```
> restart:
```

The rolling ball with friction.

```
>  with(liesymm):with(linalg):with(plots):with(difforms):
>
Warning, new definition for close
Warning, new definition for norm
Warning, new definition for trace
Warning, new definition for `&^`
Warning, new definition for d
Warning, new definition for mixpar
Warning, new definition for wdegree
```

```
> setup(x,theta,t,v,omega,s,px,po,pt);
```

The independent variables are on a 6 dimensional space

$$[x, \theta, t, v, \omega, s, px, po, pt]$$

```
>
> defform(theta=0,omega=0,x=0,v=0,s=0,t=0,px=0,po=0,pt=0,a=const,b=const,c=const,k=const,mu=const,m=const,beta=const,lambda=const):
> Lagrange:=(1/2*beta*m*lambda^2*omega^2+1/2*m*v^2);
```

$$Lagrange := \frac{1}{2}\beta\, m\, \lambda^2\, \omega^2 + \frac{1}{2} m\, v^2$$

Note that the Lagrangian is defined as the sum of the rotational energy and the translational energy. !!!!

```
> Ham:=diff(Lagrange,omega)*omega+diff(Lagrange,v)*v-Lagrange;
```

$$Ham := \frac{1}{2}\beta\, m\, \lambda^2\, \omega^2 + \frac{1}{2} m\, v^2$$

Momenta defined canonically,

```
> Action:=(wcollect((diff(Lagrange,omega)*d(theta)+diff(Lagrange,v)*d(x)-Ham*d(t)+m*s*(lambda*d(theta) -d(x)))));
```

The Action is defined in terms of the rotational and potential energy. The momenta are presumed to be canonical Ptheta=beta*m*lambda^2*omega, Px=m*v. The constraint of no slip is lambda*d(theta)-d(x)) and its Lagrange multiplier is s. The Lagrange function is defined as L=1/2*beta*m*lambda^2*omega^2+1/2*m*v^2)

$$Action := (m\, v - m\, s)\, d(x) + (\beta\, m\, \lambda^2\, \omega + m\, s\, \lambda)\, d(\theta) + \left(-\frac{1}{2}\beta\, m\, \lambda^2\, \omega^2 - \frac{1}{2} m\, v^2\right) d(t)$$

```
> pxx:=m*(v-s);poo:=(beta*m*lambda^2*omega+m*s*lambda);
```

$$pxx := m\,(v - s)$$

$$poo := \beta\, m\, \lambda^2\, \omega + m\, s\, \lambda$$

```
> v1:=solve(px-pxx,v);v2:=solve(poo-po,omega);
>
```

$$v1 := \frac{px + m\, s}{m}$$

$$v2 := -\frac{m\, s\, \lambda - po}{\beta\, m\, \lambda^2}$$

```
> HH:=-factor(-1/2*beta*m*lambda^2*v2^2-1/2*m*v1^2);
```

$$HH := \frac{1}{2}\frac{m^2 s^2 \lambda^2 - 2\, m\, s\, \lambda\, po + po^2 + \beta\, \lambda^2\, px^2 + 2\, \beta\, \lambda^2\, px\, m\, s + \beta\, \lambda^2\, m^2\, s^2}{\beta\, m\, \lambda^2}$$

```
> factor(diff(HH,px));factor(diff(HH,po));factor(diff(HH,s));
>
>
```

$$\frac{px + m\, s}{m}$$

$$-\frac{m\, s\, \lambda - po}{\beta\, m\, \lambda^2}$$

$$\frac{m\, s\, \lambda - po + \beta\, \lambda\, px + \beta\, \lambda\, m\, s}{\lambda\, \beta}$$

```
>
> DA:=d(Action);
```

$$DA := m\, ((\mathrm{d}(v))\ \&\wedge\ (\mathrm{d}(x))) - m\, ((\mathrm{d}(s))\ \&\wedge\ (\mathrm{d}(x))) + \beta\, m\, \lambda^2\, ((\mathrm{d}(\omega))\ \&\wedge\ (\mathrm{d}(\theta))) + m\, \lambda\, ((\mathrm{d}(s))\ \&\wedge\ (\mathrm{d}(\theta))) - \beta\, m\, \lambda^2\, \omega\, ((\mathrm{d}(\omega))\ \&\wedge\ (\mathrm{d}(t))) - m\, v\, ((\mathrm{d}(v))\ \&\wedge\ (\mathrm{d}(t)))$$

```
>
```

