Retrodictive Determinism

R. M. Kiehn, Physics Department, University of Houston
from Int. J. Engng. Sci 1976 **14** pp. 249-754

Abstract: With respect to irreversible non-homeomorphic maps, contravariant and covariant tensor fields have distinctly different natural covariance and transformational behavior. For thermodynamic processes which are non-adiabatic, the fact that the process cannot be represented by a homeomorphic map emphasizes the logical arrow of time, an idea which encompasses a principle of retrodictive determinism for covariant tensor fields.

INTRODUCTION

The idea that a dissipative process is associated with a change in topology [1] implies that non-adiabatic, radiative transitions *cannot* be described by a homeomorphism. Accordingly, the transformational behavior of physical fields with respect to continuous, but not necessarily homeomorphic, maps is of some interest to researchers who study non-adiabatic transitions. For tensor fields and configurations, the classical transformation rules can, and must, be modified slightly to produce a useful concept of natural, or intrinsic, covariance with respect to maps which may not even have invertible Jacobians. Surprisingly enough, the techniques presented below indicate that for covariant tensor fields there exists a natural sense of retrodictive determinism which does not exist for contravariant tensor fields. Moreover, the logical structure is not symmetric, nor even dualistic, with the ideas of prediction of tensor fields. It appears that a system described by a tensor field may be predictively statistical, but retrodictively deterministic.

A major purpose of this article is to sharpen the perception of researchers to the transformational asymmetries associated with tensor fields, such that these ideas may be fruitfully exploited and incorporated into physical theories. A table summarizing the transformation properties of tensor rules w.r.t. maps of varying degrees of invertibility and differentiability is presented, along with a few abstract applications directed towards subtle points in the theory of thermodynamics and hydrodynamics.

INTRINSIC COVARIANCE AND THE CLASSIFICATION OF MAPS

The basic ideas to be utilized are straightforward, but appear only sporadically, if at all, in the engineering literature. (For a partial, mathematical treatment, see Lang[2].) The definition of a contravariant tensor is taken to be the classic one which uses the transformation rule,

$$X^A \to Y^\mu = J^\mu_A X^A$$

to define contravariant quantities. The Jacobian matrix, $J^\mu_A(X)$, is constructed in terms of the partial derivatives, $\partial\phi^\mu/\partial x^A$ of the map $\phi : x^A \to y^\mu$, all of which are assumed to exist, but which need not be continuous.

However, a covariant tensor is defined herein to be an object which obeys the transformation rule,

$$Y_\mu J^\mu_K = X_K \leftarrow Y_\mu$$

which, contrary to the usual tensor analysis definition, does not make use of a Jacobian inverse (the base spaces need not be of the same dimension). These two rules may be used to develop the ideas of natural, or intrinsic, covariance w.r.t. maps for which the Jacobian inverse does not (globally) exist, with the ideas agreeing with classic results when the transformations are restricted to orthogonal transformations or diffeomorphisms.

Scattered through the literature of differential topology are discussions of various maps which may or may not have inverses and whose Jacobians may or may not be invertible. A summary of such maps is given in Table 1, where more importantly, the intrinsic covariant behavior of tensor fields with respect to each class of maps is also presented. A tensor field is considered as a set of functional rules over a domain, with values in some range. The intrinsic covariance problem considered herein (and with results presented in Table 1) relates to the global solubility, and uniqueness of solubility, of the *rules* (not just their values) w.r.t. various transformations of the base spaces, as domains. For example: given two domains, x and y, with a map ϕ between them from x to y, is it possible to uniquely determine a tensor rule over the final state in terms of a tensor rule given over the initial state? The answer is no, if the map ϕ does not have an inverse. Surprisingly enough, a retrodictive version of this question obtains a favorable response for co-variant tensor fields: Is it possible to retrodictively determine a co-tensor rule over an initial state, given a co-tensor rule over a final state? An affirmative answer requires only that the Jacobian coefficients of map $\phi : x^A \to y^\mu$ exist.

