Quantum Dynamical Manifolds - 3a. Cold Dark Matter and Clifford's Geometrodynamics Conjecture

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The purpose of this paper is to explore the concepts of gravitation, inertial mass and spacetime from the perspective of the geometrodynamics of "mass-spacetime" manifolds (MST). We show how to determine the curvature and torsion 2-forms of a MST from the quantum dynamics of its constituents. We begin by reformulating and proving a 1876 conjecture of W.K. Clifford concerning the nature of 4D MST and gravitation, independent of the standard formulations of general relativity theory (GRT). The proof depends only on certain derived identities involving the Riemann curvature 2-form in a 4D MST having torsion. This leads to a new metric-free vector-tensor fundamental law of dynamical gravitation, linear in pairs of field quantities which establishes a homeomorphism between the new theory of gravitation and electromagnetism based on the mean curvature potential. Its generalization leads to Yang-Mills equations. We then show the structure of the Riemann curvature of a 4D MST can be naturally interpreted in terms of mass densities and currents. By using the trace curvature invariant, we find a parity noninvariance is possible using a de Sitter imbedding of the Poincaré group. The new theory, however, is equivalent to the Newtonian one in the static, large scale limit. The theory is compared to GRT and shown to be compatible outside a gravitating mass in the absence of torsion. We apply these results on the quantum scale to determine the gravitational field of leptons. On a larger scale, they are used to determine the cold dark matter (CDM) distribution of a simple galaxy model. The model involves the relative contraction of a 3D MST manifold about a galactic center. This raises the possibility of CDM sinks at galaxy cores and CDM sources at other places. Other possible observational consequences of this model are explored. Thus, the theory gives an extension of Newton's static gravitation theory to a dynamical one with observational consequences additional to those of GRT and a solution to at least part of the galactic missing mass problem.

I. INTRODUCTION

The question of the *nature* of cold dark matter (CDM), its *distribution*, and its *effect* on spacetime is important for understanding the evolution of galaxies, the Universe, and the theory of gravitation, especially since CDM is thought to constitute up to 90-95% of the "mass" of these objects. The presence of the hypothetical CDM leads to anomalous motion of stars in galaxies [1]. That is, the known distribution of stars and other mass sources within the Newtonian paradigm cannot explain the rotational characteristics of a galaxy's spiral disk [2]. The possible presence of CDM also can be cause for questioning our basic assumptions about the structure and dynamics of mass-spacetime (MST). The problem is being approached from several directions within conventional paradigms where CDM is the source of MST curvature. This involves, e.g., the search for *missing mass*. On the macro scale, this includes the search for massive halo compact objects (MACHOs). On the micro scale, this involves searches for weakly interacting massive particles (WIMPS), light massive neutrino flux, etc. See ref. [3] for a recent review. Some of this research comes under the topic of studies on the properties of the physical vacuum. If we broaden our investigation further, we find ourselves examining the relation of quantum mechanics to gravitation.

In this paper we show it is fruitful to reorient our thinking about the problem of mass and spacetime by taking a geometrodynamical perspective so that the macro curvature and torsion of an evolving MST manifold is interpreted as the source of the CDM instead of visa versa. Since the evolution of the MST beginning with the big bang continues into the present epoch, we search for sources of the *missing curvature and torsion* in the MST itself. Thus, the anomalous motion of stars can be considered as providing Lagrangian marker particles tracing the evolution of the combined mass-spacetime of a galaxy. In doing this we are led to develop *a geometrodynamical theory of the evolution of unified mass-spacetime manifolds.*[1] On the other hand, starting at the micro scale, we find it fruitful to first compute the mass densities and fluxes in order to determine the curvature and torsion potentials. This naturally leads to the self-consistency condition that the microscopic MST flux-density be the same as the macroscopic one. That is, the geometry must be consistent from the macro through the micro scale. To achieve this we find it necessary to use more geometrically complete spaces than those used in classical General Relativity theory (GRT). Namely, our spaces have torsion because they are firmly grounded in quantum dynamical manifolds having such properties. Most of the spaces we consider have higher dimension than those considered in classical gravitation theory because the many-body configuration in a quantum system has to be dealt with at an elementary level. Our 4D MST manifolds are consequently obtained via a spacetime embedding process from evolving 3D ones obtained from projections from higher dimensional ones (which are often flat). Our densities and currents are calculated quantum dynamically and provide sources for the curvature and torsion. This relates the quasi-particle wave functions to the matter flux-density. With these caveats, the gravitation theory we derive is, surprisingly, based solely on the geometric properties of the resulting projected, non Riemannian 4D mass-spacetime and is motivated by the geometrodynamical conjecture of W.K. Clifford in 1876 [5] which we restate in modern terms and prove. The result extends Newton's static gravitational theory in a way that is significantly different from Einstein's general relativity theory (GRT) but, for reasons discussed, it agrees with the latter in the "classical" experimentally verified cases [6] and in general overall philosophy that matter curves and twists spacetime and visa versa.

In the paper we also introduce several key concepts, *Maxwell structures*, *Abelianization of Yang-Mills structures,* and *trace curvatures and potentials.* These allow us to move from what is ostensibly a non Riemann space with a 2-form curvature matrix, $\left(K_j^i\right)$, to one which is globally Riemann for the average curvature 2-form $k \equiv t_r\left(K_j^i\right) = k_{\mu\nu} dx^\mu \wedge dx^\nu = dA$, having metric tensor $g_{\mu\nu}$. That is, we consider the average curvature as defining the metrical properties of a new global space. This gives us an *induced Riemannian geometry.* This average curvature, however, in the presence of spacetime torsion, is not parity invariant. For a 4D MST, development of these considerations yields four equations for the average (gauge) potential 4-vector, A, and mass-current, J, plus six more and finally

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[1] Our approach is minimalist in the sense that we merely assume that MST is a differential manifold. This assumption can be weakened by using the fact that a continuous manifold can be approximated as closely as needed by a sequence of differentiable manifolds. We call these MST manifolds *quantum dynamical manifolds* (QDMs) to emphasize their unity, from a geometrical and topological standpoint, with the manifolds generated from an underlying quantum dynamics. We have found that QDMs have a quantum mechnanics generalizing the Dirac and Maxwell equations. This dynamics is applied in Parts 4 and 5 [4]. The overall theory is called quantum dynamical manifold theory (QDMT). The motion of a moving frame and its dual can be used to entirely determine the geometry of the manifold, see Fig.1. In an evolving quantum mechanical system there is a field of these frames that evolves making a 3D quantum dynamical manifold.

four specifying Riemann metricity as follows:

$$\Box A = -4\pi J, \tag{1}$$

$$k_{ls} = \frac{\partial A_s}{\partial x^l} - \frac{\partial A_l}{\partial x^s} = \frac{\partial \Gamma^m_{ms}}{\partial x^l} - \frac{\partial \Gamma^m_{ml}}{\partial x^s}, \tag{2}$$

$$\nabla_i(\tilde{R}^i_j - \frac{1}{2}\tilde{g}^i_j\tilde{R}) = 0. \tag{3}$$

Here $\Box$ is the d'Alembertian or second order wave operator in the induced Riemannian geometry and ∇ is a covariant derivative in the induced Riemannian geometry. The middle set of six partial differential equations (2) allows us to transfer information from the first equation set (1) concerning the currents, J, and consequently the 4-potential, A, to the average curvature 2-form k. A gauge fixing equation has to be appended to Eq.(1) as in electrodynamics. Combining the last two sets of equations (2-3) then provides ten equations for determining the ten metrical coefficients of the induced Riemann space. The middle set of equations is null for the special case where the trace connection coefficients are derivable from a single potential, $\Gamma^m_{ms} = \phi$, as is often the case in applications of GRT. In this case, if a symmetric connection is derivable from a such a single potential, the corresponding Ricci tensor $\tilde{R}^i_j$ is symmetric. In general, even for Riemann spaces, however, k_{ls} is non-vanishing. These equations rest on our ability to specify the mass-currents in the first equation. These can be calculated in a variety of ways. For examples, we show how to compute these quantities for a quantum system, in particular the lepton series, for a Friedmann-Lemaître-Robertson-Walker (FLRW) model galaxy, and for the case of stellar evolution. The metric found can be multiple valued as these equations are nonlinear, so in general one has to consider them as evolution equations. The set of equations (1-3) can also be derived for suitably Abelianized Yang-Mills systems of equations.

The most interesting application of Eq.(1-3) uses the FLRW model of a galaxy's mass-spacetime (MST) manifold, previously used in cosmology, to elucidate the nature and distribution of CDM for the galaxy model from the geometry of the spacetime. In our approach the mass-spacetime of a galaxy is considered to be an evolving compact object, an S^3 sphere. That is, we start with a certain, simple, 3D-static model geometry, then let it evolve generating the basis for the resulting evolving mass-spacetime. The stars are small contributors to the geometry, they are as mere raisins in a pudding. Here we do not extensively specify the underlying dynamics by solving quantum dynamical manifold equations as we have done previously[2] as this involves microscale fluctuations in the geometry and torsion.[3] However, it is argued that the homogeneity and isotropy of the S^3 model and other such models typically used in gravitation calculations are a result of an average over quantum fluctuations yielding a macroscopic, average, curvature. Since the theory yields results homeomorphic to the Maxwell structure equations, a $u(1)$-gauge theory, we call this an Abelianization of the gauge. Thus the new theory of gravitation developed here fits exactly into the quantum dynamical manifold theoretic picture and the wider geometrodynamical one [8], [9]. We also show how, in the evolution of MST manifolds, topologies can change and, extending a theory being developed by Kiehn, provide means of discovering the creation of topological defects when the deviatoric curvature form (i.e., the curvature form

[2]A (differential) manifold can be imagined as a continuous n-dimensional space on which a distance function is defined and which may be curved and twisted. It has the property that sufficiently close to every point the manifold can be approximated as required by a flat n-dimensional space, R^n. Quantum Dynamical Manifold Theory (QDMT) uses the quasi-particles of a quantum system to define the geometry of a quantum dynamical manifold (QDM). This geometry must be consistent with the effective (gauge) potentials of the quantum system generating it. The methods generalize to the evolution of mass-spacetimes, nuclear matter and a range of other systems that can be treated by the differential geometry of manifolds having an internal symmetry and specified by evolution equations exhibiting this internal symmetry. These systems determine what we call a *dynamical braiding* of Lie algebras. The differential manifolds are assumed to have all (an infinite level if needed) of the differentiability properties needed as a continuous manifold can be approximated by convergent sequences of differential ones. A differential manifold comes equipped with local systems of co-ordinates and maps between local open neighborhoods in regions of overlap. These local neighborhoods are called patches.

[3]For example, as in Parts 4 and 5. Our mathematical methods are generally standard ones used in differential geometry and differential topology. Hence we have endeavoured to provide references and explanations from a physical applications viewpoint. We do this in the hope that this method of approaching the problems of quantum mechanics and gravitation will be more easily understood. The manifold paradigm we use is quite different from the particle-wave one used in quuantum mechanics, and more like the outlook used in GRT, however, without the necessity of a metric. Thus a major part of the paper is explaining in physical terms what can be done with differential manifolds and how the mathematics can be used to deterimine results of physical interest. The most unusual aspect of the mathematics we use is that it applies to the microscopic quantum through the cosmological range of scales.

obtained by subtracting its trace) is not closed. Not surprisingly, quantum dynamical effects provide a natural method to create these defects.

The main results are presented as Theorems 4, 25, 26 and 27. The first theorem restates Clifford's conjecture about the nature of mass-spacetime (geometrodynamics) in modern terms and is based on Lemma 1, which states some identities for a Minkowski signatured spacetime; Lemma 2, showing the existence of diagonal components to a curvature 2-form; and on Lemma 3, concerning the trace of the curvature 2-form of any affinely connected manifold.[4] We are seeking a rather weak, globally defined force, and we find it in the trace curvature. The derivation of Lem.1 uses the exterior Calculus of É. Cartan in a way often considered as yielding a geometric derivation of Maxwell's equations. However, we argue that this derivation describes 4D MST manifolds in general since it can be shown instead that the Maxwell equations arise from the description of the dynamics of color symmetric fields having $so(3)|_3$ Lie algebraic representation color symmetry. Since the result depends solely on the geometry of an evolving 4D MST, it allows us to introduce a definition of CDM as the divergence of a gravitational field G. The G-field is associated with another field which we call a "whirl" field W. The next major theorem, Thm.25, determines the gravitational potential of the lepton series (e, μ, τ) using the understanding gained from Thm.4 and Lem.2. Even though the topology of the lepton series is shown to be S^1, S^3 or S^7, they are found to behave gravitationally and electrodynamically as a point masses or charges under a delocalization approximation in k-space. The discussion surrounding Thm.25 shows how to compute the mass-density and mass-current matrix of a quantum dynamical system. This allows us to demonstrate that the trace gravitational curvature of such systems is non null and of the correct form for studies of gravitation, i.e., yielding a $1/r$-potential. We show Thm.4 and its supporting lemmas can be generalized to the form of Yang-Mills equations in Thm.10. The next main result, Thm.26, gives the results of the calculation of the evolution dependent gravitational field of the model galactic MST. The proof of Thm.26 uses the Calculus of differential forms applied to the geometry of the 3D-S^3 model embedded into a 4D MST, i.e., into a 4D space with Minkowski signature. These results depend on a single model parameter, the relative shrinking rate of the galaxy, and provide a method for parametrizing the observed dispersion velocities of marker stars in galaxies. These velocities have been found to be greater than can be attributed to the observed mass in the Newtonian or GRT approximations [1]. The result of the computation of the gravitational field, G, "whirl" field, W, and CDM distribution of the evolving galaxy are summarized as well in the Figures.[5] Finally the subject of stellar structure and gravitational collapse is considered using the new theory. These results are presented as Thm.27.

In keeping with the necessity to give evidence that the new theory has new practical applications, we show that the trace curvature is non vanishing in principle and can be used to provide an induced Riemann geometry that is effectively an Abelianization of the underlying geometry and this is useful for applications to computing the gravitational field of elementary particles, the evolution of galaxies and the nature of cold dark matter, and the evolution of stars. Our main point of departure from GRT lies in the fact we introduce the dynamics without breaking the Bianchi identities. For a number of classical "experimental proofs of GRT", the new theory and GRT provide identical results; these include the Birkhoff theorem, advance of the perihelion, gravitational red shift, and bending of light rays.

II. RESTATEMENT AND PROOF OF THE CLIFFORD CONJECTURE

A. Clifford's Conjecture

In the latter part of the 19^{th} century, W.K. Clifford formulated a vision of the structure of MST well in advance of A. Einstein's independent theory of gravitation [7]. Clifford's 1876 conjecture [5] about the evolution of what we

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[4] Our results are generally limited to compact manifolds that are subsequently embedded into a spacetime. In fact, the known 3D Universe, although large, is finite in extent. Therefore, it is topologically compact. The symmetries associated with noncompact Minkowski spacetime are thus approximate or restricted to mathematical tangent spaces. In previous Parts we have argued that mass-spacetime (MST) in its lowest level representation, is topologically a multiply connected, evolving 3D manifold called a *handle-body*, standardly denoted with g-handles as M^3_g. These can be embedded into a 4D manifold as a MST through a process of embedding the 3D manifold into a spacetime locally having Minkowski signature (-1,1,1,1). This is called re-dynamicization.

[5] The topology of more complicated quantum dynamical manifolds suitable for more realistic models - multiply connected and unoriented ones - and their embeddings arising from their dynamics is described in some detail in the sequel. Only the simplest of these topologies, S^3, is considered in this paper. More complicated, multiply connected (g-holes), topologies, called *handle-bodies*, M^3_g, are constructed by carving our parts of S^3 by a process called Heegaard splitting and twisting the parts giving them a so-called Dehn twist before reinserting them. The M^3_g can also be considered as multiple sheeted covers of S^3.

now call mass-spacetime is

> *That small portions of space are in fact of a nature analogous to little hills on a surface which is on the average flat; namely that the ordinary laws of geometry are not valid on them. That this property of being curved or distorted is constantly being passed from one portion of space to another after the manner of a wave. That this variation of curvature is what really matters in the phenomenon which we call the motion of matter, whether ponderable or etherial. That in the physical world nothing else takes place but this variation subject (possibly) to the law of continuity.*

To this we further add torsion, dynamics and a spectral analysis allowing a quantum dynamical description described in detail elsewhere. We find it helpful to consider evolving spaces that are non Riemannian, so-called spaces of affine connection [10, p.143] and to ultimately bring in quantum dynamics in order to find a non zero mass density. In the usual applications of a Riemann spaces, the curvature matrix of 2-forms $(K^{\mu}_{\nu[jk]}dx^j \wedge dx^k)$ is antisymmetric in the set of indices $\Lambda = \{\mu, \nu\}$ making the matrix of 2-forms (K^{μ}_{ν}) have null trace [11]. In general, this is not the case and, further, torsion contributes to the removal of this triviality as we will see below. We show in an example below for the lepton series that the mass-density and mass-flux of quantum dynamical systems have non-null trace. Other examples are given in Appendix B.

B. Restatement and Proof

The modernization of the Clifford conjecture is the content of our Theorem 4. Its proof requires three lemmas. The first involves proving some identities that are equivalent to the Maxwell equations. In other words, we derive topological constraints[6] for the evolution of manifolds in the form of identities involving other combinations of the curvature tensor compared to the Bianchi identities (See Appendix). The second lemma is essentially a corollary of the celebrated Chern-Weil theorem.

Lemma 1 *Maxwell Structure Equations. For a given differential manifold M^n having a single matrix of (Riemann) curvature 2-forms $K(\Lambda) = (K^{\mu}_{\nu[jk]}dx^j \wedge dx^k)$, defined for the tensor index set $\Lambda : (\mu, \nu)|_{\mu,\nu=0,1,2,3} = \{(\mu, \nu)\}$, the following definition and identities hold on a co-ordinate patch*

$$d^{\star}K^i_j \equiv 4\pi {J_g}^i_j, \tag{4}$$

$$d{J_g}^i_j = 0, \tag{5}$$

$$4\pi^{\star}{J_g}^i_j = {}^{\star}d^{\star}K^i_j \equiv {}^{\star}d^{\star}dA^i_j = ({}^{\star}d)^2 A^i_j. \tag{6}$$

with the last definition holding for the Abelianization of the potential structure $\{(A^i_j)_\mu \equiv A_\mu\}^m_{\mu=1}$ when there is more than one potential, $m > 1$.

Proof of Lem.1: The first equation is a definition of the 3-form mass-current density J_g in terms of the curvature. Its consistency lies in the usefulness of the physical results produced. The second equation follows from the Poincaré

[6] These are constraints on the evolution of a manifold in the same way that the Bianchi constraints

$$dK^k_i = [K^k_j, \omega^j_i] \quad and \quad d\tau^k = K^k_i \wedge \theta^i - \omega^k_i \wedge \tau^i$$

are identities derived from the definition of curvature, K^k_j, torsion, τ^k, the moving frame, e_i, and its dual, θ^i, and connection, ω^j_i, as follows

$$\begin{aligned} de_i - \omega^j_i e_j &= 0, \\ d\theta^i - \theta^j \wedge \omega^i_j &= \tau^i, \qquad (\tau^i) \equiv (T^i_{[jk]}\theta^j \wedge \theta^k), \\ d\omega^i_j - \omega^m_j \wedge \omega^i_m &= R^i_j, \qquad (R^i_j) \equiv (R^i_{j[kl]}\theta^k \wedge \theta^l) \end{aligned}$$

and the use of the exterior calculus of Cartan. Our use of square-bracketed tensor subscripts implies we take only one ordering of, e.g., j, k in $[j, k]$ so as to avoid double counting or having to introduce a factor of $\frac{1}{2}$. In these equations the nontensor label g is used to remind us that these are gravitational quantities, not the usual electromagnetic ones occuring in similar formal contexts.

lemma ($d^2\omega = 0$): $4\pi dJ_{g\,j}^{\,i} = d^{2\star}K_j^i = 0$. The quantity $\left(J_{g\,j}^{\,i}\right)$ is conserved. Here (A_m^v) is the matrix of 1-forms constituting the local connection, A. In electrodynamics, where there is a single (4-vector) potential A and a single curvature, the 2-form $F = (F_{[jk]}dx^j \wedge dx^k)$ satisfying $dF = 0$, by the converse of the Poincaré lemma,[7], there are *local gauge fields*, $A[\Lambda] \,|\, A \equiv A_\nu dx^\nu$ defined up to addition of a gauge field $d\varphi$ such that $F = dA$. This encourages us to write the third relation, Eq.(6), involving the gauge potential A.

This can be achieved for a (torsionless) Riemann space where K is automatically closed for a single potential 1-form, i.e., $dK_j^i = [K_v^j, A_m^v] = 0$,[8] or when there is more than one gauge potential matrix but $(A_j^i)_\mu = k_\mu 1$ so $[A_\mu, A_\nu] = 0$ (i.e., for Abelian gauges). For any algebra of potentials $\{A_\mu\}$, a procedure which maps them into commutative group is called an *Abelianization.* The trace operation does this and consequently $[t_r A_\mu, trA_\nu] = [k_\mu, k_\nu] = t_r[A_\mu, A_\nu] = 0$. The map $(A_j^i)_\mu \rightarrow k_\mu 1$ is an Abelianization. The Abelianization can leave traces of the original symmetry in the $\{k_\mu\}$. In general, when we have a single matrix, A, we can diagonalize it (perhaps at the expense of introducing complex numbers) or tridiagonalize if it is real-antisymmetric. However, we cannot, in general, diagonalize/tridiagonalize several matrices, $\{A_\mu\}$ simultaneously. This occurs, for instance, when the matrices form a representation of a Lie algebra. The algebra provides a torsional obstruction[9] to the diagonalization and to allowing $dK_\mu = [K_\mu, A_\mu] = 0$. Thus we would like to show that there are spaces with consistent geometry which have diagonal curvature 2-forms and we would like to know something about them. This is the content of Lem.2 proved below. The result will also be important in applications involving use of quantum dynamical currents. Thus, we assume the result; there exist (at least locally, we will find a global results - Lemma 3 below) gauge fields with matrix components A_ν^μ, such that

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[7] If ω is a p-form in E^n such that $d\omega = 0$,then there is a $(p-1)$-form α such that $\omega = d\alpha$. (Exception: $p = 0$. Then $\omega = f$ is a scalar and $df = 0$ simply implies f is a constant.) [10, p.27]. We can use the topological invariance of differential forms to (try to) extend this to a manifold, M^n, more general than E^n. However, the result is only local as counter-examples show. The converse is true only for special manifolds without topological defects as discussed further below.

[8] This statement is the identity for Riemann spaces that

$$\sum_{(jkl)} \nabla_m K_{jkl}^i = \nabla_m K_{jkl}^i + \nabla_k K_{jlm}^i + \nabla_l K_{jmk}^i = 0$$

We use the letter K rather than R for the Riemann curvature throughout to emphasize our quantities are derived from non-Riemannian geometries.

[9] Torsion provides us with a picture of a manifold in terms of a Fourier series of symmetric spaces connected via spontaneous, torsion-mediated transitions between them. to see this consider a scaler ϕ^0 and a vector with components ϕ^l . We have from the following two equations the definition of a symmetric-space Ricci operator given in the third

$$\sum_{l=1}^{n} \left(\nabla_k \nabla_l - \nabla_l \nabla_k + T_{kl}^i \nabla_i\right) \phi^l = R_{kl}\phi^l$$

$$\left(\nabla_k \nabla_l - \nabla_l \nabla_k + T_{kl}^i \nabla_i\right) \phi^0 = 0\phi^0$$

$$\sum_{l=1}^{n} \left(\nabla_k \nabla_l - \nabla_l \nabla_k\right) \phi^l = \Lambda_{kl}\phi^l = (\widehat{R}_{ic})_{kl}\phi^l$$

The second equation for the torsion acting on a scalar field clearly shows T_{kl}^i is antisymmetric. The symmetry/antisymmetry of R_{kl} is not determined by the first because of the summation over l. When torsion is absent the antisymmetry of R_{kl} follows from the manifest antisymmetry of $(\nabla_k \nabla_l - \nabla_l \nabla_k)$. The equations above tell us that for manifolds with local curvature of a single signature, there is induced a handedness to the dynamics via the induced torsion. Considering the third equation as an eigen-problem leads to the concept of spaces with constant Riemann curvature, so-called *symmetric spaces* , i.e. spaces with an ambivalent parity. The scalar case corresponds to a zero eigenvalue. One can consider these to be a "Fourier-series" basis resulting in the picture of general spaces being built of symmetric spaces spontaneously connected by the torsion which provides a perturbation to the symmetric picture. In this way torsion is responsible for the change of curvature. This picture is discussed further in Part 1a. Further on, associated with the second equation above, we will derive the following expression for the gauge derivative commutator

$$[D_\mu, D_\nu] = F_{\mu\nu}^a \gamma_a$$

where the square matrices defining the gauge algebra satisfy $[\gamma_a, \gamma_b] = f_{bc}^a \gamma_c$. The f_{bc}^a are called the structure constants of the Lie algebra and the $F_{\mu\nu}^a$ are called field strengths.

$K[\Lambda] = dA[\Lambda]$ for every component of the single matrix of 2-forms, $K[\Lambda]$.[10] These 2-forms could vary from one local coordinate patch of the manifold to another, differing by a tensor transformation, the potentials differing by a gauge transformation.

To complete the proof of Lem.1, in the first part of Eq.(6), take the Hodge-$\star$ dual to convert a p-form in an m-dimensional space to a $(m-p)$-form.[11] Substitute ${}^\star K[\Lambda] = {}^\star dA[\Lambda]$ and then take the exterior derivative giving $d^\star K[\Lambda] = d^\star dA$. Taking the dual once again yields: ${}^\star d^\star dA \equiv ({}^\star d)^2 A^i_j$. These results hold for any dimension. If we further restrict our attention to a 4D MST, then ${}^\star K[\Lambda]$ is a matrix of 2-forms and if the tangent space metric is Minkowski, then $({}^\star d)^2 \to -\Box$, the second order wave operator or d'Alembertian.[12] This completes the proof. We now show these potentials and curvatures actually exist by proving the second lemma:

Lemma 2 *A space with a diagonal curvature* 2*-form* $K = k1$*, is related to a flat space by a gauge shift.*

Proof: Let the curvature 2-form K have a diagonal representation, then

$$K = dA - A \wedge A = k1 \tag{7}$$

and one has

$$\begin{aligned} dK &= d^2A - dA \wedge A - A \wedge dA \\ &= -(k1 + A^2)A + A(k1 + A^2) = 0 \end{aligned} \tag{8}$$

This curvature is a closed 2-form. So locally, by the converse of Poincaré's lemma, there is a potential a satisfying

$$k1 = da1 = dA - A \wedge A. \tag{9}$$

Now consider the deviation, Ω, of the connection A about this average curvature by introducing a *gauge shift* defined by

$$\Omega \equiv A - a1. \tag{10}$$

Then

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[10]The components $4\pi L^l_m \equiv {}^\star K^l_m$ which are closed ($dL^l_m = 0$) for which this does not hold ($4\pi L^l_m \neq dA^l_m$), i.e., ones which are inexact differential forms are the most interesting as they involve topological changes. An example where globally $d^\star K = 0$ does not imply ${}^\star K = dA$ occurs in a space with wormholes (J.A. Wheeler, in *Batelle Recontres*, *Lectures in Math. Physics,* eds. C.M. DeWitt and J.A. Wheeler, W.A Benjamin, 1968, p.266)

[11]The Hodge-$\star$ depends on the metric g_{ij} locally defined by the moving frame which is defined everywhere. The manifolds we consider do not necessarily have a globally defined (holonomoc) frame field. For instance, we always have to keep track of the sheet of the manifold we are on. Spaces with global frame fields are said to be parallelizable and their torsion vanishes.

[12]Using the definition of the coderivative of a p-form [15, p.337]

$$\delta\omega = (-)^{n(p+1)+1}\,{}^\star d({}^\star\omega),$$

the property ${}^\star\delta K = (-)^p d({}^\star K)$ and the Lorentz gauge $\delta A = 0$, we have from $K = dA - A \wedge A$ that

$$\begin{aligned} {}^\star\delta K &= {}^\star\delta dA - {}^\star\,\delta(A \wedge A), \\ (-)^p d({}^\star K) &= {}^\star(\delta d + d\delta)A - {}^\star\,\delta A \wedge A + {}^\star\,A \wedge \delta A \\ &= (-)^\star \Delta A. \end{aligned}$$

From which, taking into account the definitions in Eq.(6), we obtain ($p = 2$)

$$\Delta A = -4\pi\,{}^\star J$$

Here Δ is called the Laplace-Beltrami operator. This equation states, given J we can compute the potential. We have, in these terms a wave equation in any dimension, $n > 1$. Instead of Eq.(6) we obtain in addition ($\delta = {}^\star d^\star$)

$$\delta K^l_m = 4\pi\,{}^\star J^l_{gm}, \quad dK^l_m = 0, \quad dJ^l_{gm} = 0$$

$$
\begin{aligned}
d\Omega - \Omega \wedge \Omega = dA - A \wedge A - da1 \\
-a1 \wedge A - A \wedge a1 + a \wedge a \\
= 0
\end{aligned} \tag{11}
$$

So the connection Ω is for a flat space. In this case there is a matrix of functions B such that $\Omega = B^{-1}dB$, where B is an an element of $O(n,m)$ for a real space or is a unitary matrix in the case of a complex space and $\Omega \in o(n,m)$. This completes the proof of the lemma.

Our result describes the situation whereby removing the source potential α from the original gauge potential leaves a flat space. We note that Ω, itself, is not diagonal. We further note, that a superposition of Abelianized potentials, defined by $\sum k_\mu = \sum da_\mu$, can form an approximation to a curvature 2-form from its constituent parts. Below, for a concrete example, we will show that spaces such as these having diagonal curvature 2-form matrices arise in the computation of the gravitational field derived from the quantum dynamical manifolds of electrons.

In general, however, we have $K = dA - A \wedge A$, and $dK = [K, A]$, a Bianchi identity, and not just $K = dA$ as in the lemma. In fact, when the A are a collection of *matrix* 1-forms, we show below that $K = K_{[\mu,\nu]}dx^\mu \wedge dx^\nu = \partial_\mu A_\nu - \partial_\nu A_\mu - [A_\mu, A_\nu]$ which is non diagonal in general. As also shown below, to regain the oft used structure $K = dA$ we must introduce gauge-covariant derivatives, as stated in the Yang-Mills Structure equations. This, however, does not lead to the closure of K ($dK = 0$) unless the gauge is $u(1)$, i.e., $[A_\mu, A_\nu] = 0$.[13] This in turn is related to obstructions to Frobenius integrability as related further below.

The third lemma requires us to introduce some of the notation inherent in the statement of Clifford's conjecture. In fact, this lemma allows us to put some of the pieces of MST fields together and compute a "center of mass or average" curvature

Lemma 3 *(Flanders [10, p.149]) Let M^n be affinely connected with curvature 2-form $K(\Lambda) = (K^\mu_{\nu[jk]}dx^j \wedge dx^k)$ defined for the tensor index set $\Lambda : (\mu,\nu)|_{\mu,\nu=0,1,2,3} = \{(\mu,\nu)\}$. Then the "average" curvature 2-form*

$$
k(\Lambda) = \sum_{i\in\Lambda}{}' (\tilde{K}^i_i + \check{K}^i_i) = \sum_{i\in\Lambda}{}' \tilde{K}^i_i[\Lambda] \equiv t_r(K[\Lambda]) \tag{12}
$$

is defined globally over M^n, independent of the local frame, is closed ($dk = 0$), and is exact ($k = da$).[14] *Here $\tilde{K}^i_i$ and $\check{K}^i_i$ are the symmetric and antisymmetric parts of $K[\Lambda]$. The primed summations indicate the appropriate normalization for Minkowski or ordinary Euclidean M^n signature for the tangent spaces is to be used.*[15,16]

Proof of Lem.3: We give an outline here. By Lem.2 the trace can be non-null. It may arise, for instance from a diagonal curvature 2-form $k1 = k_{\alpha\beta}dx^\alpha \wedge dx^\beta 1$. For motivation, a local result can be obtained immediately depending essentially on the converse of Poincaré's lemma: locally every closed form is exact. We then find a local form of Lem.3 follows from a generalization of the converse of the Poincaré lemma for a contractible manifold:

$$
\begin{aligned}
a_g(\eta,x,y,z) = (\int_0^1 G_x(\varepsilon\eta,\varepsilon x,\varepsilon y,\varepsilon z)\varepsilon d\varepsilon \wedge (\eta dx - x d\eta) + \cdots \\
+ W_x \varepsilon d\varepsilon \wedge (ydz - zdy) + \cdots) + \text{closed forms}
\end{aligned} \tag{13}
$$

---

[13]Local results, such as those obtained from the converse of Poincaré's lemma, are suitable for fields which are naturally compact such as electric (when counter-balanced so total charge is zero) and magnetic fields. Thus, the basis of electrodynamics can be considered to be Poincaré's lemma with $u(1)$ gauge.

