```
> restart:
```

The rolling ball with friction.

ROLLBALL.MWS

```
>  with(liesymm):with(linalg):with(plots):with(difforms):
>
Warning, new definition for close
Warning, new definition for norm
Warning, new definition for trace
Warning, new definition for `&^`
Warning, new definition for d
Warning, new definition for mixpar
Warning, new definition for wdegree
```

```
> setup(x,theta,t,v,omega,s);
```

The independent variables are on a 6 dimensional space

$$[x, \theta, t, v, \omega, s]$$

```
>
> defform(theta=0,omega=0,x=0,v=0,s=0,t=0,a=const,b=const,c=const,k=const,mu=const
,m=const,beta=const,lambda=const):
> Lagrange:=(1/2*beta*m*lambda^2*omega^2-1/2*m*v^2);
```

$$Lagrange := \frac{1}{2}\beta\, m\, \lambda^2\, \omega^2 - \frac{1}{2} m\, v^2$$

Note that the Lagrangian is defined as the Difference between the rotational energy and the translational energy. !!!!

```
> Ptheta:=evalm(diff(Lagrange,omega));Px:=evalm(diff(Lagrange,v));Ham:=Ptheta*omeg
a+Px*v-Lagrange;
```

$$Ptheta := \beta\, m\, \lambda^2\, \omega$$

$$Px := -m\, v$$

$$Ham := \frac{1}{2}\beta\, m\, \lambda^2\, \omega^2 - \frac{1}{2} m\, v^2$$

Momenta defined canonically,

```
> Action:=wcollect(Ptheta*d(theta)+Px*d(x)-Ham*d(t)+s*(lambda*d(theta) -d(x)));
```

The Action is defined in terms of the rotational and potential energy. The momenta are presumed to be canonical Ptheta=beta*m*lambda^2*omega, Px=m*v. The constraint of no slip is lambda*d(theta)-d(x)) and its Lagrange multiplier is s. The Lagrange function is defined as L=1/2*beta*m*lambda^2*omega^2+1/2*m*v^2)

$$Action := (-m\, v - s)\, \mathrm{d}(x) + (\beta\, m\, \lambda^2\, \omega + s\, \lambda)\, \mathrm{d}(\theta) + \left(-\frac{1}{2}\beta\, m\, \lambda^2\, \omega^2 + \frac{1}{2} m\, v^2\right)\mathrm{d}(t)$$

```
> DA:=d(Action);
```

$$DA := -m\,((\mathrm{d}(v))\ \&\hat{}\ (\mathrm{d}(x))) - ((\mathrm{d}(s))\ \&\hat{}\ (\mathrm{d}(x))) + \beta\, m\, \lambda^2\,((\mathrm{d}(\omega))\ \&\hat{}\ (\mathrm{d}(\theta))) + \lambda\,((\mathrm{d}(s))\ \&\hat{}\ (\mathrm{d}(\theta))) - \beta\, m\, \lambda^2\, \omega\,((\mathrm{d}(\omega))\ \&\hat{}\ (\mathrm{d}(t))) + m\, v\,((\mathrm{d}(v))\ \&\hat{}\ (\mathrm{d}(t)))$$

The two form is of rank 3 hence when the no slip condition fails. Hence the two form defines a non-compact symplectic space.

```
> Hel:=Action&^DA;
```

$$Hel := \left(-\frac{1}{2} m^2\, v^2 - m\, v\, s - \frac{1}{2} m^2\, \beta\, \lambda^2\, \omega^2\right)\&\hat{}(\mathrm{d}(x), \mathrm{d}(v), \mathrm{d}(t)) + (m\, v + s)\, \beta\, m\, \lambda^2\, \omega\, \&\hat{}(\mathrm{d}(x), \mathrm{d}(\omega), \mathrm{d}(t)) - (m\, v + s)\, \beta\, m\, \lambda^2\, \&\hat{}(\mathrm{d}(x), \mathrm{d}(\omega), \mathrm{d}(\theta)) + (-\lambda\, m\, v + \beta\, m\, \lambda^2\, \omega)\, \&\hat{}(\mathrm{d}(x), \mathrm{d}(s), \mathrm{d}(\theta))$$