Map	Assumed to exist	Covariant field rule	Contravariant field rule
Continuous	$\phi, d\phi$	R	-
Submersion	$\phi, d\phi_{onto}$	R_{unique}	-
Immersion	$\phi, d\phi_{1-1}$	R	R_{unique}
Local Inverse	$\phi, d\phi_{onto}, d\phi_{1-1}$	R_{unique}	R_{unique}
Cont, disc. Inverse	$\phi, \phi^{-1}, d\phi$	R	P
Disc., Cont Inverse	$\phi, \phi^{-1}, d\phi^{-1}$	P	R
Submanifold	$\phi, \phi^{-1}, d\phi_{1-1}$	R	$(R \& P)_{unique}$
Quotient manifold	$\phi, \phi^{-1}, d\phi_{onto}$	$(R \& P)_{unique}$	P
Homemorphisms	$\phi, \phi^{-1}, d\phi, d\phi^{-1}$	$R \& P$	$R \& P$
Embedding	$\phi, \phi^{-1}, d\phi_{1-1}, d\phi^{-1}_{onto}$	$R \& P$	$(R \& P)_{unique}$
Projection	$\phi, \phi^{-1}, d\phi_{onto}, d\phi^{-1}_{1-1}$	$(R \& P)_{unique}$	$R \& P$
Diffeomorphism	$d\phi \& d\phi^{-1}$ 1-1 & onto	$(R \& P)_{unique}$	$(R \& P)_{unique}$

Table 1. Retrodictive and predictive behavior of tensor fields w.r.t. continuous maps with different invertibility and differential structure. R = Retrodictive, P =Predictive

A summary of the transformational solubility (and uniqueness of solutions) of tensor fields with respect to various maps is presented in Table 1. Certain details are enumerated in Appendix A.

The logical asymmetry exhibited by the table is remarkable, as is the fact that for non-homeomorphic maps (which are necessary to represent dissipative transitions) there exists the possibility of a retrodictive determinism – but not a predictive determinism. There appears to be an arrow of time built into the transformational behavior of tensor fields with respect to non-homeomorphic maps. A recognition of this built in logical asymmetry should be taken into account by those theories which treat irreversible processes.

PHYSICAL APPLICATIONS

There seems to exist a predilection in physics to obtain a description of nature which is predictive. The classic problem in point mechanics is a problem in prediction: given initial data, what is the future trajectory? Watanabe states that ”Every closed system of physical laws must include a time-dependent law from which it is possible to deduce a predictive statement [3]... ”. . . physical theory is preeminently a predictive instrument” [4] . . . Now it is not apparent to this author that predictive descriptions are the only way to understand nature, and a study of the transformation properties of tensor fields given in Table I gives support to the position that although a deterministic, predictive, analysis for dissipative systems is impossible, surprisingly often the opposite point of view, based on a deterministic *retrodictive* analysis, is possible for dissipative systems. There is a definite asymmetry in dissipative processes and according to Table I this asymmetry persists in the analytical description of such systems.

The map classification table is striking in that it points out and emphasizes the natural retrodictive logical structure for tensor fields, especially covariant fields, with respect to maps that are not homeomorphisms. Recently it has been emphasized that non-adiabatic systems imply changing topology [l]. For such systems, Table 1 indicates that a retrodictive analysis is appropriate, and moreover, it is the only analysis that is deterministic, if the system process is irreversible (ϕ^{-1} does not exist). Perhaps the fundamental reason that Cartan’s theory of differential forms, built on alternating covariant tensor fields, is so powerful is due to the retrodictive solubility of differential forms with respect to C^1 maps. It can be shown that the specification of a system of differential forms is equivalent to specifying a topology and the utilization of Pfaffian expressions (which are differential forms of degree one) in the science of thermodynamics was an early recognition of the need for injection of topological concepts into thermodynamic theories. Caratheodory’s use of neighborhoods [5], Landsberg’s use of restricted continuity and the frontier of a set [6] and the modern work of Boyling [7] are examples of the utility of a topological approach in thermodynamics. However, it has been emphasized only recently that dissipation and nonequilibrium thermodynamics are related to changing topology [1]. It is here that Cartan’s mathematics of differential forms demonstrates its power, for according to the previous discussion, differential forms are well behaved (at least retrodictably) with respect to non-homeomorphic, topology-changing maps. Cartan’s theory of exterior differential forms appears to be the appropriate mathematical theory for studying dissipative systems. This conclusion is a first result of the theory.