[14]It is also the case that the r^{th} power of K, K^r, satisfies $dK^r = K^r\Omega - \Omega K^r$ and $d[t_r(K^r)] = 0$ [10, p. 149]. The trace of these powers and the sums of these quantities form what are called invariant polynomials. These invariant polynomials can be used to topologically classify the manifolds because of this invariance with respect to transformations. An example are Chern, Pontrajagin, and Euler classes [13, p.380]. In the electrodynamics of 4D spacetime manifolds, where F is the 2-form of the electromagnetic field and ${}^\star F$ its Hodge-$\star$ dual, which in a space of up to n-forms converts a p-form into an $(n-p)$- form, $dF = 0$, states that no flux tubes ever end - F does not have compact support, except for the T^2 torus or the Klein bottle, K^2. The equation $d^\star F = 4\pi^\star J$ states the number of flux tubes of ${}^\star F$ ending in an elementary volume is equal to the amount of mass in that volume. Thus ${}^\star F$ can have compact support [8, p. 105].

It is the parts of K^l_m which are closed ($dK^l_m = 0$) but not exact ($K^l_m \neq dA^l_m$) that are the source of irreversibility or non-integrability ($A \wedge dA \neq 0$). We have integrability, in general, only for the average, k.

[15]The trace operator t_r alternatively used is assumed to include the normalization for M^n effected by division by n, or in the case of Minkowski space including a division by $n-2$ so the trace of a diagonal matrix $(-1,1,1,1)$ in 4D Minkowski space is 2.

[16]See Appendix B.

Thus, for a contractible-to-a-point patch of a manifold, there are curvature potentials as claimed. But, this is not necessarily true globally.[17] As already noted, if we restrict ourselves to Riemann spaces of vanishing torsion, the trace curvature 2-form, vanishes, $k(\Lambda) = 0$, so it is the cases where the torsion does not vanish which are interesting.

The proof of the global results first establishes that the trace of a power of the curvature form is closed. The global existence of potentials for invariant polynomials[18] of the curvature 2-form such as the trace then follows as this is a special case of the general one for invariant polynomials, $P_r(K)$, of the curvature 2-form. The proof of this follows as a corollary of the Chern-Weil theorem [13, Ch.11], [14, p.530] see also [15, p.160].

We are now prepared for our restatement of the Clifford conjecture in modern terms:

Theorem 4 *Gravitational Field Potential Equations. Every affinely connected 4D MST manifold with Minkowski signature tangent space, has a local (4-vector) curvature field potential matrix,* $A(\Lambda) = (A^i_j), \{(\mu,\nu)\} \in \Lambda$, *such that* $K = dA$, *density matrix* $\rho_g(\Lambda)$, *and associated MST 4-current matrix,* $J_g(\Lambda)$, *with evolution locally constrained by inhomogeneous wave equations*

$$\Box A^i_j = -4\pi^\star J_{g}{}^{i}_{j}, \tag{14}$$

and current conservation laws

$$\nabla \cdot J_{g}{}^{i}_{j} + \frac{\partial \rho_{g}{}^{i}_{j}}{\partial \eta} = 0. \tag{15}$$

Furthermore, these relations are valid over the whole manifold for the trace $a_g[\Lambda] = t_r(A^i_j)$ *and* $k = t_r K = da_g[\Lambda]$ *is exact. Here* $\Box = -\frac{\partial^2}{\partial \eta^2} + \nabla^2$ *is the d'Alembertian or second order partial differential wave-operator;* η *is the evolution parameter.*[19]

Proof of Thm.4: The proof depends on the lemmas described above applied to any differential manifold having a Riemann curvature 2-form matrix $K[\Lambda] : (K[\Lambda]^l_{m[ik]} dx^i \wedge dx^k)$, with non-null trace, 3-form densities $J_g[\Lambda]^l_m$ and local 1-form potentials $A[\Lambda]^l_m$. Here we use the Hodge-$\star$ operator to convert a p-form in an m-dimensional space to a $(m-p)$-form. By restriction to a 4D Minkowski signature tangent space and using the definition of the Hodge-$\star$, and the 4D expression for $({}^\star d)^2 = -\Box$ in a Minkowski space, one has $4\pi^\star J_{g}{}^{\mu}_{\nu} = ({}^\star d)^2 A^\mu_\nu = -\Box A^\mu_\nu = {}^\star d^\star K^\mu_\nu$.[20] Thus, for a Minkowski manifold, Eq.(6), gives Eq.(14) of the Thm.4. Furthermore, at least locally, there are 4D- *MST waves* when $J_g[\Lambda] = 0$. The presence of non null J_g retards the expansion of the waves.[21,22] We then take the trace of this

---

[17]For instance, each closed -form on S^2 is exact, but any closed 2-form ω on S^2 is exact only under certain conditions. It is only so if the integral $\int_{S^2} \omega = 0$. This is similar to the statement that a potential is a point function. Thus, the inverse Poincaré lemma does not tell us that it is globally exact. Interestingly for the projective plane $P = RP^2$, if ω is a closed 1- or 2- form it is exact.

[18]A polynomial of r^{th} degree $P_r(x)$ is invariant if $P_r(A) = P_r(B^{-1}AB)$. Examples of these are the trace and determinants of matrices of forms and simple functions of these, such as $det(1 + i\alpha A)$, transforming as tensors.

[19]We note that once the mean curvature 2-form, $k = t_r(K^l_m)$, of a quantum system is obtained, we can compute the gravitational field of the quantum system using this theorem. How to compute the dynamically induced curvature 2-form matrix K^l_m and the dynamically induced torsion 2- form vector τ^i is developed in Part 1c for the quasi-particle dynamical geometry. We use the subscript g to remind ourselves that these relations involve gravitational quantities. The tensor indices belong to the set $\Lambda : (\mu,\nu)|_{\mu,\nu=0,1,2,3} = \{(\mu,\nu)\}$

[20]The co-ordinate free generalization of the d'Alembertian, $\Box$, is the Laplace-Beltrami operator.

[21]This defines a maximum speed of evolution via the wave equation $\Box A(\Lambda) = 0$. The Minkowski signature of the tangent spacetime arises from embedding or lifting of the evolution of a 3D space manifold into a spacetime so that on the light cone (defined by the co-ordinate u and the kinematical condition $du^2 = -v^2 d\eta^2 + ds^2 = 0$), $ds/d\eta = v$, the speed of free waves. One usually uses for the free wave speed $\max(v) = c$ as this speed depends only on the flat structure of the tangent space and the units of evolution and space chosen. All wave equations in the tangent space have the same form, so all the speeds of light are the same for a connected manifold. Thus the wave equation describes the geometry. Thus, it is the geometry that determines the single value of the free wave propagation. This geometry ultimately is determined by a local symmetry in the tangent space.

[22]The equations $4\pi J_G[\Lambda] = d^\star K[\Lambda]$ and $dK = 0$ for a 4D Minkowski space are homeomorphic to Maxwell's equations or a quantum dynamical manifold equation (QDME) for the fundamental (3D) representation of the color algebra $so(3)$ [4]. *Homeomorphism* does not imply identity. Only the topological structures and the structure of the dynamics are (topologically) equivalent-up to a smooth map.

result. According to Lem.3, this trace is an invariant polynomial of the Riemann curvature 2- form and is defined over the whole manifold. Thus, if we furthermore define $4\pi^\star t_r(J_g{}^\mu_\nu) \equiv 4\pi^\star j_g(\Lambda)$, we obtain for $a_g[\Lambda] \equiv t_r(A[\Lambda]^l_m)$ the relation $4\pi^\star j_g(\Lambda) = (^\star d)^2 a_g(\Lambda) = -\Box a_g(\Lambda) = {}^\star d^\star k(\Lambda)$ globally, over the whole manifold for the 2-form k.[23] Thus, locally we have a vector-tensor theory of gravitation. Globally only a vector potential, $t_r A = a_g$, exists. This is generally non null only for non Riemannian spaces. Note also that on the right-hand side of Eq.(14) is a current, not the symmetric energy-momentum tensor, T^i_j, or its *deviatoric part*: $T^i_j - g^i_j t_r T^i_j$. The existence of a non null, globally defined, 2-form connection or in physical terms, a potential, a_g, from which we can find the average curvature, is very important from a physical standpoint; it allows us the freedom to define a global field of frames for the whole manifold. Depending on the evolution, when a_g asymptotically vanishes so we have a flat space, the global frame field at each spacetime point can be shown to be an element of an orthogonal group, e.g., $O(n, 1)$. This connection can be related to Lorentz transformations. When we include the action of displacements the result can be related to the Poincaré group of special relativity. As shown in Lem.2 the global potential can be removed from a space with diagonal curvature or one can Abelianize the gauge to remove the local curvature thereby obtaining a locally Lorentz invariant space. To complete the proof of the theorem, which states a property of gravitational fields, we need to specify the source of the currents. This is deferred until we determine their generation from an underlying dynamics.

The structure of Thm.4 admits a variational formulation:

Corollary 5 *Every affinely connected 4D MST manifold with Minkowski signature tangent space, has a local (4-vector) curvature field potential matrix,* $A(\Lambda) = (A^i_j), \{(\mu,\nu)\} \in \Lambda$, *such that* $K = dA$, *density matrix* $\rho_g(\Lambda)$, *and associated MST 4-current matrix,* $J_g(\Lambda)$, *whose components satisfy a variational problems with actions*

$$S[\Lambda] = \int (-\frac{1}{8\pi} K \wedge {}^\star K - J_g \wedge A) \tag{16}$$

$$= \int (-\frac{1}{16\pi} K_{\mu\nu} K^{\mu\nu} + j_\mu A^\mu) d\eta \wedge dx^1 \wedge dx^2 \wedge dx^3 \tag{17}$$

Proof: This follows along conventional lines of electromagnetism for each component. It is essential that Eq.(15) be satisfied in order for the action to be invariant with respect to gauge changes. The action S is the integral of a polynomial which is invariant with respect to gauge transformations. When the topology of an evolving manifold changes, the continuity equation Eq.(15) does not hold. The result holds for the 2-form average curvature k as well.

For spaces without torsion, the matrix of curvature 2-forms is antisymmetric and therefore has exactly vanishing trace and trace potential. That is, for a Riemann space (a space with vanishing torsion), we have the symmetry $(K^l_{m[ik]} dx^i \wedge dx^k) = -(K^l_{m[ik]} dx^i \wedge dx^k)^T$, consequently the trace of the curvature 2-form matrix vanishes identically, so a Riemann space is on the average flat, perhaps in the sense Clifford imagined. If we consider a central region having nonvanishing torsion, surrounded by one with vanishing torsion, we encounter the possibility of discontinuous jumps in the trace curvature potential. That is the existence of trace curvature potential wells. The trace potential is only defined for the curvature up to the addition of a gauge shift, $d\varphi$, i.e., $k = d(a_g + d\varphi) = da_g$, so the same average curvature is obtained despite the gauge shift. Thus, we can adjust the potential a_g so that it is continuous across boundaries. By combining the two lemmas and the theorem above we then have

Corollary 6 *In regions of vanishing torsion (Riemann spaces)*

$$d^\star K^i_j \equiv 4\pi J_g{}^i_j, \quad dJ_g{}^i_j = 0, \quad dK^i_j = [K^j_v, \omega^v_m] = 0 \tag{18}$$

$$k = t_r(K^i_j) = 0, \quad t_r(A^i_j) \equiv a_g \doteq 0, \tag{19}$$

$$4\pi^\star J_g{}^i_j = {}^\star d^\star K^i_j = {}^\star d^\star dA^i_j \equiv (^\star d)^2 A^i_j. \tag{20}$$

The first two sets of equations are global, the last set holds locally, by the converse of the Poincaré lemma. The last equation states that the A^i_j are harmonic functions. In the static limit, locally, these satisfy a Poisson equation. Since (K^i_j) is an antisymmetric matrix of 2-forms in the torsion-free region by the corollary each component satisfies the Maxwell structure equations. We call this a $F_{\mu\nu}$ *structure*. For Riemann spaces, the trace curvature potential is of no interest since it vanishes identically because the Riemann curvature tensor is also antisymmetric in its first two indices. This is sufficient reason to formulate a physical theory on another basis that this special symmetry.

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[23] On an odd-degree form in a Minkowski space $(^{\star\star}) \to -1$.

We now push the mathematical Maxwell structure relations further by noting that in 4D Minkowski signatured spaces, the antisymmetric factor $dx^i \wedge dx^j$ appearing in any component of the curvature 2-form, $K[\Lambda]$, or in its trace, $k = t_r(K^i_j) = k_{[\mu\nu]} d\xi^\mu \wedge d\xi^\nu$, allows us to write any component of the curvature 2-form matrix or the trace in a fashion analogous to the electromagnetic field in what we call a *Maxwell structure* or a $F_{\mu\nu}$*-structure*:

$$F_{\mu\nu} = \begin{pmatrix} 0 & G_1 & G_2 & G_3 \\ -G_1 & 0 & W_3 & -W_2 \\ -G_2 & W_3 & 0 & W_1 \\ -G_3 & W_2 & -W_1 & 0 \end{pmatrix}, \tag{21}$$

thereby allowing the introduction of currents, ρ, and densities, j, (ref. [8, p.113], ref. [9])[24]

$$d^\star K[\Lambda] = \sum_{\mu=0}^{3} \frac{\partial K^{[j}_{k]}}{\partial x^\mu}(dx^\mu \wedge dx^j \wedge dx^k) \equiv 4\pi(\rho dx^1 \wedge dx^2 \wedge dx^3$$
$$-j_1 d\eta \wedge dx^2 \wedge dx^3 - j_2 d\eta \wedge dx^3 \wedge dx^1 - j_3 d\eta \wedge dx^1 \wedge dx^2) \tag{22}$$
$$\equiv 4\pi dL. \tag{23}$$

For the full matrix MST field 2-form $K = (K^j_l)$, this involves writing as a matrix each component of the tensor indexed by the set $\Lambda = \{j, l\}$. For the trace, the ρ and j_i are not matrices. In Eq.(22) we have used the definition of the Hodge-$\star$ operator in an m-dimensional space for converting a p-form into an $m-p$ one, using a local index raising tensor, g^{ij}. Since there is a (pseudo-) orthogonal moving frame at every point (the local Minkowski tangent space moving frame), it is used to define the g^{ij} needed in evaluating the $(m = n+1)$D = 4D-MST fluxes $4\pi dL = d^\star K[\Lambda]$. In the case of multiple sheeted manifolds, we must keep track of the sheet we're on. The quantity $d^\star K \equiv 4\pi J_g[\Lambda] \equiv 4\pi dL$ on the left of Eq.(22) is the divergence of physical components of a force.[25] On the right side is the resulting MST flux or mass-flux-density, the 3-form, J_g. The 3-form density J_g is written as a 4-vector (ρ_g, j_g) in an abuse of notation to emphasize the differing physical nature of the spatial density and the current terms respectively of J_g. Below we find that the *gravitational-fields* or G-fields are analogous to electric fields (E) and the *whirl fields* or W-fields are analogous to magnetic fields (H). The resulting curvature trace field tensor is Lorentz invariant also. Therefore, we write a component of the 2-form matrix $K[\Lambda]$ in the same way as the electromagnetic field tensor $F_{\mu\nu}$ is written in terms of E- and H-fields. we next consider the average curvature 2-form $k = t_r(K^i_j) = k_{[\mu\nu]} d\xi^\mu \wedge d\xi^\nu$ as defining the curvature tensor of a Riemannian metric space. As such we can compute the metric of this space using standard tensor analysis. The equations above give us a way of relating the energy-momentum of the space to the globally defined potentials. We therefore introduce the following definition

Definition 7 *Underlying dynamical (non Riemannian) geometry. Let a manifold be given with a curvature tensor,* K^l_{mik} *such that* $K^l_{ijl} \equiv K_{ij}$ *and* $\tilde{g}^{ij} K_{ij} \equiv K$*, a torsion tensor,* T^j_{lh}*, and torsion vector* $t_l \equiv T^j_{jl}$*, defining the* underlying non Riemannian geometry *then the antisymmetric average curvature tensor components,* k_{lh}*, are given by [12, p.94] the following two groups of equations and definition of mass current, J:*

$$k_{lh} = K_{hl} - K_{lh} + \nabla_j T^j_{lh} + \nabla_h t_l - \nabla_l t_h + t_j T^j_{lh}, \tag{24}$$
$$= G_{hl} - G_{lh} + \nabla_j T^j_{lh} + \nabla_h t_l - \nabla_l t_h + t_j T^j_{lh}, \tag{25}$$
$$G_{hl} \equiv G_{hl} - \frac{1}{2}\tilde{g}_{hl} K. \tag{26}$$

Here the tensor G_{hl} is a generalized Einstein tensor whose expression here is strictly natural only for a 4D MST. Then, these lead to an average gauge potential 1-form $A = A^l dx^l$ and induced average MST 4-current, J are given by the following:

---

[24] The following arguments also apply to other 2-forms, e.g., the torsion $T^i = T^i_{jk} dx^j \wedge dx^k$. One then finds a conservation law for torsion. Analogously, if components of the local torsion potential exist, one finds $4\pi {}^\star J_T{}^\mu = ({}^\star d)^2 H^\mu = -\Box H^\mu = {}^\star d^\star T^\mu$. Locally, given the curvature, we can find the torsion by solving the Bianchi identities.

[25] $4\pi L = {}^\star K$ and $K = dA$ differ significantly in 4D. The 2D domain of support for K cannot be compact without boundary. The torus, T^2, and the Klein bottle, K^2, are the only exceptions. We show elsewhere that the topology of the underlying dynamical manifold of a photon is homeomorphic to a K^2. This is consistent with the field collapse requirements of the photoelectric effect or photon absorption in finite volumes of spacetime. The domain of support of L can be non compact without boundary or compact with boundary as in electromagnetic theory.

$$k_{ls} = \frac{\partial A_s}{\partial x^l} - \frac{\partial A_l}{\partial x^s} = \frac{\partial \Gamma^m_{ms}}{\partial x^l} - \frac{\partial \Gamma^m_{ml}}{\partial x^s}, \tag{27}$$

$$\Box A = -4\pi J. \tag{28}$$

As a result of this computation, the particle paradigm can be invoked by defining density and velocity vectors

$$J \equiv (\rho u^i) = (\frac{c\rho}{\sqrt{1-\beta^2}}, \frac{\rho \mathbf{u}}{\sqrt{1-\beta^2}}). \tag{29}$$

The density here can be considered to be constant, but the volume of MST changes. We are thus led to propose the following

Definition 8 *Induced Riemannian geometry. Given* 4*-currents and consequently* $J \equiv (\rho u^i)$ *obtained from an underlying dynamical geometry, we consequently have a MST current due solely to "gravitational effects" which is due to the evolution of the spacetime. Thus, it has force-free geodesical equations of motion for Lagrangian marker particles:*

$$\frac{\partial\left(\rho u^i\right)}{\partial t} + \tilde{\Gamma}^i_{jk}\frac{\partial\left(\rho u^j\right)}{\partial t}u^k = 0 \tag{30}$$

These define the geodesics of an induced Riemannian geometry *where the* 2*-form curvature* $k_g = t_r(K^i_j) = k_{[\mu\nu]}dx^\mu \wedge dx^\nu$ 1*-form gauge potential,* A*, and* 3*-form current* j_g *satisfy the Riemann space requirements*

$$d^\star k_g \equiv 4\pi j_g, \quad dj_g = 0, \quad dk_g = 0 \tag{31}$$

$$\kappa_g = t_r(k_g) = 0, \tag{32}$$

$$4\pi^\star j_g = {}^\star d^\star k_g = {}^\star d^\star dA \equiv ({}^\star d)^2 A. \tag{33}$$

The symmetric connection coefficients, $\widetilde{\Gamma}^m_{ik}$, used here are defined consistently by a metric, $\tilde{g}_{ij}$, via the usual means [19, p.310][26] and for which one has the Bianchi identity used in GRT for mass free regions and other identities for Riemann spaces:

$$\widetilde{\Gamma}^m_{ik} = \frac{1}{2}\tilde{g}^{jm}(\partial_k\tilde{g}_{ij} + \partial_i\tilde{g}_{kj} - \partial_j\tilde{g}_{ik}) \tag{34}$$

$$\tilde{\nabla}_j\tilde{g}^{jm} = 0, \tag{35}$$

$$\nabla_i(\tilde{R}^i_j - \frac{1}{2}\tilde{g}^i_j\tilde{R}) = 0, \tag{36}$$

$$\tilde{R}^j_{ikl} = -\tilde{R}^j_{ilk}, \tag{37}$$

$$\sum_{(ikl)}\tilde{R}^j_{ikl} = 0, \tag{38}$$

$$\sum_{(mkl)}\tilde{\nabla}_m\tilde{R}^j_{ikl} = 0, \tag{39}$$

$$\tilde{R}_{ijkl} = -\tilde{R}_{jikl}, \tag{40}$$

$$\tilde{R}_{ijkl} = \tilde{R}_{klij} \tag{41}$$

In 4D spacetime in GRT we use the equation $(\tilde{R}^i_j - \frac{1}{2}\tilde{g}^i_j\tilde{R}) = 0$, and taking into account all symmetries noted above we find ten partial differential equations for the ten metrical coefficients g_{ij}. Such a system essentially is a constraint on the geometry to obey the relevant Bianchi constraints. The constraint $\nabla_i(\tilde{R}^i_j - \frac{1}{2}\tilde{g}^i_j\tilde{R}) = 0$, however, removes four of these equations, giving us only six. These are customarily augmented by the harmonic coordinate condition g^{ik} $\widetilde{\Gamma}^m_{ik} = 0$. The effects of actual mass curving spacetime is not included unless it comes from boundary conditions.

[26] As noted in Appendix B, in Riemann spaces where frame transport by $e_i(g_{ij})$ in left- and right-handed loops is different, then $[e_i(g_{ij}), e_j(g_{ij})] = c^k_{ij}e_k(g_{ij})$. This implies that we have two, slightly different metrics and hence connections, one for left-handed and one for right-handed motions. The equations Eq.(34-41) must then be used for each of these cases. In the case where the left- and right-handed systems are different, we have a natural connection between this parity noninvariance and the existence of gravity in the sense that k_{ij} is non null.

These equations in GRT are called the mass-free solutions. For instance, by assuming spherical symmetry and static metric coefficients, Schwarzschild found a metric system which has found use in understanding the geometry induced by a spherical inclusion in a spacetime.

Because of the existence of Maxwell structures for k and K^i_j, we see that the curvature, itself, produces what can be interpreted as forces acting on "inertial particles". This notion can be made more exact if we write the Lorentz force equation[27] for a test particle of mass m_p directed in its motion along an average velocity field, u^ν, determined as a Frenet-Serret equation:

$$\frac{m_p g_N}{c} k^\mu_\nu u^\nu = \frac{m_p g_N}{c} t_r(K)^\mu_\nu u^\nu \equiv \hat{\Omega}^\mu_\nu u^\nu \equiv F^\mu = m_p \dot{u}^\mu, \quad dK = 0 \tag{42}$$

Here $(u^\mu) = (\frac{c}{\sqrt{1-\beta^2}}, \frac{\mathbf{v}}{\sqrt{1-\beta^2}})$. This equation in 4D relates velocity-space spacetime geometry on the left to the concept of a force field on the right. The latter is a generalization of the notion arising from the (Newtonian) dynamics of point particles in accelerated motion along paths in 3D space. We can embed the motion of a point particle of mass m_p into a velocity-spacetime by introducing a new co-ordinate, u, such that $du^2 = -v(x)^2 dt^2 + dx^2$. The geodesics in the resulting space, $du^2 = 0$, $u = const.$, correspond to the satisfaction of the kinematical constraint $\frac{dx}{dt} = \pm v$. In order to relate the theory of the motion of point particles to the manifold approach, we imagine a velocity field is defined over the manifold. This field is to be chosen to allow the insertion of appropriate rationalizing constants for adjusting the units chosen for the gravitational and inertial effects of the motion of a hypothetical point particle along a curve. This is done so we have the convenience of having the same quantity m_p, whether measured using the force applied to accelerate particles by gravitational forces or using the force applied by springs, etc. In mathematical terms, this implies using Newton's 2^{nd} law and Eq.(42),

$$\frac{d}{dt}\frac{m_p \mathbf{v}}{\sqrt{1-\beta^2}} = m_p g_N (\mathbf{G} + \frac{\mathbf{v}}{c} \times \mathbf{W}), \tag{43}$$

$$\frac{d}{dt}\frac{m_p c}{\sqrt{1-\beta^2}} = m_p g_N \mathbf{G} \cdot \mathbf{v}/c, \quad \beta^2 = (v/c)^2\,, \tag{44}$$

$$-\left(\frac{c}{\sqrt{1-\beta^2}}\right)^2 + \left(\frac{\mathbf{v}}{\sqrt{1-\beta^2}}\right)^2 = -c^2. \tag{45}$$

Here we have deviated from our usual notation by bolding the 3-vectors for emphasis. The last equation introduces a kinematical constraint chosen so that the 4-velocity vectors have constant magnitude and thus are geodesics, $dc = 0$, in the spacetime. It is clear that we can include the proportionality constant, g_N, relating the trace curvature fields (G, W) to the motion of hypothetical point particles in the definition or scaling of these fields to eliminate its appearance in these equations. Since G contains the factor of the gravitational mass of the source, M, if we normalize to M, then the factor g_N in these equations must be modified. One replaces it with Newton's universal gravitational constant G_N. In the first two equations above, if we compute the left-hand side quantum mechanically $j = \rho\mathbf{v}$, then compute the right-hand side using the currents from another quantum dynamical body interacting with the firse in a frame where the quantum system is being accelerated by the mutual gravitational field, we should be able to compute the mass of the source. As the gravitational constant is an invariant over all such interactions (asymptotically at a large distane), we should be able to extract it also.

[27] We can form the Maxwell stress tensor

$$\begin{aligned}\tau_{ij} &= -G_i G_j - W_i W_j + \frac{1}{2}\delta_{ij}(\|G\|^2 + \|W\|^2), \quad i,j = 1,2,3 \\ \tau_{00} &= \frac{1}{2}(\|G\|^2 + \|W\|^2), \\ \tau_{0i} &= (G \times W)_i.\end{aligned}$$

Then the Lorentz force is

$$f^\mu = \nabla_\nu \tau^{\nu\mu}.$$

To properly include the time dependence of the propagation of gravitational waves for the case of a relatively moving mass (i.e. looking at the body not centered on its moving frame), we must use the Liénard-Wiechert potentials [22, p.475]:

$$\varphi = \frac{1}{4\pi}\frac{1}{r\,(1-v_r/c)}, \quad \mathbf{A} = \frac{1}{4\pi}\frac{\mathbf{v}}{r\,(1-v_r/c)}. \tag{46}$$

Here v_r is the radial speed, $\mathbf{v}$ is the relative velocity and c is the speed of light in a vacuum. This indicates that away from the moving frame reference we will need to include the currents which vanish relative to the moving frame. Next we extend the definitions introduced above for the mass and current to the general case where the currents and density are simply due to the behavior of the MST itself.

C. Definition of Induced Currents and Densities of a Spacetime

Using the definitions, equations for the G- and W- fields are obtained from Eq.(22):

$$\begin{aligned}
4\pi\rho_g &= +\left(\frac{\partial G_x}{\partial x} + \frac{\partial G_y}{\partial y} + \frac{\partial G_z}{\partial z}\right) = \nabla\cdot G \\
4\pi j_x &= -\left(\frac{\partial G_x}{\partial \eta} - \frac{\partial W^z}{\partial y} + \frac{\partial W^y}{\partial z}\right) = -\frac{\partial G_x}{\partial \eta} + (\nabla\times W)_x \\
4\pi j_y &= -\left(\frac{\partial G_y}{\partial \eta} - \frac{\partial W^x}{\partial z} + \frac{\partial W^z}{\partial x}\right) = -\frac{\partial G_y}{\partial \eta} + (\nabla\times W)_y \\
4\pi j_z &= -\left(\frac{\partial G_z}{\partial \eta} - \frac{\partial W^y}{\partial x} + \frac{\partial W^x}{\partial y}\right) = -\frac{\partial G_z}{\partial \eta} + (\nabla\times W)_z
\end{aligned} \tag{47}$$

By this stratagem the components of (K^i_j) can be interpreted as being non trivially analogous to an electromagnetic field, that is mapped into a $F_{\mu\nu}$ structure, i.e., satisfying the Maxwell structure equations. Furthermore, in regions of nonvanishing torsion, even the nonvanishing trace nontrivially satisfies the Maxwell structure equations and Eq.(47). Furthermore, each element of the remaining, deviatoric part, i.e., with vanishing trace, $K^i_j - g^i_j t_r(K^i_j)$, can be mapped into a $F_{\mu\nu}[\Lambda]$ structure. In regions where the torsion vanishes, (K^i_j) is an antisymmetric matrix of 2-forms, and we can go further by mapping all its parts into a *single* $F_{\mu\nu}$ structure by a suitable matrix basis. In a 4D MST this consists of 4×4 anti-symmetric real valued matrices. In each case we can satisfy Eq.(47). Since these equations hold for all tensor indices and the Bianchi identities hold in every region, in the region of vanishing torsion, the Schwarzschild metric can be used.

To further understand these results, we give a physical interpretation of the gradient term in Eq.(47), so that we can use the usual Newtonian force-mass language, by introducing for the induced geometry

Definition 9 *Mass-density. Given a gravitodynamical field consisting of induced Riemann G-vector and W-vector fields in a 4D MST, the induced mass-density, $\rho_g(x,y,z,\eta)$, is found from the Poisson equation*[28]

$$\nabla\cdot G[\Lambda] = -\nabla^2\Phi[\Lambda] = 4\pi\rho_g(x,y,z,\eta)[\Lambda] \tag{48}$$

According to Eq.(15) the induced mass-density satisfies a continuity equation. Thus, together with Eq.(48), the gravitodynamical mass-density, $\rho_g[\Lambda]$, satisfies the usual conditions for a conserved mass-density. Therefore, Eq.(48) provides a consistent definition of a mass-density. As a consequence ρ_g (or MST) can expand or contract, but a MST can only leak on its boundaries (evanescent waves, tunneling, black holes or big bangs). These boundaries define the genus of the multiply connected manifold. A result of this definition is that if there can are regions of the gradient of G having both signs then the induced mass-density can have both signs. Thus, in applying this definition to understand

[28]In the Lorentz gague -$\nabla\Phi = \mathbf{G} - \frac{\theta\mathbf{A}}{\partial\eta}$, hence, -$\nabla^2\Phi = \nabla\cdot\mathbf{G} - \frac{\theta\nabla\cdot\mathbf{A}}{\partial\eta}$ so one could argue for a "displacement" mass as well. But one often assumes $\nabla\cdot\mathbf{A} = 0$. We have chosen to have the outward normal positive in the definition as occurs in the standard convention used for compact mass. Note that $G[\Lambda] \longleftrightarrow G^j_m$ can be used to define a cross-density involving $j \neq m$. There are corresponding cross-currents $J_{gm}^{\ j}$ that may not vanish. These are associated with quantum mechanical effects, not present in the mean field quantities involving the trace of the curvature 2-form.

the "properties" of CDM, we must be prepared for the surprise of negative mass. Therefore, we emphasize that this definition is an interpretation of a mathematical result that a gradient has a source. We find below that compact densities computed from the quantum dynamics are positive.