$$+\lambda(\beta m \lambda \omega + s) m v \,\&\text{^}(d(\theta), d(v), d(t)) + \left(-\frac{1}{2}\beta^2 m^2 \lambda^4 \omega^2 - \beta m \lambda^3 \omega s - \frac{1}{2} m^2 \beta \lambda^2 v^2\right)\&\text{^}(d(\theta), d(\omega), d(t))$$
$$-\lambda(\beta m \lambda \omega + s) m \,\&\text{^}(d(\theta), d(v), d(x)) - \frac{1}{2} m(\beta \lambda^2 \omega^2 - v^2)\lambda \,\&\text{^}(d(t), d(s), d(\theta))$$
$$+\left(\frac{1}{2}\beta m \lambda^2 \omega^2 - \frac{1}{2} m v^2\right)\&\text{^}(d(t), d(s), d(x))$$

The Helicity 3-form above does not vanish

```
> DADA:=wcollect((DA&^DA));
```

$$DADA := 2 m^2 \beta \lambda^2 \omega \,\&\text{^}(d(v), d(x), d(\omega), d(t)) - 2 m^2 \beta \lambda^2 \,\&\text{^}(d(v), d(x), d(\omega), d(\theta))$$
$$-2 m \lambda \,\&\text{^}(d(v), d(x), d(s), d(\theta)) + 2 \beta m \lambda^2 \omega \,\&\text{^}(d(s), d(x), d(\omega), d(t)) - 2 m v \,\&\text{^}(d(s), d(x), d(v), d(t))$$
$$-2 \lambda^3 \beta m \omega \,\&\text{^}(d(s), d(\theta), d(\omega), d(t)) + 2 \beta m^2 \lambda^2 v \,\&\text{^}(d(\omega), d(\theta), d(v), d(t))$$
$$+2 \lambda m v \,\&\text{^}(d(s), d(\theta), d(v), d(t)) - 2 \beta m \lambda^2 \,\&\text{^}(d(s), d(x), d(\omega), d(\theta))$$

```
> Torsion:=wcollect(factor((Hel&^DA)));
```

$$Torsion := m^2 \lambda(-v^2 - \beta \lambda^2 \omega^2 + 2 \lambda v \beta \omega) \,\&\text{^}(d(x), d(v), d(t), d(s), d(\theta))$$
$$+ m^2 \lambda(m \beta^2 \omega^2 \lambda^3 - m v^2 \beta \lambda - 2 \lambda \beta v s + 2 \lambda^2 \beta \omega s) \,\&\text{^}(d(x), d(v), d(t), d(\omega), d(\theta))$$
$$+ m^2 \lambda(-\beta^2 \omega^2 \lambda^3 + 2 v \beta \lambda^2 \omega - v^2 \beta \lambda) \,\&\text{^}(d(x), d(\omega), d(t), d(s), d(\theta))$$

```
> Vol:=factor(DA&^DA&^DA);
```

$$Vol := 6 m^2 \beta \lambda^2 (-v + \lambda \omega) \,\&\text{^}(d(v), d(x), d(\omega), d(t), d(s), d(\theta))$$

```
> scale:=scalarpart(factor(DA&^DA&^DA));
```

$$scale := 6 m^2 \beta \lambda^2 (-v + \lambda \omega)$$

```
>
```

The Torsion 5-form has 6 components but 3 of them are zero.

```
> Tv:=-(scalarpart(Torsion&^d(v))*3/scale);
```

$$Tv := -\frac{1}{2} \frac{-\beta^2 \omega^2 \lambda^3 + 2 v \beta \lambda^2 \omega - v^2 \beta \lambda}{\lambda \beta(-v + \lambda \omega)}$$

```
> To:=-(scalarpart(Torsion&^d(omega))*3/scale);
```