For physical applications the two most important principle maps are the immersion and the submersion. Ordinarily, these maps are to be used in the field sense. That is, the physical object is considered to be the base space which is immersed or submersed into a Euclidean tensor space. The submersion induces on the physical object, as a manifold, a set of covariant vector lines which form an orthogonal field (on the object manifold) w.r.t. the fibers of the submersion. The ”gradient” vector to the spherical surface $\phi : R^3 \rightarrow R = \{x^2 + y^2 + z^2 = 1.0\}$ is an example; the fiber is the spherical surface itself. The orthogonal field spans the compliment to the fiber space created by the submersion.

On the other hand, the immersion of the object manifold into a Euclidean space, induces a global covariant metric field, $g_{\mu\nu}$, on the base space, which permits a norm to be created for contravariant vectors on the manifold. The idea of distance along a line is well defined by the immersion. The notion of distance between ”surfaces” may not be well defined by an immersion (consider the bi-refringent crystal); this concept requires a reciprocal metric field that the object manifold may not support, globally.

For physical objects which are manifolds there exist theorems that imply that they always may be immersed into a Euclidean space of sufficiently large dimension [8]. The implication is that manifolds always support a global metric field, whose covariant columns form a global, linearly independent set of differential forms. The determinant of the induced metric field is never zero, functionally, but can take on both positive and negative values, *discontinuously*. However, if the induced map, $d\phi$, of the immersion is continuous, as well as being 1-1, then the determinant of the metric field is never zero and must be definite.

In a topological sense, those manifolds which are immersed in a continuously differentiable manner must be orientable. A non-orientable manifold cannot support a covariant metric field with definite determinant. This subtle point is at the basis of Caratheodory's proof of the second part of the second law of thermodynamics [9]. Entropy is positive definite only on orientable manifolds. A set of points, or states, whose topology excludes, or makes inaccessible, another set of points, or states, supports a monotonic function only if the topology is orientable. A Mobius strip is a model of a topology (infinitely extendable) which produces inaccessible states, but one for which the entropy function is not globally definite ($S > 0$). Therefore, an immediate application of the point of view discussed above is to demonstrate a subtle and usually not expressed assumption in the theory of thermodynamics: The phase space of Gibbs must be an *orientable* sub-manifold of state space, if entropy is definite.

For contravariant considerations of a physical object as a manifold, the next most important maps are the Submanifold map and the Quotient manifold map. These mappings in physical situations are usually *from* a Euclidean space to the body manifold are made in such a way as to permit contravariant fields to be induced on the body manifold. (The submersion and immersion described above were *from* the body manifold to the Euclidean space, and induced covariant, not contravariant, fields on M) The classic example is the submanifold map, ϕ, which carries the unit interval into a curve in M. The induced differential map, $d\phi$, defines a tangent (contravariant) vector on M, in the sense of Lagrange, which spans the submanifold of M. This notion is distinct from the submersive (Hamiltonian) case which defines a covariant wave vector field on M.

If the map to M is a quotient manifold map, then a reciprocal metric field is induced on M which permits a distance between ”surfaces” concept to be defined globally. Dual to the immersive case, a distance between ”points” may not be admissable. The idea is that in the immersive case a contravariant measure is induced; in the quotient manifold case, a covariant measure is induced. The measure *coefficients* are covariant and contravariant, respectively, for the above mentioned measures. The two measure fields have different transformation properties for non homeomorphic dynamical transformations. Physically, the notion of strain is related to the covariant measure coefficients, while the notion of stress is related to the contravariant measure coefficients.