As mentioned in the introduction, to find the induced geometry, first the quantum dynamical manifold is computed, then the average curvature 2-form $k = t_r(K^{\mu}_{\nu[\alpha\beta]}dx^{\alpha} \wedge dx^{\beta})$. This determines a Riemann space allowing the computation of the macroscopic gravitational G- and W-fields and, finally, the induced (cold dark matter) densities, $\rho_g[\Lambda]$. The unaveraged dynamical extension of gravitation theory, Thm.4, holds regardless of this definition. There is certainly a difference in the manifolds defined by the average curvature of the induced geometry compared to the local curvature. The average curvature, k, defining the induced gravitational field satisfies a global continuity equation and has a globally defined 4-potential. It is therefore *the* average field which is most closely related to the classical definition of a gravitational field. The fact that the average curvature is a trace invariant leads to the consequence that substantially different materials can have the same gravitational fields, in accord with experiment. Furthermore, in the rest frame, relative to a global Lorentz transformation, this trace is the same. Because the average curvature field, k, and its potential a_g satisfy the equations $k = da_g$ and $d^{\star}k = -4\pi j_g$ in 4D spacetime, thus the topological results of Gauss, Ampere and Biot-Savart apply to gravitational fields. The first is in accord with experiment, the latter extends gravitation theory into the realm where mass-spacetime have currents, J_g. The fact that torsional structures can be spontaneously created if the manifold has sufficient curvature as shown below suggests possible mechanisms for big bang phenomena, particle and local pair creation.[29]

D. Generalization to Yang-Mills Equations for Quantum Geometry

Here we generalize the preceding field equations by determining how the non exact case can be written in a form that appears exact. This will enable us to connect the geometry of MST to contemporary approaches to the problem of understanding the structure of MST. It will also allow us to relate the gauge potentials and currents having other than Abelian $u(1)$-symmetry. We do this also so that we can appreciate the importance of the Abelian case for the generation of currents and potentials. In the case studied so far we have, at least locally for the trace curvature,

$$t_rK = k = da_g \tag{49}$$

For the other components of the curvature and in the case when K is not exact we have only the definition with a local connection matrix A

$$K = dA - A \wedge A. \tag{50}$$

This arises also in a space whose points are matrices signalling the presence of a preserved symmetry. In this case there are a collection of matrices $A = (A^j_l)$ of 1-forms, and we can expand A in terms of the bases of the matrix algebra with 1-form elements. In this case we write $A = A_{\mu}dx^{\mu}$ where the components of the gauge potential A_{μ} are matrices. We have

$$\begin{aligned} A \wedge A = A_{\mu}A_{\nu}dx^{\mu} \wedge dx^{\nu} &= \frac{1}{2}(A_{\mu}A_{\nu} - A_{\nu}A_{\mu})dx^{\mu} \wedge dx^{\nu} \\ &= \frac{1}{2}[A_{\mu}, A_{\nu}]dx^{\mu} \wedge dx^{\nu} \equiv [A, A]. \end{aligned} \tag{51}$$

In tensor components we have defined

$$K = K_{[\mu,\nu]}dx^{\mu} \wedge dx^{\nu} = (\partial_{\mu}A_{\nu} - \partial_{\nu}A_{\mu} - [A_{\mu}, A_{\nu}])\, dx^{\mu} \wedge dx^{\nu}. \tag{52}$$

In order to fit Eq.(50) into the scheme defined in Eq.(49), we consider Eq.(49) to represent an Abelian group case with diagonal matrix representation. One can verify, under a transformation of basis functions by B, that

$$K' = BKB^{-1} \tag{53}$$

[29] The fact that underlying the average curvature there is a true quantum dynamical manifold suggests that the usual quantum mechanical tunneling and other phenomena can occur on the quantum scale via the expected mechanisms, albeit extended to a full theory of quantum dynamical manifolds.

if and only if

$$A' = BAB^{-1} + B^{-1}dB. \tag{54}$$

The reasonableness of this result encourages one to introduce a gauge covariant derivative defined by

$$(D_A\omega)^l = d\omega^l + \sum_{m=1}^{n} A^l_m \wedge \omega^m, \qquad l = 1, ..., n. \tag{55}$$

$$= \nabla_\mu \phi^l(x)dx^\mu \equiv (\partial_\mu + A^c_\mu \gamma_c)\phi^l(x)dx^\mu, \qquad \gamma_c \in g. \tag{56}$$

The matrix of 1-forms (A^l_m) is a representation of the g-valued 1-form A under which ω transforms. Note this assumes a specific symmetry for the form ω.

We can express this in terms of differential forms, if we write

$$A = A^c_\mu \gamma_c dx^\mu, \tag{57}$$

then, defining as in Eq.(50)

$$K = dA - A \wedge A = K_{[\mu\nu]}dx^\mu \wedge dx^\nu, \tag{58}$$

we can define a covariant derivative so that

$$D_A K = dK - [A, K] = 0 \tag{59}$$

because of the first Bianchi relation. This is the analog to the equation $dF = 0$ in electromagnetism, a $u(1)$-gauge theory.

If many symmetries are simultaneously present, then we must employ a generalization to capture the invariance of the manifold with respect to transformations of these other symmetries. This leads to more terms in the gauge covariant derivative:

$$(D_A\omega)^l = \nabla_\mu \phi^l(x)dx^\mu, \tag{60}$$

$$\nabla_\mu \phi^l(x) = \partial_\mu \phi^l(x) + \sum_{q=1}^{n_s} \left(\frac{\kappa_q}{\hbar c}\right) \sum_{c=1}^{n_q} \sum_{k=1}^{d} A^c_{\mu,q}(x)(\gamma^{(d)}_{c,q})^l_k \phi^k(x) \tag{61}$$

Here q includes the gauge groups, numbering n_s, d, is the dimension of the group, and n_q is the order of the group. The quantities $\left(\frac{\kappa_q}{\hbar c}\right)$ are coupling constants which must be inserted to account for the relative strength of different group symmetries present.

Furthermore, we have, using Eq.(57) in Eq.(52)

$$K^a_{\mu\nu}\gamma_a = \left(\partial_\mu A^a_\nu - \partial_\nu A^a_\mu - c^a_{bc}A^b_\mu A^c_\nu\right)\gamma_a \tag{62}$$

Here γ_c is an element of (a matrix representation of a generator of) the Lie algebra g and the c^a_{bc} are the structure constants of the Lie algebra: $[\gamma_a, \gamma_b] = c^a_{bc}\gamma_c$. If we consider each element of the multiplet labeled by a separately and write

$${}^{(a)}\Gamma^\kappa_{\lambda\nu} \equiv (A^a_\nu)^\kappa_\lambda, \tag{63}$$

then Eq.(52) or Eq.(62) can be written as

$$\begin{aligned}({}^{(a)}K_{\mu\nu})^\kappa_\lambda &= \left(\partial_\mu {}^{(a)}\Gamma^\kappa_{\lambda\nu} - \partial_\nu {}^{(a)}\Gamma^\kappa_{\lambda\mu} - c^a_{bc}\, {}^{(b)}\Gamma^\kappa_{\varepsilon\mu}\, {}^{(c)}\Gamma^\varepsilon_{\lambda\nu}\right) \\ &= \left(\partial_\mu {}^{(a)}\Gamma^\kappa_{\lambda\nu} - \partial_\nu {}^{(a)}\Gamma^\kappa_{\lambda\mu} + {}^{(a)}\Gamma^\varepsilon_{\lambda\nu}\, {}^{(a)}\Gamma^\kappa_{\varepsilon\mu} - {}^{(a)}\Gamma^\kappa_{\varepsilon\mu}\, {}^{(a)}\Gamma^\varepsilon_{\lambda\nu} + \cdots\right)\end{aligned} \tag{64}$$

Where we introduce for each space labeled by (a) each with co-ordinates labeled by $(\kappa, \lambda, ...)$, we can consider this expression as one for the components of the curvature tensor, $({}^{(a)}(K_{\mu\nu})^\kappa_\lambda = {}^{(a)}K^\kappa_{\lambda\mu\nu}$of the manifolds (a). The dimension of the manifold is the dimension of the representation matrices. The ellipsis includes the interaction of the spaces twisting or entangling them together. This interpretation is in keeping with the notion in GRT that matter fields curve (and twist) spacetime. Notice that in these expressions the connection coefficients, $\Gamma^\kappa_{\varepsilon\mu}$, have no special symmetry in the lower indices. In fact, for a Lie group (here $[\gamma_\alpha, \gamma_\beta] = c^\delta_{\alpha\beta}\gamma_\delta$ (and in particular for $SU(2)$ where the $c^\delta_{\alpha\beta} = \epsilon_{\alpha\beta\delta}$−the totally anti-symmetric Levi-Civita symbol in 3D- are anti-symmetric in all indices) these spaces have a torsion, $T^\alpha_{\beta\delta}\,(= -2\epsilon^\alpha_{\beta\delta}$ or $+2\epsilon^\alpha_{\beta\delta})$, and a curvature $R^\alpha_{\beta\delta\kappa}(= 0)$. We can also define the Killing metric[30] (for $SU(2)$) $g_{\alpha\beta} = c^\gamma_{\alpha\delta}c^\delta_{\beta\gamma}(= -2\delta_{\alpha\beta})$ on these spaces, but in terms of this metric, the torsion vanishes. However, such symmetry is broken when we examine physical particles in detail.

Nevertheless, we have the following

Theorem 10 *Yang-Mills Structure Equations. In terms of the gauge-covariant derivative the curvature* 2*-form satisfies*

$$K = D_A A, \tag{65}$$

$$D_A\, {}^\star K = 4\pi J, \tag{66}$$

$$D_A K = dK - [A, K] = 0 \tag{67}$$

Proof: The first three lines are the covariant derivative generalization of Thm.4. As indicated, these equations for the geometric curvature are the famous Yang-Mills equations. Here a MST geometric interpretation is used – this is the only unique aspect of this result. The current J has components, say, J^a_μ which represent a matrix current or a multiplet current.

It is important to realize that we don't have a Poincaré lemma, since $D^2_A H = d(dH - A \wedge H) - A \wedge (dH - A \wedge H) = -(dA - A \wedge A) \wedge H = -K \wedge H \neq 0$. Keeping this in mind and recalling the definition of currents used above, $d^\star K = 4\pi J$, so automatically $dJ = 0$, but considering $D_A\, {}^\star K = d\, {}^\star K - [A, {}^\star K]$, we would like to define a new current that is conserved. This we do by setting

$$d^\star K = D_A\, {}^\star K + [A, {}^\star K] = 4\pi J + [A, {}^\star K] = 4\pi \widetilde{J}. \tag{68}$$

Poincaré's lemma assures $d\widetilde{J} = 0$. Alternatively, in the case of Eq.(64), the first four terms for each element labeled by (a) defines a non interacting current for which we can conserve $4\pi^{(a)}J = d^{2\, \star(a)}K = 0$. Now let us take the trace on $(\kappa\lambda)$. We obtain:

$$\begin{aligned} ({}^{(a)}K_{\mu\nu})^n_n dx^\mu \wedge dx^\nu &= \left(\partial_\mu{}^{(a)}\Gamma^n_{n\nu} - \partial_\nu{}^{(a)}\Gamma^n_{n\mu} - c^a_{bc}\,{}^{(b)}\Gamma^n_{\varepsilon\mu}\,{}^{(c)}\Gamma^\varepsilon_{n\nu}\right) dx^\mu \wedge dx^\nu \\ &= \left(\partial_\mu{}^{(a)}\phi_\nu - \partial_\nu{}^{(a)}\phi_\mu\right) dx^\mu \wedge dx^\nu - \left(c^a_{bc}\,{}^{(b)}\Gamma^n_{\varepsilon\mu}\,{}^{(c)}\Gamma^\varepsilon_{n\nu}\right) dx^\mu \wedge dx^\nu \qquad (69) \\ &= d({}^{(a)}\phi_\mu dx^\mu) - \left({}^{(b)}\Gamma^n_{\varepsilon\mu}\, c^a_{bc}\,{}^{(c)}\Gamma^\varepsilon_{n\nu}\right) dx^\mu \wedge dx^\nu \qquad (70) \end{aligned}$$

This expression gives the curvature 2-form in terms of the exterior derivative of a 1-form $\{\alpha = {}^{(a)}\phi_\mu dx^\mu\}$ and cross-contamination terms (not expressible as the exterior derivative of a 1-form) arising from the algebra. We see that the cross-contamination destroys the free (massless) currents analogous to the photon arising from the Maxwell structure. There is no guarantee that the potentials are actually scalers as they represent contractions of a connection. Given a curvature we can use the expression above to make a least squares best fit. The physical picture is a collection of interacting currents having a certain algebra that is approximately satisfied. We will not be using these results but it is interesting to see how such *Abelianized Yang-Mills currents* arise and how one can introduce interactions between them slowing them. In the quantum dynamical example studied below the symmetry algebra is $su(2)$ but there is a natural way to form a conserved scaler-vector current from the gauge fields which simultaneously Abelianizes the potential.

We can extend the results of the theorem to a variational formulation as follows. Let $D_A J = 0$, for gauge fields A and scaler fields φ, we then have the following variational action generalizing the Abelian case (for a single non Abelian gauge field: $A(x) = \sum A^a_\mu(x)\gamma_a dx^\mu$) [16]

$$S = \int (\frac{1}{\kappa^2} tr(K \wedge {}^\star K) + (D_A\varphi)^{\dagger\, \star}\,(D_A\varphi) - m^2\varphi^\dagger \wedge {}^\star\varphi) \tag{71}$$

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[30] The structure constants, $c^\alpha_{\beta\delta}$, of the algebra define what is callet the adjoint representation of the algebra. Evidently the matrices $(c^\alpha_\beta)_\delta$ have dimension of the group. Writing $ad(g_\delta) = (c^\alpha_\beta)_\delta$, the Killing metric can be written as $g_{\alpha\beta} = tr\,(ad(g_\alpha)ad(g_\beta))$.

If we adopt for the Lie algebra, g, the following notation for the representation matrices

$$[\gamma_a, \gamma_b] = c^c_{ab}\gamma_c, \quad tr(\gamma_a\gamma_b) = -\frac{1}{2}\delta_{ab}, \quad \{\gamma_a\} \in g \tag{72}$$

then, by variation, we find a Klein-Gordon equation

$$^\star D_A\, ^\star D_A \varphi - m^2\varphi = 0 \tag{73}$$

and a more complicated equation for $^*K$ [16].

$$tr((D_A\, ^\star K)\gamma_a) = \frac{1}{2}\kappa^2\, ^\star(\varphi^\dagger\gamma_a\,(D_A\varphi) - (D_A\varphi)^\dagger\gamma_a\varphi), \quad a = 1, ..., \dim g \tag{74}$$

On the left and right of the first three equations of the theorem, we have the same gauge symmetry invariance. We can introduce the conserved currents by adding a term $j_\mu A^\mu$ to the action. We note that the covariant derivative hides the presence of obstructions to integrability (topological defects), which occur when $K = dA$ but $A \wedge dA \neq 0$ so the flux of A, dA, gets folded into A. This lack of Frobenius integrability is directly associated with the existence of non Abelian symmetries to the MST; a fact that will be exploited below. Thus the Frobenius theory is obscured by this formulation but the underlying symmetry is highlighted.

In the usual electromagnetic field theory there are no magnetic monopoles and $\nabla \cdot B = 0$. If we allow $\nabla \cdot B = 4\pi\rho_B$, then, at least in $R^3 - \{0\}$, we have singular potentials A_μ. In the non-Abelian gauge theory the corresponding equations are $\partial_i B^{ai} = c^a_{bc} A^b_i B^{ci}$. The right-hand side is non null and we have non Abelian "magnetic monopoles". For $SU(2)$ we have the BRST instanton.

So, based on our understanding of the Clifford conjecture, we have determined that there is a variational procedure coupled to a "Fourier" expansion of the manifold in terms of an underlying symmetry. The Dirac equation is an example of these kinds of equations (as we will show), as well as are the Maxwell structures. In the Dirac case, we do not employ the variational procedure with this exact form as the last equation is for bosons.[31] This indicates a subtle difference between the Maxwell and Dirac cases. This difference in terms of quantum dynamical manifold equations is found in terms of using different color symmetry algebras $\{\gamma_a\}$. More importantly, we have shown that non Abelian structures lead to obstructions in forming a globally defined potential. Thus one must find other means of defining currents and potentials than direct using the Yang-Mills approach.

E. Relation to Birkhoff Theorem

To relate the new theory of gravitation to the phenomenological GRT, one should say something about its relation to the Birkhoff theorem as done in the following: In a torsionless region of space, say, outside a static spherical distribution of matter, the Birkhoff theorem [17, p.253] states that the Schwarzschild metric satisfying the Bianchi identity, Eq.(75) below, for vanishing torsion and, in this case consequently the equations of GRT $G^\ell_m = R^\ell_m - \frac{1}{2}g^\ell_m R = 0$, with Newtonian boundary conditions, is unique. Almost immediately one then has the theorem:

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[31] Iinstead we define

$$A^\kappa_\nu(x) = (i\gamma_\mu T^\mu)^\kappa_\nu,$$
$$\nabla_\mu\Psi = (\partial_\mu - igA^\kappa_\mu(x)\gamma_\kappa)\Psi$$

This allows the definition of a torsion (also known as the curvature of the connection and as a field strength)

$$T^\kappa_{\mu\nu} \equiv \partial_\mu A^\kappa_\nu - \partial_\nu A^\kappa_\mu + gc^\kappa_{ij}A^i_\mu A^j_\nu$$

Here $[\gamma_a, \gamma_b] = c^c_{ab}\gamma_c$ defines the structure constants of the Lie algebra. We find the following invariant under transformations of the gauge group

$$S \equiv -\frac{1}{4}T^i_{\mu\nu}T^{\mu\nu}_i + \Psi^\dagger(i\gamma^\mu\nabla_\mu - gA^\kappa_\mu\gamma_\kappa T^\mu - 1m)\Psi$$

Theorem 11 *The new gravitational field equations outside a static spherical mass distribution in the absence of torsion satisfy the Birkhoff theorem.*

Proof: This follows by noting that for the gravitational field outside a static spherical distribution of matter, the Birkhoff theorem assumes zero torsion (i.e., a Riemann, symmetric, space or an induced Riemannian geometry) and the W-field vanishes. The Schwarzschild metric follows on solving either the new gravitational field equations or the ones of GRT since both are restatements of the Bianchi constraint amounting in this case to the vanishing of the Ricci curvature $R_{\mu\nu}$ (See ref. [18]) and the requirement to fit onto a spherical boundary. In fact, the equations of GRT in this case are just the first Bianchi constraints. Thus the new gravitation theory and GRT merely reflect the structure of the Riemann curvature in different ways.

A bit more formally: The proof follows from use of the Bianchi identities, which in tensor component form are [19, p.306]:[32]

$$\sum_{(jkl)} \nabla_j R^m_{ikl} = \sum_{(jkl)} T^h_{kj} R^m_{ihl} \text{ and } \sum_{(jli)} (\nabla_j T^k_{li} - T^m_{ji} T^k_{ml}) = \sum_{(jli)} R^k_{ijl}. \tag{75}$$

Using these relations, in a torsionless region of space, say, outside a static spherical distribution of matter, one has the lemma that $\sum_{(mkl)} \nabla_m R^j_{ikl} = 0$, which, in the standard manner, leads to $\nabla_l \left(R^l_h - 1/2 g^l_h R \right) = 0$ and to the Birkhoff theorem which states that the Schwarzschild metric solution of Eq.(75), with Newtonian boundary conditions, is unique. This is proved by noting that the Ricci tensor, R^l_h, vanishes under these conditions, as does its trace. By introducing the centrosymmetric metric $ds^2 = B(r)dt^2 - A(r)dr^2 - r^2 d\theta^2 - r^2 \sin^2\theta d\phi^2$ and solving the resulting ordinary differential equations for A and B, one obtains the Schwarzschild metric, $A = (1 - \frac{2MG_N}{r})^{-1} = B^{-1}$ as the unique exterior solution meeting the Newtonian asymptotic conditions. Almost immediately one then has the theorem above.

The final part of the proof follows from the lemma by noting that for the gravitational field outside a static spherical distribution of matter, the Birkhoff theorem assumes zero torsion (i.e. a Riemann space) and the W-field vanishes. The Schwarzschild metric follows (See ref. [18, Sec. 11]). We then have the result from Thm.4 that the asymptotic behavior of the static G-field is exactly Newtonian.

F. Relation to GRT

The results in Thm.4 are different from those found in GRT.[33] As shown, the average curvature 2-form k is defined over the whole manifold, that is, it has a universal character. In addition to a similar philosophical basis − mass curves (and twists) spacetime but our theory emphasizes the dynamics of spacetime evolution − the present theory and GRT are related in other ways. In the present theory the gravitational field is obtained from the average curvature which is a trace invariant. The underlying space has torsion. Since the induced curvature 2-form k is also a 2-form it

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[32]From the second of Eq.(75), in the absence of curvature, one sees that there exists a new kind of non linear torsional wave in a 4D spacetime. These (spinning) torsional structures are an essential ingredient in developing the generalization of quantum mechanics compatible to the gravitation theory presented here [4].

[33]GRT is a theory that uses the phenomenological equation $G^l_h = R^l_h - 1/2 g^l_h R = 8\pi G_N \tau^l_h$, where G_N is Newton's gravitational constant and τ^l_h is the phenomenological energy-momentum tensor, see, e.g., [18], [8]. In GRT, instead of satisfying each of Eq.(75), we derive the *restricted Bianchi identity* $\nabla_\ell (R^\ell_m - \frac{1}{2} g^\ell_m R) = 0$ from the first of Eq.(75) for vanishing torsion. The definitions $R_{mn} = R^\ell_{mn\ell}$ and $R = R^m_m$ are used. We then use the fact that the connection coefficients can be derived from the metric tensor in a Riemannian space, $\Gamma^m_{ik} = \frac{1}{2} g^m_j (\partial_k g_{ij} + \partial_i g_{kj} - \partial_j g_{ik})$ to obtain, on substituting the expression for $R_{mn}(\Gamma^m_{ik})$, non-linear second order partial differential equations for the metric tensor. The g_{ij} then serve as a potential functions in the theory.

The Einstein gravitational equations for the metric tensor $g = (g_{\mu\nu})$ can be written in a form similar to the Clifford form. Namely, $\Box g + f_1(g) + f_2(g) = \kappa\tau$. Here $f_{1,2}$ are nonlinear operators depending on g and its first two derivatives and κ is a constant. In the weak field limit this equation becomes Eq.(77). Once the curvature and torsion are computed, the Einstein tensor, $G^l_h = R^l_h - 1/2 \delta^l_h R$, generalized for torsion, can be computed using the Ricci energy momentum-flux equation

$$\nabla_l G^l_h = 1/2 g^{jp} \{ T^m_{hl} R^l_{jmp} + T^m_{lp} R^l_{jmh} + T^m_{ph} R^l_{jml} \}$$

which is derived from the full set of Bianchi identities.

defines a Riemann structure whose induced geometry is described in Eqs.(1-3), i.e., Thm.4 plus the constraint that the metric be Riemannian. These equations point out the remaining difference between the two theories. In GRT we phenomenologically equate the deviatoric part of the Ricci tensor to the energy-momentum tensor. In the present theory this is not done, the average curvature potential is obtained by specifying the currents. The latter can be computed e.g., from quantum dynamics. We now discuss another difference between the theories, namely related to gravitational waves. One can provide a complete analysis of the kinds of waves possible in the GRT outlook by examining the Riemann curvature partitioned into electric and magnetic parts as follows [19, p.339]

$$E_{ij} = R_{i0j0} \qquad \textit{and} \qquad H_{ij} = \frac{1}{2}\epsilon_{ikl}R_{klj0} \tag{76}$$

We will not do this here, preferring to consider the case of the weak field limit in GRT where small deviations, $h_{\mu\nu}$, from the Minkowski metric, $\eta_{\mu\nu}$, are phenomenologically related to the symmetric energy-momentum tensor of special relativity, $T_{\mu\nu}$, through its deviatoric part defined by $S_{\mu\nu} \equiv T_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}T^{\lambda}_{\lambda}$, one can derive an equation [18, p.254]

$$\Box h_{\mu\nu} = -16\pi G_N S_{\mu\nu}. \tag{77}$$

This is similar to Eq.(14). Note that on the right-hand side of Eq.(14) is a current, not the deviatoric part of the symmetric energy-momentum tensor, T^i_j, as it is in Eq.(77). This is important since it allows the special relativistic energy-momentum $T^j_m = \eta^{j\nu}T_{m\nu}$ to be independently conserved and there still be a non-zero current matrix $J^j_m = \eta^{j\nu}J_{m\nu}$, a problem with previous vector-tensor theories [8, p.186]. Thus our equations are more like the equations of GRT for small amplitude waves. In the present case the current is evaluated quantum mechanically. Equations (14 and 77) both have solutions of helicity ± 2, ± 1, and 0 [18, p.254-257]. In GRT one transforms away the helicity ± 1 and 0 solutions by finding a suitable choice of coordinates. This depends on the symmetry $S_{\mu\nu} = S_{\nu\mu}$ which has only 10 independent components constrained by plane-wave relations involving the polarization tensor $\varepsilon_{\mu\nu}$ and the 4-wavevector, $k_\mu \varepsilon^\mu_\nu = \frac{1}{2}k_\nu \varepsilon^\mu_\mu$ where $k_\mu k^\mu = 0$; i.e., it has really only $10 - 4 = 6$ independent components. These waves are thought to have small amplitude and hence are difficult to detect; they have yet to be experimentally verified. An implication of Thm.4 is that the global spin-2 gravitons from GRT are not the result of the dynamics of the globally defined average curvature potentials, a_g, currents j_g, and densities, ρ_g, appearing in $\Box a_g = -4\pi {}^\star j_g$ and $\nabla \cdot j_g[\Lambda] + \frac{\partial \rho_g[\Lambda]}{\partial \eta} = 0$. They might not exist as global quantities since they have no global potential. If one carries over the same argument to the present theory, applied to the symmetric part of the current, there also remains only the helicity ± 2 (spin-2) *gravitons* for the free plane-wave case but we know in general this is only a local result. Furthermore, by choosing special transformations so as to eliminate the helicity ± 1 and 0 solutions we loose general co-ordinate system invariance (covariance). If we retain covariance, then by Thm.4, the global free wave (${}^\star j_g = 0$) radiation includes a spin-1 average curvature wave in addition to the spin-2 case . We conclude the only long range gravitational waves are those involving the symmetric solutions of Eq.(14); exactly those with spin-1 associated with the trace curvature Eq.(14) not the weak field Eq.(77). A spinning, oscillating, gravitational system still radiates energy locally and this can be a spin-2 wave but by the foregoing arguments a spin-1 global wave is also possible. The helicity ± 2 and 0 solutions, being part of the spectrum of the d'Alembertian of a tensor quantity, however, contribute to the inhomogeneous solutions. Furthermore, to synthesize the complete local solution, antisymmetric tensor solutions of Eq.(14) must be included. Thus the present theory is consistent, compared to previous vector-tensor theories and because the Bianchi identities are satisfied, the Schwarzschild metric applies to centrosymmetric spacetimes. The advance of the perihelion of Mercury results of GRT (due to the Schwarzschild metric) then hold. The gravitational redshift due to light motion outside a mass also agrees, as a simple argument shows [8, p.187]. Thus the present theory almost agrees with GRT in the three classical tests, except for the prediction of the existence of spin-1 waves and the claim that GRT spin-2 waves are not globally defined. The global theory is a $u(1)$-gauge theory and renormalizable as in quantum electrodynamics (QED). We should mention that the present theory is a manifold theory and the gravitational waves described in Eq.(14) are for weak fields only. In addition local nonsymmetric waves are predicted by the new theory. The fact that the extended theory of gravitation supports local manifold structures with spin and helicity implies that the MST continuum can support the observed spinning systems such as electrons, solar systems and galaxies. These structures of MST are different from the phenomenological structures of the older theories of gravitation which assume first a mass or energy-momentum distribution, then find the resulting gravitational fields in order to describe the evolution of bodies. In the present case, the objects are simply identifiable parts of an evolving MST with their own special symmetry and concomitant dynamics. The new theory and GRT share the most important aspect that mass curves mass-spacetime. The details, however, differ. We expect the difference to be most marked for so-called interior solutions especially of galaxies. However, we also find that we can compute the gravitational potential of the electron using Thm.4.

G. Relation to Ricci Energy-Momentum Flux Equations

Thus far we have found propagation laws, conservation laws, and globally defined potentials associated with the Riemann curvature 2-form matrix (K^l_m). As important as this is, we can go further to analyze the potentials themselves. The properties of these curvature and spin/helicity structures can be examined by forming a sequence of products of the differential 1-form $a = a_m dx^m$. Thus, in order to examine the curvature potentials locally or globally, we consider a general 1-form of the 4-potentials being a representative from $a \in \{A^l_m, a\}$ and its derivatives to form what is called called a Pfaff sequence [35, p.47]. This allows us to introduce the Pfaff dimension of the domain which is the rank of the largest non-zero element of this sequence. One thereby obtains some new quantities, the topological helicity, spin and the topological parity. These quantities do not explicitly depend on the geometric torsion, although they are related to the torsion via such things as the Bianchi identities. As related in Part 1, the torsion provides a vehicle for the transition of the manifold from one mass-spacetime configuration to another.

Before introducing these quantities, however, we want to recall that MSTs can, in general have sources/sinks. The compatibility of the torsion tensor, T^m_{hl}. and curvature tensor, R^l_{jml}, in this situation is expressed by the following theorem

Theorem 12 *Ricci Energy-Momentum Flux Equations. The source for the generalized Einstein tensor is the flux of torsion-curvature or energy-momentum as given in the following equation:*

$$\begin{aligned}\nabla_l G^l_h &\equiv \nabla_l (R^l_h - 1/2\delta^l_h R),\\ &= 1/2 g^{jp}\{T^m_{hl}R^l_{jmp} + T^m_{lp}R^l_{jmh} + T^m_{ph}R^l_{jml}\}.\end{aligned} \tag{78}$$

The interpretation of this equation is that the spacetime gradient of the generalized (allowing anti-symmetric parts) Einstein tensor, G^l_h, is given by the right-hand side. The right-hand side is then interpreted as the source of a flux of energy-momentum.

Proof: See Part 1a. It amounts to manipulating the inhomogeneous Bianchi constraints, Eqs.(75).

In a fluid mechanical system imagined as a continuum with energy-momentum leaks, this equation describes the geometric effects of the exchange of energy-momentum with the molecular basis of the fluid (viscosity). The right-hand side is not an energy momentum, it is an energy-momentum flux. In general, because torsion of a spacetime is small, we expect the right-hand side to be small except at boundaries of the manifold. In a MST Eq.(78) describes the local balance of energy-momentum flux with the spin structures of submanifolds created/destroyed during the manifolds expansion/contraction. These processes occur at a scale smaller that the average needed for the definition of a universal gravitational potential. The possibility of such energy-momentum flux perforating a MST manifold implies the possibility that the MST can support the creation of the topological structures introduced next. As the average curvature has vanishing torsion, the gravitational field appears to be created by the curvature of a Riemannian (vanishing torsion) space.