$$To := -\frac{1}{2} \frac{-v^2 - \beta \lambda^2 \omega^2 + 2 \lambda v \beta \omega}{\lambda \beta(-v + \lambda \omega)}$$

```
> Ts:=factor(scalarpart(Torsion&^d(s))*3/scale);
```

$$Ts := \frac{1}{2} \frac{\beta m \lambda^2 \omega^2 - m v^2 - 2 v s + 2 \lambda \omega s}{-v + \lambda \omega}$$

```
> SS:=solve(Ts,s);
```

$$SS := -\frac{1}{2} \frac{m(\beta \lambda^2 \omega^2 - v^2)}{-v + \lambda \omega}$$

```
>
```

The Torsion 5-form has 3 non-zero components on the six dimensional space and 3 zero components.. The non-zero components are orthogonal to the spatial coordinates {x,theta,t} in the 6D space{x,theta,t,v,omega,s}

```
>
```

The above 6-form is the volume element of the 6 dimensional space. Note that th e volume vanishes at the no slip condition. This 6-dimensional space is the top Pfaffian. It is NOT compact without boundary.

```
> AT:=factor(Hel&^DADA);
```

As this 7 form vanishes, it is proved that the Pfaff dimension of the Action 1-form is 6, with the coordinates {x,theta,t,v,omega,s}

Note that the 6-form vanishes when the noslip condition prevails. The space then reduces to a contact manifold of 5 dimensions. However, initial conditions do not necessarily satisfy the no slip constraint. Therefore the system decays (irreversibly) until the no slip condition is satisfied. Thereafter the dynamics is that of a Hamiltonian system with no irreversible dissipation.

$$AT := 0$$

```
>
> AVEC:=[Px-s,Ptheta+s*lambda,-Ham,0,0,0];
```

$$AVEC := \left[-m\,v - s,\ \beta\,m\,\lambda^2\,\omega + s\,\lambda,\ -\frac{1}{2}\beta\,m\,\lambda^2\,\omega^2 + \frac{1}{2}m\,v^2,\ 0,\ 0,\ 0\right]$$

```
> TORS:=[0,0,0,Tv,To,Ts];
```

$$TORS := \left[0,\ 0,\ 0,\ -\frac{1}{2}\,\frac{-\beta^2\,\omega^2\,\lambda^3 + 2\,v\,\beta\,\lambda^2\,\omega - v^2\,\beta\,\lambda}{\lambda\,\beta\,(-v+\lambda\,\omega)},\ -\frac{1}{2}\,\frac{-v^2 - \beta\,\lambda^2\,\omega^2 + 2\,\lambda\,v\,\beta\,\omega}{\lambda\,\beta\,(-v+\lambda\,\omega)},\ \frac{1}{2}\,\frac{\beta\,m\,\lambda^2\,\omega^2 - m\,v^2 - 2\,v\,s + 2\,\lambda\,\omega\,s}{-v+\lambda\,\omega}\right]$$

```
> test:=innerprod(TORS,AVEC);
```

$$test := 0$$

```
> DA;
```

$$-m\,((\mathrm{d}(v))\ \&\wedge\ (\mathrm{d}(x))) - ((\mathrm{d}(s))\ \&\wedge\ (\mathrm{d}(x))) + \beta\,m\,\lambda^2\,((\mathrm{d}(\omega))\ \&\wedge\ (\mathrm{d}(\theta))) + \lambda\,((\mathrm{d}(s))\ \&\wedge\ (\mathrm{d}(\theta))) - \beta\,m\,\lambda^2\,\omega\,((\mathrm{d}(\omega))\ \&\wedge\ (\mathrm{d}(t))) + m\,v\,((\mathrm{d}(v))\ \&\wedge\ (\mathrm{d}(t)))$$

```
> WORK:=wcollect(factor((m*Tv*(d(x)-v*d(t))+beta*m*lambda^2*To*(d(theta)-omega*d(t
  ))+Ts*((lambda*d(theta)-d(x))))));
```