These results emphasize the differences between Lagrangian (contravariant-particle) and Hamiltonian (covariant-wave) mechanics [l0] — differences which become evident only for dissipative systems that do not admit global metrics and reciprocal metrics. For the dissipative case, there must exist two sets of physical laws: one for the covariant ideas, one for the contravariant ideas. The differential form statements for Maxwell's equations are the foremost example of such dual behavior. The first Maxwell pair of equations involving Faraday's law and covariant **E** and **B** intensities is one statement. The distinctly different second Maxwell pair of equations involves contravariant quantities, **D** and **H** and is the second statement. There is a fundamental difference between physical *intensities*, such as the (covariant) **E** and **B** fields of electromagnetism, when compared to physical *quantities* such as the (contravariant densities) **D** and **H** fields [11] . The differences are degenerate unless the system is irreversible, a fact that implies that all physical phenomena which can be deduced from the behavior of **E** and **B** fields can be metrically deduced from the behavior of **D** and **H** fields, in non-dissipative systems. For a space that does not support both a global metric field and a reciprocal metric field, one set of equations does not uniquely imply the other. (The dual to the Einstein equation for the covariant metric field is unknown.)

A study of these results should guide the development of physical theories for dissipative systems. Such theories, without a dynamical inverse, are not amenable to predictive determinism. These systems involve changing topology, but nevertheless, such dynamical systems (if describable by continuous maps) will yield a retrodictive determinism but only for a covariant (wave) formalism. Contravariant (particle-trajectory) formalisms are never predictive if ϕ^{-1} does not exist. The physicist, for dissipative field problems, should adopt the view: Given the final data, what was the initial state from which it came? (This statement is dual, but not reciprocal to the usual Cauchy statement: given initial data, what is the final state? Curiously enough, this point of view seems to have been taken by Hadamard [12]). Such questions and their answers, although not predictive in style, also yield an understanding of nature. Moreover the methods are deterministic, not statistical and are employed in a retrodictive sense.

Perhaps the most obvious physical example of a continuously dissipative system is the turbulent fluid. The deterministic theory of a turbulent fluid has yet to be formulated, apparently because of a predilection for a predictive theory. Moreover, from the arguments given above, as the dissipative turbulent flow does not admit an inverse, a predictive deterministic theory in terms of velocity fields is impossible. Again, the point of view discussed above has led to an immediate application by proving once and for all that a predictive, non-statistical theory of turbulence is impossible. Since the time of G. I. Taylor, turbulence theories that have made any progress have been predictively statistical and non-deterministic. However, the alternate point of view, based upon the deterministic retrodiction of differential forms, is just beginning to be utilized. Early results of the theory have demonstrated that (1) if the system is dissipative, topology must change and (2) a turbulent system *cannot* be described by Hamilton's equations of motion; i.e., a Hamiltonian analysis of a turbulent fluid is impossible. Moreover, if a flow is to be diffusively dissipative (an intuitive requirement of turbulent flow) then the Liouville theorem must fail [l3]. The topological criteria imply that not only are groups not admissable (ϕ^{-1} does not exist), but also semi-groups are not admissable in a turbulent flow [l]. Also it has been demonstrated that the Navier-Stokes equations for a viscous fluid cannot be derived from a strictly Hamiltonian analysis, but indeed are representable by the covariant concepts embodied in the theory of differential forms [14]. The purpose of this article is to focus attention on the logical basis of the statement that, indeed, the physics of fields is deterministically a retrodictive science. The permissibility of physics being deterministically predictive is not the usual case and demands the special constraints of a non-dissipative system. A more detailed application of the concepts discussed above to the science of thermodynamic processes will appear elsewhere.

Acknowledgment

This work was supported in part by a NASA-Ames Research Center Grant NCA2-OR-295-502.

