A Topological Perspective of Electromagnetism.

R. M. Kiehn

Emeritus, Physics Dept., Univ. Houston
http://www.cartan.pair.com

Abstract

Modern electromagnetic theory recognizes that the classic Maxwell-Faraday and Maxwell-Ampere field equations belong to two thermodynamically distinct topological categories. The classic Maxwell PDE's can be deduced from a Cartan exterior differential system, based on two topological postulates, F-dA=0, and J-dG=0, independent from any geometrical constraints imposed by a metric, or connection, or gauge, and for any geometric dimension greater than 3. Both the potentials, A, and the field excitations, G, are not uniquely defined by the field equations, a fact which leads to topological defects and coherent structures, in both equilibrium conservative and turbulent dissipative systems. The non-equilibrium topological coherent structures discovered from a topological perspective of electromagnetism are represented by the 3-forms of Topological Torsion, A^F, and Topological Spin, A^G.

1. Introduction

1.1. Modern Electrodynamics as a topological theory

In this abbreviated summary article [1] it is recognized that the *modern* theory of electromagnetism can be understood as a topological theory expressed in terms of an exterior differential system of two postulates [2]:

$$\text{1. The postulate of conserved flux} \quad : \quad F - dA = 0, \tag{1.1}$$

$$\text{2. The postulate of conserved currents} \quad : \quad J - dG = 0. \tag{1.2}$$

The exact 2-form of thermodynamic field "intensities" $F(E,B)$ is defined in terms of inexact 1-form of potentials, A, in units of $\hbar/$e. The exact 3-form of charge current density, J, is defined in terms of the inexact 2-form density of thermodynamic field quantities, or "excitations", $G(D,H)$ in units of $\hbar$. The two form, F, historically is associated with forces, and the 2-form density, G, historically is associated with sources.

From these two topological postulates, it is possible (without the geometric constraints, or choice, of metric, connection, or gauge) to deduce, abstractly, two sets of classical PDE's . The first set is known as the Maxwell-Faraday equations, and the second set is known as the Maxwell-Ampere equations. In Engineering format:

$$\text{Maxwell Faraday PDE's} \quad \{curl\,\mathbf{E} + \partial\mathbf{B}/\partial t = 0, \qquad div\,\mathbf{B} = 0\}, \tag{1.3}$$

$$\text{Maxwell Ampere PDE's} \quad \{curl\,\mathbf{H} - \partial\mathbf{D}/\partial t = \mathbf{J}, \qquad div\,\mathbf{D} = \rho\}. \tag{1.4}$$

It is also true that the abstract form of these eight PDE's is universally the same, without additional features, or terms, in all geometric dimensions of 4 or greater. In short, the "modern" and the "classical" electromagnetic theory have the *same* base of partial differential field equations.

The differences between the "modern" theory and the "classical" theory can be ascribed to the fact that the Maxwell-Faraday equations of "intensities" and the Maxwell-Ampere equations of "quantities" are recognized to belong to two thermodynamic categories which are topologically distinct. In the first category, the 1-form of potentials can have non-unique but topologically closed components, A_c, which do not contribute to the 2-form of intensities, F, as $dA_c \Rightarrow 0$. Similarly, in the second category, the 2-form density of excitations can have non-unique but closed components, G_c, which do not contribute to 3-form density of charge-currents, J, as $dG_c \Rightarrow 0$. In the historic literature of the last 50 years, the non-uniqueness of the potentials has led to a large industry called gauge theories. The non-uniqueness of the excitations has not been so well developed.

These concepts of non-uniqueness can effect the topological properties of the solutions to the Maxwell electrodynamic system. The non-uniqueness can appear as discontinuities in solution amplitude and derivatives (electromagnetic signals), as a solution multi-valuedness (polarization), as envelope (Huygen wavelet or Cherenkov) solutions, and in many other physically recognizable topological properties related to the continuity, integrability, differentiability, compactness, and reality of the solutions. In short, the non-uniqueness of both the potentials,

A, and the field excitations, G, lead to the concept of topological defects as regions where topological refinements of uniqueness, integrability, differentiability, compactness, or reality fail. Of particular note is the fact that the solutions to the PDE's which involve topological defects need not be defined in a locally linear, and therefor unique, manner. The defects are sets upon which the solutions do not behave as tensors, for tensors require that different neighborhoods are related by diffeomorphisms and therefor impose uniqueness by linearity. The use of exterior differential forms circumvents these limitations of uniqueness demanded by the theory of tensor analysis, for differential forms are functionally well behaved in a retrodictive pullback sense [3] relative to differential maps which are not one to one diffeomorphisms. Exterior differential forms are well behaved with respect to evolutionary processes that include topological change, and the production of topological defects.

Topological defects find representation in terms of closed, but not exact, homogeneous differential forms of topological dimension M immersed in spaces of geometric dimension $N \succeq M$. The integrals over closed integration chains (closed cycles which may not be boundaries) of such closed, but not exact, exterior differential forms are the basis of deRham cohomology theory [4]. The values of such closed integrals have rational ratios and provide a topological basis for "quantization". Integrals of exact k-forms over closed cycles or boundaries are always zero. Integrals of closed but not exact k-forms over cycles which are not boundaries are not zero. Integrals of closed but not exact k-forms over boundaries have zero values. In a topological theory of electromagnetism, the closed, but not exact, components of the 1-form of potentials, A, lead to 1-dimensional period integrals and the concept of the flux quantum as a 1-dimensional topological defect. Similarly, closed but not exact components of the 2-form of excitation densities, G, lead to 2-dimensional period integrals and the concept of the charge quantum as a 2-dimensional topological defect. These period integral concepts have been discussed elsewhere [5].

Most of the emphasis in this article however is placed upon the non-classical, closed but not exact, exterior differential forms, which can be constructed from A and G and their exterior derivatives and exterior products. The two most important of these constructions, in a domain of 4 geometric dimensions, is defined in terms of the Topological Torsion 3-form, $A\hat{}F$, and the Topological Spin 3-form density, $A\hat{}G$. The 3-dimensional topological defects associated with the closed but not exact components of these 3-forms are defined as the topological torsion-helicity quantum and the topological spin-chirality quantum, respectively.

It can be demonstrated that flux quantum period integrals are elements of topological defects in equilibrium thermodynamic systems, but the torsion-helicity quanta and the spin-chirality quanta are elements of topological defects in *non-equilibrium*, dissipative, thermodynamic systems, and can be created as artifacts of irreversible processes. It is the discovery of these non-equilibrium properties that demonstrates the utility of the topological perspective of electromagnetism.