$$WORK := (-m\,v - s)\,\mathrm{d}(x) + (\beta\,m\,\lambda^2\,\omega + s\,\lambda)\,\mathrm{d}(\theta) + \left(-\frac{1}{2}\beta\,m\,\lambda^2\,\omega^2 + \frac{1}{2}m\,v^2\right)\mathrm{d}(t)$$

```
> factor(Action&^WORK);
> Action;
```

$$0$$

$$(-m\,v - s)\,\mathrm{d}(x) + (\beta\,m\,\lambda^2\,\omega + s\,\lambda)\,\mathrm{d}(\theta) + \left(-\frac{1}{2}\beta\,m\,\lambda^2\,\omega^2 + \frac{1}{2}m\,v^2\right)\mathrm{d}(t)$$

The virtual work is proportional to the Action. The virtual work 1-form is not zero until the no-slip condition is obtained.

```
> Wx:=factor(scalarpart(WORK&^d(theta)&^d(t)));
>
```

$$Wx := -m\,v - s$$

```
> Wo:=factor(simplify(scalarpart(WORK&^d(t)&^d(x))));
```

$$Wo := \lambda\,(\beta\,m\,\lambda\,\omega + s)$$

```
> Wt:=factor(scalarpart(WORK&^d(x)&^d(theta)));
```

$$Wt := -\frac{1}{2}m\,(\beta\,\lambda^2\,\omega^2 - v^2)$$

The Work 1-form relative to the Torsion vector is proportional to the Action with the factor 2*lambda^3*m^2*beta

```
> DWORK:=wcollect(simpform(d(WORK)));
```

$$DWORK := -m\,((\mathrm{d}(v))\ \&\wedge\ (\mathrm{d}(x))) - ((\mathrm{d}(s))\ \&\wedge\ (\mathrm{d}(x))) + \beta\,m\,\lambda^2\,((\mathrm{d}(\omega))\ \&\wedge\ (\mathrm{d}(\theta))) + \lambda\,((\mathrm{d}(s))\ \&\wedge\ (\mathrm{d}(\theta))) - \beta\,m\,\lambda^2\,\omega\,((\mathrm{d}(\omega))\ \&\wedge\ (\mathrm{d}(t))) + m\,v\,((\mathrm{d}(v))\ \&\wedge\ (\mathrm{d}(t)))$$

```
> IRR:=wcollect(WORK&^DWORK);
```

$$IRR := \left(-\frac{1}{2} m^2 v^2 - m\, v\, s - \frac{1}{2} m^2 \beta \lambda^2 \omega^2\right) \&\wedge(d(x), d(v), d(t)) + (m\, v + s)\, \beta\, m\, \lambda^2\, \omega \,\&\wedge(d(x), d(\omega), d(t)) - (m\, v + s)\, \beta\, m\, \lambda^2 \,\&\wedge(d(x), d(\omega), d(\theta)) + (-\lambda\, m\, v + \beta\, m\, \lambda^2\, \omega) \,\&\wedge(d(x), d(s), d(\theta)) + \lambda\, (\beta\, m\, \lambda\, \omega + s)\, m\, v \,\&\wedge(d(\theta), d(v), d(t)) + \left(-\frac{1}{2} \beta^2 m^2 \lambda^4 \omega^2 - \beta\, m\, \lambda^3\, \omega\, s - \frac{1}{2} m^2 \beta \lambda^2 v^2\right) \&\wedge(d(\theta), d(\omega), d(t)) - \lambda\, (\beta\, m\, \lambda\, \omega + s)\, m \,\&\wedge(d(\theta), d(v), d(x)) - \frac{1}{2} m\, (\beta \lambda^2 \omega^2 - v^2)\, \lambda \,\&\wedge(d(t), d(s), d(\theta)) + \left(\frac{1}{2} \beta\, m\, \lambda^2 \omega^2 - \frac{1}{2} m\, v^2\right) \&\wedge(d(t), d(s), d(x))$$

```
> W:=(factor(WORK));
```