REFERENCES

[1] R. M. KIEHN, J. Math. Phys. 15, 9 (1974).
[2] S. LANG, Introduction to Differentiable Manifolds. Interscience, New York (1962).
[3] S. WATANABE, Rev. Mod Phy. 26, 26 (1955).
[4] S. WATANABE, Rev. Mod Phys. 27, 179 (1955).
[5] J. KESTIN, A Course in Thermodynamics, Vol. I, p. 460. Blaisdell, Massachusetts (1966).
[6] P. T. LANDSBERG, Thermodynamics, Interscience, New York (1961).
[7] J. B. BOYLING, Commun. Math. Phys. 10, 52 (1968).
[8] The Whitney embedding theorem is described in many texts, e.g. S. Sternberg, Lectures on Differential Geometry. Prentice Hall, New Jersey (19ti4).
[9] C. CARATHEODORY, Math. Ann. 67, 355 (1909).
[10] The referee kindly pointed out that the differential geometric view of Lagrangian vs Hamiltonian mechanics is nicely put in D. Godbillion, Geometric Differentiale et Mecanique. Herman, Paris (1969). Note also should be taken of H. Rund, The Hamilton-Jacobi Theory in the Calculus of Variations. Van Nostrand, New York (1966).
[11] E. J. POST, The Formal Structure of Electromagnetics. North Holland, Amsterdam (1957).
[12] J. HADAMARD, Lectures on Cauchy's Problem, p. 51. Dover Press, New York (1952).
[13] The notion that the Liouville theorem must fail disallows the utilization of a symplectic geometry approach (better said in 1997: symplectic transformations cannot be utilized to study turbulence, but there exist uniquely defined thermodynamic irreversible processes on the symplectic manifold!), e.g. R. Abraham, Foundations of Mechanics. Benjamin, New York (1967).
[14] R. M. KIEHN, InL J. Engng Sci. 13, 941 (1975).
(Received 10 December 1975)

APPENDIX A

Transformational covariance

To demonstrate the transformation asymmetry of tensor fields this article considers maps between spaces of different dimensionality, ϕ”$M \rightarrow N$, from points x^A in the domain to points y^μ in the range. It is assumed that a physical system can be described as a tensor field, i.e. by a map $\alpha : M \rightarrow \tau^A$ in the initial state and a map $\bar{\alpha} : N \rightarrow \bar{\tau}^\mu$ in the final state. Both of these maps are from the base space (M or N) to the contravariant tensor space (τ^A or $\bar{\tau}^\mu$). An alternative field description can be made in terms of the maps $\beta : M \rightarrow \tau_K$ and $\bar{\beta} : N \rightarrow \tau_\upsilon$, which are from the base spaces to the covariant tensor spaces. The contravariant and covariant fields behave differently with respect to predictive and retrodictive deterministic solubility and part of the purpose of this article is to demonstrate these differences in behavior, *even though the field values may be related by a metric*.

For purposes herein it will be assumed that ϕ exists andthe Jacobian matrix of partial derivatives $\partial\phi^\mu/\partial x^A = J^\mu_A$ exists; i.e., unless otherwise specified, the map ϕ is continuous. The Jacobian matrix induces two linear maps, $d\phi : \tau^A \rightarrow \bar{\tau}^\mu$ and $\widetilde{d\phi} : \tau_K \leftarrow \tau_\upsilon$ between the tensor spaces. The direction of the arrows is important; they demonstrate that the Jacobian matrix always permits the values of the contravariant fields, as numbers, to be predicted (but not necessarily retrodicted) and similarly the covariant values may be retrodicted w.r.t. ϕ (but not necessarily predicted). The usual rules are expressible in coordinate language as,

$$d\phi : V^A(x) \to V^\mu(x) = \sum_A J^\mu_A(x) V^A(x) \qquad \text{A1}$$

and

$$\widetilde{d\phi} : A_K(x) = \sum_\mu A_\mu(y(x)) J^\mu_K(x) \leftarrow A_\mu(y) \qquad \text{A2}$$