1.2. Non-uniqueness, discontinuities, topological defects

The topological postulates, hence the PDE's they generate, are valid in any frame of reference. So why is the Lorentz system of transformations so dominant in classical electromagnetic theory? In a remarkable piece of work [6] published in 1932, V. Fock demonstrated that the point set upon which the solutions to the Maxwell PDE's are not uniquely defined defines a propagating discontinuity, and this point set is defined in terms of solutions to the Eikonal equation. The Eikonal equation is a non-linear partial differential equation that consists of a (canonical) quadratic form with signature (+,+,+,-) or (-,-,-,+):

$$\text{Eikonal} \quad (\pm\partial\varphi/\partial x)^2 \pm (\partial\varphi/\partial y)^2 \pm (\partial\varphi/\partial z)^2 \mp 1/c^2(\partial\varphi/\partial t)^2 = 0. \tag{1.5}$$

The *only linear* set of transformations that preserves the Eikonal null quadratic form is the Lorentz system of transformations. Hence a propagating discontinuity (the Fock definition of an electromagnetic signal) appears as a propagating discontinuity (a signal with finite universal propagation speed, c) to all Lorentz equivalent observers. It is not the PDE's of Maxwell that give importance to the Lorentz transformations, it is the *singular "characteristic" solutions of propagating discontinuities (a topological defect)* that insure the importance of the Lorentz transformations. Fock also demonstrated that the non-linear fractional Moebius transformation also preserves the eikonal equation. The projective mapping permits the propagation speed of the discontinuity to be less than or greater than c.

It is perhaps of more interest to realize that the components of the Eikonal quadratic form can be interpreted as an isotropic null vector, for which E. Cartan gave the name Spinor [7]. This topological result of modern electromagnetism emphasizes the fact that spinors play a role in wave phenomena at all scales, from the micro world of quantum mechanics to the macro world of cosmology. The Eikonal equation is a quadratic form with the same signature as the Minkowski metric line element. Einstein's specification that light travels along a null geodesic follows from the constraint of a metric geometry. However, the characteristic solutions representing propagating discontinuities follow from the system of PDE's with out regard to a metric. Spinor solutions for the characteristics are not tensors, for spinors have an ambiguity under reflections with respect to a sign. Superposition of (conjugate) spinors can be arranged to produce vectors, which are tensors. These facts are realized in spinor states of circular polarization which can be combined into vector states of linear polarization.

Applications of constitutive equations and their topological features to the generation of the singular solution sets of the PDE's (describing domains of propagating non-uniqueness) and the interpretation in terms of electromagnetic signals are given in [8]. A notable result is the demonstration of exact (quaternion - spinor) solutions for which the 4-fold degeneracy of propagating signals is broken, not only for polarization, but also for propagation direction. The propagation speed of an electromagnetic signal outbound is not necessarily the same as the propagation speed inbound! The signal speed need not be the same in both directions, in direct confrontation with an axiom of special relativity. The results were verified in dual polarized ring laser experiments [9].

2. Topological Spin and Topological Torsion

The exterior differential forms that make up the electromagnetic system on a geometric domain of 4 dimensions consist of the primitive 1-form, A, and the primitive N-2 form density, G, their exterior derivatives, and their algebraic intersections defined by all possible exterior products. The complete Maxwell system of exterior differential forms (the Pfaff sequence for the Maxwell system on 4 geometric dimensions) is given by the set:

$$\{A; F = dA, G; J = dG, A\hat{}F, A\hat{}G, A\hat{}J; F\hat{}F, G\hat{}G\}. \tag{2.1}$$

These differential forms and their unions may be used to form a topological base on the domain of independent variables. The Cartan topology constructed on this system of forms has the useful feature that the exterior derivative may be interpreted as a limit point, or closure, operator in the sense of Kuratowski [10]. The exterior differential systems that define the Maxwell-Ampere and the Maxwell-Faraday equations above are essentially topological constraints of closure.

The complete Maxwell system of differential forms (which assumes the existence of A and G and C2 differentiability) also generates two other exterior differential systems.

$$(F\hat{}G - A\hat{}J) - d(A\hat{}G) = 0, \quad (2.2)$$
$$F\hat{}F - d(A\hat{}F) = 0. \quad (2.3)$$

The two 4-forms $(F\hat{}G - A\hat{}J)$ and $(F\hat{}F)$ are exact and have closed integrals which are evolutionary invariants of continuous deformations. The closed integrals therefor describe topological properties.

The first 3-form density, $A\hat{}G$, with physical units of $\hbar$, is called the topological spin (or chirality) [5] and the second 3-form, $A\hat{}F$, with physical units of $(\hbar/e)^2$, is called the topological torsion (or helicity) [11]. These two exterior 3-forms, $A\hat{}G$ and $A\hat{}F$ are not usually found in discussions of classical electromagnetism. The 3-forms are abstractly defined (on a space of 4 geometric dimensions with a volume element, $\Omega_4 = dx\hat{}dy\hat{}dz\hat{}dt$) in terms of exterior multiplication, but can be given realization in terms of 4 component engineering variables, $\mathbf{S}_4$ and $\mathbf{T}_4$.

$$\text{Topological Spin density} \quad : \quad A\hat{}G = i(\mathbf{S}_4)\Omega_4 \quad (2.4)$$
$$\mathbf{S}_4 = [\mathbf{A} \times \mathbf{H} + \mathbf{D}\phi, \mathbf{A} \circ \mathbf{D}], \quad (2.5)$$

$$\text{Topological Torsion vector} \quad : \quad A\hat{}F = i(\mathbf{T}_4)\Omega_4 \quad (2.6)$$
$$\mathbf{T}_4 = -[\mathbf{E} \times \mathbf{A} + \mathbf{B}\phi, \mathbf{A} \circ \mathbf{B}]. \quad (2.7)$$

These constructions should be compared with the exact charge current 4-vector density, J, with a 4 component engineering representation, $\mathbf{J}_4 = [\mathbf{J}, \rho]$. The concepts of the topological Spin density (current) and the topological Torsion vector have had almost no utilization in applications of classical electromagnetic theory. Each construction depends explicitly on the existence of the 1-form of Action-potentials.