The two form is of rank 3 hence when the no slip condition fails. Hence the two form defines a non-compact symplectic space.

```
> Hel:=wcollect(Action&^DA);
```

$$\begin{aligned} Hel :=\ & (m^2\, \lambda\, v + \beta\, m^2\, \lambda^2\, \omega)\ \&\wedge(\mathrm{d}(x), \mathrm{d}(s), \mathrm{d}(\theta)) - m^2\, (-v + s)\, \beta\, \lambda^2\ \&\wedge(\mathrm{d}(x), \mathrm{d}(\omega), \mathrm{d}(\theta)) \\ & + \left(-\frac{1}{2}\, m^2\, v^2 + m^2\, v\, s + \frac{1}{2}\, m^2\, \beta\, \lambda^2\, \omega^2\right) \&\wedge(\mathrm{d}(x), \mathrm{d}(v), \mathrm{d}(t)) + \frac{1}{2}\, m^2\, (\beta\, \lambda^2\, \omega^2 + v^2)\ \&\wedge(\mathrm{d}(t), \mathrm{d}(s), \mathrm{d}(x)) \\ & - \frac{1}{2}\, m^2\, (\beta\, \lambda^2\, \omega^2 + v^2)\, \lambda\ \&\wedge(\mathrm{d}(t), \mathrm{d}(s), \mathrm{d}(\theta)) - m^2\, \lambda\, (\beta\, \lambda\, \omega + s)\, v\ \&\wedge(\mathrm{d}(\theta), \mathrm{d}(v), \mathrm{d}(t)) \\ & + \left(-\frac{1}{2}\, \beta^2\, m^2\, \lambda^4\, \omega^2 - \beta\, m^2\, \lambda^3\, \omega\, s + \frac{1}{2}\, \beta\, \lambda^2\, m^2\, v^2\right) \&\wedge(\mathrm{d}(\theta), \mathrm{d}(\omega), \mathrm{d}(t)) + m^2\, (-v + s)\, \beta\, \lambda^2\, \omega\ \&\wedge(\mathrm{d}(x), \mathrm{d}(\omega), \mathrm{d}(t)) \\ & + m^2\, \lambda\, (\beta\, \lambda\, \omega + s)\ \&\wedge(\mathrm{d}(\theta), \mathrm{d}(v), \mathrm{d}(x)) \end{aligned}$$

The Helicity 3-form does not vanish

```
> DADA:=wcollect(simpform(DA&^DA));
```

$$\begin{aligned} DADA :=\ & 2\, m^2\, \beta\, \lambda^2\ \&\wedge(\mathrm{d}(v), \mathrm{d}(x), \mathrm{d}(\omega), \mathrm{d}(\theta)) + 2\, m^2\, v\ \&\wedge(\mathrm{d}(s), \mathrm{d}(x), \mathrm{d}(v), \mathrm{d}(t)) \\ & + 2\, m^2\, \lambda\ \&\wedge(\mathrm{d}(v), \mathrm{d}(x), \mathrm{d}(s), \mathrm{d}(\theta)) - 2\, \beta\, m^2\, \lambda^2\, v\ \&\wedge(\mathrm{d}(\omega), \mathrm{d}(\theta), \mathrm{d}(v), \mathrm{d}(t)) \\ & - 2\, m^2\, \beta\, \lambda^2\, \omega\ \&\wedge(\mathrm{d}(v), \mathrm{d}(x), \mathrm{d}(\omega), \mathrm{d}(t)) - 2\, m^2\, \lambda^3\, \beta\, \omega\ \&\wedge(\mathrm{d}(s), \mathrm{d}(\theta), \mathrm{d}(\omega), \mathrm{d}(t)) \\ & - 2\, m^2\, \lambda\, v\ \&\wedge(\mathrm{d}(s), \mathrm{d}(\theta), \mathrm{d}(v), \mathrm{d}(t)) - 2\, m^2\, \beta\, \lambda^2\ \&\wedge(\mathrm{d}(s), \mathrm{d}(x), \mathrm{d}(\omega), \mathrm{d}(\theta)) \\ & + 2\, m^2\, \beta\, \lambda^2\, \omega\ \&\wedge(\mathrm{d}(s), \mathrm{d}(x), \mathrm{d}(\omega), \mathrm{d}(t)) \end{aligned}$$

```
> DADADA:=wcollect(DA&^DADA);
```