III. EVOLUTION OF MST TOPOLOGIES

This section begins the study of manifold evolution that exceeds the bounds of the induced Riemann manifold. Through the quantum mechanics providing non-average MST we have ways to change the topology of the manifold. In a physical sense, we can create different topology excited states and can create pairs of charge conjugate particles. For simplicity, we begin by examining the Riemann case. We will find that even in this simplest of cases significant topological change can be characterized. Associated with the system, $dk = 0$, because k is exact ($k = da_g$), are two other exterior differential systems of equations, so the complete system, adding to that of Lem.1 is

Theorem 13 *(Kiehn [34]) The complete set of exterior differential equations for the* 1*-form of average action a_g (curvature potential, when it exists, e.g., for the average curvature) is*

$$\begin{aligned}k &= da_g,\\ d\ell &\equiv {}^\star j,\\ d(a_g \wedge \ell) - (k \wedge \ell - a_g \wedge {}^\star j) &= 0,\\ d(a_g \wedge k) - k \wedge k &= 0\end{aligned} \tag{79}$$

Proof: The first line is a consequence of the proof of the reformulated Clifford conjecture, Thm.4. The second is a definition. Using the property of the exterior derivative $d(\lambda \wedge \mu) = d\lambda \wedge \mu + (-)^{deg(\lambda)} \lambda \wedge d\mu$ of a wedge product of forms and the definitions on the first two lines of Eq.(79) we find for 1-form a_g and 2-form ℓ in the third line $d(a_g \wedge \ell) = da_g \wedge \ell - a_g \wedge d\ell = k \wedge \ell - a_g \wedge {}^\star j$. By invoking Poincaré's lemma ($d^2 = 0$) in the fourth line we find for 1-form a_g and 2-form k that $d(a_g \wedge k) = da_g \wedge k - a_g \wedge dk = k \wedge k - a_g \wedge d^2 a_g = k \wedge k$.

To evaluate these, we note that the globally defined 1-form $t_r(A^l_m) \equiv a_g = a_{gm} dx^m$ of the Thm.4. has units of action and defines a 4-vector field over the manifold. We can remove a certain amount of dependencies on the exact geometric behavior of the action by normalizing it. The natural way of normalizing such quantities is to use the Minkowski norm. This has become "natural" because of our familiarity with the unification of space and time introduced by Einstein, Poincaré and Weyl. The present theory, nonetheless sheds additional light on this approach. By introducing a (pseudo-) Gauss-map in a Minkowski spacetime ($a_g \to a_g / \|a_g\|$), defined immediately below, we obtain a *topological action,* and we can then define the following Lorentz transformation invariant global quantities: da_g, a topological vorticity, $a_g \wedge da_g$ or $a_g \wedge {}^\star da_g$, topological helicity, and $da_g \wedge da_g$ or spin $da_g \wedge {}^\star da_g$ which can be accorded the meaning of topological parities [36]. We can also define analogous quantities for the local A^l_m.[34] Adopting a Minkowski tangent space or relativistic 4-vector notation, a pseudo-norm condition can be written in terms of "physical components of the topological action" as ($A_g \in \{A^l_m,\, a_g = t_r(A^l_m)\}$)

$$\widehat{A}_g[\Lambda] = \sum_{i=0}^{3} A_{gi}[x] dx^i \equiv \sum_{i=0}^{3} \frac{v_i[x]}{\sqrt{(1-\beta^2)}} dx^i, \qquad \beta = \frac{\|v[x]\|_3}{c}. \tag{80}$$

Here:[35]

$$\widehat{A}^4_{g0} = \frac{c}{\sqrt{1-\beta^2}} \tag{81a}$$

$$\left\|\widehat{A}_g\right\|_4 = -\frac{c^2}{(1-\beta^2)} + \frac{\sum_{i=1}^3 (v_i)^2}{(1-\beta^2)} = -c^2 \tag{81b}$$

In Eq.(81b) we see the effect of the Minkowski norm in producing the normalized topological action (choosing units where the maximum speed of propagation (free wave propagation) is $c = 1$) is to make the magnitude constant. This means $d\widehat{A} = 0$. Such motion is said to be by parallel displacement. In this way we avoid self-intersections of surfaces of constant $\widehat{A}$. In a 4D Euclidean space, a Gauss-map of M^3_g maps onto the 3D sphere, S^3. Thus, in general a multiply connected oriented 3-manifold can be regarded to form a branched cover of the S^3 sphere with the self-intersections restricted to the branch sets (knots).[36] Because a_g is defined globally, we can use its Gauss-map to determine (using

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[34]This is called a Pfaff sequence [35, p.47]. Given the sequence of forms constructed from a 1-form, its exterior derivative , its products and ordered by differential form degeee, e.g., $\{A, dA, A \wedge dA, ...\}$, the degree of the highest non zero form is called the Pfaff degree. The import is that a given differential form A defined on, say, m dimensions may require fewer independent functions for its description. The set $\{A, F = dA, L = {}^\star F, J = dL\}$ forms the basis for a complete Pfaff sequence of differential forms: in various combinations of products and exterior derivatives limited by the fact that in 4D there can be at most 4-forms.

[35]The kinematical constraints defining the motion of a point in 4D Minkowski space give the standard or cannonical kinematics of point-particles in SRT ($\beta = v/c$)

$$\frac{dx}{v_x/\sqrt{1-\beta^2}} = \frac{dy}{v_y/\sqrt{1-\beta^2}} = \frac{dz}{v_z/\sqrt{1-\beta^2}} = \frac{-cdt}{v/\sqrt{1-\beta^2}}$$

These are equivalent to the standard non-relativistic kinematic constraints as $\beta \to 0$. These equations derive from the Pffafian differential equation $\omega = dv = 0$, i.e., that the motion be a geodesic. Such a vector function v, is said to lead to motion by parallel displacement. In is interesting that in the present theory cannonical motion requires an integrability condition. The condition $dv = 0$ is also interpreted as the statement that the motion is force free. In a space with torsion such geodesical motion can occur along geometric helices. We also have the constraint that on the light cone ($\|v\| = c$) that $du^2 = -c^2 dt^2 + ds^2 = 0$.

[36]In Parts 2a and b we found matrix solutions to the Dirac and Maxwell equations that could be interpreted topologically as projective spaces. These arise from the dynamics leading to solutions mapping $(z_1, z_2, .., z_{n-1}, a)$ to $(z_1/a, z_2/a, .., z_{n-1}/a)$. These projective spaces are topologically obtained by identifying antipodal points. For instance the unoriented space RP^2 is obtained from the sphere S^2 in R^3 by identifying points opposite along a diameter. These create to what are called cross-caps. In odd-dimensional spaces, e.g., R^3, a moving frame has its orientation reversed on traversing through one of these. These spaces can be used to connect others having the same (Right or Left) orientation. The projective solutions can be considered to perforate the otherwise benign M^3_g spaces discussed here.

"Morse theory") the number of times it covers the sphere S^3. This provides a topological "charge", i.e., another invariant with respect to smooth changes in the geometry. We can also find the structure of this branched-over-a-knot cover and obtain a classification of the potential manifold. Each cover of S^3 can be considered analogous to a separate Riemann sheet. Motion of points coming from different directions onto the bifurcation can lead to transition to different sheets at the branch set. The matrix vector potential components, A^l_m, moreover, are not necessarily defined globally as evidenced by the case of the vector potential for the Dirac- and BRST- monopoles. The Pfaff sequence and its interpretation for a 4D manifold leads to the following definition[37]

Definition 14 (*Kiehn [36]*) Gauss-Mapped topological quantities. *Given a topological (Gauss-mapped) action $\hat{a}_g$, the following invariants can be defined for a manifold:*

$$\begin{aligned}
\hat{a}_g &= a_{gm} dx^m && \text{topological action,}\\
\omega &= d\hat{a}_g = \omega_{\mu\nu} dx^\mu \wedge dx^\nu && \text{topological vorticity,}\\
s_4 &= \hat{a}_g \wedge d\hat{a}_g = s_{\mu\nu\sigma} dx^\mu \wedge dx^\nu \wedge dx^\sigma && \text{topological helicity-current,}\\
\bar{s}_4 &= \hat{a}_g \wedge {}^\star d\hat{a}_g = \tau_{\mu\nu\sigma} dx^\mu \wedge dx^\nu \wedge dx^\sigma && \text{topological spin-current,}\\
p_4 &= d\hat{a}_g \wedge d\hat{a}_g = p_{\mu\nu\sigma\tau} dx^\mu \wedge dx^\nu \wedge dx^\sigma \wedge dx^\tau && \text{helicity orientation,}\\
\bar{p}_4 &= d\hat{a}_g \wedge {}^\star d\hat{a}_g = \bar{p}_{\mu\nu\sigma\tau} dx^\mu \wedge dx^\nu \wedge dx^\sigma \wedge dx^\tau && \text{spin orientation.}
\end{aligned} \tag{82}$$

The topological orientations are 4-forms and thus volume elements in the 4D space. The normalized volume can have either a positive or a negative sign, if non vanishing, depending on the orientation of the manifold, hence the fact that the 4-forms of volume are called parities. When unnormalized, thereby gaining a geometric aspect, they give a measure of the growth or shrink rate of the MST manifold. The change of these quantities relative to an observer travelling with points affixed in the manifold (Lagrangian perspective) can be evaluated using the Lie derivative. As a consequence we have a powerful method for tracking the change of topological features of the evolving MST in a way that respects the viewpoint of a co-moving observer..

Finally we complete the thought introduced at the beginning of this section concerning special relativity theory (SRT). In Eq.(82), when a_g is the average topological action, it evolves, within a sheet, by parallel displacement, $da_g = 0$. Thus SRT achieves a situation where there is no topological vorticity, helicity, etc. This extends to the case where a varies with position. The same can be said about the ordinary velocity components used in SRT. In SRT, then we have difficulties describing processes such as pair-creation. In the present theory, the average potential is obtained from the curvature 2-form and is unnormalized, so there is the possibility of topology change, and the unnormalized versions of Eq.(82) are appropriate.

Reverting back to unnormalized quantities and by an abuse of notation, letting A stand for A^l_m or its trace as indicated by context, we write ($A \in \{A^l_m,\, a = t_r(A^l_m),\, A^l_m - g^l_m t_r(A^l_m)\}$),

$$A = \Phi d\eta + A_x dx + A_y dy + A_z dz, \tag{83}$$

then we find

$$\begin{aligned}
F = dA = &\left(\frac{\partial \Phi}{\partial x} - \frac{\partial A_x}{\partial \eta}\right) dx \wedge d\eta + \left(\frac{\partial \Phi}{\partial z} - \frac{\partial A_z}{\partial \eta}\right) dz \wedge d\eta + \left(\frac{\partial \Phi}{\partial y} - \frac{\partial A_y}{\partial \eta}\right) dy \wedge d\eta\\
&+ \left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right) dy \wedge dz + \left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right) dz \wedge dx + \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right) dx \wedge dy,
\end{aligned} \tag{84}$$

$$\begin{aligned}
{}^\star F = {}^\star dA = &-\left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right) dx \wedge d\eta - \left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right) dy \wedge d\eta - \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right) dz \wedge d\eta\\
&+ \left(\frac{\partial \Phi}{\partial x} - \frac{\partial A_x}{\partial \eta}\right) dy \wedge dz + \left(\frac{\partial \Phi}{\partial z} - \frac{\partial A_z}{\partial \eta}\right) dz \wedge dx + \left(\frac{\partial \Phi}{\partial y} - \frac{\partial A_y}{\partial \eta}\right) dx \wedge dy.
\end{aligned} \tag{85}$$

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[37] The sequence terminates with the 4-form of volume as higher forms necessarily vanish. On a 4D domain, the element in this sequence $da \wedge da$ is a 4-form volume, which, if it vanishes on a set, reduces to at most a 3-form. This odd-degree domain converts the symplectic domain to a contact one. The vanishing of the Pfaff form leads to a conservation law. Irreversible processes will be on a symplectic manifold of Pfaff dimension 4, of non vanishing $da \wedge da$. Topological defects appear as vanishing points of lesser Pfaff dimension [34]. The Chern-Simmons form, $a \wedge da - \frac{2i}{3} a \wedge a \wedge a$, is a popular 3-form clearly associated with a Pfaff dimension of three and the topological torsions. From the lemma for the potential of the curvature trace it is defined over the whole manifold.

These can be written respectively as as[38]

$$F = G_x dx \wedge d\eta + G_y dy \wedge d\eta + G_z dz \wedge d\eta$$
$$+W_x dy \wedge dz + W_y dz \wedge dx + W_z dx \wedge dy, \tag{86}$$
$$^\star F = -W_x^* dx \wedge d\eta - W_y^* dy \wedge d\eta - W_z^* dz \wedge d\eta$$
$$+G_x^* dy \wedge dz + G_y^* dz \wedge dx + G_z^* dx \wedge dy. \tag{87}$$

Thus, the 3-form helicity, $s_4 = A \wedge dA$, (which vanishes for an integrable Pfaffian system) and the spin, $\bar{s}_4 = A \wedge {}^\star dA$, (which may not vanish for an integrable Pfaffian system) can be computed using Eqs.(83-85). Calling $\mathbf{A}$ the 3-vector part of A, so that $A = (\Phi, A_x, A_y, A_z) = (\Phi, \mathbf{A})$ we can express the 3-form results for the helicity, as in Eq.(22) in terms of a helicity-density, ρ_s, and a helicity 3-current, $\mathbf{j}_s = (j_x, j_y, j_z)$, in terms of components as follows:

$$4\pi\rho_s \equiv \mathbf{A} \cdot (\nabla \times \mathbf{A}), \tag{88a}$$
$$4\pi \mathbf{j}_{s|i=x,y,z} \equiv 4\pi(\frac{\partial \mathbf{A}}{\partial \eta} \times \mathbf{A} + (\mathbf{A} \times \nabla)\Phi + \Phi(\nabla \times \mathbf{A}))_{i=x,y,z}. \tag{88b}$$

If one imposes the Lorentz condition $\partial_\mu A^\mu = 0$, uses $\nabla \times \mathbf{A} = \mathbf{W}$ and $\mathbf{G} = -\nabla\Phi + \partial\mathbf{A}/\partial\eta$, then the equations above may be rewritten so we find the helicity density, ρ_s, and the helicity 3-current, $\mathbf{j_s}$, as follows:

$$4\pi\rho_s \equiv \mathbf{A} \cdot \mathbf{W}, \tag{89a}$$
$$4\pi \mathbf{j}_{s|i=x,y,z} \equiv 4\pi(\mathbf{G} \times \mathbf{A} + \Phi\mathbf{W}) \tag{89b}$$

By employing the Hodge-$\star$ operation on the differential form version of these equations, we find the spin-density, ρ_s, and a spin 3-current, $\mathbf{j}_{\bar{s}}$ as derived from an expression like, Eq.(22): [15, p.153]

$$4\pi\rho_{\bar{s}} \equiv \mathbf{A} \cdot \mathbf{G}, \tag{90a}$$
$$4\pi \mathbf{j}_{\bar{s}|i=x,y,z} \equiv 4\pi(\mathbf{A} \times \mathbf{W} + \Phi\mathbf{G}) \tag{90b}$$

The 3-form $\bar{s}_4 \equiv a_g \wedge \ell = a_g \wedge {}^\star k = a_g \wedge {}^\star da_g$ is a 4-*spin- current* and the 3-form $s_4 \equiv a_g \wedge k = a_g \wedge da_g$ is a or 4-*helicity current.*[39] Each of these combine a scaler density and a 3-vector current. The two have similar geometric

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[38]In a 4D spacetime we write a 2-form as

$$F = E_x dx \wedge d\eta + E_y dy \wedge d\eta + E_z dz \wedge d\eta$$
$$+B_x dy \wedge dz + B_y dz \wedge dx + B_z dx \wedge dy$$

Using the relations ${}^\star(dx \wedge dy) = -dz \wedge d\eta$, etc., and ${}^\star(dx \wedge d\eta) = dy \wedge dz$, we obtain

$$^\star F = -B_x^* dx \wedge d\eta - B_y^* dy \wedge d\eta - B_z^* dz \wedge d\eta$$
$$+E_x^* dy \wedge dz + E_y^* dz \wedge dx + E_z^* dx \wedge dy$$

The dual of the two form is obtained by switching coefficients between the temporal and the spatial parts accompanied by a change in sign of the temporal part. Note that that one more application of the Hodge-$\star$ reverses the sign in the last term so that ${}^{\star\star}F = -F$. By making suitable definitions, derived from the Maxwell equations based on the distinction of intrinsic and extrinsic components: $E_i^* = D^i$ and $B_i^* = H^i$ and using a properly rescaled evolution parameter $d\eta = cdt$, the last equation can be written as

$$^\star F = -H^x dx \wedge d\eta - H^y dy \wedge d\eta - H^z dz \wedge d\eta$$
$$+D^x dy \wedge dz + D^y dz \wedge dx + D^z dx \wedge dy$$

In free space everything simplifies since $E = D$ and $B = H$.

[39]The condition $s_4 = a_g \wedge da_g = 0$ is an integrability condition that the Pfaffian differential equation $a_g = 0$ has a solution. Thus the non-vanishing of the helicity implies this non integrability. Non integrability is a topological condition. It is the closed ($d\omega = 0$) non exact ($\omega \neq d\alpha$) quantities that reflect the emergence of multiple connectedness of the gravitational system. It is significant that the average (trace) gravitational curvature is exact. Thus the off-diagonal and even some of the diagonal terms to K^l_m are of interest for evolutionary topology change. The fact that $a_g \wedge da_g \neq 0$ implies that the Pfaff dimension [35] is at least three. The integrability condition $a \wedge da = 0$ defines a laminar flow globally.

structure in a 4D MST because $^\star k$ and k are both 2-forms there but they have different values. These conditions do not necessarily hold globally for the individual components K_i^m since we have the Bianchi identity $dK_i^m = [K_j^m, \Omega_i^j]$ which implies that the full matrix, K_i^m, is not closed. For the average gravitodynamical curvature matrix, k or $^\star k$, however, we have two new kinds of globally defined gravitational currents. The divergence of this 3-form is a 4-form satisfies, (using Eq.(79)):

$$4\pi d\, s_4 = 4\pi d(a \wedge da) = 4\pi da \wedge da = 4\pi(k \wedge k) = \left(\frac{\partial \rho_s}{\partial \eta} + \nabla \cdot j_s\right) d\eta \wedge dx \wedge dy \wedge dz = 4\pi p_4. \tag{91}$$

$$4\pi d\, \bar{s}_4 = 4\pi d(a \wedge {}^\star da) = 4\pi(k \wedge {}^\star k - a \wedge {}^\star j) = \left(\frac{\partial \rho_{\bar{s}}}{\partial \eta} + \nabla \cdot j_{\bar{s}}\right) d\eta \wedge dx \wedge dy \wedge dz = 4\pi\,(\bar{p}_4 - a \wedge d^\star k). \tag{92}$$

Equations (89-90) define 3-forms. The exterior derivative of these is a 4-form MST density, which can be computed as a 4-divergence of the vectors of Eqs.(89-90). The vanishing of $a \wedge da$ is a Frobenius integrability condition for Pfaff dimension three, implying that the partial differential equation $a = 0$ is integrable. However, they may not vanish. That is, unlike the case of the induced curvature current and density conservation laws, the (unnormalized) topological parity in Eqs.(91-92) is not necessarily zero for the individual 1-form potentials A_m^l nor for the trace $a \equiv t_r(A_m^l)$, because $dt_r A = da = k$ and the mean curvature, k, derived from the average potential, are generally nonzero. The larger this mean curvature (the smaller or more point-like the manifold), the greater the 4D helicity/spin induced spontaneous creation/annihilation. This is in accordance with experience in elementary particle experiments at high energy where we can slam manifolds together creating high curvature MSTs and consequently new vortical structures - particle manifolds. For curved manifolds, the implication is [36] "that 3-dimensional defects of helicity current can be created or destroyed spontaneously within the medium ...[mass-spacetime manifold]. The production of such defects is an entropy increasing process, and therefore these processes, if continuous, must be irreversible." If $\bar{p}_4$ is an invariant, then its divergence, according to Eq.(92) will fluctuate during the evolution of the manifold. Equations (91-92) show that the helicity/spin fluxes depend on the volume 4-form. The conservation of this form implies that the helicity/spin flux is zero, but the spin flux depends on the vector potential. Thus, there are Aharanov-Bohm effects possible. Equations (91-92) also provide an easy way to compute these fluxes as the product of 2-forms.

The (unnormalized) spin and helicity current divergences in Eqs.(91-92) are not necessarily zero for the individual 1-form potentials A_m^l nor for the trace $a_g \equiv t_r(A_m^l)$, because $dt_r(A_m^l) = da_g = k$ and the average curvature 2-form, k, derived from the average potential, are generally nonzero as discussed previously. The larger this average curvature (the smaller or more point-like the manifold), the greater the 4D helicity/spin induced spontaneous creation/annihilation. This is in accordance with experience in elementary particle experiments at high energy where we can slam manifolds together creating high curvature MSTs and consequently new torsional structures - particle manifolds. For curved manifolds, the implication is [36] "that 3-dimensional defects of helicity current can be created or destroyed spontaneously within the medium ...[mass-spacetime manifold]. The production of such defects is an entropy increasing process, and therefore these processes, if continuous, must be irreversible." If $\bar{p}_4$ is an invariant, then its divergence, according to Eq.(92) will fluctuate during the evolution of the manifold. Equations (91-92) show that the helicity/spin fluxes depend on the volume 4-form. The conservation of this form implies that the helicity/spin flux is zero, but the spin flux depends on the vector potential. Thus, there are Aharanov-Bohm effects possible. Equations (91-92) also provide an easy way to compute these fluxes as the product of 2-forms. These results are necessary conclusions based on the simple assumption that 4D MST is an evolving differential manifold. These results clearly depend on the vector potential. Until the demonstration of the Aharanov-Bohm effect, it was thought that measuring effects due directly to the 4-vector potential would always lead to ambiguous results

We obtain the following kinds of expression for the components of the vector potential 2-form, $a(t, x, y, z)$, for a contractible region U by using standard methods associated with the converse of Poincaré's lemma [10, p.30]:

$$a(t, x, y, z) = (\int_0^1 G_x(\varepsilon t, \varepsilon x, \varepsilon y, \varepsilon z)\varepsilon d\varepsilon \wedge (tdx - xdt) + \cdots) + \text{closed forms} \tag{93}$$

The fluxes computed using this equation, while not directly geometric torsions,[40] lead to geometric torsion whenever there is non-zero curvature as shown in the second Bianchi identities, $d\tau = \sigma K - \tau\Omega$, relating the gradient of the torsion

[40] In ref. [36] and [34] s_4 and $\bar{s}_4$ are called topological torsions. We reserve "torsion" for the torsion 2-form or its tensor coeffcients.

vector (of 2-forms) $\tau^i = \tau^i_{jk} dx^j \wedge dx^k$ to the connection 1-form Ω and the full curvature 2-form, K.[41] Computing the helicity current flow lines from $\frac{dx}{j_x} = \frac{dy}{j_y} = \frac{dz}{j_z} = \frac{d\eta}{\rho}$,[42] one can find the paths of fiducial Lagrangian Marker Particles (LMPs). Therefore, instead of adopting the usual approach of positing the densities and currents (as in Newtonian and GRT gravitational theory), then finding the fields, we first determine the manifold's connection. This leads to its curvature. From this we can find the torsion by solving a partial differential equation. From these quantities we find the gravitational density and flux 4-vector $(\rho, \mathbf{j})$ and concomitant helicity/spin 4-vectors. The details of the geometry precisely reflect the details of the evolving connection on the manifold. By splitting the source term, $\mathbf{j}$, in the wave equation for the vector potential, into longitudinal and transverse parts, $\mathbf{j} = \mathbf{j}_\ell + \mathbf{j}_t$, we find that the source for $\mathbf{A}$ is j_t, i.e., $\nabla^2 \mathbf{A} - \frac{1}{c^2}\frac{\partial^2 \Phi}{\partial t^2} = -\frac{4\pi}{c}\mathbf{j}_t$. Here $\nabla \cdot \mathbf{G} = 0$. In this transverse gauge, by Fourier transforming this last relation, we find that the $\mathbf{k}$-vector is perpendicular to the corresponding Fourier wave component of $\mathbf{G}$, i.e., $\mathbf{k} \cdot \mathbf{G}(\mathbf{k}) = 0$. Along these same lines of thought, the meaning of the spin and helicity have been developed in the analogous case of electromagnetic theory by Kiehn [37].

Therefore, we transfer this to the gravitational idiom where we have proved the existence of a global vector potential for the induced Riemann manifold and the zero of potential is referenced to the beginning of the universe as follows. The vanishing of the Pfaff form $a \wedge {}^\star da$ is a topological constraint permitting transverse G-waves: the vector potential

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[41] We have (2^{nd} Bianchi identity)

$$\begin{aligned} d\tau^\ell &= K^\ell_i \wedge dx^i - \omega^\ell_i \wedge \tau^i = \frac{1}{2}\sum d\left(\tau^\ell_{ij} dx^i \wedge dx^j\right) \\ &= \frac{1}{3!}\sum \left(\frac{\partial \tau^\ell_{ij}}{\partial x^k} + \frac{\partial \tau^\ell_{jk}}{\partial x^i} + \frac{\partial \tau^\ell_{ki}}{\partial x^j}\right) dx^i \wedge dx^j \wedge dx^k \\ &= \frac{1}{3!}\sum_{(ijk)} \left(R^\ell_{ijk} + T^m_{ki} T^\ell_{mj}\right) dx^i \wedge dx^j \wedge dx^k \end{aligned}$$

We have used $\Omega = (\omega^i_j) = (\sum_k \Gamma^i_{jk} dx^k)$. This can be written in terms of the covariant derivative of the torsion tensor leading to the second Bianchi identity as displayed previously. So the second Bianchi identity yields a set of linear partial differential equations for the torsion in terms of the curvature 2-form K and the connection 1-form Ω. For each of the elements of the torsion matrix vector τ^i we can define an antisymmetric matrix

$$\sigma_{\mu\sigma} = \begin{pmatrix} 0 & \varepsilon_1 & \varepsilon_2 & \varepsilon_3 \\ & 0 & \mu_3 & -\mu_2 \\ & & 0 & \mu_1 \\ & & & 0 \end{pmatrix} = -\sigma_{\sigma\mu}$$

We can also define a spin vector with components $s^\mu = 1/2\epsilon^{\mu\nu\rho\sigma} p_\nu \sigma_{\rho\sigma}$ where $\epsilon^{\mu\nu\rho\sigma}$ is the completely antisymmetric Levi-Civita tensor, p_ν are the components of the 4-momentum tensor and $\sigma_{\rho\sigma}$ can be found, using the properties of the Levi-Civita tensor, that $\sigma_{\mu\nu} = 1/2\epsilon^{\mu\nu\rho\sigma} p_\sigma s_\rho$. Thus the second Bianchi identity can be written in terms of a local 4-momentum, p_σ and a set of spin vectors $(s_\rho)^i$. If the (16) spin parameters are considered given as well as the four momenta components, then the torsion is characterized by only flour parameters and can be determined from the second Bianchi identity, assuming the components of the curvature are known. From these expressions we conclude that the presence of curvature, even an average curvature, produces torsion. A physical model is an expanding or contracting region of spacetime. Such a manifold, according to these equations, can support the creation/annhilation of vortical structures. Such structures on a quantum scale can be periodic stationary states as shown in subsequent parts.

[42] A Pfaffian system

$$\omega = Pdx + Qdy + Rdz = 0$$

is integrable if there exists an 1-parameter family of surfaces orthogonal to the 2-parameter system of curves determined by

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

Another way to expresss this is in terms of the necessary and sufficient (Frobenius integrability) condition

$$\omega \wedge d\omega = T \cdot (\nabla \times T) = 0$$

Where $T = (P, Q, R)$.

$\mathbf{A}$ is orthogonal to $\mathbf{G}$: $\mathbf{A} \cdot \mathbf{G} = 0$.The vanishing of $a \wedge {}^{\star}da$ is a topological constraint permitting transverse W-waves $\mathbf{A} \cdot \mathbf{W} = 0$. When both of these quantities vanish, one can have ordinary GW-waves. These correspond respectively to transverse electric (TE), transverse magnetic (TM), and electromagnetic waves (EM).

When we have a quantum dynamical calculation of the connection, as we will have in the sequel, the geometry computed will define the mass density and currents associated with gravitation in terms of the average gravitational potential, but it will define helicity/spin 4-currents as well.

Based on extensions of this analysis we can prove the following

Corollary 15 *(Kiehn [36]) If the four components of the* 3*-form helicity current and helicity density vanish over a domain, then the manifold evolution satisfies the Frobenius complete integrability theorem [10, Sec.7.3], and the evolution is never chaotic, braided, nor knotted.*

It follows that the concept of helicity/spin is necessary for understanding turbulent or chaotic manifold evolution. Furthermore, it is seen that the lack of integrability is a topological condition that is related to the production of helicity. We therefore conclude that

Corollary 16 *Dynamically evolving spacetime manifolds induce the creation of mass, helicity, and spin densities in the form of vortices that can be viewed as defects in an otherwise curved spacetime.*[43]

We, therefore, have provided a more concrete realization of the interpretation of Eq.(78). Namely, that not only can a manifold's evolution be interpreted in terms of the creation of helicity/spin structures creating a generalized Einstein tensor containing antisymmetric parts, but there are conditions where these structures have conserved densities and currents. Furthermore, the evolution of these structures involves the introduction of different topological quantities into the manifold along with its average expansion/contraction. This expansion/contraction may be termed *cosmological* and leads to non zero chirality for the manifold and the non integrability of Pfaffian differential equations describing the evolution of current flows.

Thus in this subsection we again find interesting physical results including the fact that an evolving manifold can generate torsion-curvature currents, but we do not have yet a way to generate such manifolds with curvature or torsion. We merely have another way of analyzing them. Equations (14 and 15) merely describe the internal dynamics possible for spaces having certain local Minkowski symmetry or geometry. It is significant that these conservation laws arise from the evolution of a manifold for this may be viewed as the mathematical origin of the physically observed conservation laws in spacetime. Also significant is that these results depend on the topological dimension of four and a $3+1$ Minkowski local metric. It should be clear that the results depend on only the geometry of the space which may be help explain the small strength of gravity outside a compact mass. The space is relatively flat; only the Bianchi identities are operant to insure a nice manifold. We conclude this subsection by restating the fact proven above

Corollary 17 *The mean curvature, $k(\Lambda)$, of a mass-spacetime (MST) manifold obeys dynamical and conservation laws isomorphic to the laws of electromagnetism.*

Thus the study of gravitation can take advantage of the well developed study of electromagnetism and the analysis of the differential geometry and topology manifolds that can subsequently be embedded into 4D MSTs. The basic idea is that the trajectory of a single probe, called a Lagrangian Marker Particle (LMP) provides a response that can be used to extricate the dynamics of all parts of the manifold, experimentally those parts coupled strongly enough to the LMP to influence its motion. This subject is been examined experimentally and theoretically for evolving 3D fluid manifolds giving a characterization of turbulence. The manifolds found can be characterized as being homeomorphic to evolving handle-bodies embedded in a spacetime.