2.1. Topologically Transverse waves

The vanishing of the topological Spin 3-form is a topological constraint on the domain that defines topologically transverse electric (TTE) waves: the vector potential, $\mathbf{A}$, is orthogonal to $\mathbf{D}$, in the sense that $\mathbf{A} \circ \mathbf{D} = 0$. The vanishing of the topological Torsion 3-form is a different topological constraint on the domain that defines topologically transverse magnetic (TTM) waves: the vector potential,

$\mathbf{A}$, is orthogonal to $\mathbf{B}$, in the sense that $\mathbf{A} \circ \mathbf{B} = 0$. (In fluid dynamics, $\mathbf{A} \circ \mathbf{B} = 0$ is called the helicity). When both 3-forms vanish, the topological constraint on the domain defines topologically transverse (TTEM) waves. The geometric definitions of transverse waves may or may not be equivalent to the topological definitions. For classic real fields this double constraint would require that vector potential, $\mathbf{A}$, is collinear with the field momentum, $\mathbf{D} \times \mathbf{B}$, and in the direction of the wave vector, $\mathbf{k}$. Theoretical examples (and dielectric waveguide experiments) indicate that a TTEM solution does not radiate. The result gives a possible explanation for the paradox that an electron in the Bohr model is accelerated, yet does not seem to radiate. This conjecture will be discussed elsewhere.

Note that if the 2-form F was not exact, such topological concepts of transversality would be without meaning, for the 3-forms of Topological Spin and Topological Torsion depend upon the existence of the 1-form of Action-Potentials, A. The torsion vector $\mathbf{T}_4$ and the Spin vector $\mathbf{S}_4$ are "associated" vectors to the 1-form of Action in the sense that

$$i(\mathbf{T}_4)A = 0 \quad and \quad i(\mathbf{S}_4)A = 0, \tag{2.8}$$

a result will prove to be of importance in the description of a topological basis for superconductivity.

2.2. The Poincare Topological Invariants

The exterior derivatives of the 3-forms of topological Spin and topological Torsion produce two exact 4-forms, $F\hat{}G - A\hat{}J$ and $F\hat{}F$, whose closed integrals are topological objects which generalize the conformal invariants of a Lorentz system, as discovered by Poincare and Bateman. Note that these topological properties of invariance with respect to continuous deformations are valid even in the domain of dissipative charge-currents and radiation.

In the format of independent variables $\{x, y, z, t\}$, with a volume element Ω_4, the exterior derivative, acting on the 3-forms as a topological limit point generator, can be related to the classic 4-divergence of the 4-component Spin and Torsion vectors, $\mathbf{S}_4$ and $\mathbf{T}_4$.

$$\begin{aligned} Poincare\ 1 &= d(A\hat{}G) = F\hat{}G - A\hat{}J \\ &= \{div_3(\mathbf{A} \times \mathbf{H} + \mathbf{D}\phi) + \partial(\mathbf{A} \circ \mathbf{D})/\partial t\}\ \Omega_4 \\ &= \{(\mathbf{B} \circ \mathbf{H} - \mathbf{D} \circ \mathbf{E}) - (\mathbf{A} \circ \mathbf{J} - \rho\phi)\}\ \Omega_4. \end{aligned} \tag{2.9}$$

$$\begin{aligned} Poincare\ 2 &= d(A\hat{}F) = F\hat{}F \\ &= \{div_3(\mathbf{E}\times\mathbf{A}+\mathbf{B}\phi) + \partial(\mathbf{A}\circ\mathbf{B})/\partial t\}\ \Omega_4 \\ &= \{2\mathbf{E}\circ\mathbf{B}\}\ \Omega_4. \end{aligned} \tag{2.10}$$

The Poincare invariants are, in effect, the evolutionary source terms for the 3-forms of topological spin, $A\hat{}G$, and topological torsion, $A\hat{}F$. When the Poincare invariants are zero, the closed integrals of the electromagnetic 3-forms of $A\hat{}G$ and $A\hat{}F$ become additional topologically coherent configurations invariant with respect to all evolutionary processes of continuous deformation.

The first term in the first Poincare invariant has a coefficient function which represents twice the difference between the magnetic energy density and the electric energy density of the electromagnetic field in a Lagrangian sense:

$$\text{Topological Field Lagrangian:}\quad F\hat{}G = (\mathbf{B}\circ\mathbf{H} - \mathbf{D}\circ\mathbf{E})\ \Omega_4. \tag{2.11}$$

The second term in the first Poincare invariant has a coefficient function which is defined as the interaction energy density:

$$\text{Topological Interaction:}\quad A\hat{}J = (\mathbf{A}\circ\mathbf{J} - \rho\phi)\ \Omega_4. \tag{2.12}$$

In Lagrangian variational methods, the 4-form $F\hat{}F$, which defines the second Poincare invariant, has been related to the concept of topological Parity:

$$\text{Topological Parity:}\quad F\hat{}F = +\{2\mathbf{E}\circ\mathbf{B}\}\ \Omega_4. \tag{2.13}$$

The pre-image 3-form of topological Torsion has been associated with Chern-Simons terms in fiber bundle theory.

2.3. Topological Torsion and Spin quanta

When either Poincare deformation invariant vanishes, the corresponding closed 3-dimensional integrals of $A\hat{}G$ and $A\hat{}F$ become deRham period integrals. The closed, but not exact, components of each 3-form can be put into correspondence with "quantized" topological defects.

The topological Spin quantum is defined as the closed integral of those closed but not exact components of the 3-form $A\hat{}G$ (which represent the kernel of the first Poincare 4-form),

$$\text{Spin quantum} = \iiint_{Z3} A\hat{}G \text{ with units n } \hbar. \tag{2.14}$$

The period integrals $\iiint_{Z3} A\hat{}G$ are deformation invariants (hence define a topological property) with rational ratios. $Z3$ is a closed integration chain defined in regions where $d(A\hat{}G) = 0$.

Similarly, when the second Poincare invariant vanishes, the closed integral of the 3-form of Torsion-Helicity becomes a deformation invariant with quantized values:

$$\text{Torsion quantum} = \iiint_{Z3} A\hat{}F \text{ with units m } (\hbar/e)^2. \tag{2.15}$$

The period integrals $\iiint_{Z3} A\hat{}F$ are deformation invariants (hence define a topological property) with rational ratios. In this case, $Z3$ is a closed integration chain defined in regions where $d(A\hat{}F) = 0$.

It is important to realize that the topological conservation laws (deformation invariants with respect to homeomorphisms) are valid in a plasma as well as in the vacuum, subject to the conditions of zero values for the Poincare invariants. On the other hand, topological evolution and transitions between "quantized" states of Spin-chirality or Torsion-helicity require that the respective Poincare invariants are not zero.

