take on values ± 1. There are two cases corresponding to left and right handed "polarizations": $a = b$ or $a = -b$. (There actually are 6 cases to consider, by cyclically permuting the variables, and these can be combined to represent spinor solutions.[2]) The details of the calculation are presented elsewhere [4], but the results of the similarity curvature invariants are summarized below. Both the Mean curvature and the Adjoint curvature of the implicit Hopf hypersurface in 4D vanish, for any choice of a or b. The Gauss curvature is non-zero, positive, real and is equal to the square of the radius of a 4D euclidean sphere. The cubic interaction energy density is zero.

$$Mean\ Curvature\ (linear\ sum\ of\ \ curvatures) = 0, \tag{1.7}$$

$$Gauss\ Curvature\ (quadratic\ in\ curvatures) > 0 \tag{1.8}$$

$$A\hat{\ }J_s \;\; Adjoint\ Curvature\ (cubic\ in\ curvatures) = 0 \quad (1.9)$$

$$Topo\log ical\_Torsion = A\hat{\ }dA \neq 0 \quad (1.10)$$

$$J_s\ the\ Adjoint\ Current \approx A\hat{\ }dA \neq 0 \quad (1.11)$$

$$Topo\log ical\_Parity = dA\hat{\ }dA \neq 0 \quad (1.12)$$

The computations indicate that the Hopf implicit surface has three curvature eigen values, $\{0, +i\omega, -i\omega)$. Hence the Hopf surface is a 3D imaginary *minimal* hyper surface in 4D, has two non-zero imaginary curvatures, and is of positive Gauss curvature! Real minimal surfaces in 3D have negative Gauss curvature. Strangely enough, the charge-current density is not zero, but is proportional to the topological Torsion vector that generates the 3 form $A\hat{\ }F$. The Topological Torsion vector has a direction field proportional to the radius of a 4D sphere, representing an expansion (or contraction) of space-time. The Topological Parity 4 form is not zero, and its coefficient $(4ba)$ depends on the sign of the coefficients a and b. In other words the 'handedness' of the different components of the 1-form of Action determines the orientation of the normal field with respect to the implicit surface. From section 1.4 it is known that a process described by a vector proportional to the Topological Torsion vector, in a domain where the topological parity $(4ba)$ is non-zero, is thermodynamically irreversible.

1.7. Applications to p-brane theories and gravity

Every Pfaffian 1-form whose coefficients are functionally homogeneous of degree zero can be used to describe the normal field to an implicit surface. The equation 1.1 can be put into correspondence with the principle of equivalence, where $F\hat{\ }G$ plays the role of the gravitational field and where $A\hat{\ }J$ plays the role of inertial energy density. When the Topological Spin is closed (has zero divergence) then the gravitational energy density is equivalent to the inertial energy density. The curvature similarity invariants can be computed from the Jacobian matrix of the homogeneous 1 form. For those p-branes which are 3 dimensional implicit surfaces in 4 dimensions, the interaction (inertial) energy density of is exactly the cubic curvature similarity invariant of the implicit hypersurface. As the curvature radii get smaller and smaller, the electromagnetic interaction energy - being proportional to the cube of the curvatures - could conceivably prevent, if not impede, gravitational collapse. It seems intuitive that a collapsing mass system generates high temperatures, which intuitively would ionize the matter to produce an electromagnetic plasma. Certainly such terms involving electromagnetic interaction should be included in the dynamics of collapsing mass systems. Note that this effect, like the Bohm-Aharanov effect, does not depend explicitly upon the field strengths, **E** and **B**. Such considerations appear to have been neglected in metric based curvature theories that claim to generate black holes.