A. Quantization of Manifold Structures

The average curvature 2-form, k, is both closed ($dk = 0$) and exact ($k = da$), therefore it has a certain simplicity as will be shown shortly. Nonetheless, as shown above even this simplicity contains a richness of possible structures

[43]At this juncture, we mention that S^3 can be characterized as Euclidean flat with torsion as shown in the Appendix of Part 1a. Its evolution induces dynamical curvature leading to the formation of helicity defects. A static spacetime universe can be modeled by S^3. The properties of evolving S^3 manifold geometries are studied in Part 3.

including loops, knots, expansion/contraction and multiple connectedness. Here we introduce a measure of the topology of MST manifolds which are not necessarily contractible. It is further recalled that the general components to the Riemann curvature 2-form, which may not have global potentials, however, can have local potentials and piecing these together reflects the structure of the underlying manifold better. To put some of this in perspective, we recall that in electrostatics, all is well if we integrate along paths that do not encircle a hole in the manifold. In complex variable theory, on the other hand, we recall the Cauchy residue theorem that $\oint f(z)dz = 2\pi i\Sigma residues$. This formula is derived by triangulating the complex plane and encircling the poles of $f(z)$. This suggests a deeper theory is needed to explain the topological structures in multiply connected MSTs. The measure we describe requires us to delve more deeply into integration theory of differential forms. The basic idea depends on the possibility of dividing a compact manifold into elementary parts which can be mapped into simplexes. The idea is best illustrated by an example. A differential manifold on a plane can be covered by triangles, which are called 2D simplices. We use this in the proof of the Cauchy residue theorem. In 3D the elementary 3-simplex is a tetrahedron. In 3D the boundary of a compact manifold, M, denoted by ∂M can be represented as a summation of the boundary 2simplex triangles. To a simplex there can be associated an orientation induced by the order of its vertices. It is clear, with appropriate maps, ϕ, of elementary simplices, s_i^n, in an nD manifold with, say, unit lengths yielding "curved" simplices, σ_i^n, that we can tightly fill or represent any closed manifold. Furthermore, we can associate a weight or density to each term in a summation of these simplices and form the expression

$$c^n = \sum_i w_i \sigma_i^n \tag{94}$$

This defines what is called a weighted chain decomposition of the manifold. If the manifold is of infinite extent this chain may or may not be defined. If we denote the oriented boundary of a curved simplex by $\partial\sigma_i^n$, then, accounting for the cancellation of adjacent but oppositely oriented boundaries in the boundary sum, we have

$$\partial c^n = \sum_i w_i \partial\sigma_i^n \tag{95}$$

It is clear that there are many ways to divide the manifold into chains with the same result, so we must consider all these together as making one equivalence class. If we call this class $C^n(M)$, we then have a general definition of a boundary operator acting on these classes, namely:

$$\partial : C^n(M) \mapsto C^{n-1}(M). \tag{96}$$

Use the definitions of a cycle and a boundary cycle [44] we can define integrals of forms on manifolds [10, p.63]. As a start we use our notation of chains to define the division of the manifold into elementary parts and thereby an integral into easy to compute parts:

$$\int_{C^p} \omega = \sum_i \int_{\sigma_i^p} \omega w_i \sigma_i^p. \tag{97}$$

We use the map ϕ which maps neighborhoods $\bar{s}_i^p$ (s-flat) of simplexes in E^p to curved simplices, σ_i^p of M. If we restrict ourselves to finite manifolds, then we do not have to deal with the question of interchanging limits in the above sum. This provides the rule

$$\int_{\sigma_i^p} \omega = \int_{\bar{s}_i^p} \phi^* \omega. \tag{98}$$

Here ϕ^* is the map of forms induced by the map ϕ. On the right-hand side we have an ordinary p-fold integration. We can use this to state Stokes theorem in arbitrary dimensions. Let ω be a p-form on a manifold, M, and C be a $(p+1)$-chain, then Stokes theorem is

$$\int_{\partial C} \omega = \int_C d\omega \tag{99}$$

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[44] *Cycle:* A cycle is a chain z^n whose boundary vanishes ($\partial z^n = 0$).
Boundary cycle: Given a chain $C^n(M)$, the quantity $b^{n-1}(M) = \partial C^n(M)$ is called a boundary cycle.
The latter is consistent because $\partial b^{n-1}(M) = \partial^2 C^n(M) = 0$ since the boundary of a boundary vanishes. Thus $b^{n-1}(M)$ is a cycle ($\partial b^{n-1} = 0$).

At an elementary simplex level we have as well

$$\int_{\sigma_i^p} d\omega = \int_{\bar{s}_i^p} \phi^*(d\omega) \tag{100}$$

With these preliminaries out of the way we are ready to introduce a final definition and then state the important De Rham theorems. This will be followed by an analysis of their implications relative to MST manifolds.

Definition 18 *Period of a form. Given a closed p-form ω, to each p-cycle z^p on a manifold M, there corresponds a number called its period*

$$per(\omega) = \int_{z^p} \omega. \tag{101}$$

If z^p is a boundary $z^p = \partial C^{p+1}$, then the period vanishes. We can then state two De Rham theorems

Theorem 19 *(1st DeRham [10, p.66]) A closed form ($d\omega = 0$) is exact ($\omega = d\alpha$) if and only if all its periods vanish ($per(z_i^p)(\omega) = 0$).*

The integral of an exact form over a cycle vanishes and conversely if a closed form ($d\omega = 0$) has all its periods zero, then it is an exact form ($\omega = d\alpha$). The inexactness (periods that do not vanish) gives a measure of the failure of the form to have a potential.

Theorem 20 *(2nd DeRham [10, p.66]) Let each period z_i^p be assigned a number, $per(z_i^p)$, subject to the rule that whenever*

$$\sum_i w_i z_i^n - \text{ a boundary, then } \sum_i w_i per(z_i^p) = 0 \tag{102}$$

Then there exists a closed p-form ω on M for each p-cycle, z_i^p, which has the periods

$$\int_{z_i^p} \omega = per(z_i^p). \tag{103}$$

Armed with this computational tool, we consider the problem of counting the number of topological non exact defects in a MST manifold. Our first step is to realize, by proper choice of the map $\phi(\phi^*)$, that the weights w_i can be chosen to be integers. We then have a way of counting the quantities in the elementary n-hedra. We see that in this sense that a MST is quantized. Another approach to counting is to consider normalized ratios of a total weighted sum to the unweighted sum. These ratios, however, may be only rational numbers. As a further application, we can show that to each p-form on a manifold there is a potential theory for these periods. The implication, under certain conditions, is that there is a potential function from which we can determine an integral measure of a form over a p-cycle. This is contained in a corollary to the following version of the Hodge decomposition theorem.

Theorem 21 *(Hodge decomposition theorem [10, pp138-139]) For a closed (compact) Riemann manifold if ω is any p-form, then there is a $(p-1)$-form α, a $(p+1)$-form β, and a harmonic p-form γ such that*

$$\omega = d\alpha + \delta\beta + \gamma. \tag{104}$$

The forms $d\alpha$, $\delta\beta$, and γ are unique.

If ω is an arbitrary closed p-form ($d\omega = 0$), the term $\delta\beta$ is absent, and $\omega = d\alpha + \gamma$ and $\int_{z^p} d\alpha = 0$. Furthermore, if z^p is any p-cycle, then, by Stokes theorem

$$\int_{z^p} \omega = \int_{z^p} \gamma = \int_{\partial z^p} d\gamma \tag{105}$$

That is, the harmonic p-form γ has the same periods as does ω. The harmonic form γ can then serve as a surrogate for the closed form ω as far as integral measures go. Thus

Corollary 22 *For a closed (compact) Riemann manifold, given a p-form $4\pi j$ there is a p-form A satisfying*

$$\Delta A = -4\pi j, \tag{106}$$

if and only if the γ and j are orthogonal, $(\gamma, j) = 0$, for every harmonic function γ.

One can show $\Delta = d\delta + \delta d$ in a flat space is the d'Alembertian, $\Box$, so this defines the conditions under which there is a solution to the wave equation of Thm.4 for a general curved manifold. We can call the parts of j which are not orthogonal to harmonic functions γ its anharmonic parts. We have seen in the derivation of the wave equation, that j satisfies a continuity equation, but in the case of nonintegrability, e.g., $A \wedge dA \neq 0$, there are corresponding currents that are not conserved. In these cases the current contains anharmonic parts.

Therefore, we see that there are effective means to measure the topological defectiveness of manifolds and these measures are nontrivial just when potential theory breaks down; namely when the base form or potential is not integrable. These cases are ones in which the associated streamlines are chaotic or all points in submanifolds are accessible along a path in the sense of Carathéodory. We conjecture that the topological defects described here and embedded in an overall curved MST are exactly the compact mass objects described by the quantum dynamical manifold equations introduced previously. In other words, topological defects are created in such processes as pair-creation, big bangs and the relinking of manifolds. The proof of this will have to await the detailed computation of pair creation.

B. Creation of Topological Defects

We briefly indicate the quantum dynamical basis for the creation of topological defects introduced in the last section. Define the full deviatoric curvature 2-form potential

$$\Delta^l_m \equiv A^l_m - g^l_m t_r A^l_m. \tag{107}$$

Here (l, m) label a quasi-particle basis in a suitable Hilbert space of square integrable functions. Recall that in the case of Minkowski space including a division by $n-2$. The quantity Δ^l_m does not always exist and this has physical implications as will be now explained.

By the converse of the Poincaré lemma, in a contractible patch of a manifold, this object exists locally when the curvature is closed, i.e., where K^l_m satisfies

$$dK^l_m = [K^n_m, \Omega^l_n] = 0. \tag{108}$$

This equation expresses a conservation law associated with a Riemann space. Each pair (l, m) provides a (scattering) channel for defect transmutation in a local sense, when K is closed: $dK^l_m = d^2 A^l_m = 0$. The scattering matrix obtained by solving $K^l_m = dA^l_m$ is an orthogonal matrix or, when a complex base space is used, it is unitary. When $dK^l_m \neq 0$, then we don't have a local conserved potential. The closure of K is thus a condition for the existence of local geometric potentials replacing quantum dynamics for infinitesimal test bodies (i.e., ones that don't change the geometry). The average curvature has a potential, and is hence exact. The replacement of the system response by curvature and torsion automatically including all forces involved in the underlying dynamics, thus has an underlying potential in only special cases. This is different from arbitrarily equipping a manifold with a closed 2-form to describe gravity, say, in the presence of a Maxwell or other quantum field, because the 2-form in the present case is the curvature 2-form itself. Furthermore, we have shown that such a closed and exact 2-form of the curvature does not exist in general, but only for the average curvature which is nonvanishing for non Riemannian spaces. For example, in a molecular dynamics problem, where the reacting species are distant and where there are no changes in topology, we have an average or mean field that slowly changes. At sufficient separation distance we have gravitational interaction. At somewhat smaller distances we have a smooth transition to electrostatic and even polarization fields in the quantum system which overwhelm the gravitational fields. At this point we begin also to have semi-classical effects as the quantum mechanics begins to be felt. In addition, when one molecule is considered as quantum dynamical probe we have quantum interference effects. Even though we have $a \wedge da = 0$, so the Pfaffian differential equation $a = 0$ can be integrated to give the average behavior of the system, we can have $A^l_m \wedge dA^l_m \neq 0$, so that quantum topological defects can be created. Generally these amount to different representations of a symmetry becoming important. For instance we know that $su(2)|_{n>2} \approxeq M^3_g$ has holes in it and even the excited states of $\Psi = R_{nl}Y_{lm}$ of simple atomic theory have nodes (and in 3D nodal surfaces). In addition, the bifurcation behavior of the average potential (when it exists) comes into play more strongly at the quantum level; any real path of evolution is well defined, and an observer following this evolution would not see the bifurcation directly. He would "tunnel" right through it. The lack of globally defined potentials A^l_m (average connection) can be assuaged by embedding the local potentials (requiring $dK^l_m = 0$) into a sufficiently large space accounting for the connection of the different paths and their local geometry (gauge potential Ω). However, we don't always have $dK^l_m = 0$. Equation (108) does, however, tell us when we don't have even a local potential. Thus, it is clear that, to describe the dynamics close in to a quantum dynamical body, more is needed than an average curvature field and its global potential in 4D Minkowski space.

On a larger, galactic scale, we can have the creation of defects and can explore this using simple models. We have noticed in this part that, with the connection, we can pass through the singularity of galactic collapse and recreation. In the following Part 3b we discuss the linking and relinking of turbulent manifolds in which topology changes. In Part 6 we show how one can measure underlying dynamics using test particles and apply this to turbulent manifolds. With the latter machinery in place we can therefore study the creation and relinking of various hypothetical models of universes.

IV. APPLICATIONS AND COMPARISONS

We have observed that the presence of torsion in the underlying non Riemannian geometry is required for non null trace curvature 2-form, k. This curvature form can be used to induce a Riemannian geometry. By itself, its components have a physical interpretation being related to forces as in the Lorentz force equations. Here we examine first the case of quantum particles derived from geometric solutions to the Dirac equation. Then we use the method of specifying a reasonable induced 2-form k to determine the evolution of a galaxy. Because the evolution of the spacetime induces gravitation-like effects, we find the distribution of cold dark matter (CDM) for the model and mentioned in the paper's title. Finally we investigate the present method's application to stellar evolution.

A. Derivation and Extension of the Gravitational Inverse Square Law for an Electron

A source of torsion is the evolution of a quantum manifold's mass flux-density matrix as we now show. On completion of the computation of the flux-density, we will have completed the proof of Thm.4 by exhibiting an example of a curvature 2-form matrix with non-null trace. Now, we have shown that the density and mass current can be related to the curvature potential using that theorem. Thus, given a density and current obtained in another way, we can compute its potential. We use the so-called $su(2)|_2$ color algebra quantum dynamical manifold equations (QDMEs) which provide matrix solutions of the Dirac equations determining the mass-density and mass-flux. We will need two lemmas. The first provides the matrix solutions, the second derives the mass conservation law isomorphic to the one stated in Thm.4. On imposing the self-consistency result that the two currents are the same, we find the result desired. We will next proceed to prove the theorem showing the existence of a gravitational potential for these solutions. We will then re-examine the steps in obtaining the results finding reasons necessitating renormalization of the results as is required in QED. Finally we will compare the present approach to the one obtained from a combination of the Yang-Mills and Einstein approaches. But first the lemmas.

Here we begin with the Dirac equation for an electron-positron manifold. Because of time invariance degeneracy, we have pairs of solution vectors which can be arranged so that matrix solutions are obtained. The equation for these matrices are what we call the Dirac quantum dynamical manifold equations (QDMEs). Because of the spatial derivative term contains $\sigma \cdot \nabla$, with $\{\sigma_i\}$ being the 2×2 Pauli matrices, we also call these equations $su(2)|_2$-QDMEs. We call $su(2)|_2$ the *color algebra* representation for these two-line QDMEs.. Their matrix solutions, called *flavor symmetric* ones, correspond to pairs of 2D moving frames held in dynamical contact (a 2-spinor of moving 2D frames). In this scheme the electron $E > 0$ and the positron $E < 0$ emerge as homeomorphic to S^3 spheres with inward and outward directed normals. The following two lemmas and Thm.25, then constitute a derivation of Newton's law of gravitation from the geometry induced by the quantum particle. Corrections to this law are indicated when there is relative motion between an observer.

Lemma 23 *The $su(2)|_2$ QDMEs*

$$(i\hbar\partial_t - e\varphi - \alpha(-ic\hbar\nabla + eA))\Psi = \beta mc^2\Psi, \tag{109}$$

have projective matrix solutions given in Fourier-space and for vanishing gauge potentials by

$$\Phi_b(k) = \frac{c\hbar\sigma \cdot k}{E_\pm + mc^2}\Phi_a(k), \quad \Phi_a(k) = \frac{c\hbar\sigma \cdot k}{E_\pm - mc^2}\Phi_b(k), \quad E_\pm^2 = m^2c^4 + (c\hbar\,\|k\|)^2 \tag{110}$$

Here

$$\Psi = \begin{pmatrix} \phi_a \\ \phi_b \end{pmatrix}, \quad \phi_{a,b} = \begin{pmatrix} A_{a,b} & -B_{a,b}^* \\ B_{a,b} & A_{a,b}^* \end{pmatrix}, \quad \beta = \begin{pmatrix} 1_2 & 0_2 \\ 0_2 & -1_2 \end{pmatrix}, \quad \alpha^i = \beta\gamma^i \tag{111}$$

where the γ^i are the Dirac matrices

$$\gamma^0 = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}, \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}, \gamma^5 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{112}$$

and the $\sigma^i$are the Pauli matrices

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \sigma^2 = -i \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{113}$$

Proof: Use $E = \hbar\omega$, for null gauge potentials $(\varphi, \mathbf{A})-$ i.e., ignoring the self-consistent self-gauge field produced by the lepton. This means we have not guaranteed that the quantum state is locally invariant with respect to changes in phase. We then have for the Fourier representation of Eq.(109)

$$\begin{pmatrix} E & -c\hbar\sigma \cdot k \\ -c\hbar\sigma \cdot k & E \end{pmatrix} \begin{pmatrix} \Phi_a(k) \\ \Phi_b(k) \end{pmatrix} = mc^2 \begin{pmatrix} \Phi_a(k) \\ -\Phi_b(k) \end{pmatrix} \tag{114}$$

The matrix solutions given in Eq.(110) are now obtained from Eq.(114) by solving the indicated simultaneous equations for $\Phi_b(k)$ and $\Phi_a(k)$, completing the proof.

Note that by using the solutions given in Eq.(114) one finds

$$\Phi_b(k) = \frac{c\hbar\sigma \cdot k}{E_\pm + mc^2} \frac{c\hbar\sigma \cdot k}{E_\pm - mc^2} \Phi_b(k). \tag{115}$$

This implies

$$\frac{c\hbar\sigma \cdot k}{E_\pm + mc^2} \frac{c\hbar\sigma \cdot k}{E_\pm - mc^2} = 1. \tag{116}$$

If we use the fact $(\sigma \cdot k)^2 = k^2$, then Eq.(116) can be solved leading to the last relation given in Eq.(110) verifying the relativistic energy-wavevector relation. We now define

$$\Sigma_+(k) = \frac{c\hbar\sigma \cdot k}{E_\pm + mc^2}, \quad \Sigma_-(k) = \frac{c\hbar\sigma \cdot k}{E_\pm - mc^2}. \tag{117}$$

In this notation

$$\Phi_b(k) = \Sigma_+(k)\Phi_a(k), \quad \Phi_a(k) = \Sigma_-(k)\Phi_b(k). \tag{118}$$

By rewriting Eq.(114) for the adjoints we find

$$\Phi_b^\dagger = \Phi_a^\dagger \Sigma_-, \quad \Phi_a^\dagger = \Phi_b^\dagger \Sigma_+. \tag{119}$$

Equation (116) can then be written as

$$\Sigma_+(k)\Sigma_-(k) = \Sigma_-(k)\Sigma_+(k) = 1. \tag{120}$$

We then obtain a result used below:

$$\Phi_b^\dagger(k)\Phi_b(k) + \Phi_a^\dagger(k)\Phi_a(k) = \Phi_a^\dagger(\Sigma_-\Sigma_+ + 1)\Phi_a = 2\Phi_a^\dagger\Phi_a. \tag{121}$$

If Φ_a is an orthogonal or a unitary matrix, then $\Phi_a^\dagger\Phi_a = 1$ and an extra factor of 2 arises in the normalization convention for (Φ_a, Φ_b) derived from Eq.(121). If instead we assigned a normalization of $\sqrt{1/2}$ to each component matrix this factor of 2 would disappear. A solution of the x-space equations (109) uses a linear superposition of these solutions.

We are then in a position to show that the $su(2)|_2$-QDME has flavor symmetric solutions and to derive a current/mass conservation equation for these manifolds. We employ a method that avoids squaring the QDME operator which in the case of the Dirac QDME would lead to the d'Alembertian.

Lemma 24 *[4, Part 2b] The mass-density matrix $\rho_m(x,t)$ and the mass-flux matrix $J_m(x,t)$ associated with the flavor symmetric (matrix) solutions to the Dirac equation obey a continuity equation:*

$$\partial_t \rho_m(x,t) + \nabla \cdot J_m(x,t) = 0 \tag{122}$$

where

$$\rho_m(x,t) \equiv \Phi_a^\dagger \Phi_a + \Phi_b^\dagger \Phi_b \tag{123}$$

$$J_m(x,t) \equiv \left(\Phi_a^\dagger, \Phi_b^\dagger\right) \alpha \begin{pmatrix} \Phi_a \\ \Phi_b \end{pmatrix} = (\Phi_a^\dagger \sigma \Phi_b + \Phi_b^\dagger \sigma \Phi_a) \tag{124}$$

Here $\rho_m(x,t)$ is a 2×2 matrix that can be interpreted as a mass-density matrix of the quantum system. These relations hold for the trace of Eqs.(122-124).

Proof: The current conservation law for the matrix solutions is derived by first writing the equation and its adjoint in the standard Dirac notation, where (for the Dirac QDME $\gamma^i = \sigma^i$)

$$\alpha^i = \begin{pmatrix} 0 & \gamma^i \\ \gamma^i & 0 \end{pmatrix} \text{ and } \beta = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

but with a 2-spinor containing frame fields matrices as elements

$$i\hbar \partial_t \begin{pmatrix} \Phi_a \\ \Phi_b \end{pmatrix} + i\hbar c \alpha \cdot \overrightarrow{\nabla} \begin{pmatrix} \Phi_a \\ \Phi_b \end{pmatrix} - \beta m c^2 \begin{pmatrix} \Phi_a \\ \Phi_b \end{pmatrix} = 0, \tag{125}$$
$$-i\hbar \partial_t \left(\Phi_a^\dagger, \Phi_b^\dagger\right) - i\hbar c \left(\Phi_a^\dagger, \Phi_b^\dagger\right) \alpha^\dagger \cdot \overleftarrow{\nabla} - mc^2 \left(\Phi_a^\dagger, \Phi_b^\dagger\right) \beta = 0.$$

The first equation set describes the particle, the second a time reversed particle, or anti-particle. So, multiplying the first equation by the adjoint spinor and the second by the spinor (using $\beta^\dagger = \beta$ and the notation $\overleftarrow{\nabla}$ operates to the left) there is obtained

$$i\hbar \left(\Phi_a^\dagger, \Phi_b^\dagger\right) \partial_t \begin{pmatrix} \Phi_a \\ \Phi_b \end{pmatrix} + i\hbar c \left(\Phi_a^\dagger, \Phi_b^\dagger\right) \alpha \cdot \overrightarrow{\nabla} \begin{pmatrix} \Phi_a \\ \Phi_b \end{pmatrix} - mc^2 \left(\Phi_a^\dagger, \Phi_b^\dagger\right) \beta \begin{pmatrix} \Phi_a \\ \Phi_b \end{pmatrix} = 0, \tag{126}$$

$$-i\hbar \left(\partial_t \left(\Phi_a^\dagger, \Phi_b^\dagger\right)\right) \begin{pmatrix} \Phi_a \\ \Phi_b \end{pmatrix} - i\hbar c \left(\Phi_a^\dagger, \Phi_b^\dagger\right) \alpha^\dagger \cdot \overleftarrow{\nabla} \begin{pmatrix} \Phi_a \\ \Phi_b \end{pmatrix} - mc^2 \left(\Phi_a^\dagger, \Phi_b^\dagger\right) \beta \begin{pmatrix} \Phi_a \\ \Phi_b \end{pmatrix} = 0. \tag{127}$$

Subtracting the second equation from the first and collecting terms one has proved

$$i\hbar \partial_t \left(\Phi_a^\dagger \Phi_a + \Phi_b^\dagger \Phi_b\right) + i\hbar c \left(\Phi_a^\dagger, \Phi_b^\dagger\right) \left(\alpha^\dagger \cdot \overleftarrow{\nabla} + \alpha \cdot \overrightarrow{\nabla}\right) \begin{pmatrix} \Phi_a \\ \Phi_b \end{pmatrix} = 0, \tag{128}$$

i.e., Eq.(122). Using the definitions given in Eqs.(123-124), and the fact that for the Dirac QDME the matrix $\alpha^\dagger = \alpha$ because of its $su(2)|_2$ content is self-adjoint we can complete the proof.

One can also show for this problem that the flavor symmetric $SU(2)$-solutions to the Dirac equation in a $1/r$ potential representing excited states of the electron density yield a $SU(2)$-structure with nodes defined by the zeroes of the radial and angular functions. Therefore, we can compute the curvature induced by these excited states. Furthermore, the multi-particle solutions belong to $SU(2)|_n$. By examining the structure of the determinantal relation $P_n(w,r) = \det(SU(2)|_n)$, these can be shown to be homeomorphic to n-sheeted covers, $P_n(w,r)$, of S^2 branched at $n(n-1) = 2g$ points having the general form

$$n-even: P_n(w,r) = (w^2 + n^2 r^2)!! \equiv \quad (w^2 + 1^2 r^2)(w^2 + 3^2 r^2) \cdots (w^2 + (n-1)^2 r^2) = 1 \tag{129}$$
$$n-odd: P_n(w,r) = (w^2 + n^2 r^2)!! \equiv w(w^2 + 2^2 r^2)(w^2 + 4^2 r^2) \cdots (w^2 + (n-1)^2 r^2) = 1 \tag{130}$$

Here $r^2 = x^2 + y^2 + z^2$, so the product represents a map of $P_n(w, r = \pm\sqrt{x^2+y^2+z^2})$ onto S^3, and g is the genus of the 2D surface $P_n(w,r) = 1$ when w is considered complex and is also the degree of the map onto S^3.

Notice that for the projective spacetime solutions of the 2-line $su(2)|_2$ QDMEs, the density matrices, $\rho_g(x,t)$, of Eq.(123) have one of three possible topologies as deduced by the following:[45]

$$\begin{aligned}
\Phi_{a,b} &= \begin{pmatrix} A_{a,b} & -B^*_{a,b} \\ B_{a,b} & A^*_{a,b} \end{pmatrix} \in SU(2)|_2 \approxeq S^1,\, S^3 & (131)\\
&\Longrightarrow \left(\rho_{m}{}^{j}_{k}(x,t)\right) \equiv \Phi_b^\dagger \Phi_b + \Phi_a^\dagger \Phi_a, & (132)\\
&= \left(|A_a|^2 + |B_a|^2 + |A_b|^2 + |B_b|^2\right) 1 \approxeq S^1,\, S^3,\, S^7 & (133)\\
\rho_m(x,t) &= t_r\left(\rho_{m}{}^{j}_{k}(x,t)\right) = \left(|A_a|^2 + |B_a|^2 + |A_b|^2 + |B_b|^2\right) & (134)
\end{aligned}$$

That is, the $su(2)|_2$-color algebra QDME has matrix solutions which can be classified according to this scaler structure – none of which is localized. We have previously identified the different topologies in Eq.(131) with the electron, S^1, muon, S^3, and tauon, S^7. We further note that the Hopf map (projection) $S^3 \to S^2 \times S^1$ produces the structure of the Dirac monopole (shown in another context by the last figure of this Part) and that the map $S^7 \to S^4 \times S^3$ gives the BRST instanton. In the case of S^7 we get a new structure. Each of the (S^1, S^3, S^7) is associated with a group structure $(U(1) \approxeq SO(2), SU(2), SU(3))$. However, when we compute the mass-density according to Eq.(132), the resulting gravitational potential appears point-like. Any finite amount of localization, even to the size of the universe, introduces an renormalization effect. This localization also arises from the necessity of the body to interact with its own self-field to preserve its $u(1)$ gauge color symmetry and modifies the right-hand side of Eq.(133) as discussed elsewhere. The scaler electrodynamic potential obtained from $-\nabla^2\varphi = 4\pi\rho_m(x,t)$ modifies the time derivative by a gauge term, $\partial_t \to \partial_t + i\left(e^2/\hbar c\right)\varphi$, renormalizing the mass of the electron-positron. Here e^2 is a coupling constant. In the frame of the electron, considered static, the interaction with the self-vector field $\mathbf{A}$ must also be included by the gauge transformation $\nabla \to \nabla + i\,(e/c)\,\mathbf{A}$. Similar terms are needed for the gravitational self-energy correction but these, of course, are small.

Note the normalization condition, Eq.(121), assures us that $t_r\left(\rho_{m}{}^{j}_{k}(x,t)\right)$ is non-negative. The non-negativity of $\rho_m(x,t)$ is evident from the right-hand side of Eq.(134). Because the frame field solutions are topologically equivalent to spheres or objects carved out from these on a pointwise basis, the frame fields can be pointwise normalized. This avoids the subtle non-relativistic process of instantly integrating over the whole manifold as required using the standard, Dirac interpretation of the 4-spinor wave function as a generator of a probability density. The continuity Eq.(122) is the same form as that found for the gravitational field and this provides our mass-density and mass-flux. We can use this to compute a vector potential for the trace curvature of a manifold. More generally as the quantities are matrices of, in general, complex quantities we can employ a realization map to obtain the corresponding real space description of the geometry.

Equation (128) is somewhat more general than Eq.(122) in that non self-adjoint representations can be used. It is clear that the procedure just developed can be extended to n-spinors with suitably extended definitions and to a complete density matrix by employing direct products. This is shown by writing the equations for self-adjoint representations and proceeding as follows (β is a diagonal matrix commuting with the other matrices):

$$\begin{gathered}
\alpha \cdot \overrightarrow{\nabla} = \left(\partial_t 1 + \gamma^i \nabla_i\right), \\
i\alpha \cdot \overrightarrow{\nabla}\Psi = m\beta\Psi, \qquad -i\Psi^\dagger\alpha^\dagger \cdot \overleftarrow{\nabla} = m\beta\Psi^\dagger, \\
\Psi^\dagger \otimes \alpha \cdot \overrightarrow{\nabla}\Psi = m\beta\Psi^\dagger \otimes \Psi, \qquad -i\Psi^\dagger \otimes \alpha^\dagger \cdot \overleftarrow{\nabla}\Psi = m\beta\Psi^\dagger \otimes \Psi, \\
\partial_t\left(\Psi^\dagger \otimes \Psi\right) + \Psi^\dagger \otimes \left(\alpha^\dagger \cdot \overleftarrow{\nabla} + \alpha \cdot \overrightarrow{\nabla}\right)\Psi = 0, \\
\partial_t\left(\Psi^\dagger \otimes \Psi\right) + \nabla_i \Psi^\dagger \otimes \gamma^i \Psi = 0. \qquad (135)
\end{gathered}$$

The physical meaning of this direct product can be understood by examining the $2 \otimes 2$ case in terms of the following direct product matrix

[45]Notice the signature of the manifold, S^1, S^3 or S^7 in the last expression has been lost because of the square $\rho \geq 0$. The underlying dynamics respects the signature, but for the mass-density it is lost.

$$(\rho_{m}{}^{j}_{k}) = \left(\Psi^\dagger \otimes \Psi\right) = \begin{pmatrix} \Phi_a^\dagger \\ \Phi_b^\dagger \end{pmatrix} \otimes \begin{pmatrix} \Phi_a \\ \Phi_b \end{pmatrix} = \begin{pmatrix} \Phi_a^\dagger \Phi_a & \Phi_a^\dagger \Phi_b \\ \Phi_b^\dagger \Phi_a & \Phi_b^\dagger \Phi_b \end{pmatrix} \tag{136}$$

$$= \frac{1}{2}(\Phi_a^\dagger \Phi_a + \Phi_b^\dagger \Phi_b)1 + \begin{pmatrix} \Phi_a^\dagger \Phi_a - 1/2(\Phi_a^\dagger \Phi_a + \Phi_b^\dagger \Phi_b) & \Phi_a^\dagger \Phi_b \\ \Phi_b^\dagger \Phi_a & \Phi_b^\dagger \Phi_b - 1/2(\Phi_a^\dagger \Phi_a + \Phi_b^\dagger \Phi_b) \end{pmatrix} \tag{137}$$

The first term contains the average density; the second one the fluctuations about it. We can imagine assigning a charge of 1/2 to each term here as discussed above, but we see that the symmetry adaptation is the source of the factor. However, as the nature of charge is an integer topological quantity this is ill advised. As these quantities are $SU(2)$, the direct product corresponds to the representation $SU(2)|_2 \otimes \overline{SU(2)}|_2 = 2 \otimes \overline{2} = 1 \oplus 3$. That is a scaler-vector decomposition. We can use the density matrix, $(\rho_{m}{}^{j}_{k})$, and the currents $\left(j^j_{mk}\right)$ to determine the geometric-potential (A^j_k) due to the full density/current matrix, not just the trace, from Eq.(14):

$$\Box A^j_k = -4\pi J_{m}{}^{j}_{k}, \tag{138}$$

where we have removed the spatial density coefficients $dx^1 \wedge dx^2 \wedge dx^3$, etc., of the 3-forms.[46] It is significant, as long as the norm $\Phi_a^\dagger \Phi_a + \Phi_b^\dagger \Phi_b$ is positive semidefinite, that we have non repulsive potentials. As these potentials curve spacetime, the motion of similar test particles in it will be attracted to a compact source. Such a density matrix can be obtained for any, multiparticle, quantum system.[47] In general, we have discovered

$$J^j_{gk} = J^j_{mk} \tag{139}$$

A two step process is thus involved in obtaining the gravitational field from the quantum dynamical density: first we find the solutions to some quantum dynamical manifold equations. Then we construct the density (and current) from which we determine the induced Riemann gauge potential. Note that we apparently do not have to use Newton's gravitational parameter G_N. The quantum dynamical mass-density-flux simply curves the MST.