$$DADADA := (-6\, m^3\, \lambda^3\, \beta\, \omega + 6\, m^3\, \beta\, \lambda^2\, v)\ \&\wedge(\mathrm{d}(v), \mathrm{d}(x), \mathrm{d}(s), \mathrm{d}(\theta), \mathrm{d}(\omega), \mathrm{d}(t))$$

```
> scale:=factor(scalarpart(DADADA));
```

$$scale := 6\, m^3\, \lambda^2\, \beta\, (-\lambda\, \omega + v)$$

```
> DADA&^Action;
```

$$\begin{aligned} & (-2\, m^3\, \lambda^3\, \beta\, \omega\, v - m^3\, \beta^2\, \lambda^4\, \omega^2 + m^3\, \beta\, \lambda^2\, v^2)\ \&\wedge(\mathrm{d}(s), \mathrm{d}(\theta), \mathrm{d}(\omega), \mathrm{d}(t), \mathrm{d}(x)) \\ & + (m^3\, \beta^2\, \lambda^4\, \omega^2 + m^3\, \beta\, \lambda^2\, v^2 - 2\, \beta\, m^3\, \lambda^2\, v\, s + 2\, m^3\, \lambda^3\, \beta\, \omega\, s)\ \&\wedge(\mathrm{d}(v), \mathrm{d}(x), \mathrm{d}(\omega), \mathrm{d}(\theta), \mathrm{d}(t)) \\ & + (2\, m^3\, v\, \beta\, \lambda^2\, \omega - m^3\, \lambda^3\, \beta\, \omega^2 + m^3\, \lambda\, v^2)\ \&\wedge(\mathrm{d}(s), \mathrm{d}(x), \mathrm{d}(v), \mathrm{d}(t), \mathrm{d}(\theta)) \end{aligned}$$

```
>
```

```
>
> Torsion:=wcollect(factor(simplify(Hel&^DA)));
```

$$Torsion := m^3\,\lambda\,(2\,\lambda\,\beta\,v\,s - 2\,\lambda^2\,\beta\,\omega\,s - \beta^2\,\omega^2\,\lambda^3 - \beta\,v^2\,\lambda)\ \&\text{^}(\mathrm{d}(x), \mathrm{d}(\omega), \mathrm{d}(\theta), \mathrm{d}(v), \mathrm{d}(t))$$
$$+ m^3\,\lambda\,(-\beta^2\,\omega^2\,\lambda^3 - 2\,v\,\beta\,\lambda^2\,\omega + \beta\,v^2\,\lambda)\ \&\text{^}(\mathrm{d}(x), \mathrm{d}(s), \mathrm{d}(\theta), \mathrm{d}(\omega), \mathrm{d}(t))$$
$$+ m^3\,\lambda\,(-v^2 - 2\,\lambda\,v\,\beta\,\omega + \beta\,\lambda^2\,\omega^2)\ \&\text{^}(\mathrm{d}(x), \mathrm{d}(s), \mathrm{d}(\theta), \mathrm{d}(v), \mathrm{d}(t))$$

```
> Tv:=factor(scalarpart(Torsion&^d(v)));
```

$$Tv := m^3\,\lambda^2\,\beta\,(-\beta\,\lambda^2\,\omega^2 - 2\,v\,\lambda\,\omega + v^2)$$

```
> To:=-factor(scalarpart(Torsion&^d(omega)));
```

$$To := m^3\,\lambda\,(v^2 + 2\,\lambda\,v\,\beta\,\omega - \beta\,\lambda^2\,\omega^2)$$

```
> Ts:=factor(scalarpart(Torsion&^d(s)));
```