2. Summary

A well defined procedure has been implemented to deduce a consistent exterior differential system in Maxwell-Electromagnetic format, starting from a homogeneous 1-form of Action, A. The 1-form is used to generated a Jacobian matrix, $[\mathbb{J}]$, and its adjoint, which is used to generate a globally closed 3-form of charge current, J_s. It follows that the exterior differential system $F - dA = 0$ is always equivalent to the system of PDE's known as the Maxwell-Faraday equations, and the exterior differential system, $J_s - dG_s = 0$, is equivalent to the system of PDE's known as the Maxwell-Ampere equations. No constitutive or duality constraints have been subsumed. It is known that elements of the two combined exterior differential systems lead to an N-1 form, previously defined as Topological Spin, $A\hat{}G$, and a 3-form of Topological Torsion, $A\hat{}F$. Exterior differentiation leads to two more exterior differential systems, $d(A\hat{}G) - F\hat{}G + A\hat{}J = 0$, and $d(A\hat{}F) = F\hat{}F$. The N form defined as the interaction energy density, $A\hat{}J_s$, can be evaluated in terms of the $(N-1)^{th}$ similarity curvature invariant of the Jacobian matrix, $[\mathbb{J}]$, and is equal to $Trace[\mathbb{J}]^{adjoint}$. As the trace of the adjoint matrix for a hypersurface in 4D is a polynomial cubic in the curvatures, it would appear that concept of the interaction energy between the charge current density and the potentials can be related to an expression cubic in the curvatures of the implicit hypersurface.

3. Appendix: Generalized Implicit hypersurfaces.

The classic implicit surface is generated by assigning a constant value to a function, $\phi(x, y, z..)$. It is important to recall that an implicit surface, in contrast to a parametric surface, can consist of more than one disconnected components. The gradient field to the given function represents a normal field to the surface, and tangent vectors which reside on the surface are orthogonal to the normal field at all points. As the normal field for the classic implicit surface is a gradient field, its associated 1-form is exact. If this normal gradient field is rescaled by a factor such that it is homogeneous of degree zero in its functional arguments, then the Jacobian matrix of the rescaled normal field can be used to generate the curvatures of the implicit surface.

This procedure can be extended to the study of generalized implicit surfaces whose normal field is not representable by an exact 1-form. The 1-form representing the normal field can have arbitrary Pfaff dimension. If the Pfaff dimension of the 1-form is greater than 2, then the implicit surface can support topological torsion, $A\hat{}dA \neq 0$. It is necessary that the Pfaff dimension be greater than 2 if the implicit surface admits an envelope.

3.1. The Holder norm and similarity curvature invariants.

After division by a suitable function of the coefficient potentials, λ, an original 1-form of Action

$$A_0 = (U(x,y,z,...)dx + V(x,y,z,...)dy + W(x,y,z,...)dz...), \qquad (3.1)$$

can be made homogeneous of degree zero in terms of those coefficient functions that define the potentials. It is to be emphasized that the homogeneity condition is not on the arguments of the coefficients, but on the coefficient functions themselves. The scaling function of choice, $\lambda,$ is a Holder norm and is defined in terms of the covariant coefficients of the 1-form:

$$\lambda = (aU^p + bV^p + cW^p + ...)^{n/p}. \qquad (3.2)$$

The index n will be defined as the homogeneity index; the index p will be described herein as the isotropic index, and the constants $(a, b, c...)$ are constant scale factors whose signs determine the signature. By choosing the index n to be unity, $n = 1$, the 1-form, A, defined as

$$A = A_0/\lambda = (Udx + Vdy + Wdz...)/\lambda = A_k dx^k \qquad (3.3)$$

becomes homogeneous of degree zero in its coefficients. That is if every coefficient function is increased by a factor β then the coefficient function A_k does not change. This homogeneous degree zero 1-form, A_0/λ, is used to define an implicit hypersurface in the variety, whose geometrical properties can be expressed classically in terms of the similarity invariants of the associated *singular* doubly covariant Jacobian dyadic (or matrix), $[\mathbb{J}] = [\partial A_m/\partial x^n]$. Classically these similarity invariants are "symmetric" functions of the surface curvatures. Examples may be found at [4].