It is clear that this density and similar ones have non null trace and they lead to an interesting MST geometry.[48,49]

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[46]Because $t_r\left(A^j_k\right)$and $t_r\left(J_{m}{}^{j}_{k}\right)$ have identical tensor indices (none), we have as many equations as unknowns here. When we convert these results to an induced Riemann space using $k = da$, we must insure $\nabla_\mu g^{\mu\nu} = 0$. These equations hold in any dimension space, but in particular for a 4D MST the first set are six relations and the Riemann constraint gives four more, for a total of ten; exaclty the number required.

[47]This is different from the densities defined for the cold dark matter based on the gradient of the curvature interior to an S^3 sphere.

[48]When we extend this process to three lines of equations for a $su(3)|_3$ color algebra we obtain a decompostion for $3 \oplus \overline{3} = 8 \oplus 1$:

$$\left(\Psi \otimes \Psi^\dagger\right) = \begin{pmatrix} u \\ d \\ s \end{pmatrix} \otimes \begin{pmatrix} u^\dagger \\ d^\dagger \\ s^\dagger \end{pmatrix} = \begin{pmatrix} u \cdot u^\dagger & u \cdot d^\dagger & u \cdot s^\dagger \\ d \cdot u^\dagger & d \cdot d^\dagger & d \cdot s^\dagger \\ s \cdot u^\dagger & s \cdot d^\dagger & s \cdot s^\dagger \end{pmatrix}$$

$$= \frac{1}{3}\left(u \cdot u^\dagger + d \cdot d^\dagger + s \cdot s^\dagger\right)$$

$$+ \begin{pmatrix} u \cdot u^\dagger \text{-} \frac{1}{3}\left(u \cdot u^\dagger \text{+} d \cdot d^\dagger \text{+} s \cdot s^\dagger\right) & u \cdot d^\dagger & u \cdot s^\dagger \\ d \cdot u^\dagger & d \cdot d^\dagger \text{-} \frac{1}{3}\left(u \cdot u^\dagger \text{+} d \cdot d^\dagger \text{+} s \cdot s^\dagger\right) & d \cdot s^\dagger \\ s \cdot u^\dagger & s \cdot d^\dagger & s \cdot s^\dagger \text{-} \frac{1}{3}\left(u \cdot u^\dagger \text{+} d \cdot d^\dagger \text{+} s \cdot s^\dagger\right) \end{pmatrix}$$

It is apparent that the nine elements of the direct product give an eight-fold way decomposition of a vector space.

[49]To demonstrate the non vanishing torsion note the matrix homeomorphism

$$\left(\Psi^\dagger \otimes \Psi\right)_{ij} = x_1 1 + i\sigma \cdot x \leftrightarrow \begin{bmatrix} x_1 & x_2 & x_3 & x_4 \\ -x_2 & x_1 & -x_4 & x_3 \\ -x_3 & x_4 & x_1 & -x_2 \\ -x_4 & -x_3 & x_2 & x_1 \end{bmatrix}, \qquad i,j = 1,2$$

This geometry has non vanishing implicit torsion defined in terms of the Levi-Civita completely antisymetric tensor ε^j_{ik} even when $r^2 = x_1^2 + x_2^2 + +x_3^2 + x_4^2 = const.$ as follows:

$$T^j_{ik} = (\Gamma^j_{ik} - \Gamma^j_{ki}) = -2\varepsilon^j_{ik}, \text{ e.g.,} \quad T^3_{12} = -2$$

since the matrices are $SU(2)$. Furthermore, the geometry is doubled $(i,j = 1,2)$.

Moreover, in the bargain we get the equation for the off-diagonal parts. If we include a vector potential in our equations ($u(1)$-gauge derivatives), then we will have off diagonal currents. The self-gauging of the quantum dynamical manifolds produces additional effects on the attendant gravitational field. In the present theory the self-energy of the gravitational field is thus included in contrast to traditional GRT. In concert with GRT, however, the density-current provides the potential for determining the curved MST. The potentials include not only the renormalization self-interaction with the particles electromagnetic field arising from the fact that $SU(2)$ is a double cover of $SO(3)$, but "external fields" as well. As it is, we show currents vanish without such a potential.

As we wish to determine the average current from Eq.(124), we write

$$t_r\left(J_m{}^j_k\right)|_i = j_i(x,t) = t_r(\Phi_a^\dagger\sigma_i\Phi_b + \Phi_b^\dagger\sigma_i\Phi_a), \quad i=1,2,3. \tag{140}$$

Now, using the properties of the trace, and

$$\Phi_{a,b} \equiv \left(\begin{array}{cc} A_{a,b} & -B_{a,b}^* \\ B_{a,b} & A_{a,b}^* \end{array}\right), \tag{141}$$

we find

$$\begin{aligned} t_r[\Phi_a^\dagger\sigma_i\Phi_b + \Phi_b^\dagger\sigma_i\Phi_a] &= t_r[(\Phi_a^\dagger\Phi_b + \Phi_b^\dagger\Phi_a)\sigma_i] \\ &= t_r\left[\left(\begin{array}{cc} \begin{array}{c} A_a^*A_b + A_aA_b^* \\ +B_a^*B_b + B_aB_b^* \end{array} & 0 \\ 0 & \begin{array}{c} A_a^*A_b + A_aA_b^* \\ +B_a^*B_b + B_aB_b^* \end{array} \end{array}\right)\sigma_i\right] \\ &= (A_a^*A_b + A_aA_b^* + B_a^*B_b + B_aB_b^*)\, t_r[\sigma_i] \\ &= 0 \end{aligned} \tag{142}$$

So we have proved that the average currents vanish in the frame of the equation. By the continuity equation, we then have average density conservation:

$$\frac{\partial\rho_m}{\partial t} = 0. \tag{143}$$

In further discussion, therefore, we can ignore the currents and concentrate only on the density. The procedure described above does not involve the squaring of the operator acting on the spinor of frames. It manages to transform a non Abelian gauge theory into one for a diagonal matrix. We call this an *Abelianization of the potential.*

Since $\rho_m = t_r(\Phi_a^\dagger\Phi_b + \Phi_b^\dagger\Phi_a) = \left(|A_a|^2 + |B_a|^2 + |A_b|^2 + |B_b|^2\right) \approxeq (S^1, S^3, S^7)$, we should examine the (S^1, S^3, S^7) solution topologies for which we determine what we call their *inner potential structure*. Surprisingly, these cases can be distinguished by examining their Yang-Mills content as defined next.

1) For the trace $\rho_m : \left(|A_a|^2 + |B_a|^2 + |A_b|^2 + |B_b|^2\right) \approxeq S^1 \approxeq U(1)$, we can determine a $U(1)$-gauge potential for the inner potential structure:

$$k = da = k_{\mu\nu}dx^\mu \wedge dx^\nu, \tag{144}$$

$$A = A_\mu dx^\mu, \tag{145}$$

as we know from electromagnetic field theory. The global (anti-symmetric) average curvature is of the form of an electromagnetic field

$$k_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \tag{146}$$

and this satisfies the Maxwell structure equation

$$dk = 0 \tag{147}$$

Because of this, by the converse of Poincaré's lemma, we have a local potential. The trace, ρ_m, is non zero as the mass density does not vanish. As the trace is an invariant polynomial, we have a global potential. The density in the S^1 case has an inner potential structure leading to a Yang-Mills equation for the Abelian group $U(1)$. The multiparticle states amount to the projection of a helix segment onto a plane with the number of cycles in the helix corresponding to the number of electrons. This inserts a phase factor and in the Dirac QDME leads to the introduction of a $u(1)$

self-gauge potential. The above case was for the case of a R^4 base space and an S^1 gauge group. If the manifold of definition is multiply connected we have a non trivial base space. In the case of R^3 with a hole at the origin the potential computed for $R^3 - \{0\}$ leads to a description of the Dirac monopole [13, p.355]. For assemblages of these, there is a quantized total flux.

2) From the trace $\rho_m : \left(|A_a|^2 + |B_a|^2 + |A_b|^2 + |B_b|^2\right) \approxeq S^3 \approxeq SU(2)$ we can define a $SU(2)$ matrix , $(\widetilde{\rho}_{m}{}^{j}_{k})$, and use it to determine the inner potential structure by solving $\Box \widetilde{A}^j_k = -4\pi \widetilde{\rho}_m{}^j_k$ for the local $SU(2)$-gauge potential 1-form $\widetilde{A} = \widetilde{A}^c_\mu \gamma_c dx^\mu$. From this we can find the curvature matrix of 2-forms $\widetilde{K} = d\widetilde{A} - \widetilde{A} \wedge \widetilde{A} = \frac{1}{2}\widetilde{K}_{\mu\nu} dx^\mu \wedge dx^\nu = \frac{1}{2}(\partial_\mu \widetilde{A}_\nu - \partial_\nu \widetilde{A}_\mu - [\widetilde{A}_\mu, \widetilde{A}_\nu]) dx^\mu \wedge dx^\nu = \frac{1}{2}(\partial_\mu \widetilde{A}^c_\nu - \partial_\nu \widetilde{A}^c_\mu + \varepsilon^c_{ab} \widetilde{A}^a_\mu, \widetilde{A}^b_\nu)\gamma_c dx^\mu \wedge dx^\nu$.[50] One finds an invariant

$$S = -\frac{1}{4}\int tr(\widetilde{K}_{\mu\nu}\widetilde{K}^{\mu\nu}) = \frac{1}{2}\int tr(\widetilde{K} \wedge^\star \widetilde{K}) \tag{148}$$

This is the Yang-Mills action whose variation with respect to the potential yields the Y-M structure relation $D_A \widetilde{K} = 0$ as expected. As this equation is simply the first Bianchi equation, we have

$$d\widetilde{K} = [\widetilde{K}, \widetilde{A}] \tag{149}$$

this is not $d\widetilde{K} = 0$ so we do not have a closed differential form. This means that only for the invariant trace of this system do we have global potential. Nonetheless, the density in the S^3 case has an inner potential structure leading to a Yang-Mills equation for the non Abelian group $SU(2)$. In this case one can also consider a potential satisfying $A_\mu \to g(x)^{-1}\partial_\mu g(s)$ as $|x| \to \infty$ in R^4. This leads to the instanton potential [13, p.359].

3) From the trace $\rho_m : \left(|A_a|^2 + |B_a|^2 + |A_b|^2 + |B_b|^2\right) \approxeq S^7 \approxeq SU(3)$ we can represent the inner potential structure using $SU(3)$ matrices. Again we have a Yang-Mills gauge theory for the gauge 1-form which we can use to evaluate the only global trace curvature 2-form. Nonetheless, the density in the S^7 case has an inner potential structure leading to a Yang-Mills equation for the non Abelian group $SU(3)$.

For emphasis, we note that only the S^1 case has its total potential specified by its gauge algebra. It is the only manifold of the three ($e \approxeq S^1, \mu \approxeq S^3, \tau \approxeq S^7$) which we can associate with a stable particle.

In the above derivations we have used a flux-density derived from both Abelian and non Abelian symmetry for the density but we have non-zero trace to K because of the inherent flavor symmetry of the quantum dynamics. It was seen that this symmetry depends on the color algebra in a sensitive fashion. In this case the flavor symmetry is $SU(2)$. It appears that the non Abelian symmetry in the flavor space is what is required for the existence of torsion. Furthermore, while electromagnetism (Abelian gauge, $u(1)$) has a globally defined average potential, as evidenced by the Maxwell structure equation $dK = 0$, in general, the Yang-Mills structure does not have such a global potential completely determining the curvature. That is we have only $D_A K = 0$ (Bianchi constraint). However, the trace curvature is still defined and leads to a global potential. It is this global Abelian part, i.e., the Abelianized component, that we associate with gravitation.

We can use the requirement that the QDME lead to a conserved density for selecting from all possible QDMEs those that are physically reasonable. It is clear that by first Fourier transforming Eqs.(125), then performing the operations of taking adjoint and subtraction we would find, instead of Eq.(123), a different definition of the Fourier components of the mass-density, namely,

$$\rho_m(k) \equiv \Phi^\dagger_a(k)\Phi_a(k) + \Phi^\dagger_b(k)\Phi_b(k). \tag{150}$$

In terms of this mass-density (which is not the Fourier transform of the spatial mass-density, Eq.(123)) we have

Theorem 25 *The gravitational field of the Dirac electron is inverse square in the absence of self-interaction (renormalization) effects.*

Proof. We first concern ourselves only with the scaler potential φ since the currents vanish for the coevolving frame field. The gravitational field equation, given by Eq.(14), $\Box A^j_k = -4\pi\, {}^\star J^j_{mk}$, for the S^1 trace, which has a global potential, then leads to the Poisson equation

$$-\nabla^2\varphi(\mathbf{x}) = 4\pi\rho_m(\mathbf{x}). \tag{151}$$

[50] We can also directly map the $SU(2)$ matrix of 2-forms, $\widetilde{K}$, into the form of an electromagnetic ($F_{\mu\nu}$) tensor, a fact used in the next section.

The Fourier transform is

$$k^2\varphi(k) = 4\pi\rho_m(k) \tag{152}$$

We now must decide the value of $\rho_m(k)$. If we choose k-space normalization according to Eq.(121) of Lem.23 and we drop the factor of 2 (this amounts to assigning a "charge" of 1/2 to each of the $su(2)$-quarks, $\Phi_{a,b}$, i.e., $\|\Phi_{a,b}\|^2 = \frac{1}{2}$ in the two-line QDME) we find

$$\varphi(k) = \frac{4\pi}{k^2}. \tag{153}$$

We note the solution given in Eq.(153) represents the spatial mass-density of a point particle $\rho(\mathbf{x}) = \delta(\mathbf{x})$ for a spacetime point $x = (t, \mathbf{x})$ and this corresponds to a completely de-localized k-space density (n.b. we are ignoring self-field effects here). We have thus proved that the quantum dynamical body has a gravitational curvature potential given by the classical results, in the absence of self-interaction effects. This completes the proof.

As ordinary Riemann spaces (i.e., spaces without torsion) have null trace curvatures, the present curvature arises from inherent torsion of the quantum dynamical system which is evident in the helical evolution paths of the quasi-particle system in spacetime. This torsion arises, as shown above from the tandem evolution of a pair of 2D moving frames, $\Phi_{a,b}$. This leads to the conclusion that to compute the dynamical curvature and torsion for a geometric interpretation of quantum dynamics one uses the quasiparticle amplitudes. To compute the global gravitational potential, one uses the quantum dynamical density matrix. The lepton produces a field that can be measured by observers moving relative to the body and leads to the usual retarded and advanced potentials.

The topological structure of the charge density matrix obtained directly from the space-time solutions of the QDMEs is different from those obtained by Fourier-transforming the mass-density of the lepton and then proceeding. The results developed above, thus show the order of operations in Fourier-transform space and ordinary space are important. The first approach, using a point-like structure has produced a kind of average-over-quantum-fluctuation result. It is clear that such a total de-localization would lead to problems in defining the interaction region of a charge with a finite body in finite time. It ultimately leads to a renormalization requirement for the gravitational mass just as in QED. We have nonetheless confirmed an important aspect of the structure of a microscopic particle, its gravitational field. Because of the isomorphism of electromagnetism and gravitational theory demonstrated here, the lepton also has an electromagnetic field, especially a magnetic moment. Furthermore, we notice the inertial mass parameter, does not appear directly, so the result is a structural one resulting from a certain kind of dynamics. The masses have to be determined in another fashion – in fact, using the standard methods of quantum electrodynamics or self-consistent self-gauge quantum dynamical manifold theory – we can compute the renormalized mass of the lepton interacting, not just with its own electromagnetic field, but also with its own (self-gauge) gravitational filed. We thus compute the gravitational mass of the lepton and find that it is the same as the inertial mass obtained by examining the dynamics of the lepton as $\hbar \to 0$ via a Foldy-Wouthuysen transformation.

By examining the structure of the $SU(2)$ frame field solutions, $\Phi_{a,b}$, to the $su(2)$-QDMEs given above we can show that the solutions correspond to parts identifiable via conjugation with the two-sheeted cover of an ordinary sphere. This, of course, is related to the two-fold cover of $SO(3)$ by $SU(2)$. We begin by writing an element of $SU(2)$ in a flat space form as follows

$$A(w,r) = \begin{pmatrix} w+iz & -y+ix \\ y+ix & w-iz \end{pmatrix} = w1 + i\sigma\cdot x. \tag{154}$$

Here the $\{\sigma_i\}$ are the three Pauli matrices. We introduce the flat space parametrization as follows

$$\begin{gathered} w = a\cos\chi, \\ z = a\sin\chi\cos\theta,\ x = a\sin\chi\sin\theta\cos\phi,\ y = a\sin\chi\sin\theta\sin\phi, \\ 0 \le \chi < \pi,\ 0 \le \theta < \pi,\ 0 \le \phi < 2\pi, \\ \det(A) = a^2 = w^2 + x^2 + y^2 + z^2. \end{gathered} \tag{155}$$

The last equation demonstrates the assertion that the $SU(2)$ matrix in Eq.(154) is homeomorphic to a flat space S^3 as this equation is the equation of a sphere in R^4. We next observe that w changes sign between the ranges $0 \le \chi < \pi/2$ and $\pi/2 \le \chi < \pi$. This can be used to define two parts of the representation or equivalently of S^3 as being the result of an operation $\hat{T}$ which reverses the sign of w, $\hat{T}\,A(w,r) = A(-w,r)$. We then ask the question "What are other operations on the space and how are these two parts related by them?" To answer this we define the Wigner conjugation transformation operator:

$$c_w = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad c_w^{-1} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \tag{156}$$

There can be then defined the conjugate representation of $su(2)$ obtained by taking complex conjugates of the original representation matrices. This operation can also be determined via a Wigner similarity transformation of the Pauli matrices:

$$\sigma_x^* = -c_w \sigma_x c_w^{-1}, \quad \sigma_y^* = -c_w \sigma_y c_w^{-1}, \quad \sigma_z^* = -c_w \sigma_z c_w^{-1}. \tag{157}$$

In Eq.(154) we see that changing the sign of (only) w (i.e., reversing only the direction of the w-axis thereby changing the orientation of the space; reversing all co-ordinates, introducing an operator $\hat{P}$ which changes the sign of (w, x, y, z), $\hat{P}\, A(w, r) = A(-w, -r)$, does not change the volume element's sign, thereby not changing the orientation. We can relate the operation of charge conjugation to complex conjugation by the following computation

$$\begin{aligned} c_w A(w,r) c_w^{-1} = w1 + i c_w \sigma c_w^{-1} \cdot x = w1 - i\sigma^* \cdot x = (w1 + i\sigma \cdot x)^* \equiv A^*(w,r), \\ A^* = c_w A c_w^{-1} = \hat{C}\, A \end{aligned} \tag{158}$$

By the properties of complex conjugation

$$\hat{C}^2 = 1. \tag{159}$$

We still have the operation of complex-conjugation. Combining these operations allows the following

$$\begin{aligned} \hat{T}\, A &= -w1 + i\sigma \cdot x \\ \left(\hat{P}\,\hat{T}\right) A &= w1 - i\sigma \cdot x = w1 + i c_w^{-1} \sigma^* c_w \cdot x = c_w^{-1}(w1 + i\sigma^* \cdot x) c_w \\ &= \hat{C}^{-1}(w1 + i\sigma^* \cdot x) \end{aligned} \tag{160}$$

We can consider the representation of $su(2)$ to be equivalent to that of $su(2)^*$, written $su(2) \approxeq su(2)^*$, since they have the same commutation relations, thus, according to the last equation above,

$$\hat{C}\,\hat{P}\,\hat{T} \approxeq 1. \tag{161}$$

So the effects of $\hat{P}\,\hat{T}$ and complex conjugation, $\hat{C}\,\hat{P}\,\hat{T}$, is the identity. It is natural to call $\hat{T}$ a time reversal operator and $\hat{P}$ a parity reversal operator. Note, in addition to $\hat{C}^2 = 1$, that $\hat{T}^2 = 1$ and $\hat{P}^2 = 1$ so all three operators are idempotent. Clearly, they commute among themselves. Thus, using Eq.(161), we have

$$\hat{T} \approxeq \hat{C}\,\hat{P} \tag{162}$$

Time reversal is then equivalent to the result of parity reversal and charge conjugation. If we identify points related by $\hat{T}$, then the discontinuous transformations $\hat{T}$ or $\hat{C}\;\hat{P}$ allow us to naturally relate the particle, E_+ to the anti-particle, E_- solutions of the $su(2)$-QDMEs for all values of $x = (x, y, z)$. This is the same as forming a projective space by identifying antipodal points and then operating with charge conjugation which changes the sign of the outward normal to enveloping spheres. Here we must consider charge to be related to the signature of mass. These also are related to an orientation reversal. We turn to larger systems after briefly comparing the present approach to one involving the use of Einstein's field equations and the Yang-Mills equations.

B. Comparison to the Eienstein-Yang-Mills Approach

By equating the deviatoric part of the Ricci tensor, $R_{\alpha\beta} - 1/2 g_{\alpha\beta} R$, to the energy-momentum, $\tau_{\alpha\beta}$, computed from Yang-Mills theory, one obtains the Einstein-Yang-Mills (EYM) equations:

$$R_{\alpha\beta} - 1/2 g_{\alpha\beta} R = \tau_{\alpha\beta}, \tag{163}$$

$$\tau_{\alpha\beta} = F_{\alpha\lambda} F_\beta^\lambda - 1/4 g_{\alpha\beta} F_{\lambda\nu} F^{\lambda\nu}, \tag{164}$$

$$A = A_\lambda^\alpha \gamma_a dx^\lambda, \tag{165}$$

$$F = dA - [A, A]. \tag{166}$$

The first equation is the Einstein gravitational field equations. The quantity $\tau_{\alpha\beta}$ whose computation is indicated in the following equation is the Yang-Mills contribution and it has well defined properties, but its substitution in the first line breaks the Bianchi identities. The present approach uses the quantum dynamical and Yang-Mills current to obtain the potential for the curvature. We have local existence of these potentials when the trace curvature is closed, $dk = 0$, and global existence for invariant polynomials of the connection. In the case examined above the S^1 topology mass density with an Abelian gauge group satisfies the condition of closedness and invariance for the trace.

In the present theory we have used the following to relate the covariant derivatives, ∇_k, to the curvature, R^i_{jkl}, and torsion, T^i_{kl}, of a manifold

$$\sum_{l=1}^{n} \left(\nabla_k \nabla_l - \nabla_l \nabla_k + T^i_{kl} \nabla_i\right) \phi^l = R_{kl} \phi^l,$$
$$\left(\nabla_k \nabla_l - \nabla_l \nabla_k + T^i_{kl} \nabla_i\right) \phi^0 = 0 \phi^0.$$

In the absence of torsion we can write the first equation in the form

$$[\nabla_k, \nabla_l] = R_{ijkl} \sigma^{ij} \tag{167}$$

Here σ^{ij} spans a two-dimensional space, i.e., $dx^i \wedge dx^j$. Associated with the Riemann curvature tensor R^i_{jkl} are the contractions $R_{jk} = R^i_{jki}$ and $R = g^{ij} R_{jk}$. If M^n is an n-dimensional manifold and $g_{ij}(\alpha)$ is a metric depending smoothly on the parameter α, such that outside a compact region g_{ij} does not depend on α, then the following general (Gauss-Bonnet [20, v.5,ch.13]) expression holds

$$\frac{d}{d\alpha} \int_{M^n} \sqrt{-g} R dx = \int_{M^n} \left(R_{ij} - 1/2 g_{ij}(\alpha) R\right) \frac{dg^{ij}(\alpha)}{d\alpha} \sqrt{-g} dx. \tag{168}$$

The quantity

$$\frac{1}{16\pi G_N} \int_{M^n} \sqrt{-g} R dx \tag{169}$$

defines the Einstein-Hilbert action. We have used the following expression for the gauge derivative commutator using Eq.(56) $D_\mu \equiv (\partial_\mu + A^c_\mu \gamma_c)$,

$$[D_\mu, D_\nu] = F^a_{\mu\nu} \gamma_a$$

where the square matrices defining the gauge algebra satisfy $[\gamma_a, \gamma_b] = f^a_{bc} \gamma_c$ and it is recalled that f^a_{bc} are called the structure constants of the Lie algebra. The quantities $F_{\mu\nu} = F^a_{\mu\nu} \gamma_a$ can be used to define an invariant action which can be combined with that of Eq.(169) to give an expression of the following form for an EYM theory

$$S = -\frac{1}{16\pi G_N} \int_{M^n} \sqrt{-g} R dx - \int \left(\frac{1}{16\pi} F_{\mu\nu} F^{\mu\nu} + j_\mu A^\mu\right) \sqrt{-g} dx \tag{170}$$

The first term gives the Lorentz group invariance and the second the gauge invariance. To introduce strings we define the gauge-covariant derivatives relative to the (universal) string group into this expression [14, ch.8]. Through this brief description we see that both the Yang-Mills approach and the string approach neglect the torsion found to be critical to our development of gravitational effects.

C. Cold Dark Matter

1. An S^3 Model of a Galaxy

A globally symmetric Riemann space (vanishing torsion) gives rise to pairs (ℓ, s) of Lie algebras, ℓ, and isometries, s, that are categorized as compact, Euclidean, or noncompact. The compact case corresponds to positive sectional Riemann curvature and compact spaces; the negative to non compact spaces [21, Ch.V]. These two types explain the formal analogy between the compact and non compact matter in the definition above and the source of single positive signature (outward normal) of ordinary compact matter. The presence of torsion can be shown to dynamically provide the "glue" between these two types of spaces and to be a source of new topological defects. Thus it is reasonable to extend the definition of "mass" to this larger realm. Consequently, the cold dark matter (CDM), whose definition is introduced above, is potentially different from the compact matter normally encountered as the dynamics produces both positive and negative G-field gradients, and hence positive and negative CDM. Normal compact matter acts as positive curvature entities (positive outward normal) that have shrunk around a local origin as vacancies in a solid continuum. A closed, oriented manifold can be considered a hypersurface and the number of times it envelopes an origin can be counted by computing the degree, d, a Gauss map of the manifold considered as a cover of an appropriately dimensioned sphere: $2d = \sum_{\alpha=1}^{N} \text{sign}\det(J)$, where J is the Jacobian of the map. With this definition, 4D MST can be considered a multiply connected 4D continuum of negative or zero curvature with positive curvature "holes" or other defects. However, because of the signature properties, it is unlike the usual 3D continua encountered in mechanics and therefore represents a new state of matter unless ordinary matter also has multi-signed densities such as that allowed by Eq.(48).[51]

As we have noted, GRT is obtained from the phenomenalization of the Bianchi identities, namely, by the insertion of observable (non CDM) energy-momentum into the identities. This, of course, breaks or invalidates the identities. We also note the only non-trivial solutions to GRT tested observationally are those essentially involving the exterior Schwarzschild solution in a case where these identities fortuitously are not broken. This implies only a curvature induced potential theory has been observationally verified in GRT and Newtonian gravitational theory [6]. Furthermore, spin-2 gravitational waves predicted from GRT having an inhomogeneous term, have not been observed. Thus, the gravitodynamical equations just derived, applied to the missing mass problem interior to a galaxy, offer a test of the new dynamical gravitation theory. The reader should also be made aware of the fact that classical GRT, even though it is a theory of gravitation, does not include the energy-momentum of its own gravitational field. In fact, it gives no account of the properties of mass at all except that it can curve spacetime. We now turn to possible observational tests of the new gravitation theory.

It has been found that galaxies and groups of galaxies have dispersion velocities greater than can be attributed to the observed mass in the Newtonian or GRT approximations [1] and there is a possible anomalous [23] expansion of the Universe. The former is a $\geqslant 90\%$ effect since the stars and gas account for only $\leq 10\%$ of the mass required to bind the observed rotating systems. This has been called the *missing mass effect* and is also interpreted as the presence of *cold dark matter* (CDM). The so-called CDM neither emits nor absorbs light - it is an ideal candidate for a dynamically curved MST. Since we are interested in the dynamics of a galaxy, we need to find a model for its mass-spacetime. The simplest is a shrinking spherical model. This implies that as compact matter is devoured by a central black hole in at least some galaxies the 90% part, the CDM part, is also devoured. That is, the galaxy is a sink for CDM as well as ordinary compact matter. However, there is a conservation law obeyed so the only place that one can actually loose MST is along a boundary-perhaps provided by a central black hole.

The manifold we choose is a 3D sphere S^3. This also provides a simple model for the entire Universe in an expansionary mode. To construct the MST we embed the evolving 3D S^3 into a 4D spacetime. This manifold also has torsion necessary for fitting spin structures into it. The picture is that the relatively compact galaxy with built-in torsion develops a spin as its environment expands. This defines a relative contraction which may be augmented by other contractive processes. We assume that the S^3 model has already included the averaging over local quantum fluctuations in the metric. This leaves an average 2-form k. Averaging the metric or averaging the curvature are, of course, different. For our model calculation this subtlety is ignored.

The surprising consequence of the theorem developed next is that, by using a FLRW $SU(2)|_2 \approxeq S^3$ gauge model [24] of a homogeneous isotropic($SO(3) \subset SU(2)$) QDM having a relative contraction/expansion to its evolution, one finds "extra" vortex effects in the MST.[52] This provides a solution to at least part of the galactic missing mass

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[51]The CDM differs from ordinary continua in another respect: it has no constitutive relations relating stress to strain. Instead, it has underlying quantum dynamical manifold equations (QDMEs) [4]. This is found also to be the case for ordinary compact matter such as electrons and protons, etc. The equations derived above, resembling the Maxwell equations, can in fact be solved using the eigenspectrum from either the Dirac or Maxwell operators with finite mass eigenvalues. This gives a spectral resolution of these kinds of mass-spacetime manifolds into a collection of interacting electron-positron pairs and massive photons $\{e^{\pm}, \gamma\}$. This can be extended to S^7 and the entire fermionic, spin-1/2 lepton series and also to the $\{W^{\pm}, Z^0, (\overline{Z}^0)\}$ spin-1 boson-system. Thus, the FLRW model mass-spacetime developed next is incomplete–there are no baryons. Furthermore, the scale for the existence of these particles must span the very small to galactic distances. These problems can be solved by using the entire category of quantum dynamical manifold equations and the resolution of a QDM into Lie group symmetric parts interacting via geometric torsion [4]. Then, one must average out most of the quantum details to obtain an average curvature as seen by a single test particle supported by a 4D base spacetime as indicated in the proof of Thm.4.

[52]The flat space S^3 has Euclidean differential line element $ds^2 = dx^2 + dy^2 + dz^2 + dw^2$ as embedded in R^4. The fact that $SU(2) \approxeq S^3$ is a double cover of $SO(3)$, the ordinary symmetry of isotropic space, is contained in the Euler relation that the squares of four numbers equals the square of three: $(x^2 + y^2 + z^2 + w^2) = (a^2 + b^2 + c^2)^2$.

By using the transformation from the Euclidean space $(y^i; i = 1, 2, 3, 4)$ with metric $\delta_{\mu\nu}$ in S^3 as embedded in R^4, the following change of coordinates $y^\alpha = \frac{x\alpha}{r}\sin\frac{1}{2}r$, $y^4 = \cos\frac{1}{2}r$ gives the following metric $(r = \delta_{\alpha\beta}x^\alpha x^\beta)$ [19, p.189] problem and a possible explanation of the appearance of jets in stellar or other QDM evolution. The approach is to consider the mass-spacetime manifold to be analogous to a 3D elastic continuum with one extra dimension [25], [26]. Such manifolds can support torsion that appears as a swerve or spin along geodesics in a mass-spacetime [27], [30]. We next describe the basic 3D Robertson-Walker geometries possible, and select S^3 as the candidate for a galaxy.