$$Ts := m^3\,\lambda^2\,\beta\,(2\,v\,s - 2\,\lambda\,\omega\,s - \beta\,\lambda^2\,\omega^2 - v^2)$$

```
> P:=factor(Tv+lambda*To)/(lambda^2*m^3);M:=factor(-Tv+lambda*To)/(lambda^2*m^3);
```

$$P := (\beta + 1)\,(-\beta\,\lambda^2\,\omega^2 + v^2)$$
$$M := 4\,\lambda\,v\,\beta\,\omega + \lambda^2\,\beta^2\,\omega^2 - \beta\,v^2 + v^2 - \beta\,\lambda^2\,\omega^2$$

```
>
```

The Torsion 5-form has 3 non-zero components on the six dimensional space and 3 zero components.. The non-zero components are orthogonal to the spatial coordinates {x,theta,t} in the 6D space{x,theta,t,v,omega,s}

```
> VOL:=wcollect(factor(DA&^DA&^DA));
```

$$VOL := 6\,m^3\,\lambda^2\,\beta\,(-\lambda\,\omega + v)\ \&\text{^}(\mathrm{d}(s), \mathrm{d}(x), \mathrm{d}(v), \mathrm{d}(t), \mathrm{d}(\omega), \mathrm{d}(\theta))$$

```
> DA;
```

$$m\,((\mathrm{d}(v))\ \&\text{^}\ (\mathrm{d}(x))) - m\,((\mathrm{d}(s))\ \&\text{^}\ (\mathrm{d}(x))) + \beta\,m\,\lambda^2\,((\mathrm{d}(\omega))\ \&\text{^}\ (\mathrm{d}(\theta))) + m\,\lambda\,((\mathrm{d}(s))\ \&\text{^}\ (\mathrm{d}(\theta)))$$
$$- \beta\,m\,\lambda^2\,\omega\,((\mathrm{d}(\omega))\ \&\text{^}\ (\mathrm{d}(t))) - m\,v\,((\mathrm{d}(v))\ \&\text{^}\ (\mathrm{d}(t)))$$

```
> Work:=wcollect(m*Tvv*(d(x)-v*d(t))-m*Tss*(d(x)-lambda*d(theta))+beta*m*lambda^2*
  Too*(d(theta)-omega*d(t)));
```

$$Work := (m\,Tvv - m\,Tss)\,\mathrm{d}(x) + (m\,Tss\,\lambda + \beta\,m\,\lambda^2\,Too)\,\mathrm{d}(\theta) + (-m\,Tvv\,v - \beta\,m\,\lambda^2\,Too\,\omega)\,\mathrm{d}(t)$$

The work 1-form in the direction of the Torsion vector is equal to the heat 1-form (no internal energy change!) fx is the friction force? fo is the dissipative torque. ft is the dissipative power. For force DEFINED AS the time rate of change of momentum, then the decay of the momenta are exponentially proportional to the scale factor.

```
> fx:=factor(m*Tv-m*Ts);fo:=factor(m*Ts*lambda+beta*m*lambda^2*To);ft:=factor((-m*
  Tv*v-beta*m*lambda^2*To*omega));Action;AA:=(wcollect(fx*d(x)+fo*d(theta)+ft*d(t)
  ));AAA:=wcollect(factor(3*AA/scale));scale;
> solve(fx,v);solve(fo,v);
```