2.4. Thermodynamics and the Pfaff Topological dimension

The realization that electromagnetism can be expressed in terms of an exterior differential system permits Cartan's magic formula to be applied to the problems of continuous topological evolution [12], with the destruction or creation of electromagnetic topological defects. For any physical system that admits description in terms of a 1-form of Action, A, and for evolutionary processes that can be described in terms of singly parametrized vector fields, V, Cartan's magic formula of continuous topological evolution becomes the abstract equivalent to the first law of thermodynamics.

$$L_{(\mathbf{V})}A = \{i(\mathbf{V})dA + d(i(\mathbf{V})A)\} = \{W + dU\} = Q. \tag{2.16}$$

Note that the virtual work 1-form, W, for an electromagnetic system is in effect the work induced by the Lorentz force law:

$$W = i(\mathbf{V})dA \Rightarrow \{\mathbf{E} + \mathbf{V} \times \mathbf{B}\} \circ d\mathbf{r} - \{\mathbf{E} \circ \mathbf{V}\}dt. \tag{2.17}$$

Different evolutionary dynamics belong to equivalence classes in different topological categories. These categories are defined in terms of the Pfaff topological dimension of the 1-forms of Action, A, and the induced 1-forms of Work, W, and Heat, Q. The Pfaff topological dimension, or class, of a 1-form is determined from the number of successive non zero elements of its Pfaff sequence. For the 1-form of Action, A, the Pfaff sequence on a geometric space of 4 dimensions is given by the set $\{A, dA, A\hat{}dA, dA\hat{}dA\}$. The Pfaff topological dimension is the irreducible number of functions M (that define submersive maps from the geometric dimension N to the topological space of dimension, M) that encode the topological properties of the specific exterior differential 1-form. The Pfaff topological dimension is less than or equal to the geometric dimension of the domain of interest.

2.4.1. Equilibrium systems and Reversible processes

The thermodynamic interpretation of Cartan's magic formula depends upon both the elements of a physical system, encoded as an exterior differential system, and the evolutionary process, encoded as a vector field, $\mathbf{V}$. Equilibrium systems will be defined as domains where the topological dimension of the 1-form of Action is 2, or less. Caratheodory's concept of reversible processes requires that the heat 1-form is integrable, such that the Frobenius integrability theorem is satisfied: $Q\hat{}dQ = 0$. Hence, for reversible processes, the Pfaff topological dimension of Q is 2, or less [14]. Adiabatic processes are such that $i(\mathbf{V})Q = 0$, but can be reversible or irreversible.

Extremal Hamiltonian processes are such that the Pfaff dimension of the work 1-form, W, is zero ($W = i(\mathbf{V})dA \Rightarrow 0$). The Pfaff dimension of the work 1-form, W, is unity for both Bernoulli-Casimir processes, (where the virtual work 1-form is exact: $W = i(\mathbf{V})dA \Rightarrow d\Theta$), and for the Helmhotz-Stokes-Symplectic processes (where the virtual work 1-form is closed but not exact: $dW = di(\mathbf{V})dA \Rightarrow 0$). All such processes are reversible processes, as the Pfaff dimension of the heat 1-form, Q, (for such processes) is at most 1.

As the Pfaff dimension of the 1-form, A, of an equilibrium system is at most 2, the equilibrium 1-form satisfies the Frobenius conditions of unique integrability, $A\hat{}dA = 0$. Such equilibrium systems do not support a non-zero topological torsion vector, $A\hat{}F \Rightarrow 0$, but the Pfaffian equation, $A = 0$, admits a local integrating factor. A closed, but not exact, 1-form of Action, A_c, in a domain of Pfaff dimension 2 will admit description in terms of two independent functions of the independent geometrical variables, x^k, at most. A classic example is given by the expression:

$$A_c = (y\ dx - x\ dy)/(x^2 + y^2) \tag{2.18}$$

In a space of 3 geometric dimensions, $\{x, y, z\}$, this 1-form is closed in an exterior differential sense ($dA = 0 \supset$ zero curl) everywhere except along the z axis. The 1-form is of Pfaff dimension one almost everywhere, except along the z-axis. The 1-form becomes singular along the z axis, and it is this singularity that defines the topological defect.

The vector field of components $[y/(x^2+y^2), -x/(x^2+y^2), 0]$ describes a circulation about the z axis, without vorticity, and yet generates a non-zero value for a "circulation" line integral $\oint A_c$ when the path of integration encircles the z axis. Note that the singularity is an open 1-dimensional line. It is now common place in many disciplines to call this topological line singularity a "vortex" - even though the vorticity in a hydrodynamic sense is zero exterior to the singularity. Such 1-dimensional defects are ubiquitous in type II superconductors. Those regions where A_c is closed, but not exact, expel the magnetic fields (Meissner repulsion).

As another example in 3D geometrical space, consider the closed but not exact 1-form

$$A_c = (\beta\ dz - z\ d\beta)/(z^2 + \beta^2), \text{ with } \quad \beta = (\sqrt{x^2 + y^2} - a). \tag{2.19}$$

The 1-form, A_c, is closed, but not exact, almost everywhere, except on the circle $x^2 + y^2 = a^2$ in the $z = 0$ plane. The singularity (topological defect) is a closed 1 dimensional circle, which could be compared to a "smoke ring vortex". There exists a finite circulation integral $\oint_{z1} A_c$ for any cycle that links the ring of radius a in the $z = 0$ plane.

The two one forms can be combined to yield a "torus" configuration with two cycles, one enclosing the z axis and the other enclosing the ring. More complicated "vortices" admit spiral torsion projections in 3D geometrical domains. In summary, closed but not exact 1-forms are topological defects (holes and handles) in equilibrium systems of Pfaff topological dimension 2.

2.4.2. Turbulent dissipative systems and irreversible processes

If $dA\hat{}dA = +2\ \mathbf{E} \circ \mathbf{B}\ \Omega_4 \neq 0$, the electromagnetic 1-form, A, is of Pfaff topological dimension 4, which is maximal in regions of geometric dimension 4. Such non-equilibrium domains support the topological torsion vector of 4 geometric components, $\mathbf{T}_4$, but do not require that the magnetic field be topologically transverse, (as $A\hat{}F \neq 0$). By direct application of Cartan's magic formula of continuous topological evolution, relative to a direction field given by the topological torsion vector, $\mathbf{T}_4$, it follows that $L_{(\mathbf{T}_4)}A = \sigma A = (\mathbf{E} \circ \mathbf{B})A$, and

$$L_{(\mathbf{T}_4)}A\hat{}L_{(\mathbf{T}_4)}dA = Q\hat{}dQ = (\mathbf{E} \circ \mathbf{B})^2 A\hat{}dA. \tag{2.20}$$

Recall that the classic thermodynamic criteria for irreversible processes is given by the constraint, $Q\hat{}dQ \neq 0$ [15]. Hence processes containing a component proportional to $\mathbf{T}_4$ are thermodynamically irreversible when the 1-form of Action is of Pfaff dimension 4, for then $\mathbf{E} \circ \mathbf{B} \neq 0$, $A\hat{}dA \neq 0$ and therefor $Q\hat{}dQ \neq 0$. When $dA\hat{}dA = 2(\mathbf{E} \circ \mathbf{B})\Omega_4 \Rightarrow 0$, but $A\hat{}dA \neq 0$, the Pfaff topological dimension of A becomes equal to 3, and the process defined by the direction field $\mathbf{T}_4$ is reversible. As the Pfaff dimension of the Action is not 2, such processes are not "equilibrium" processes, but represent reversible non-equilibrium (or "far from equilibrium") processes, which may be chaotic.