The possible 3D Robertson-Walker homogeneous-isotropic spaces based on a (symmetric) metric, γ_{ij}, are defined by the line elements

$$\begin{aligned} ds^2 &= \kappa^{-1}(d\chi^2 + \sin[\chi]^2(d\theta^2 + \sin[\theta]^2 d\phi^2)), \quad S^3(\kappa > 0) \\ ds^2 &= (d\chi^2 + \chi^2(d\theta^2 + \sin[\theta]^2 d\phi^2)), \quad E^3(\kappa = 0) \\ ds^2 &= -\kappa^{-1}(d\chi^2 + \sinh[\chi]^2(d\theta^2 + \sin[\theta]^2 d\phi^2)), \quad H^3(\kappa < 0) \end{aligned}$$

The signature of these line elements is defined by $\sigma = \kappa[\eta]/|\kappa[\eta]| \in \{-1, 0, 1\}$. Here κ is the curvature parameter. These can all be brought into the same form by defining

$$\begin{aligned} \Sigma &= \sin[\chi], \quad \text{if } \sigma = +1 \\ \Sigma &= \chi, \quad \text{if } \sigma = 0 \\ \Sigma &= \sinh[\chi], \quad \text{if } \sigma = -1 \end{aligned} \tag{171}$$

One can introduce a radius function $a[\eta]$ by $\kappa = \frac{\sigma}{a[\eta]^2}$, so that, by a change from Cartesian to spherical co-ordinates in the first of Eq.(173):

$$\begin{aligned} x^1 &= x = a[\eta]\Sigma[\chi] \sin\theta \cos\phi \\ x^2 &= y = a[\eta]\Sigma[\chi] \sin\theta \sin\phi \\ x^3 &= z = a[\eta]\Sigma[\chi] \cos\theta \end{aligned} \tag{172}$$

one has[53]

$$\gamma_{ik} dx^i dx^k = a[\eta]^2 \Sigma^2[\chi](d\chi^2 + \Sigma[\chi]^2(d\theta^2 + \sin[\theta]^2 d\phi^2)). \tag{173}$$

The expansion factor, $a[\eta]$, was introduced by Friedmann [24] in his study of General Relativistic Cosmology. The Friedmann–Lemaître-Robertson-Walker (FLRW) [24] geometry has the flexibility needed to describe an evolving galactic MST having both positive and negative curvatures. The galaxy can have compact regions ($\kappa > 0$) or non compact ones ($\kappa \leq 0$) depending on initial conditions and interactions with neighbors. The parameter $a[\eta]$ is not restricted to have a maximum speed of evolution $\dot{a}[\eta]$ as the speed of light. The situation, where all local velocities satisfy the constraints of Special Relativity ($\kappa \leq 0$) is described in a subsequent Part. These corrections are only important at the beginning of the big bang and in the vicinity of a black hole and in high energy collisions of elementary particles.

An ingredient needed in proving the theorem, obtained from the FLRW model in an orthogonal frame basis $\{\xi^i\}$, is the globally defined "average-isotropic" MST curvature form (a single 2-form matrix $k = t_r(K^i_{j[kl]} dx^k \wedge dx^l) \equiv$

$$g_{\mu\nu} = -4\delta_{\mu\nu}\frac{1-\cos r}{r^2} - \frac{2x_\mu x_\nu}{r^2}\left(1 - 2\frac{1-\cos r}{r^2}\right)$$

The torsion tensor of S^3 has components $T^j_{ik} = -2\varepsilon^j_{ik}$ ($\varepsilon-$ the completely antisymmetric Levi-Civita tensor and $\Gamma^j_{ik} = -\varepsilon^j_{ik}$ $, i, j, k = 1, 2, 3$). The curvature, R^j_{ikl}, can be shown to vanish by using the Jacobi identity. By an appropriate change in the definition of the local connection, $\tilde{\Gamma}^i_{jk} \equiv \Gamma^i_{jk}$ - $\frac{1}{2}T^i_{jk}$, we can always eliminate the torsion $\tilde{T}^i_{jk} = \tilde{\Gamma}^i_{jk} - \tilde{\Gamma}^i_{jk} = T^i_{jk} - \frac{1}{2}T^i_{jk} + \frac{1}{2}T^i_{kj} = 0$, replacing a space with torsion by one with curvature. In order to maintain the helical paths one must introduce a metric, leading to curvature. The choice of whether or not to do so depends on the physical description we wish to employ. We implicity employ such a transformation below in order to compare to the classical theories of gravitation.

[53]The standard FLRW spacetime with line element of the form

$$ds^2 = \sigma^2 d\eta^2 + S^2(\eta) d\Sigma^2$$

is hyperoblic with $\sigma^2 = -1$. Our compact spacetime line element has $\sigma^2 = 1$.

$k_{[\mu\nu]} d\xi^\mu \wedge d\xi^\nu$ [54]) but expressed as a matrix labeled by the 1-form components by [8, p. 357]:

$$t_r K[\Lambda] = (k_{[\mu\nu]}) : ({}^{(3)}R_j^k) = \left(\frac{1+\dot{a}^2}{a^2} \xi^{\hat{k}} \wedge \xi^\mu \right), \quad R_{\hat{k}}^{\hat{\eta}} = \frac{\ddot{a}}{a} \xi^{\hat{\eta}} \wedge \xi^{\hat{k}}, \quad \hat{k} = 1, 2, 3. \tag{174}$$

Here the galaxy radius parameter is $a[\eta]$. Galactic evolution is modelled by $\dot{a}[\eta] = -c_r \cdot \sin[\eta] < 0$. The QDM size-collapse rate parameter, c_r, is empirical. The spatial part of this curvature is homogeneous and isotropic for each instant of the evolution parameter. The spatial-temporal part is similarly homogeneous and isotropic, but the prefactors differ between the evolutionary and spatial parts. For comparison, a totally homogeneous spacetime curvature is a single matrix of 2-forms $k^{ij} = K\xi^i \wedge \xi^j$ where K is a constant over the whole manifold [10, p. 148]. The expressions in Eq.(174) are only slightly more complicated. Theorem 4, therefore applies directly and trivially. Using Eq.(21) and the cited values for the curvature form, $k[\Lambda]$, in Eq.(174), the Cartesian components of the extra G- and W-fields can be computed. In the S^3 MST continuum, an observer may not realize that the homogeneous and isotropic MST in which he lives is evolving, since everything appears flat. But as one can show, even at a local level, the space can support torsion so the dynamics, while on the average appearing to obey Newton's law that objects free of external forces travel in straight lines, actually permits internal spinning of the mass-spacetime. This apparently applies to both the microscopic as well as the macroscopic scales. This torsion can be arbitrarily replaced by an appropriate curved manifold but at the risk of confounding the physics as we now understand it. Interestingly, an observer tied to the S^3 might notice, as his galaxy shrinks, that distant stars in his galaxy are getting closer (a kind of reverse Hubble effect). The same observer, using a force paradigm to interpret the results of his observations would find gravitational fields caused by CDM. The resulting G and W- fields he would infer for the S^3 galaxy can be computed and yield:

Theorem 26 *The FLRW model G- and W-fields, providing the missing mass effect (cold dark matter (CDM)) modifying the classical G-fields of Newton in Cartesian coordinates are:*

$$\begin{array}{ll} G_1 = \frac{\ddot{a}xE}{r^2} - \frac{H}{r^2\rho}(yr - zx) & W^1 = \frac{xT}{r^2\rho} - \frac{S(y\rho+xz)}{r\rho^2} \\ G_2 = \frac{\ddot{a}yE}{r^2} + \frac{H}{r^2\rho}(xr + zy) & W^2 = \frac{yT}{r^2\rho} + \frac{S(x\rho-yz)}{r\rho^2} \\ G_3 = \frac{\ddot{a}zE}{r^2} - \frac{H\rho^2}{r^2\rho} & W^3 = \frac{zT}{r^2\rho} + \frac{\rho^2 S}{r\rho^2} \end{array} \tag{175}$$

Here $a[\eta]$ is the instantaneous "radius" of the galaxy at evolution epoch η.

And further

$$E[\eta,\chi] \equiv \frac{r/a}{\cos\chi}, H[\eta,\chi] \equiv \Sigma[\chi](\ddot{a} - \left(1+\dot{a}^2\right) \frac{\dot{a}}{a} \tan\chi), \tag{176}$$

$$S[\eta,\chi] \equiv \frac{\left(1+\dot{a}^2\right)\Sigma[\chi]}{a\cos\chi}, T[\eta,\chi] \equiv \left(1+\dot{a}^2\right)\Sigma^2[\chi]\frac{\rho}{r}$$

$$E[\eta,\chi] \equiv \frac{\pm(r/a)}{\sqrt{1-(r/a)^2}}, H[\eta,\chi] \equiv \frac{r}{a}(\ddot{a} \mp \left(1+\dot{a}^2\right)\frac{\dot{a}}{a}\frac{(r/a)}{\sqrt{1-(r/a)^2}}), \tag{177}$$

$$S[\eta,\chi] \equiv \frac{\pm\left(1+\dot{a}^2\right)(r/a)}{a\sqrt{1-(r/a)^2}}, T[\eta,\chi] \equiv \left(1+\dot{a}^2\right)\left(\frac{r}{a}\right)^2\frac{\rho}{r}$$

The last pair of equations was obtained from the first pair by using $r = a\sin\chi$. The *upper* sign applies to $0 < \chi \leq \pi/2$ and the *lower* sign to $\pi/2 < \chi \leq \pi$. $\rho^2 = x^2 + y^2$ and $\Sigma[\eta] = \sin[\chi] = r/a[\eta]$, for S^3, etc. in the FLRW model. Because of Eq.(158) and the surrounding discussion we can relate these two parts to time reversed parts. Equivalently, we can say that the parts for $\pi/2 < \chi \leq \pi$ are parity and charge reversed compared to the $0 < \chi \leq \pi/2$ parts.

Proof: This amounts to embedding the evolution of the 3D object S^3 into a Minkowski spacetime, finding the components G_i and W^k from the (pre-averaged) trace curvature 2-form, k, and obtaining the resultant cold dark

[54] Recall that for $k = t_r(K_\nu^\mu)$ there exists a potential $da = k$ which is globally defined and for orthogonal transformations $O(n-1,1)$ in the Minkowski tangent space and there are several other quantities defined by the Pfaff sequence $\{a, da, a \wedge da, da \wedge da\}$. The interpretation of these quantities gemnerates two more kinds of gravitational structures and the possibility of irreversible (topology changing) evolution.

matter distribution $\rho_g(x,y,z,\eta) = (4\pi)^{-1}\nabla\cdot G$. This allows us to display Cartesian planar slices of the G- and W-fields. In this we note for the evolving S^3 that we start with a 4-coordinate representation of the curvature 2-form $k = t_r(K^i_j(\Lambda))$, we then allow the radius $a[\eta]$ to decrease. The development is lengthy and only a summary is given here [33].

We first verify that the standard results for S^3 expressed in an orthonormal co-ordinate system are correct. This requires setting up the co-ordinate system,

$$\xi^{\hat\eta} = d\eta, \quad \xi^{\hat\chi} = a[\eta]d\chi, \quad \xi^{\hat\theta} = a[\eta]\sin\chi d\theta, \quad \xi^{\hat\phi} = a[\eta]\sin\chi\sin\theta d\phi. \tag{178}$$

in which the radius $a[\eta]$ varies with the epoch parameter, η. We then compute the moving frame

$$\begin{aligned} e_0 &= e_\eta = (1,\ 0,\ 0,\ 0)\\ e_1 &= e_r = (0,\ \sin[\theta]\cos[\phi],\ \sin[\theta]\sin[\phi],\quad +\cos[\theta])\\ e_2 &= e_\theta = (0,\ \cos[\theta]\cos[\phi],\ \cos[\theta]\sin[\phi],\ -\sin[\theta])\\ e_3 &= e_\phi = (0, -\sin[\phi],\ \cos[\phi],\ 0) \end{aligned} \tag{179}$$

$$\xi^0 = d\eta, \quad \xi^1 = dr, \quad \xi^2 = r d\theta, \quad \xi^3 = r\sin\theta d\phi \tag{180}$$

This allows computing the connection

$$\Omega = (\omega^j_i[dx^l]) = \begin{pmatrix} 0 & -\frac{\cot\chi}{a}\xi^{\hat\theta} & -\frac{\cot\chi}{a}\xi^{\hat\phi} & -\frac{\dot a}{a}\xi^{\hat\chi}\\ \frac{\cot\chi}{a}\xi^{\hat\theta} & 0 & -\frac{\cot\theta}{a\sin\chi}\xi^{\hat\phi} & -\frac{\dot a}{a}\xi^{\hat\theta}\\ \frac{\cot\chi}{a}\xi^{\hat\phi} & \frac{\cot\theta}{a\sin\chi}\xi^{\hat\phi} & 0 & -\frac{\dot a}{a}\xi^{\hat\phi}\\ \frac{\dot a}{a}\xi^{\hat\chi} & \frac{\dot a}{a}\xi^{\hat\theta} & \frac{\dot a}{a}\xi^{\hat\phi} & 0 \end{pmatrix}, \quad \kappa > 0 \tag{181}$$

From this we can compute the average curvature $k = d\Omega - \Omega\wedge\Omega$ or the induced Riemann space and interpret the following as the pre-averaged 2-form trace displayed as a matrix of 2-forms, i.e. as a Maxwell structure:

$$k = t_r K[\Lambda]^l_m = \left(k[\Lambda]_{ik} d\xi^i \wedge d\xi^k\right) = \begin{pmatrix} 0 & k_{01} & k_{02} & k_{03}\\ -k_{01} & 0 & k_{12} & -k_{31}\\ -k_{02} & -k_{12} & 0 & k_{23}\\ -k_{03} & k_{31} & -k_{23} & 0 \end{pmatrix} \tag{182}$$

$$\approxeq \begin{pmatrix} 0 & \begin{matrix}\ddot a\times\\ d\eta\wedge d\chi\end{matrix} & \begin{matrix}\ddot a\Sigma\times\\ d\eta\wedge d\theta\end{matrix} & \begin{matrix}\ddot a\Sigma\sin\theta\times\\ d\eta\wedge d\phi\end{matrix}\\ \begin{matrix}\ddot a\times\\ -d\eta\wedge d\chi\end{matrix} & 0 & \begin{matrix}\left(1+\dot a^2\right)\Sigma\times\\ d\chi\wedge d\theta\end{matrix} & \begin{matrix}\left(1+\dot a^2\right)\Sigma\sin\theta\times\\ d\chi\wedge d\phi\end{matrix}\\ \begin{matrix}-\ddot a\Sigma\times\\ d\eta\wedge d\theta\end{matrix} & \begin{matrix}-\left(1+\dot a^2\right)\Sigma\times\\ d\chi\wedge d\theta\end{matrix} & 0 & \begin{matrix}\left(1+\dot a^2\right)\Sigma^2\times\\ d\theta\wedge d\phi\end{matrix}\\ \begin{matrix}-\ddot a\Sigma\sin\theta\times\\ d\eta\wedge d\phi\end{matrix} & \begin{matrix}-\left(1+\dot a^2\right)\Sigma\sin\theta\times\\ d\chi\wedge d\phi\end{matrix} & \begin{matrix}\left(1+\dot a^2\right)\Sigma^2\times\\ -d\theta\wedge d\phi\end{matrix} & 0 \end{pmatrix} \tag{183}$$

These expressions for the curvature agree with those in the literature, e.g., Eq.(174) [8, p.357], after factors of $\frac{1}{a^2}$ associated with replacement of differentials in $\xi^{\hat k}\wedge\xi^{\hat\mu}$ and $\frac{1}{a}$ for $\xi^{\hat\eta}\wedge\xi^{\hat k}$ are made. After using the transformation of the four Cartesian co-ordinates (x,y,z,w) on S^3 to the three (χ,θ,ϕ) ones:

$$\begin{aligned} w &= a[\eta]\cos\chi\\ x &= a[\eta]\sin[\chi]\sin\theta\cos\phi = r\sin\theta\cos\phi\\ y &= a[\eta]\sin[\chi]\sin\theta\sin\phi = r\sin\theta\sin\phi\\ z &= a[\eta]\sin[\chi]\cos\theta = r\cos\theta \end{aligned} \tag{184}$$

we obtain the G- and W- fields given in Eq.(175).

Results for the CDM distribution for a specific evolution model are given in Fig 1. The extra G-field, arising solely from the evolution of the 4D MST manifold, giving rise to an anomalous convection of stars, is plotted in Fig.2. The evolution of the galaxy through a big crunch is shown in Fig.3. We now discuss these results in detail after considering a few caveats.

The CDM can either be considered to compress and accumulate as is thought to occur with a black hole or to be continually lost through leaks. However, it is noted that Eqs.(175) have been numerically checked to satisfy the required continuity Eq.(15) for the total ρ- and J-fields, so, if some of the CDM is to disappear, this effect must occur as another property of the model. To this end, notice the whole model volume is swept out for $0 \leq \chi = \frac{2\pi r}{2a} < \pi$ as the other two angular variables and r cover their range. This means that the factors $\cos\chi$ and $\tan\chi$ change sign at $\chi = \pi/2$. The consequence is that during the evolution of the galaxy the cold dark matter (CDM) for $\pi/2 \leq \chi < \pi$ is either "ejected" from the galaxy or remains making the double cover as a 2-ply structure. By identifying points differing by $\pi/2$ in this space we produce the orientable 3D projective space RP^3. the space S^3 is its double cover. Here is where the distinction between compact matter and CDM is important. The CDM is just an alias for the curvature of the MST. To "eject" CDM really means "have the manifold evolve" in a certain way. Since this evolution is already specified, we have no freedom to change it, so we must be content to consider the second sheet as part of the manifold structure that coevolves with the sheet we can observe. To observe this "other part or sheet" of the CDM requires one essentially travel the whole length of the galaxy again – while the galaxy is evolving – or tunnel through. Locally, the opposite sides of this structure can be considered to have opposite orientation (they are time reversed) as shown above. Puncturing the 2-ply structure produces topological defects enabling us to insert or glue other topological objects into the over-all manifold [32, p.93]. One can also introduce unoriented submanifolds allowing connection between like oriented subspaces. For instance, since the Klein bottle, K^2, is a boundary in 4D as is $S^3 \subset R^4$, it can be glued into the resulting hole. We can use the K^2 to glue like oriented manifolds to each other. Such structures occur as solutions to the quantum dynamical manifold equations developed previously. To model the observable part of the galaxy, therefore, we use the restriction $0 \leq \chi \leq \pi/2$. Such a choice corresponds, for the lepton case described above to a viewpoint where only a particular type of particle (E_+ or E_-) and parity of leptons is normally encountered. For the total dynamics both locally and globally, however, we have to consider both sheets of the mapped S^3 structure, i.e., the whole oriented manifold RP^3. This arises as well from the fact that both $E_\pm$ eigenvector states are needed to make a complete basis set.

Examination of the results in Eqs.(175) and Eqs.(176-177) shows that the units of epoch are length units so that $\left(1 + \dot{a}^2\right)$ is homogeneously dimensioned. As the radial co-ordinate, $r = a[\eta]\sin[\chi]$, varies from zero to the galactic radius, $a[\eta]$, (using the limit on the variable χ) and the definition of Σ from the first of Eq.(171) we have

$$\Sigma[\chi] \to \sin[\chi] = r/a, \qquad 0 \leq r \leq a[\eta], \quad \kappa > 0 \tag{185}$$

in order to maintain homogeneous co-ordinates. To connect the range of χ to the galactic size we further set the co-ordinate $\chi = \pi r/a = 2\pi r/2a$. That is, the ordinary radial co-ordinate is given in units of the size of the galaxy. These steps essentially put into the co-ordinates an implicit non-dimensionalization facilitating the adoption of the spherical (r, θ, ϕ) co-ordinates. From this argument, the $S^3(\eta)$ co-ordinates (η, r, x, y, z) are expressed in units of the current epoch's galactic radius.

In Eq.(175) we can use a single range of χ, e.g., $0 < \chi \leq \pi/2$, as indicated before, or as indicated in Eq.(177) we can use both signs. Using both signs implies that at points (x, y, z) the values of the expressions that depend on the signature at χ and at $\chi + \pi/2$ cancel on summation. This leads to the following expressions for the G- and W-fields (including both $0 \leq \chi < \pi/2$ *plus* $\pi/2 \leq \chi < \pi$ parts at same point):

$$\begin{array}{ll} G_1 = -\frac{\ddot{a}}{ar\rho}(yr - zx) & W^1 = \frac{x}{r}\left(\frac{1+\dot{a}^2}{a^2}\right) \\ G_2 = +\frac{\ddot{a}}{ar\rho}(xr + zy) & W^2 = \frac{y}{r}\left(\frac{1+\dot{a}^2}{a^2}\right) \\ G_3 = -\frac{\ddot{a}\rho^2}{ar\rho} & W^3 = \frac{z}{r}\left(\frac{1+\dot{a}^2}{a^2}\right) \end{array} \tag{186}$$

The total G-field is plotted in Figs.2-3 for selected slices through the galaxy. In Fig.4 starting from the bottom on the upper-left to the bottom right the x-y–slices go through successively more northward distance from the south pole. Figure 3 gives x-z–slices. That is ones for y-varying from the back to the front. Notice the clear recirculation pattern or fountain effect. If stars are created in the plane of the ecliptic, then they will be convected in time through the fountain through the center of the galaxy. Further, we find for the model a density and an energy functional:

$$\rho(x, y, z) = (4\pi)^{-1}\frac{z}{\rho(x, y)r(x, y, z)}, \tag{187}$$

$$E(x, y, z) = \frac{1}{8\pi}\left(\|G\|^2 + \|W\|^2\right) = \frac{1}{8\pi}\left(2 + \left(\frac{1+\dot{a}^2}{a^2}\right)^2\right). \tag{188}$$

Because these terms are non-zero, the surprising result is that the CDM density will be non zero, $\nabla \cdot G = 4\pi\rho$. This is shown in Fig.4. This total CDM does not depend on c_r in the present model. In the simple model used here,

$\ddot{a}/a = -1$, that is, the G-field does not depend on the collapse rate parameter, c_r, at all. The missing mass $\rho(x,y,z)$ is really just an effect of the convection of the spacetime and it can have positive and negative values as shown in the figure. Another surprise is that there is a constant outward directed radial force from the s*w*irl field. This force does not depend on the contraction or expansion, i.e., signature of the dynamics! If we consider the balance of forces (computed by a difference because of the outward normals are opposite) at a point (x,y,z) then the terms of opposite signature add and those of the same signature cancel. The difference is a measure of the longitudinal stress in the spacetime. We will halve the result so the forces in a χ half-cycle can be obtained as sums of the form $G_i \pm \Delta G_i$. We obtain for the total G- and W-fields (including both sheets, effectively, $0 \leq \chi < \pi/2$ *minus* $\pi/2 \leq \chi < \pi$ parts at same point):

$$
\begin{aligned}
\Delta G_1 &= \tfrac{\ddot{a}xE'}{r^2} - \tfrac{H'}{r^2\rho}(yr - zx) & \Delta W^1 &= -\tfrac{S'(y\rho + xz)}{r\rho^2} \\
\Delta G_2 &= \tfrac{\ddot{a}yE'}{r^2} + \tfrac{H'}{r^2\rho}(xr + zy) & \Delta W^2 &= +\tfrac{S'(x\rho - yz)}{r\rho^2} \\
\Delta G_3 &= \tfrac{\ddot{a}zE'}{r^2} - \tfrac{H'\rho^2}{r^2\rho} & \Delta W^3 &= +\tfrac{\rho^2 S'}{r\rho^2}
\end{aligned}
\tag{189}
$$

$$
\begin{aligned}
E'[\eta,\chi] &\equiv \tfrac{(r/a)}{\sqrt{1-(r/a)^2}}, \quad H'[\eta,\chi] \equiv -\tfrac{r}{a}\left(\left(1+\dot{a}^2\right)\tfrac{\dot{a}}{a}\tfrac{(r/a)}{\sqrt{1-(r/a)^2}}\right) \\
S'[\eta,\chi] &\equiv \tfrac{\left(1+\dot{a}^2\right)(r/a)}{a\sqrt{1-(r/a)^2}}
\end{aligned}
\tag{190}
$$

We give in Fig.5 the flow direction field of $G + \Delta G$ for x-y–planes (different z-values) through the galaxy and for x-z–planes in Fig.6. The distribution of partial densities is given in Fig.7. Notice that in going through the origin, the CDM density changes sign. This completes the proof of CDM distribution theorem, Thm.26, and extends it to the calculation of the mass fluxes.

2. *Observational Consequences for the Model*

Note that the $1/r$ gravitational potential effects (e.g., due to a black hole) are neither included in Eqs.(175) nor in the Figures. Since *the gravitodynamical equations are linear*, we can superimpose the effects of such Newtonian potentials due to compact masses. Compared to the standard theories of Newton or Einstein, the present theory predicts a qualitatively different vortex induced by the S^3 centrosymmetric relative collapse, but the motion is more complicated than a simple MST swirl, for instance, showing evidence of GW-field induced axial-polar jets. In the figures we have concentrated on the G-field because the W-field terms are a factor $1/c$ smaller. thus the swirl is a first order effect, not a “relativistic” correction. This effect would enhance the rotation of the accretion disk around a black hole for instance, and probably contribute to the development of a uniaxial jet. The physical picture is that the relative collapse of an S^3 MST leads to the anomalous motion or convection of the stars in a galaxy; i.e., there is more to galactic formation than compact mass and Newtonian gravitation. Because the CDM is free of conventional mass, it neither absorbs nor emits light – it is, after all, CDM. The galactic extent of the moving CDM naturally gives rise to differential or Doppler shifted galactic lensing not predicted by GRT that can be used to verify the adequacy of the simple model. Comparison of GRT to the current theory on a large-field but small-scale may be possible using neutron star observations [31]. It is also likely that the present theory has additional implications relative to supernova evolution with gravitational collapse. In the presence of a $1/r^2$ force field due to say a sufficiently strong black hole, the axial jet predicted the model can be substantially weakened. Thus, one expects to observe the jets in cases where the black hole is small. This in turn is probable for young galaxies. With two parameters, the strength of the black hole and the galactic shrinking rate, we can, of course, obtain better fits to observation.

It is significant that the G-vector is proportional to the condensation rate, $\ddot{a}$. This should vary from galaxy to galaxy, but we see in the simple S^3-model that the total G does not depend on c_r. Instead of ΔG, we plot $G_i + \Delta G_i$ in Figs.5 and 6 as this would be observed if only a single sheet of the S^3 structure were visible. At the risk of pushing a simple model too far, we call the reader's attention to some possibly interesting dynamical effects. As the central $1/r^2$ G-field grows, the outward axial jet becomes nullified over a surface whose radial distance grows with the accretion of mass in the central disk. This leads to progressively larger sheet of MST where the G-field has no radial component. This can lead to very strange motion compared to an exclusively Newtonian model. Another dynamical effect is that stars created in the galaxy core, before the growth of a central black hole, would be ejected along the jet and appear in the galactic halo as some of the oldest stars in the galaxy. There would be a concomitant dearth of such stars in at the other pole region. It is barely possible that current observations would be able to verify these effects

[2]. As far as the structure of MST is concerned, the most important point to resolve is the 2-ply structure. The difference between the 1-ply dynamic case (the other ply being wrapped around for $\pi/2 < \chi \leq \pi$ and unseen) and the static or 2-ply case (if both plys contribute to the observation) is characterized by evolution illustrated in Fig.7 as compared to the static result of Fig.4. This comparison would require an assembly of observations of different galaxies at presumably different stages of their evolution and a normalization for size effects; not easily accomplished. Another effect may be the emergence of MST into the current Universe from other sheets as illustrated in the second half of figures in Fig.7. One may speculate that these mini big bangs are quasars. This would lead to a nice symmetry and a total dynamics that, including dissipative effects into other sheets, would lead to an observed dynamical universe with many parts as in a turbulent fluid. The results above concerning the Ricci energy-momentum flux equation and those below concerning the creation of topological defects certainly suggests that defects would be created in this high curvature-torsion environment. The effect of the W-field strengths are reduced by the factor of $1/c$ as in electrodynamics, so they are not plotted here. However, they are needed as part of the mathematical structure leading to a 4-vector potential for the 4D MST. We note that the structure of the G-field in this simple model mimics some of the effects that might be associated with a large galactic magnetic field. Finally the present model suggests that there may be anomalous motion about black holes if the S^3 structure is also found around these objects. Other situations for more complicated topologies of MST could certainly occur and possibly be observable from the measurement of the motion of stars in galaxies. It is to this general situation we now turn.

We have considered only the simplest choices for the topologies of a MST, those associated with S^3 or H^3. These are associated with the group $SL(2,C)$ through the complexification of $SU(2)$. Thus the only kind of spacetime considered is $SL(2,C)$ and the only compact particles are S^3 ones. Since $sl(2,C) = su(2) + isu(2)$, one maps the six components of K^l_m into $sl(2,C)$ along the evolutionary path. One notices immediately when the Gaussian curvature is positive that one has the compact part and when it is non positive one has the non compact spacetime. This change of topology allows gluing compact galaxies together to make a universal manifold. As shown in previous parts one can glue such manifolds together using evolution equations such as the $su(2)|_2$ quantum dynamical manifold equation which resembling the Dirac equation. This dynamical gluing is locally equivalent to a complexification of a group. The gluing occurs over both plys of the manifold as observed in ordinary spacetime. A question remaining is then: What is the compatible velocity field for a given universe radius $a[\eta]$? This is the missing mass velocity. It is not necessarily a gravitational motion. The manifold compatibility condition allowing the determination of the compatible velocity field are the Bianchi identities. Even if the S^3 universe is spatially homogeneous and isotropic, the velocity field which allows for the expansion does not need to be homogeneous and isotropic. In Fig.2-3 the G-vector fields at several different $z-$ planes illustrating a cross-section of the toroidal flow and the jets that develop because of the map. The other terms in the fluxes give details that are consistent with a modified Hopf map of S^3 for $\Sigma = \sin[\chi]$. A schematic of the results of that map are given in Fig.8. One presumably has a Hopf-type map as well when $\Sigma = \sinh[\chi]$, but of H^3. One, then, expects to see jets and influxes into the core of spinning $S^3 \to S^2 \times S^1$ galaxies. A new feature of a complexified MST is that it can smoothly go from compact $S^3 \to S^2 \times S^1$ to the non compact H^3.

Although the FLRW model is the simplest topological structure possible for a three-dimensional manifold with $g = 0$ holes, $M^3_{g=0}$, embedded in a MST, it contains a remarkable resemblance to a real galaxy when one adds the effect of a central Newtonian (black hole) core and places stars at the flat interface between positive and negative CDM ($\nabla \cdot G = 4\pi\rho_g(x,y,z,\eta) = 0$). That is, in the regions where the CDM is absent. The later equation is analogous to the Einstein gravitational equations ($(G^{\mu\nu} + 8\pi\kappa\tau^{\mu\nu})_{;\nu} = 0$) stating the conservation of ordinary energy-momentum in a flat space. Along the $\rho_g = 0$ region, as shown in Thm.26, gravitodynamical (curvature) waves can sweep through at their greatest rate, unimpeded by MST density. It can be argued further that this is a natural place for accumulating "normal" matter stars. There forms the conditions for an accretion disk. One concludes that the stars form or perhaps persist at the interface of positive and negative CDM density where (curvature) gravitational waves can more freely propagate. This does not explain globular clusters, some of the oldest objects observed. These may be simply Newtonian structured manifolds, i.e., without an accompanying non Newtonian MST component (CDM). One suspects these may not have CDM sinks as described here for the elliptic galaxies. One naturally speculates that the Universe is populated by these and other M^3_g structures. These manifolds are classified by the representations $SU(2)|_n$. If the Universe is closed by CDM, one can speculate that it, too, is an evolving M^3_g-QDM that can be embedded into a mass-spacetime. There is already strong evidence for the evolution of topological defects in MST leading to M^3_g and their consequent embedding in a spacetime. We now outline this subject [33].