$$fx := -2\,m^4\,\lambda^2\,\beta\,(-\lambda\,\omega + v)\,(-v + s)$$
$$fo := 2\,m^4\,\lambda^3\,\beta\,(-\lambda\,\omega + v)\,(\beta\,\lambda\,\omega + s)$$
$$ft := -m^4\,\lambda^2\,\beta\,(-\lambda\,\omega + v)\,(\beta\,\lambda^2\,\omega^2 + v^2)$$
$$(m\,v - m\,s)\,\mathrm{d}(x) + (\beta\,m\,\lambda^2\,\omega + m\,s\,\lambda)\,\mathrm{d}(\theta) + \left(-\frac{1}{2}\,\beta\,m\,\lambda^2\,\omega^2 - \frac{1}{2}\,m\,v^2\right)\mathrm{d}(t)$$
$$AA := -2\,m^4\,\lambda^2\,\beta\,(-\lambda\,\omega + v)\,(-v + s)\,\mathrm{d}(x) + 2\,m^4\,\lambda^3\,\beta\,(-\lambda\,\omega + v)\,(\beta\,\lambda\,\omega + s)\,\mathrm{d}(\theta)$$
$$- m^4\,\lambda^2\,\beta\,(-\lambda\,\omega + v)\,(\beta\,\lambda^2\,\omega^2 + v^2)\,\mathrm{d}(t)$$

$$AAA := \frac{1}{2}(2\,v - 2\,s)\,m\,\mathrm{d}(x) + \frac{1}{2}(2\,\lambda^2\,\beta\,\omega + 2\,\lambda\,s)\,m\,\mathrm{d}(\theta) + \frac{1}{2}(-v^2 - \beta\,\lambda^2\,\omega^2)\,m\,\mathrm{d}(t)$$

$$6\,m^3\,\lambda^2\,\beta\,(-\lambda\,\omega + v)$$

$$\lambda\,\omega,\ s$$

$$\lambda\,\omega$$

```
> To;Tv;
```

$$m^3\,\lambda\,(v^2 + 2\,\lambda\,v\,\beta\,\omega - \beta\,\lambda^2\,\omega^2)$$

$$m^3\,\lambda^2\,\beta\,(-\beta\,\lambda^2\,\omega^2 - 2\,v\,\lambda\,\omega + v^2)$$

```
> m:=1;lambda:=1.5;beta:=2/5;
```

$$m := 1$$

$$\lambda := 1.5$$

$$\beta := \frac{2}{5}$$

```
>
```

CASE 1

```
> syspp :=  diff(v(t),t)=Tv,diff(omega(t),t)=To:   fcns := {omega(t), v(t)};
  p:=dsolve({syspp,omega(0)=1.0,v(0)=0},fcns,type=numeric):
```

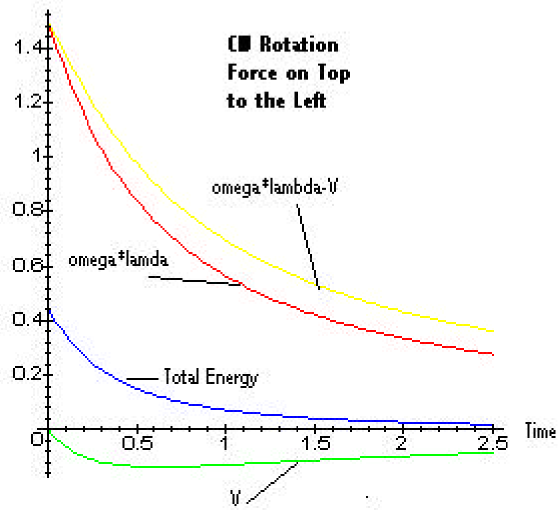
$$fcns := \{\mathrm{v}(t),\ \omega(t)\}$$

Now solve the ODE,s for the Torsion vector and make a plot of the Torsion vector dynamics. What is remarkable is that the model depends strongly upon initial conditions and the sense of positive translation relative positive rotation.

Suppose the ball is initially rotating clockwise (overspin) without translation. Clockwise is assumed positive rotation, positive translation to the right.

The Plot below of the Torsion vector dynamics indicates an interaction "force" must be acting on the TOP of the ball, trying to slow down the rotation, but at the same time accelerating the center of mass of the system to the left. The Total "energy" (rotational and translational) about CM ultimately decays to zero. The System never achieves rolling without slip. The plot yields an interpretation in terms of an "interaction or friction-like force", which points to the left and acts on the top of the ball.

```
> odeplot(p,
  [[t,lambda*omega(t)],[t,v(t)],[t,lambda*omega(t)-v(t)],[t,(m*lambda^2*beta*omega
  (t)^2+m*v(t)^2)/2]],0..2.5,numpoints=100);
```