On a geometric domain of 4 dimensions, assume that the evolutionary process generated by $\mathbf{T}_4$ starts from an initial condition (or state) where the Pfaff topological dimension of A is also 4. Depending on the sign of the divergence of $\mathbf{T}_4$, the process follows an irreversible path for which the divergence represents an expansion or a contraction. If the irreversible evolutionary path is attracted to a region (or state) where the Pfaff topological dimension of the 1-form of Action is 3, then $\mathbf{E} \circ \mathbf{B}$ becomes (or has decayed to) zero. The zero set of the function $\mathbf{E} \circ \mathbf{B}$ defines a hypersurface in the 4 dimensional space. If the process remains trapped on this hypersurface of Pfaff dimension 3, $\mathbf{E} \circ \mathbf{B}$ remains zero, and the $\mathbf{T}_4$ process becomes an extremal field. Such extremal fields are such that the virtual work 1-form vanishes, $W = i(\mathbf{T}_4)dA = 0$, and the now reversible $\mathbf{T}_4$ process has a Hamiltonian representation. The system is conservative in a Hamiltonian sense, but it is in a "excited" state on the hypersurface that is far from equilibrium, as the Pfaff dimension of the 1-form of Action is 3, and not 2. (If the path is attracted to a region where the function $\mathbf{E} \circ \mathbf{B}$ is oscillatory, the system evolutionary path defines a limit cycle.)

In effect, the topological defects represented by closed but not exact components of the 3-form $A\hat{}F$ can be generated by dissipative and irreversible processes leaving the topological defects as remnants in states far from equilibrium. A classic mechanical example is given by the skidding bowling ball (Pfaff dimension 2n+2) where the initial angular momentum and kinetic energy decay irreversibly until the state is reached where the rolling takes place without slipping (Pfaff dimension 2n+1). The subsequent evolution is reversible in a thermodynamic sense, but the state is far from the state of rest, the equilibrium state.

In summary, from a topological perspective, closed but not exact electrodynamic 3-forms can form topological defects or long-lived remnants and coherent structures, such as wakes and self organized condensates, in dissipative turbulent domains of Pfaff dimension 4.

3. The Topological Plasma

For electromagnetic systems, a particular interesting choice of specialized processes are those that leave the closed integrals (around cycles $z2$) of the 2 form of field excitations, G, a deformation (relative) integral invariant. Such processes βV which preserve the net *number* of charges, globally, are defined as elements of the category of plasma processes:

$$L_{\beta V}(\iint_{z2} G) = \iint_{z2} i(\beta V)dG + \iint_{z2} d(i(\beta V)G) = \iint_{z2} i(\beta V)J \Rightarrow 0. \tag{3.1}$$

The criteria for relative integral invariance with an arbitrary deformation parameter, β, implies that $i(\beta V)J = \beta\{(V^4\mathbf{J} - \rho\mathbf{V})^x dy\hat{}dz - ... + (\mathbf{J}\times\mathbf{V})^x dx\hat{}dt..\} \Rightarrow 0$. This constraint has expression in engineering language as,

$$\text{Plasma Processes:}\;\; \mathbf{J} = \rho\mathbf{V}, \qquad \mathbf{J}\times\mathbf{V} = 0. \tag{3.2}$$

A plasma process need not conserve energy. Again, the 3-forms, J, $A\hat{}G$ and $A\hat{}F$ are of particular interested for their tangent manifolds define "lines" in the 4-dimensional variety of space and time. Relative to plasma processes, the topological evolution associated with such lines, and their entanglements, is of utility in understanding solar corona and plasma instability.

The invariance principle that defines a plasma process on G should be compared to the Helmholtz process on F:

$$L_{\beta\mathbf{V}}(\iint_{z2} F) = \iint_{z2} i(\beta V)dF \Rightarrow 0. \tag{3.3}$$

The closed integral of electromagnetic flux is an intrinsic topological (deformation) invariant of an electromagnetic system, for the 2-form F is exact by construction (the postulate of potentials). In a plasma, for which the evolutionary processes are constrained such that $\mathbf{J} = \rho\mathbf{V}$, both the closed integrals of F and G are deformation invariants. In the sense, the plasma is a topological refinement of the complete Maxwell system.

The "perfect" plasma is defined as a process that is both a plasma process and a Hamiltonian extremal process. It follows that the virtual work 1-form must vanish. The plasma process will have the form $J = \rho[\mathbf{V}, 1]$ such that the Hamiltonian extremal criteria yields the "Work Free" equation:

$$W = i(J)dA = \rho(\mathbf{E} + \mathbf{V} \times \mathbf{B}) \circ d\mathbf{r} + \rho(\mathbf{V} \circ \mathbf{E})dt \Rightarrow 0 \tag{3.4}$$

The "perfect or ideal" plasma is therefor a "Force Free" plasma when the Lorentz force is zero. If the plasma is a "force free" plasma, then if follows that

$$\text{Force Free plasma process:} \quad \mathbf{V} \circ \mathbf{E} = \mathbf{0}, \quad \mathbf{E} \circ \mathbf{B} = \mathbf{0}, \quad \mathbf{J} \circ \mathbf{E} = 0. \tag{3.5}$$

Other plasma dynamics [16] belong to categories that depend upon the Pfaff dimension of the Work 1-form.

3.1. Extremal vs. Bernouilli plasmas

The development that follows is guided by Cartan's pioneering work, in which he examined those specialized processes for mechanical systems that leave the closed integrals of the 1-form of Action, A, a deformation (relative integral) invariant. Cartan proved that such processes always have a Hamiltonian representation. There are two classes of such Hamiltonian processes, the Extremal class and the Bernoulli class:

$$\text{Hamiltonian Processes} \tag{3.6}$$

$$L_{(\beta\mathbf{V})} \int_{z1} A = \int_{z1} i(\beta V)dA + di(\beta V)A \Rightarrow 0. \tag{3.7}$$

$$Extremal: \quad i(\beta\mathbf{V})dA = 0, \tag{3.8}$$

$$Bernoulli: \quad i(\beta\mathbf{V})dA = d\Theta. \tag{3.9}$$

The closed integration chain $z1$ is not necessarily a boundary.