Note that the existence of a Jacobian inverse has not been assumed. The action of the Jacobian is to push forward the values of contravariant fields and ”pull back” the values of the covariant fields. A diagrammatic description is given in Figure 1. Classical developments of tensor analysis assume that an inverse Jacobian matrix exists such that the rule for co-variant tensor transformation instead of being given by (A2) becomes $[d\phi]^{-1} : A_K(x) \to A_\mu(x) = [J^\mu_K(x)]^{-1} = A_K(x)$. This transition rule will not be assumed herein.

download pictures to \scinotebook\pictures

download pictures to \scinotebook\pictures

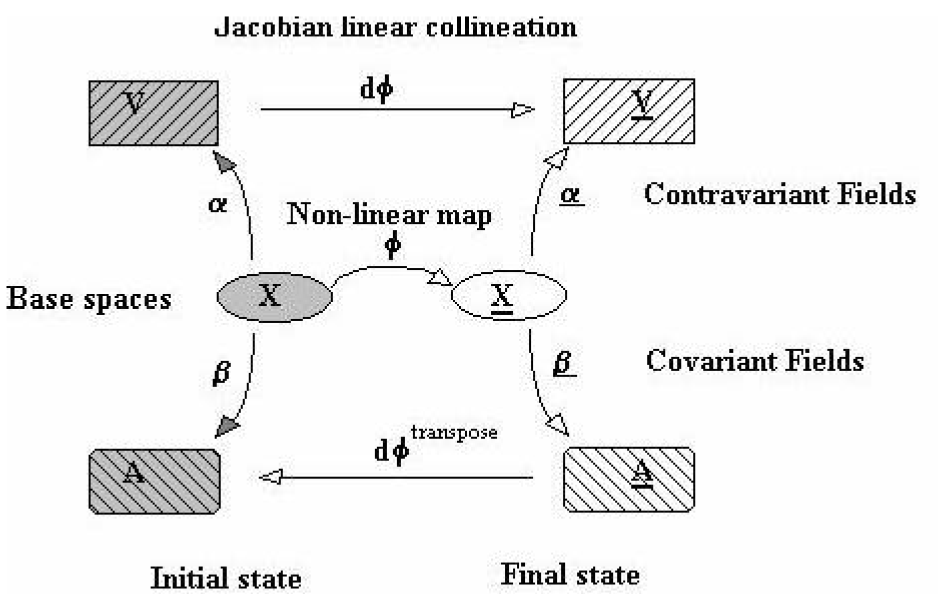

Fig. 1. Induced maps between tensor fields.

Equations (Al) and (A2) not only yield recipes for computing values of covariant and contravariant tensors, but also explicitly demonstrate the differences between covariant and contravariant fields. Equation (A2) permits the covariant rule, β, to be deduced retrodictively from the covariant rule, $\overline{\beta}$. On the other hand, equation (Al) for contravariant values does not permit the contravariant rule, $\overline{\alpha}$, to be deduced predictively from the rule α, for the functions $V^\mu(x)$ have arguments on the domain space, x and are not functions of variables, y, on the range space. This fact is the fundamental observation which distinguishes between covariant/contravariant and retrodictive/predictive analysis. Covariant fields (especially differential forms) are always retrodictive, even for irreversible maps where ϕ^{-1} does not exist. This principle result is easily deduced from the directionality of the arrows in Fig. 1. A study of the arrows in the figure will lead to a quick understanding of the transformational behavior for the tensor field rules, α and β, as more invertibility, or differentiability, structure is assumed for the map ϕ.

As an example of the analysis, consider the submersion for which $d\phi$ is onto ($\widetilde{d\phi}$ 1-1). For submersions, $d\phi$ admits a right inverse ($J \circ \tilde{J}$ is non-singular on the range space) and by modifying the arrows in Fig. I it is easy to see that the co-values are both predictive and retrodictive, the contra-values are predictive and the covariant rule β is retrodictively *unique*. (The proof can be effected easily by writing in a bi-jective arrow for $\widetilde{d\phi}$.)