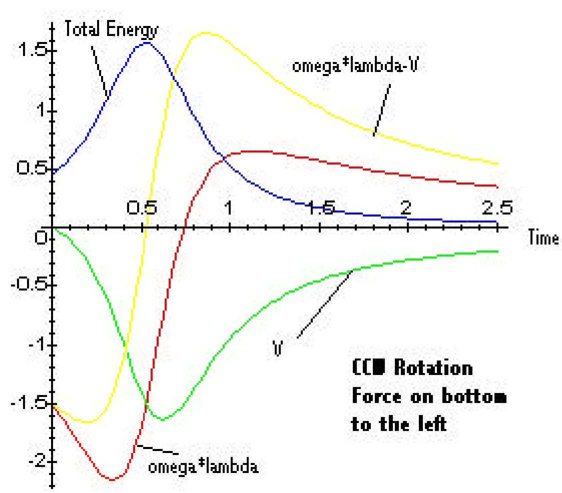
CASE 2:

```
> syspp :=  diff(v(t),t)=Tv,diff(omega(t),t)=To:   fcns := {omega(t), v(t)};
  p:=dsolve({syspp,omega(0)=-1.0,v(0)=0},fcns,type=numeric):
```

$$fcns := \{v(t), \omega(t)\}$$

In this case:.
The ball is initially rotating counter-clockwise without translation. Clockwise is assumed positive rotation, positive translation is to the right.
The Plot below yielda an interpretation in terms of an interaction "force" pointing to the left, but acting on the bottom of the ball, trying to slow down the rotation, but accelerating the center of mass of the system to the left. This circumstance can achieve rolling without slip. The system models what appears to happen when a spinning ball with underspin is placed on a friction plane in the presence of gravity..

```
> odeplot(p,
  [[t,lambda*omega(t)],[t,v(t)],[t,lambda*omega(t)-v(t)],[t,(m*lambda^2*beta*omega
  (t)^2+m*v(t)^2)/2]],0..2.5,numpoints=100);
```



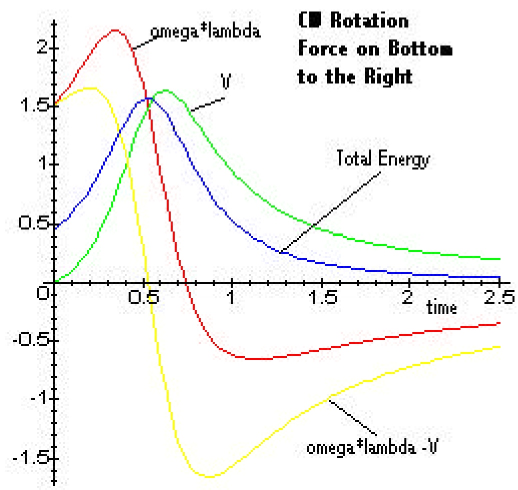
CASE 3

If the components of the Torsion vector are multiplied by minus 1, and the initial condition is no translation but overspin (clockwise rotation) then the dyanamic system models a sliding ball with an "interaction force" acting to the right on the bottom of the ball.
In this case the ball translates to the right, and at t = 0.5+ the no slip condition is reached. A "phase change" occurs. From t=0 to the critical time, the total energy about the CM increases. After the "phase change" takes place, a new Hamiltonian solution is the unique vector field on a space of 2n+1. The ball continues to rotate and translate without further change in the "kinetic energy".
Strange as it may seem, it is this situation that corresponds most closely to the oberservation of what happens when a overspinning bowling bowl is placed on a friction surface in the presence of gravity.

```
> syspp :=  diff(v(t),t)=-Tv,diff(omega(t),t)=-To:   fcns := {omega(t), v(t)};
  p:=dsolve({syspp,omega(0)=1.0,v(0)=0},fcns,type=numeric):
```

$$fcns := \{v(t), \omega(t)\}$$

```
> odeplot(p,
  [[t,lambda*omega(t)],[t,v(t)],[t,lambda*omega(t)-v(t)],[t,(m*lambda^2*beta*omega
  (t)^2+m*v(t)^2)/2]],0..2.5,numpoints=100);
>
```