An electromagnetic system has not only the primitive 1-form, A, but also the N-2 form, G, which can undergo evolutionary processes. For electromagnetic systems, a set of equations similar to those that define Hamiltonian processes can be used to define specialized processes that leave the closed integrals of the N-2=2 form, G, of field excitations, a deformation (relative integral) invariant. These special processes will be defined as Plasma processes. Such process do not create free charge, but they can cause a change in the number of charge pairs of opposite sign. The equations that must be satisfied are of the form:

$$\text{Plasma Processes} \tag{3.10}$$

$$L_{(\beta\mathbf{V})}\int_{z2} G = \int_{z2} i(\beta V)dG + di(\beta V)G \Rightarrow 0. \tag{3.11}$$

$$Extremal : i(\beta V)dG = 0, \tag{3.12}$$

$$Bernoulli : i(\beta V)dG = d\omega. \tag{3.13}$$

In the Extremal case,

$$i(\beta V)dG = \beta\{(\mathbf{J} - \rho\mathbf{V})^x dy\hat{}dz - ... + (\mathbf{J}\times\mathbf{V})^x dx\hat{}dt... \Rightarrow 0, \tag{3.14}$$

implies that the extremal Plasma process obeys the classic expressions:

$$\text{Extremal Plasma process } \mathbf{J}_E = \rho\mathbf{V}. \tag{3.15}$$

3.2. The Topological Hall effect

In the Bernoulli case of a Plasma process the integrand must be proportional to an exact 2-form, $d\omega$. There is one obvious candidate, the 2-form, F :

$$i(\beta V)dG = \beta\{(\mathbf{J} - \rho\mathbf{V})^x dy\hat{}dz - ... + (\mathbf{J}\times\mathbf{V})^x dx\hat{}dt... = \sigma_{Hall}F. \tag{3.16}$$

The conductivity coefficient σ_{Hall} in the expression must be a domain constant. Comparing the components of the equation of constraint yields the properties of a Bernoulli Plasma process:

$$\text{Bernoulli Plasma process,} \quad \mathbf{J}_B = \rho\mathbf{V} + (\sigma_{Hall}/\beta)\mathbf{B}, \tag{3.17}$$

$$\text{and,} \quad (\mathbf{J}_B \times \mathbf{V}) = (\sigma_{Hall}/\beta)\mathbf{E}. \tag{3.18}$$

$$(\sigma_{_{Hall}}/\beta)(\mathbf{J}_B \circ \mathbf{E}) \quad \Rightarrow \quad 0 \tag{3.19}$$
$$(\sigma_{_{Hall}}/\beta)(\mathbf{V} \circ \mathbf{E}) \quad \Rightarrow \quad 0 \tag{3.20}$$
$$(\sigma_{_{Hall}}/\beta)(\mathbf{E} \circ \mathbf{B}) \quad \Rightarrow \quad 0. \tag{3.21}$$

Thus the Bernoulli plasma process leads to a current $\mathbf{J}_B$ which is orthogonal to the $\mathbf{E}$ field and whose magnitude is proportional to the $\mathbf{B}$ field. To quote Landau and Lifshitz [17] "As we see, it (the Hall effect) gives rise to a current perpendicular to the electric field, whose magnitude is proportional to the magnetic field." The conclusion is that the Bernoulli Plasma process generates a Hall effect, and requires that the second Poincare coefficient must vanish. It follows that the Topological Hall effect exists in non-equilibrium systems where the 1-form A cannot be of Pfaff dimension 4. Bernoulli plasma processes are not dissipative in the sense that such that $(\mathbf{J}_B \circ \mathbf{E}) = 0$.

The appearance of a magnetic conductance, $\sigma_{_{Hall}}$, is novel to the topological format of electromagnetism as presented herein, and is deduced from the sole assumption that the Plasma current defines a process direction field that preserves the closed integrals of the 2-form, G. Plasma processes do not change the net charge within the closed integration domain. That is, charges can be produced only in equal and opposite pairs by a "Plasma process".

The conclusion is that the Hall effect is a topological property of electromagnetism, and can appear at all scales, from the microworld to the macroworld to the cosmological world. In the next section, and in domains where the non-equilibrium 3-forms of Topological Torsion and Topological Spin are closed, it is demonstrated how the Topological Hall effect can have an impedance multiplied by a rational fraction. That is, the rational fraction Hall impedance is a topological result, independent from quantum theory.

4. Topological Superconductors.

The objective is to define superconductivity in a topological manner which incorporates the "quantization" features of deRham cohomology theory. The idea follows from the recognition that the Hall impedance exhibits rational fraction behavior. This rational fraction behavior was predicted on topological grounds by E. J. Post [18]. The implication is that superconductivity is related to topological defect structures. There are three ways to construct an impedance Z (with physical dimensions h/e^2) from period integrals [19].

$$\begin{array}{rl} \text{Ordinary Superconductors :} & \text{Impedance } \mathsf{Z}_1 = \oint A / \iint_{z2} G \\ \text{Anyon High Tc ?} & \mathsf{Z}_2 = \iiint_{z3} A\hat{}G / (\iint_{z2} G)^2 \\ \text{Fractional Hall :} & \mathsf{Z}_3 = \iiint_{z3} A\hat{}F / \iiint_{z3} A\hat{}G \end{array} \qquad (4.1)$$

In order to produce rational fractions, the closed integrals must be period integrals, where the integrands are closed in an exterior differential sense over the closed domains (cycles of 1, 2 or 3 dimensions) of integration. The closure condition on the first impedance Z_1 requires that $dA \Rightarrow 0$, which implies that the domain excludes the field intensities (Meissner repulsion). The closure condition on the third impedance, Z_3 requires that both Poincare invariants must vanish, but **E** and **B** fields are permitted in the domain of integration (as is observed in the Hall effect).

The conjecture to be explored herein is that a supercurrent corresponds to the case where the electromagnetic interaction energy density vanishes in a topological sense. The motivation for such an assumption is founded upon the observation that if the 3-form of charge current density, J, was proportional to either the 3-form of topological torsion, $J = A\hat{}F$, or the 3-form of topological spin, $J = A\hat{}G$, then it follows that the interaction energy density of classical field theory will vanish, $A\hat{}J \Rightarrow 0$. Assume that a supercurrent contains components proportional to topological torsion 3-form and the topological spin 3-form. In order for the components of such a charge current 4-vector to be closed (and exact) the respective Poincare invariants of its components must be zero. Under such constraints, the closed integrals of the closed 3-forms of topological torsion and topological spin have rational (quantized) values, and become "deformable" topological invariants. Although the geometrical dimension of space time is 4, the constrained system has topological Pfaff dimension 3 and is not an equilibrium system. The evolutionary processes represented in terms of the divergence free forms of $\mathbf{T}_4$ and $\mathbf{S}_4$ are not irreversible. Such formulations, therefor, are possible candidates for non-dissipative supercurrents.