The determinant of the Jacobian matrix so constructed ($n = 1$, any $a, b, c..., p$) is always zero, indicating the existence of at least one zero eigen value (curvature or reciprocal radius). In this article, the Gauss constraint will be used ($p = 2,\ a = b = c... = 1$). The Jacobian matrix so constructed is singular, and induces a singular metric on the variety via the pullback $[g] = [\mathbb{J}]^{Transpose} \circ [\mathbb{J}]$. The zero determinant result also implies the existence of a global N-1 dimensional variety which in effect defines the implicit (hyper) surface. It is a standard geometrical procedure to construct the symmetric similarity invariants of the Jacobian matrix by forming the Cayley-Hamilton characteristic polynomial. Note that the induced symmetric metric $[g]$ does not carry the complete story of the surface properties inherent in the Jacobian dyadic, for the Jacobian matrix is not necessarily symmetric.

3.2. A globally closed current from the adjoint matrix

Next construct the doubly contravariant matrix $[\mathbb{J}]^{adjoint}$ equal to the adjoint (matrix of co-factors transposed) of the doubly covariant Jacobian matrix. This adjoint matrix exists algebraically, even though the inverse of the singular Jacobian matrix, and the inverse of the induced singular metric does not. Use the adjoint matrix to construct the contravariant vector current, $|\mathbf{J}_s\rangle = [\mathbb{J}]^{adjoint} \circ |\mathbf{A}\rangle$, and the N-1 form density, $J_s = i(\mathbf{J}_s)dx\hat{}dy\hat{}dz...$ Remarkably for any Holder norm with $n = 1$, arbitrary signature, arbitrary scale factors, and arbitrary exponent p, the N-1 form, J_s, is closed,

$$dJ_s = 0 \quad for\ \ n = 1. \tag{3.4}$$

As this closure result is global, it follows that $J_s - dG_s = 0$, which is equivalent to the Maxwell-Ampere equations. The subscript s is used to distinguish the fact that J_s has been deduced from the singular Jacobian matrix, and does not explicitly depend upon the field intensities, $F = dA$ and some arbitrary constitutive constraint between F and G. Note that given a J_s the corresponding G_s is not uniquely determined. The N-2 form density, G_s, may have closed and exact components as well as closed non-exact components, neither of which contribute to a specific charge-current density, J_s. These components can be used as the basis for various "gauge" theories that have physical meaning.

It is to be noted the induced metric is singular and therefor cannot be used to define a raising tensor as an inverse metric. It can be demonstrated that when the Holder norm is specialized to the Gauss map, then the coefficient of the interaction

N form density, $A\hat{}J_s$, is equal to the $(N-1)^{th}$ similarity invariant of the Jacobian field. For all implicit surfaces, simple or not, this similarity invariant is equal to the trace of the Jacobian Adjoint matrix and is equal to the sum of all possible products of different matrix eigen values which are of degree N-1. This similarity invariant is defined herein as the Adjoint Curvature of the implicit surface.

4. Acknowledgments

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5. References

[1] The importance of the N-1 form A^H (now written as A^G) was first anticipated in: R. M. Kiehn and J. F. Pierce, *An Intrinsic Transport Theorem* Phys. Fluids 12, 1971. The concept was further employed in R. M. Kiehn, *Periods on manifolds, quantization and gauge*, J. of Math Phys 18 no. 4 p. 614 (1977), and with applications described in R. M. Kiehn, *Are there three kinds of superconductivity,* Int. J. Mod. Phys B 5 1779. (1991)

[2] R. M. Kiehn, *The Photon Spin and other Topological Features of Classical Electromagnetism* in Vigier2000, edited by V. Amoroso and J. Hunter, ((Kluwer, Dordrecht 2001 to appear)

R. M. Kiehn, *Topological Evolution of classical electromagnetic fields and the photon*, in "The Photon and the Poincare Group", edited by V. Dvoeglazov, (Nova Science Publishers, NY 1999)

[3] http://www22.pair.com/csdc/pdf/thermo2.pdf ,

Marsden, J.E. and Riatu, T. S. (1994) "Introduction to Mechanics and Symmetry", Springer-Verlag, NY p.122.

[4] http://www22.pair.com/csdc/pdf/holder4d.pdf and .../holder3d.pdf