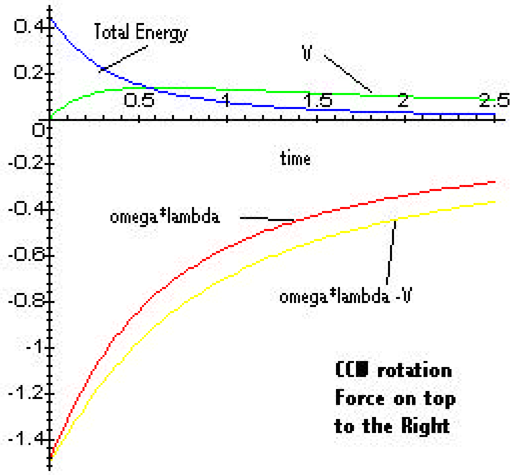


CASE 4
The last case corresponds to the initial condition of underspin, with an interaction force acting the top of the ball and pointing to the right. The rotational motion is decreased, the system is accelerated to the right, the total energy decays monatonically.

```
> syspp :=  diff(v(t),t)=-Tv,diff(omega(t),t)=-To:   fcns := {omega(t), v(t)};
  p:=dsolve({syspp,omega(0)=-1.0,v(0)=0},fcns,type=numeric):

> odeplot(p,
  [[t,lambda*omega(t)],[t,v(t)],[t,lambda*omega(t)-v(t)],[t,(m*lambda^2*beta*omega
  (t)^2+m*v(t)^2)/2]],0..2.5,numpoints=100);
```

$$fcns := \{v(t), \omega(t)\}$$




```
>
>
>
>
>
```

The above 6-form is the volume element of the 6 dimensional space. Note that the volume vanishes at the no slip condition. This 6-dimensional space is the top Pfaffian. It is NOT compact without boundary. The Gamma factor is 1/3 !

```
> AT:=factor(Hel&^DADA);
```

As this 7 form vanishes, it is proved that the Pfaff dimension of the Action 1-form is 6, with the coordinates {x,theta,t,v,omega,s}
Note that the 6-form vanishes when the noslip condition prevails. The space then reduces to a contact manifold of 5 dimensions. However, initial conditions do not necessarily satisfy the no slip constraint. Therefore the system decays (irreversibly) until the no slip condition is satisfied. Thereafter the dynamics is that of a Hamiltonian system with no irreversible dissipation.

$$AT := 0$$

```
>
> AVEC:=[Px-s,Ptheta+s*lambda,-Ham,0,0,0];
```

$$AVEC := \left[Px - s,\ Ptheta + 1.5\, s,\ -.4500000000\, \omega^2 - \frac{1}{2} v^2,\ 0,\ 0,\ 0 \right]$$

```
> TORS:=[0,0,0,Tv,To,Ts];
```

$$TORS := [0,\ 0,\ 0,\ .8100000000\, \omega^2 + 2.700000000\, v\, \omega - .9000000000\, v^2,$$
$$-1.5\, v^2 - 1.800000000\, v\, \omega + 1.350000000\, \omega^2,$$
$$-.9000000000\, v^2 + 1.800000000\, v\, s - .8100000000\, \omega^2 - 2.700000000\, \omega\, s]$$

```
> test:=innerprod(TORS,AVEC);
```

$$test := 0$$

```
>
>
```

The Work 1-form relative to the unique Torsion vector is not integrable until the no-slip condition is reached. Hence the Heat 1-form Q is not integrable. Therefor a process defined by the unique Torsion vector is irreversible, until the no slip condition is obtained.

```
>  d(v)+lambda*beta*d(omega)=0;
```

$$d(v) + .6000000000\ d(\omega) = 0$$

The first two equations admit an immediate integral during the dissipative motion along the torsion vector.

```
> v+lambda*beta*omega = (1+beta)*Vnoslip;
```

$$v + .6000000000\ \omega = \frac{7}{5}\, Vnoslip$$

```
>
```

The no-slip condition states that v=lambda*omega, so the constant of integration is 1+beta)