A third case would consider those situations where the 3-form charge current density has components proportional to those components of the 1-form of potentials which are elements of a spinor, $J = \lambda J_{spinor}$. The 3-form can always be multiplied by an integrating factor such that the rescaled spinor current has zero divergence. Similarly, suppose the 1-form A_{spinor} (to within a factor) also has the same spinor component functions. Then the interaction density vanishes, as

$$\text{Spinor London current:} \quad J_{spinor} = i(\lambda A_{spinor})\Omega \tag{4.2}$$
$$\text{Interaction Energy density:} \quad A\hat{\ }J = \lambda \langle A_{spinor} \circ A_{spinor} \rangle \ \Omega_4 \Rightarrow 0. \tag{4.3}$$

Hence, a charge current 3-form composed of three parts, such that

$$\text{Total Supercurrent } J_{\text{supercurrent}} = J_{spinor} + A\hat{\ }F/\lambda + A\hat{\ }G/\eta, \tag{4.4}$$
$$\text{With Interaction Energy density } A\hat{\ }J = A\hat{\ }J_{\text{supercurrent}} \Rightarrow 0, \tag{4.5}$$

is a candidate for a superconducting current, which intuitively has no interaction energy density.

5. Topological defects, Quarks to Galaxies

It was demonstrated in section 2.4.1 that period (circulation) integrals of the 1-form of electromagnetic Action potentials, A, lead to the concept of "vortex defect lines" . The idea is extendable to "twisted vortex defect lines" in three dimensions. The "twisted vortex defects" become the spiral vortices of a Complex Ginsburg Landau (CGL) theory , while the "untwisted vortex lines" become the defects of Ginzburg-Pitaevskii-Gross (GPG) theory [20].

Evidence of such topological defects (at the macroscopic level) can be demonstrated by the creation of Falaco Solitons in a swimming pool [21]. These experiments demonstrate that such topological defects are available at all scales. The Falaco Solitons consist of spiral "vortex defect" structures (analogous to CGL theory) on a two dimensional minimal surface, one at each end of a 1-dimensional "vortex line" or thread (analogous to GPG theory). Remarkably the topological defect surface structure is locally unstable, as the surface is of negative Gauss curvature. Yet the pair of locally unstable 2-D surfaces is *globally* stabilized by the 1-D line defect attached to the "vertex" points of the minimal surfaces.

The rotational minimal surfaces of negative Gauss curvature which form the two endcaps, like quarks, apparently are confined by the string, but if the string (whose "tension" induces global stability of the unstable endcaps) is severed, the endcaps (like quarks in the elementary particle domain) disappear (in a non-diffusive manner). In the microscopic electromagnetic domain, the Falaco soliton structure offers an alternate, topological, pairing mechanism that can be compared to the Cooper pairing in superconductors. In the macroscopic domain, the experiments visually indicate "almost flat" spiral arm structures during the formative stages of the Falaco solitons. In the cosmological domain, could these universal topological defects represent the ubiquitous "almost flat" spiral arm galaxies? Could M31 and the Milky way be connected by a topological defect thread? Take note of the recent Hubble photo entitled "Intergalactic 'Pipeline' Funnels Matter Between Colliding Galaxies" and posted at http://hubblesite.org/newscenter/archive/2001/02/.

6. Maxwell Topological Defects and Black Holes.

A typical problem in classical electromagnetism starts with an assumed distribution of conserved charges and currents, J, which are used to deduce a set of field excitations, $G(D, H)$. Then a constitutive constraint is imposed to determine the field intensities, $F(E, B)$. The excitations are not uniquely determined, and the potentials, A, are ignored !

There exists a little used "reverse" procedure that will produce a globally closed charge current density, starting from a set of potentials, but without the explicit use constitutive equations. The concept is similar to the London formulation of superconductivity (which uses the conjecture that $\mathbf{J} \sim \mathbf{A}$), but the general procedure does not depend upon a quantum argument. Starting from an arbitrary set of C1 differentiable components for a 1-form of Action potentials, A, it is possible to construct a globally closed 3-form density, J_s, by algebraic procedures. The method is an extension of the theory of the differential geometry for implicit surfaces to those surfaces where a normal field is given, but for which the normal field is not a perfect exact differential [22]. In essence, a shape matrix can be defined as the Jacobian matrix of the components of the given 1-form of Action, divided by a Holder norm as a scaling factor, $\lambda = \{\sum \pm (A_k)^p\}^{n/p}$ with homogeneous index n. The shape matrix is defined for the choice n=1, p=2, and a euclidean signature.

$$\text{Shape matrix:} \quad [\mathcal{J}] = \left[\partial(A_k/\lambda)/\partial x^j\right]. \tag{6.1}$$

The resulting shape matrix has determinant zero, hence defines a hypersurface (defect) in 4 space. The similarity invariants of the shape matrix are related to the curvatures of the hypersurface, and are determined from the Cayley-Hamilton characteristic polynomial for the shape matrix. This procedure is precisely the procedure used to describe the differential geometry of implicit surfaces, where the normal field to the surface is a gradient of the function whose zero set defines the implicit surface. The general procedure admits investigation of surfaces whose normal field is not a global gradient.

The linear Mean curvature of the hypersurface is related to the sum of the eigenvalues (curvatures) and is easily determined from the trace of the shape matrix. The quadratic Gauss curvature is related to the sum of the permuted product pairs of the curvatures. The cubic Adjoint curvature is related to the permuted product triples of the curvatures. The Adjoint curvature is easily computed as the trace of the Adjoint shape matrix. The Adjoint shape matrix, $\left[\widehat{\mathcal{J}}\right]$, is defined as the matrix of cofactors (of the shape matrix) transposed. The Adjoint shape matrix is well defined even though the shape matrix so constructed does not have an inverse.

If one constructs the contravariant tensor density

$$\mathbf{Z}_4 = \left[\widehat{\mathcal{J}}\right] \circ |A_k/\lambda\rangle \tag{6.2}$$

by multiplying the components of the rescaled 1-form of Action by the Adjoint shape matrix, the remarkable result is that the 3-form $J_s = i(\mathbf{Z}_4)\Omega_4$ is globally closed: $dJ_s = 0$. Hence, from the potentials, a candidate has been constructed (without the use of a constitutive constraint or some other assumption) that could play the role of an electromagnetic charge-current density.