If the map ϕ is an immersion such that $d\phi$ is 1-1 ($\widetilde{d\phi}$ is onto), then the map permits a left inverse of $d\phi$ ($\tilde{J} \circ J$ is non-singular on the domain space). Co-values and co-variant rules are retrodictive. Contra-values are both predictive and retrodictive, but now the contravariant rules are retrodictively *unique*.

If the map ϕ is such that $d\phi$ is both 1-1 and onto, then co- and contra-values are both retrodictive and predictive and both the contravariant rule and covariant rule are retrodictively *unique*.

None of the above maps admit an inverse function, globally; none of the rules are predictive. The "arrow of time" permits determinism only in a retrodictive sense.

A continuous map, with a discontinuous inverse, yields enough additional structure beyond the primitive cases considered above such that for the first time the contravariant rule, α, is predictive. A completely dual situation occurs for those discontinuous maps ϕ that admit an inverse ϕ^{-1} and $d\phi^{-1}$ (but no $d\phi$): the co-values and covariant rules become predictive and the contra-values and contravariant rules become retrodictive.

Of somewhat greater importance to physical systems are the maximal rank Submanifold map, for which ϕ, ϕ^{-1}' and $d\phi_{1-1}$, are valid globally, and its dual, which is the Quotient manifold map, for which ϕ, $\phi^{-1}$and $d\phi_{onto}$ are globally valid. These maps permit contravariant vector fields to be induced globally on a manifold.

None of the maps considered so far are homeomorphisms; they do not necessarily preserve topology. Subsequent maps to be considered are homeomorphisms and all admit continuous inverses. Dissipative, irreversible systems cannot be described by such maps. Dissipative systems imply a change in topology.

For the weakest homeomorphism, ϕ and $d\phi$ exist and similarly ϕ^{-1} and $d\phi^{-1}$ exist. For such maps, both contravariant values and rules, as well as covariant values and rules, are soluble in both a retrodictive and predictive manner.

If the homeomorphism is an embedding, $d\phi$ is 1-1 and $d\phi^{-1}$ is onto; the contravariant rules become uniquely soluble in both a predictive and a retrodictive sense.

If the homeomorphism is a projection, then $d\phi$ is onto and $d\phi^{-1}$ is 1-1; the covariant rules become uniquely soluble in both a predictive and retrodictive sense.

Finally if the map is a diffeomorphism (ϕ exists, ϕ^{-1} exists, $d\phi$ is 1-1 and onto, $d\phi^{-1}$ is 1-1 and onto) then both contravariant and covariant rules are uniquely soluble in a predictive and retrodictive sense. The diffeomorphism is the usual map considered in classical tensor analysis as a coordinate transformation. With respect to such maps (that is, with respect to classical tensor analysis) the different solubility features of contravariant and covariant tensor fields becomes degenerate and indistinct.

A summary of the above map classifications is presented in Table I along with the retrodictive or predictive solubility of the associated covariant and contravariant tensor fields. Note the emphasis on retrodiction of covariant fields for irreversible maps.

At this point it should be mentioned that there exists a concept distinctly different from that of a field; it is the concept of a configuration mapping. The concept of a configuration as a map from the tensor space to the base space M is completely dual to the field concept as a map from the base space M to the linear tensor space. This configuration concept is related to the mathematical notion of a vector bundle and is little used in the physics literature except in the theory of elasticity (1997 comment: How things have changed in 25 years). The retrodictive and predictive behavior of configurations is also dual to that of fields. In fact, Table I can be completely dualized for configurations by substituting ”configuration” for ”field”, and interchanging the retrodictive and predictive behavior (interchange R and P in the Table). The proofs follow quickly by merely changing the direction of the arrows for the maps α, $\bar{\alpha}$, β and $\bar{\beta}$ in Fig. 1.