$$W := -d(x)\, m\, v - d(x)\, s + d(\theta)\, \beta\, m\, \lambda^2\, \omega + d(\theta)\, s\, \lambda - \frac{1}{2} d(t)\, \beta\, m\, \lambda^2\, \omega^2 + \frac{1}{2} d(t)\, m\, v^2$$

The Work 1-form relative to the unique Torsion vector is not integrable until the no-slip condition is reached. Hence the Heat 1-form Q is not integrable. Therefor a process defined by the unique Torsion vector is irreversible, until the no slip condition is obtained.

The dynamical system defined by the Torsion vector is

```
> d(v)-Tv*d(tau)=0;d(omega)-To*d(tau)=0;d(s)-Ts*d(tau)=0;d(x)-v*d(t);d(theta)-omeg
  a*d(t)=0;d(tau)-aa*d(t)=0;
```

$$d(v) + \frac{1}{2} \frac{(-\beta^2 \omega^2 \lambda^3 + 2\, v\, \beta\, \lambda^2\, \omega - v^2 \beta\, \lambda)\, d(\tau)}{\lambda\, \beta\, (-v + \lambda\, \omega)} = 0$$

$$d(\omega) + \frac{1}{2} \frac{(-v^2 - \beta\, \lambda^2 \omega^2 + 2\, \lambda\, v\, \beta\, \omega)\, d(\tau)}{\lambda\, \beta\, (-v + \lambda\, \omega)} = 0$$

$$d(s) - \frac{1}{2} \frac{(\beta\, m\, \lambda^2 \omega^2 - m\, v^2 - 2\, v\, s + 2\, \lambda\, \omega\, s)\, d(\tau)}{-v + \lambda\, \omega} = 0$$

$$d(x) - v\, d(t)$$

$$d(\theta) - \omega\, d(t) = 0$$

$$d(\tau) - aa\, d(t) = 0$$

```
> PH:=To*d(v)-Tv*d(omega)=0;
```

$$PH := -\frac{1}{2} \frac{(-v^2 - \beta\, \lambda^2 \omega^2 + 2\, \lambda\, v\, \beta\, \omega)\, d(v)}{\lambda\, \beta\, (-v + \lambda\, \omega)} + \frac{1}{2} \frac{(-\beta^2 \omega^2 \lambda^3 + 2\, v\, \beta\, \lambda^2\, \omega - v^2 \beta\, \lambda)\, d(\omega)}{\lambda\, \beta\, (-v + \lambda\, \omega)} = 0$$

```
> dPxdt= factor(m*Tv-Ts);Action;
```

$$dPxdt = -m\, v - s$$

$$(-m\, v - s)\, d(x) + (\beta\, m\, \lambda^2\, \omega + s\, \lambda)\, d(\theta) + \left(-\frac{1}{2} \beta\, m\, \lambda^2 \omega^2 + \frac{1}{2} m\, v^2\right) d(t)$$

```
> dPthetadt:=factor(beta*m*lambda^2*To+Ts*lambda);
>
```

$$dPthetadt := \lambda\, (\beta\, m\, \lambda\, \omega + s)$$

```
>
```

Both the (New) translational momenta and angular momenta decay at the same exponential rate.!!!

New Momenta (mv+s) , (2beta*lambda^2*m*omega+s*omega) are NOT conserved.

>

The first two equations admit an immediate integral during the dissipative motion along the torsion vector.

>

$$v + \lambda \beta \omega = (1 + \beta)\, Vnoslip$$

>

>

>

The no-slip condition states that v=lambda*omega, so the constant of integration is 1+beta)