Even more remarkably, the topological component defined as the interaction energy density, when constructed with this current and potential becomes proportional to the cubic Adjoint curvature of the shape matrix:

$$\text{Interaction energy density: } A\hat{}J_s = \{trace\left[\widehat{\mathcal{J}}\right]\}\ \Omega_4. \tag{6.3}$$

In other words the interaction electromagnetic energy so computed is cubic in the hypersurface curvatures.

If one considers a collapsing system, then the geometric curvatures increase with smaller scales. If Gauss quadratic curvature is to be related to gravitational collapse of matter, then at some level of smaller scales a term cubic in curvatures would dominate. It is conjectured that the cubic curvature produced by the electromagnetic effect described above could prevent the collapse to a black hole. Cosmologists and relativists apparently have ignored such cubic curvature effects.

7. Conclusions

A topological perspective of electromagnetism demonstrates that the electromagnetic theory of Maxwell is a universal theory that goes beyond the usual presentations which impose geometric constraints of metric, connection, constitutive relations or constraints of equilibrium thermodynamics. In fact, the topological perspective leads to "quantum-like" coherent structures and topological defects, such as the "flux quantum" and the "charge quantum", without the imposition of a Copenhagen version of quantum mechanics. Moreover, the topological perspective points out that these topological coherent structures are not concepts restricted to the microphysics domain, but - as topological concepts independent from scales - are to be found at macroscopic and cosmological scales.

The topological perspective demonstrates the classical electromagnetism can be applied to non-equilibrium thermodynamic systems, and leads to the discovery of two more "quantum-like" coherent structures, which are to be found only in non-equilibrium thermodynamic systems. These new objects are the Topological Spin, $A\hat{}G$, and the Topological Torsion, $A\hat{}F$, which - in domains where the Poincare invariants vanish - have closed integrals that are deformation invariants and have rational ratios (often associated with microscopic quantum phenomena). These topological coherent structures of Pfaff topological dimension 3 can form long lived states far from equilibrium as topological defects in non-equilibrium turbulent domains of Pfaff topological dimension 4. The practical utility of these new topological electromagnetic properties has just begun.

References

[1] For more detail and many examples see http://www.cartan.pair.com

[2] R. M. Kiehn, "Topological evolution of classical electromagnetic fields and the photon" in V. Dvoeglazov (ed.), *Photon and Poincaré Group,* (Nova Science Publishers, Inc., Commack, New York, 1999) p.246-262. ISBN 1-56072-718-7 (http://www22.pair.com/csdc/pdf/photon.pdf)

[3] R. M. Kiehn, "Retrodictive Determinism", Int. J. of Eng. Sci. **14**, 749 (1976). (http://www22.pair.com/csdc/pdf/retrodic.pdf)

[4] G. deRham, *Varietes Differentiables*, (Hermann, Paris, 1960)

[5] R. M. Kiehn, "Periods on manifolds, quantization and gauge." J. Math Phy. **18**, 614 (1977), (http://www22.pair.com/csdc/pdf/periods.pdf)

[6] V. Fock, *Space, time and gravitation* (Pergamon-MacMillan, New York, 1964)

[7] E. Cartan, *The Theory of Spinors* (Dover, New York, 1966) p. 39

[8] R. M. Kiehn, G. P. Kiehn, and B. Roberds, “Parity and time-reversal symmetry breaking, singular solutions and Fresnel surfaces", Phys. Rev A **43**, 5165-5671 (1991). (http://www22.pair.com/csdc/pdf/timerev.pdf)

[9] V. E. Sanders and R. M. Kiehn IEEE J Quant Elec QE-13, 739 (1977) "Dual Polarized Ring Lasers"

[10] S. Lipschutz, *General Topology* (Schaum, New York 1965), 88.

[11] R. M. Kiehn, “Topological Torsion, Pfaff Dimension and Coherent Structures“, in: H. K. Moffatt and T. S. Tsinober (eds.), *Topological Fluid Mechanics*, (Cambridge University Press, 1990) 449-458. (http://www22.pair.com/csdc/pdf/camb89.pdf)

[12] R. M. Kiehn, (1991) “Continuous Topological Evolution“, LANL arXiv math-ph /0101032 (http://www22.pair.com/csdc/pdf/contevol3.pdf)

[13] R. M. Kiehn, ”Thermodynamic Irreversibility and the Arrow of Time” in R. Bucher et.al. (eds.), *The Nature of Time: Geometry, Physics and Perception*, (Kluwer Academic Publishers, Netherlands, 2003) 243-250. (http://www22.pair.com/csdc/pdf/arwfinal.pdf)

[14] P. Bamberg and S. Sternberg, *A course in mathematics for students of physics, 2* (Cambridge University press, Cambridge, 1992) p. 775- 780

[15] P. M. Morse, *Thermal Physics* (Benjamin, NY, 1964), p. 60, .

[16] R. M. Kiehn, (2001) "Topological-Torsion and Topological-Spin as coherent structures in plasmas", arXiv physics /0102001, http://www22.pair.com/csdc/pdf/plasmas.pdf, and ..holder3d.pdf)

[17] Landau, L. D. and Lifshitz, E. M., "Electrodynamics of Continuous Media" (Pergamon Press, London 1960) p. 97.

[18] E.J. Post, Phys Lett. A94 343 (1983)

[19] R. M. Kiehn, " Are there three kinds of superconductivity", Int. Journ. of Modern Physics, **5**, 10 p. 1779 (1991)

[20] O. Tornkvist and E. Schroeder, "Vortex dynamics in Dissipative Systems", Phys Rev. Lett. **78**, 10 , p. 1908-1911 (1997).

[21] R. M. Kiehn, (1987) The Falaco Effect as a topological defect was first noticed by the present author in the swimming pool of an old MIT friend, during a visit in Rio de Janeiro, at the time of Halley's comet, March 1986. The concept was presented at the Austin Meeting of Dynamic Days in Austin, January 1987, and caused some interest among the resident topologists. The easily reproduced experiment added to the credence of topological defects in fluids. It is now perceived that this topological phenomena is universal, and will appear at all levels from the microscopic to the galactic. LANL arXiv/gr-qc/0101098 (http://www22.pair.com/csdc/pdf/falaco97.pdf)

[22] R. M. Kiehn, "Curvature and torsion of implicit hypersurfaces and the origin of charge", Annales de la Foundation Louis de Broglie, vol **27**, p.411 (2002). LANL arXiv /gr-qc /0101109 (http://www22.pair.com/csdc/pdf/contevol3.pdf)