It is noticed that the central core G-field swirl broadens in going from the bottom to the top of the values plotted and that the motion in the (x,y)-plane is entirely circular. In a more realistic model, this would be elliptical. The interpretation is that this simple galaxy has a single jet of positive CDM density and one of negative CDM density. Within the CDM arm there is spiral motion about the z-axis. The evolution of the S^3 has apparently produced the equivalent of a Hopf map $S^3 \to S^2 \times S^1$ (Dirac monopole) with the addition of internal twist to the orbits of particles. This occurs because the solutions to the dynamical manifold equations are projective in nature. The whole assembly can move through the larger MST possibly convecting stars with it at constant velocity, like a top, perturbed by the swirl velocities and any angular rotation of the galaxy as a whole. Such assemblies are naturally connected to neighboring MSTs through the Bianchi identities. One can extend the current theory to include all parallelizable spheres S^1, S^3, and S^7 and their Heegaard splitting by the introduction of exterior derivatives gauge covariant with respect to $U(1)$, $SU(2)$ and $SU(3)$.

D. Stellar Structure

One of the significant applications of GRT is to the structure and evolution of stars. Here we briefly outline how the present theory can be used for this purpose.

Theorem 27 *Stellar Structure and Gravitational Collapse. The average curvature describes an induced Riemannian space so, in terms of a metric, $\tilde{g}^{\mu\nu}$, we can evaluate the d'Alembertian and covariant derivative in the equations*

$$\Box A = -4\pi J, \tag{191}$$

$$\nabla_i G^i_j = \nabla_i(\tilde{R}^i_j - \frac{1}{2}\tilde{g}^i_j \tilde{R}) = 0, \tag{192}$$

$$\tilde{\nabla}_j \tilde{g}^{jm} = 0, \tag{193}$$

$$\widetilde{\Gamma}^m_{ik} = \frac{1}{2}\tilde{g}^{jm}(\partial_k \tilde{g}_{ij} + \partial_i \tilde{g}_{kj} - \partial_j \tilde{g}_{ik}). \tag{194}$$

$$k_{ls} = \frac{\partial A_s}{\partial x^l} - \frac{\partial A_l}{\partial x^s} = \frac{\partial \Gamma^m_{ms}}{\partial x^l} - \frac{\partial \Gamma^m_{ml}}{\partial x^s} \tag{195}$$

Here $J \equiv (\rho u^i) = (\frac{c\rho}{\sqrt{1-\beta^2}}, \frac{\rho \mathbf{u}}{\sqrt{1-\beta^2}})$.

Proof. This amounts to finding enough equations from the set above to determine the ten metric coefficients. There are four equations in the first set (191), but because $dJ = 0$ only three of them are bound. To uniquely determine the A_i we introduce a gauge setting constraint analogous to that used in electrodynamics, the so-called Lorentz gauge: $\partial_i A^i = 0$. The 4-vector potential is used in Eq.(195) to determine six relations for the $\tilde{g}_{ij}$.

The second set of four equations (192) are four more constraints. In GRT we set

$$G^i_j = -8\pi G_N \tau^i_j \tag{196}$$

Taking into account the symmetries of the Riemann curvature tensor components, there are then ten equations in Eq.(196), constrained by the four of Eq.(192). The standard approach thus has six equations to determine ten g_{ij}. It is common to introduce the harmonic coordinate constraint $g^{ij}\tilde{\Gamma}^k_{ij} = 0$ to compensate for these problems. The addition of the four new gauge fixing constraints gives the ten equations needed. We will keep Eq.(192) as four equations and proceed along a different path.

Equations (193) follow from the vanishing of the torsion, not increasing the total of equations that can determine the metric. They do not need further accounting.

The Eqs.(194) allows us to relate the connection coefficients to the metric coefficients for a Riemann space. This allows us to convert Eq.(192) into PDEs for the metric. In Eq.(195) we note that if the Γ^m_{mk} were obtained from a single potential, i.e., $\Gamma^m_{mk} = \frac{\partial \phi}{\partial x^k}$, then we have $\frac{\partial \Gamma^m_{mk}}{\partial x^l} = \frac{\partial \Gamma^m_{ml}}{\partial x^k}$ and we would have $k_{ls} = 0$. This is the reason for keeping track of the non Riemann space's effects. We conclude that we must have $\Gamma^m_{mk} = \widetilde{\Gamma}^m_{mk}$ but they be derived from independent A_k. We net six equations from Eq.(195) which combined with the four from Eq.(192) yield the ten equations needed for determining the ten unknown metric coefficients.[55] The four Eqs.(191) are just sufficient along

[55] The Ricci tensor is not probably not symmetric, eventhough k_{ij} is antisymmetric.In terms of the constraints $\nabla_i(\tilde{R}^i_j - \frac{1}{2}\tilde{g}^i_j\tilde{R}) = 0$ this means that we must use symmetrized versions. This is related to the possibility that the MST is subtly not parity invariant. There is a difference in left, $\grave{\Gamma}^m_{ik}$ and right-handed, $\acute{\Gamma}^m_{ik}$, connections and a two-sided connection $\widetilde{\Gamma}^m_{ik}$. This difference is found by considering parallel translations going in either left or right-handed loops exclusively. In general, if we have both left and right connections, and $a + b = 1$, then $a\grave{\Gamma}^m_{ik} + b\acute{\Gamma}^m_{ik}$ is a connection also. It has not escaped our notice that the physical spacetime in which we live has a slight handedness bias.

with the gauge fixing constraint to determining the potentials needed in Eqs.(195). There are considerations related to the presence of torsion at a fundamental level discussed in Appendix B.

We now discuss methods to determine the 4-current, J. The equations of motion for the star's fluid, generating J can be, e.g., the Navier-Stokes equations plus an equation of state relating the pressure to the density

$$P = P(\rho). \tag{197}$$

For the case of a static distribution of fluid which is spherically symmetric, we can use the Schwarzschild exterior metric. This implies $\tilde{R}^i_j - \frac{1}{2}\tilde{g}^i_j\tilde{R} = 0$ everywhere, automatically satisfying Eq.(192). That is, for this static, spherically symmetric case, one uses a line element of the Schwarzschild form

$$ds^2 = -e^{\Phi}c^2dt^2 + [e^{\Lambda}dr^2 + r^2(d\theta^2 + \sin^2\theta d\varphi^2)] \tag{198}$$

with two unknown functions. We thus must have two equations to determine the two functions. Moreover, as known, the Schwarzschild singularity is just a local co-ordinate phenomenon so, by a transformation to Kruskal coordinates, we can use the Schwarzschild exterior solution throughout the manifold consistent with the Bianchi constraints for a spherically static symmetric spacetime. In this case time-independence implies that W vanishes, and we have only a scaler potential equation for Eq.(191) given by a single equation

$$\nabla^2 A_0 = -4\pi\rho \tag{199}$$

Now G provides the radial force acting on the fluid, so we use the equation of state to relate G to ρ. We relate G to the metrical coefficients via Eq.(195), choosing the orthonormal frame to be a spherical co-ordinate system, thereby obtaining a second equation.

In the description of the stellar problem we have not had to use the Einstein ansatz, Eq.(196). We avoid this by using the Navier-Stokes equations and other phenomenology to determine the currents first.

V. SUMMARY AND CONCLUSIONS

In this paper, we formulate and prove in modern terms a conjecture of W.K. Clifford about the nature of mass-spacetime. This provides a proof of the existence of a universal gravitodynamics extending the Newtonian gravitation theory without breaking the Bianchi identities (as done in GRT). We were able to prove global results only for the average curvature potential, a_g. However, we show that this (Maxwell) structure induces a Riemann curvature which can be interpreted as a resulting from a Riemann space and its metric found. The theory applies in a restricted sense to any finite dimensional manifold, but gains greater physical applicability as a global theory of gravitation when restricted to a 4D mass-spacetime (MST). In the most restrictive sense, for the average curvature to be nonvanishing there must be non vanishing torsion. In the case a quantum dynamical density comes from a non Abelian structure, the Abelianized component is used to compute the scalar and vector potential. This suggest the origin of the weakness of gravitation as compared to analogous electromagnetic interaction lies in the smaller extent of torsion. Global torsion leads to a handedness of the MST. The global results are homeomorphic to the Maxwell equations (they can be gotten one from another by a conformal map) allowing elements of the average curvature 2-form to be directly associated with a gravitational field G and a whirl field W. Some local results parallel the small curvature results of GRT. We show how to compute the mass-density and mass-current matrix of a quantum dynamical system. This allows us to demonstrate that the trace (gravitational) curvature of such systems is non null and of the correct form for gravitation effects. The import of this is we can now consider (pre-averaged) smoothed manifolds to be subject to some of the same differential geometrical and differential topological analysis as ones derived from the color dynamics of a quantum dynamical manifold equation. In fact, we are able to generalize the Clifford conjecture to include Yang-Mills equations when the underlying system is non-Abelian. The surprise is that these emerge from the study of the structure of the curvature 2-form. This provides a direct link between geometry and dynamics which has been exploited to determine the quantum dynamical manifold equations studied more extensively in other parts. This promotes the manifold gravitation theory from just being analogous to electrostatics to being homeomorphic to the full electromagnetic theory and unifies it with an underlying quantum dynamics. The surprising conclusion is just how universal this gravitation potential is; because the global theory averages over the curvature field produced, e.g., by fluctuations in a quantum dynamical system, it depends hardly at all on the exact quantum nature of the constituents. As shown, this result is solely a consequence of the geometric and topological properties of evolving MST manifolds derived from the average curvature 2-form, k, giving the gravitational G- and whirl W-fields. The average curvature, k, defining the gravitational fields, is a topological invariant satisfying a global continuity equation and has a globally defined 4-potential. Since the average curvature is a trace invariant, substantially different materials can have the same gravitational fields, in accord with experiment. It is therefore this average field which is closest to the classical definition of a gravitational field. Because the average curvature field, k, and its potential a_g satisfy the equations $k = da_g$ and $d^\star k = -4\pi j_g$ in 4D spacetime, the topological results of Gauss, Ampere and Biot-Savart apply to gravitational fields. The first is in accord with experiment, the latter extends gravitation theory into the realm where mass-spacetimes have currents, j_g. Since vortical structures can be spontaneously created, if the manifold has sufficient curvature ($k \approx 1/radius\text{-}of\text{-}curvature\text{-}squared$), the present theory suggests possible manifold mechanisms for big bang phenomena, pair creation, and other creation/annihilation effects. The consequences are significant with respect to our view of the Universe at large. For example, one can show how to compute topological characteristic classes, linking numbers, topological charges, etc., to characterize the MST manifolds.[56] We show that these kinds of structures are relevant for the structure of the electron and we computed its gravitational potential. Further, when certain non-integrability conditions for the average gravitational 4-potential are satisfied (e.g., $a_g \wedge da_g \neq 0$ or $da_g \wedge da_g \neq 0$), irreversible processes creating vortical MST structures or topological defects occur. These processes appear to be possible for the non-average parts of the Riemann curvature 2-form fields as well. These other components are present when the curvature is derived from the quantum dynamical manifold theory described elsewhere.

In this paper we show how the new gravitation theory satisfies the Birkhoff theorem by asymtotically achieving Newtonian behavior outside a massive body in the absence of torsion and cold dark matter (CDM). The new gravitation theory differs from general relativity theory since GRT uses an empirically derived energy-momentum tensor to determine the curvature of the manifold directly through the metric. The new approach, on the other hand, computes the gravitational forces (i.e., the average curvature k) directly from the geometry of the quantum dynamical MST manifold. This is a more robust approach since a global metric-potential does not have to exist as in GRT. Because there is a global average curvature-potential, the current theory has solutions where GRT may not. In the new theory the gravitating and the mass-spacetime manifolds are the same object probably leading to an equivalence between gravitational and inertial mass in a large-scale, asymptotic limit.

As a consequence of the present work, even though the Maxwell equations and the equations of the new gravitation theory are homeomorphic, the vast difference seen in the strength of the two forces is explained by the fact that gravitational interactions result from the residual or average flavor curvature remaining after the stronger, shorter-range interactions have been accounted. The fact that only the trace gravitodynamical field necessarily has a global potential further separates universal gravitation from unification with electromagnetic theory. The two forces are further distanced conceptually because the equations of electrodynamics represent an evolution of a (photon) manifold with internal color symmetry and the gravitodynamics describes the average behavior of the resulting manifold, with (possibly broken) flavor symmetry, no matter what its internal color symmetry. To reiterate, the Maxwell equations of electromagnetism are the quantum dynamical manifold equations associated with 4D flavor manifolds (with possibly broken flavor symmetry) having an internal color symmetry arising from $so(3)|_3$ and the theory of gravitation involves an average over a curvature field derived from a quasi-particle frame field. This is conventionally explained by using different coupling constants, but here it is seen the difference is more fundamental and this precludes a direct combination of electromagnetism and gravitation. Furthermore, gravitation is more of a topological property of 4-manifolds than a description of specific geometrical properties. We also show that there is the possibility of two new kinds of gravitodynamical mass-currents in addition to the conventionally recognized one and determine conditions linking MST manifold evolution having changes in topology to irreversibility in its evolution.

In conclusion, this paper provides a dynamical theory of gravitation which is asymtotically Newtonian in the absence of cold dark matter and torsion. Application of the new gravitation theory to a galaxy model gives an explanation of cold dark matter and of the anomalous motion of stars convected by its presence. The jets of matter observed in some galaxies may also be explained by the model. The effects of CDM should be observable as differential Doppler shifts in gravitational lensing of radiation around rotationally opposite limbs of galaxies. New gravitational whirl fields (W) are predicted. Because the average gravitational and electromagnetic theory are nearly identical and linear, the G-W fields can be quantized along conventional lines without the infinities appearing in approaches based on GRT. Furthermore, the new manifold theory of gravitation predicts the existence of the "missing mass" as a MST structure. That is, the "missing mass effect" is due to the evolution of the MST itself. This effect is interpretable in terms of mass-density as shown but this requires admitting negative mass-density. Since ordinary (positive density) mass is associated with compact structures, perhaps, we should search for the "*missing MST curvature effect*" instead. This effect can be detected by the anomalous convection of stars compared to the classical theories. In addition, effects due to the MST flux may be detectable near black holes with companions providing very large mass-spacetime fluxes to the black

[56] See Part 3b. These provide a startling rich picture of the possibilities of the average MST structures of the Universe. This is especially true of the possibility of linking and relinking of MST manifolds in a larger picture of the Universe.

hole. In the S^3 galaxy model explored, the overall continuity and energy conservation were maintained by developing a 2-ply structure in (x, y, z)-space between which sheets of the manifold could exchange energy-momentum. This leads one to speculate that such processes may exist in nature for the expected more complicated multiply connected (g-holed), evolving M_g^3 manifolds and be associated not only with the evolution of galaxies/quasars but the Universe as a whole. The resulting picture is that galaxies are actually evolving MST structures which have been made visible by the presence of stars. The motion of these stars depends on Newtonian gravity in part, but there is a substantial effect due to the evolution (read space-time curvature) of the MST in which they are embedded. The picture of the Universe which emerges is that it is comprised of multiple interconnected submanifolds of net positive curvature having torsion created at times of high curvature, and immersed in a background of MST which on the average has low curvature. Paraphrasing Clifford:

> *The Universe consists of small portions of mass-spacetime which are of a nature analogous to vortices embedded in an evolving multiply connected manifold which is on the average flat. In fact, the ordinary laws of (even Riemannian) geometry are not valid on them. That the property of being curved or twisted is constantly being passed from one portion of mass-spacetime to another after the manner of a wave constrained by its own structure. That this variation of curvature and torsion is what really matters in the phenomenon which we call the motion of matter, whether compact or non-compact. That in the physical world nothing else takes place but this variation. A law of continuity applies to this geometry locally and there exists a global average gravitation 4-potential.*

VI. APPENDIX A. BIANCHI IDENTITIES

We begin our exposition of manifold geometry by writing down the Frenet-Serret (1851-52) equations describing the evolution of a 3D moving frame of vectors, or triad (t, n, b) consisting of the tangent, normal and binormal attached to a point moving along a curve, $c(s)$, parametrized by arc length s, embedded in a surface [38]:

$$\frac{d}{ds}\begin{bmatrix} t \\ n \\ b \end{bmatrix} = \begin{bmatrix} 0 & \omega_1 & 0 \\ -\omega_1 & 0 & \omega_2 \\ 0 & -\omega_2 & 0 \end{bmatrix}\begin{bmatrix} t \\ n \\ b \end{bmatrix} \equiv \Omega \begin{bmatrix} t \\ n \\ b \end{bmatrix}. \tag{200}$$

This is illustrated in Fig.5. The ω_i are connection coefficients. These equations describe the geometry of a curve in 3D or a particle moving along such a curve at unit speed. Or, generalizing to nD (the essential concepts are the *differential connection* or simply the *connection* Ω and the *moving frame*, $\Theta = \{\theta_i\}$)[57]

$$\frac{d}{ds}\{\theta_i\} - \Omega\{\theta_i\} = 0 \text{ or } \quad d\Theta - \Omega\Theta = 0, \quad \Theta \in O(n), \Omega \in o(n). \tag{201}$$

Under a change of basis $\theta_i \to A\theta_i$ engendered by the transformation A, the connection, Ω, transforms as a *gauge quantity*: $\Omega \to A\Omega A^{-1} + dAA^{-1}$.[58] Élie Cartan generalized Eq.(200) and Eq.(201) for the motion of a moving frame moving over a nD manifold and calculated the torsion τ and curvature R differential 2-forms giving the

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[57]The *moving frame* is a local property of a differential manifold which can be pictured as a co-ordinate frame $\{\theta_i\}$ that is differentially connected point-to-point. The infinitesimal variation from point P, is written as (employing the Einstein summation convention) $\delta P = \sigma^i\theta_i$ and the simultaneous variation of the frame is given by (the Frenet-Serret equations) $\delta\theta_i = \omega_i^j\theta_j$. The matrix (ω_i^j) comprises the connection matrix, Ω. The infinitesimals $\{\sigma^i\}$ comprise what is called the dual frame and are often denoted dx^i or $d\omega^i$. É. Cartan has shown that from the moving frame and its dual one can obtain a complete geometric description of the manifold. This means that if we have the moving frame for a quantum dynamical manifold, we can compute all its geometrodynamical properties (See Part 1c). The moving frame approach also allows one to geometrize dynamics providing an alternative to the Newtonian one. for instance, the quantity ω_1 in the Frenet-Serret equations above gives the acceleration of a particle along a path parametrized by the differential path length ds.

The *frame field* for an evolving quantum system gives all possible frames for evolution of the manifold. The actual evolution path, depends on initial conditions and takes a specific path "transporting" the moving frame along a real path. In some systems, typified by Schrödinger wave mechanical approximations, the geometry and, therefore, the frame field is preserved by time translations (Energy is conserved). In others, there is spatial translation invariance and momentum is conserved. These are related to special geometries appropriate to simple evolution of mass-spacetimes.

[58]If we solve the middle equation for Ω, giving $\Omega = d\Theta\Theta^{-1}$, then the curvature $K = d\Omega - \Omega \wedge \Omega$ vanishes as $0 = d(d\Theta) = d\Omega\Theta - \Omega d\Theta = (K + \Omega)\Theta - \Omega(\Omega\Theta) = K\Theta$. since the frame does not vanish, $K = 0$. The matrix Θ belongs to the orthogonal

Theorem 28 *(Cartan [39]) The Moving Frame of Cartan. Equations for the moving frame $\Phi = (e_1..., e_n)$ its dual $\Theta = (\theta^1, ..., \theta^n)$, the connection 1-form $\Omega = (\omega^i_j)$, curvature 2-form R, and torsion 2-form τ, are given by*[59]

$$\begin{aligned} de_i - \omega_i^j e_j &= 0, \\ d\theta^i - \theta^j \wedge \omega^i_j &= \tau^i, \qquad (\tau^i) \equiv (T^i_{[jk]}\theta^j \wedge \theta^k), \\ d\omega^i_j - \omega^m_j \wedge \omega^i_m &= R^i_j, \qquad (R^i_j) \equiv (R^i_{j[kl]}\theta^k \wedge \theta^l) \end{aligned} \tag{202}$$

Physically spaces with torsion have motions that are locally either straight line or helical. Motion in spaces with curvature is accelerated motion. We show how to compute these quantities for a general MST QDM in Part 1b when the machinery needed for creating QDMs is in place. By exterior differentiation, one finds the following compatibility relations for the moving frame to be consistent with the geometry of the manifold

Corollary 29 *(Cartan [39], [19, p.307]) Cartan Equations of Structure. The conditions that the moving frame be consistent with the geometry of its underlying manifold relate the rate of change of the curvature, dR^k_i, to the the torsion, τ^k, the connection one-form $\Omega = (\omega^j_i)$ and the dual frame θ^i in the following way*

$$dR^k_i = [R^k_j, \omega^j_i] \quad \text{and} \quad d\tau^k = R^k_i \wedge \theta^i - \omega^k_i \wedge \tau^i. \tag{203}$$

Corollary 30 *Bianchi Identities [19, p.307-309]. The Cartan Equations of Structure are equivalent, in local tensor components, to the first and second Bianchi identities respectively:*[60]

$$\sum_{(jkl)} \nabla_j R^m_{ikl} = \sum_{(jkl)} T^h_{kj} R^m_{ihl}, \tag{204}$$

$$\sum_{(jli)} \nabla_j T^k_{li} = \sum_{(jli)} (R^k_{ijl} + T^m_{ji} T^k_{ml}). \tag{205}$$

Here the sum over (jli), etc. means over cyclic permutations of these indices. Equations (202) to (203) completely describe the geometry of such differential manifolds as DMEs and QDMEs.[61] The natural motion in such spaces is

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group $O(3)$ in this case. Generalizations to other orthogonal and unitary groups can be made. It is precisely the case where symmetry is slightly broken, that small curvature arises.

[59]Several notations are found in the literature for the Riemann curvature matrix for a natural frame $\theta^i = dx^i$, $R^j_i = R^i_{j[kl]}dx^k \wedge dx^l = \frac{1}{2}R^i_{jkl}dx^k \wedge dx^l = d\Omega^j_i + \Omega^j_m \wedge \Omega^m_i = d\Omega^j_i - \Omega^l_i \wedge \Omega^j_l$. The latter is used here. We often write $R = d\Omega - \Omega \wedge \Omega$ in matrix-of-forms notation. Expressed in an orthonormal basis, both Θ and Φ can be expressed as matrices. The differential form wedge product symbol "$\wedge$" is often omitted when no confusion is expected. The wedge product between a pair of 1-forms is antisymmetric $dx^k \wedge dx^l = -dx^l \wedge dx^k$. By extension, if α and β are respectively p-forms and q-forms, then $\alpha \wedge \beta = (-)^{pq}\beta \wedge \alpha$. The exterior differentiation operator d in Eq.(202) is idempotent $d^2 = 0$ a result arising from the equality of mixed second partial derivatives in flat spaces. The linear differential operator d generalizes and unifies the vector analysis operators of gradient, curl and divergence. For instance, if f is a 0-form, then df is the ordinary differential. When ω is a 1-form, $d\omega$ is a 2-form whose coefficients are the curl of the components of ω Other details about the methods of differential geometry can be found in ref. [10] or [18]. The last two equations can be rewritten neatly in a form displaying the tensor components of the curvature and torsion (calling the dual to the moving frame vectors ω^i instead of θ^i):

$$\begin{aligned} d\omega^i &= \omega^k \wedge \omega^i_k + T^i_{[jk]}\omega^j \wedge \omega^k \\ d\omega^j_i &= \omega^k_i \wedge \omega^j_k + R^j_{i[kl]}\omega^k \wedge \omega^l \end{aligned}$$

.These equations neatly seperate the evolution of the dual frame $d\omega^i$ depending on the torsion from the evolution of the connection $d\omega^j_i$ which depends on the curvature.

[60]The mathematical notation we adopt is to indicate components of tensors using superscripts for contravariant components and subscripts for covariant ones. We will usually denote vectors and matrices without bolding, having defined them as such previously. This is to avoid the problem of distinguishing by a separate notation vectors of operators and components, etc., in different dimensionality spaces. For instance, the point P has co-ordinates x_P, there are co-ordinate differentials dx^k, there are vectors v, and matrices M.

[61]The method of moving frames allows a co-ordinate free derivation of the geometry of manifolds, for instance, for cases without global metrics. The methods of QDMT allow a similar generalization for manifolds having internal symmetry. This approach sidesteps some of the complications involved in dealing with the infinite dimensional group of all smooth transformations of a manifold, $Diff(M^n)$, encountered in the problems of physics.

locally linear *or* helicoidal.[62] Equations (204-205) involve the curvature tensor, R^m_{ihl}, used in general relativity, the covariant derivative $\nabla_k \phi^j = \partial_{x^k}\phi^j - \Gamma^j_{kl}\phi^l$ [63] given in terms of the *connection coefficients* , Γ^j_{kl}, and the *torsion* tensor given by the anti-symmetric part of the connection coefficients: $T^k_{ml} = \Gamma^k_{ml} - \Gamma^k_{lm}$.[64] This quantity is assumed to vanish in GRT. Such restricted spaces are called Riemann spaces. For these spaces, a metric tensor exists globally and its covariant derivative vanishes. In the absence of torsion one has the usual (first) Bianchi identity used in the spacetime of GRT. For the purposes of extending GRT and QM, spaces without torsion have insufficient generality. Even though it is common to refer to the Bianchi relations as identities because of their original algebraic derivation, we find it better to view them as constraints that a consistent manifold must satisfy. As a consequence we see in Eq.(205) that curvature causes a torsion flux. This outlook is important because we generally start with a manifold, then from the differential connection, we compute the curvature and torsion.

Example systems constrained by the general Eq.(204) are quantum systems with spin, e.g., electromagnetic waves or electrons. One can deal with motion of electromagnetic waves by restriction to Riemann spaces without spin by considering only the spatial path (e.g., along the Pointing vector $\approx E \times H$) instead of a helicoidal spacetime geometry. Equation (204) can be used to show how parity-violating torsion terms are missed in GRT. Equation (205) describes nonlinear torsion waves if $R^k_{ijl} = 0$. A metric need not be defined globally in the unrestricted case of Eqs.(204-205). This gives immense generality to results determined using them. The simplicity of the result given by the Cartan equations of structure and the differential form derivation of the Bianchi constraints, Eq.(203), is totally obscured in the tensor analytic results, Eqs.(204,205), and their derivation. Much of classical physics, however, is formulated as such strictly local tensor analytic results.[65]

These results can be applied to the question of the effect of connecting, say, a spin onto a spacetime forming a mass-spacetime manifold. The inclusion of intrinsic spin has been interpreted as adding torsion to the manifold [28]. This requires a unified space for the spin, F, and a space for the spacetime, M. The combination $E \approxeq M \times F$ has the structure of a fibre bundle. The geometric glue binding these two spaces yields an interaction, of course.

VII. APPENDIX B. RIEMANN SPACES WITH UNDERLYING TORSION.

In this appendix we derive some properties of the average curvature tensor used to generate the gravitational fields. We show how an underlying torsion can be accomodated and how this is related to parity noninvariance. This indicates that parity non-conservation in beta decay or other weak interactions is connected to the new gravitation theory. We begin by averaging the Riemann curvature

$$K^i_{jkl} = \frac{\partial \Gamma^i_{lj}}{\partial x^k} - \frac{\partial \Gamma^i_{kj}}{\partial x^l} + \left(\Gamma^m_{lj}\Gamma^i_{km} - \Gamma^m_{kj}\Gamma^i_{lm}\right) \tag{206}$$

by setting $i = j = n$ yields the 2-form matrix

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[62]In the absence of curvature in a spacetime one has a nonlinear dynamical problem

$$\sum_{(jli)} \nabla_j T^k_{li} = \sum_{(jli)} T^m_{ji} T^k_{ml}.$$

Thus, torsional structures can be self-propagating, once seeded (as in the hydrodynamics of perfect fluids).

[63]As before we employ the summation convention.

[64]The definition of the connection is given in terms of the moving frame vectors e_κ and the (ordinary) covariant derivative: $\omega_{\mu\nu} = \nabla_{e_\mu} e_\nu = \Gamma^\lambda_{\nu\mu} e_\lambda$ Thus the rate of change of a basis vector (function) in the direction of a basis vector is a linear combination of basis vectors. This is a (geometrodynamical) *response function*.

[65]In General Relativity, we set $dR = [R, \Omega] = 0$, in Eq.(203) and ignore the second equation $d\tau = \Omega R - \tau\Omega$. Ironically, torsion contains much of the quantum mechanics and the dual (in the sense of a Fourier transform pair) to the equation $d\omega^i = \omega^k \wedge \omega^i_k + T^i_{[jk]} d\omega^j \wedge d\omega^k$ leads to such equations as the Dirac and the Maxwell curl equations for manifolds evolving with the appropriate underlying symmetry. Thus in GRT we do not have the conditions required for the moving frame to be consistent with the geometry of the underlying manifold for general manifolds. It is, thus, not surprising that the original General Relativity and quantum mechanics are incompatible.

$$K^n_{nkl}dx^k \wedge dx^l = k_{kl}dx^k \wedge dx^l = \left(\frac{\partial \Gamma^n_{ln}}{\partial x^k} - \frac{\partial \Gamma^n_{kn}}{\partial x^l} + (\Gamma^m_{ln}\Gamma^n_{km} - \Gamma^m_{kn}\Gamma^n_{lm})\right) dx^k \wedge dx^l \tag{207}$$
$$= \left(\frac{\partial \Gamma^n_{ln}}{\partial x^k} - \frac{\partial \Gamma^n_{kn}}{\partial x^l}\right) dx^k \wedge dx^l$$

The last line is obtained by interchanging the dummy summation indices m and n in the last term thereby cancelling the last term in parentheses of the preceeding equation. We can show that the trace is generally non zero by using the second Bianchi identity

If one performs a contraction ($k = l$) of the second Bianchi identity

$$\sum_{(jli)} (\nabla_j T^k_{li} - T^m_{ji} T^k_{ml}) = \sum_{(jli)} R^k_{ijl} \tag{208}$$

and uses the anti-symmetry of the Riemann curvature tensor in the last pair of indices, one finds again the k_{il} tensor. That is

$$k_{ij} = R_{ji} - R_{ij} + \sum_{(jli)} (\nabla_j T^k_{li} - T^m_{ji} T^k_{ml})|_{k=l}. \tag{209}$$

That is, it is the skew symmetric part of the Ricci tensor when the torsion vanishes. A particularly neat version of this identity is the following [12, p.93]

$$\sum_{(rst)} \nabla_t k_{rs} = \sum_{(rst)} T^m_{rs} k_{mt} \tag{210}$$

clearly showing that the source of the 2-form k is the torsion.

In general, k_{ls} does not vanish, even in the case of a symmetric connection. In the case where $A_k = \Gamma^m_{mk}$ we have from the result for k_{ls}

$$k_{ls} = \frac{\partial A_s}{\partial x^l} - \frac{\partial A_l}{\partial x^s} = \frac{\partial \Gamma^m_{ms}}{\partial x^l} - \frac{\partial \Gamma^m_{ml}}{\partial x^s}$$

showing it is possible to define a 1- form $A = A_l dx^l$ such that $k = dA$.

Finally we note that, since k is a 2-form, it can be written in a way that defines an induced Riemann space. For this purpose it is convenient to define a torsion vector $t_l = T^j_{jl} = -T^j_{lj}$ and to use in Eq.(209) above to find [12, p.94]

$$k_{lh} = R_{hl} - R_{lh} + \nabla_j T^l_{jh} + \nabla_h t_l - \nabla_l t_h + t_j T^j_{lh} \tag{211}$$

This equation shows that we can bury the torsion in the 2-form $k = k_{lh} dx^l \wedge dx^h$ and still, on the basis of the antisymmetry of the 2-form have an induced Riemann geometry. We remind ourselves that the Ricci tensors here R_{hl} are for an anti-symmetric underlying geometry. In addition to the clear antisymmetry in the indices in Eq.(211) we note that the torsion (pseudo-)vector, t_l, is not parity invariant as are not in general the torsions themselves.

Explicit calculation for a Riemann space with symmetric connection coefficients ($\Gamma^s_{kn} = \Gamma^s_{nk}$) in terms of a metric using Eq.(212) below shows the 2-form k to vanish. In general, we have only that $\Gamma^j_{jk} = A_k$ and k_{ij} does not vanish. This can be considered the result of combining two connections, one for the left and one for the right so that k_{ij} vanishes.

However, if there is single function ϕ, e.g., $\phi = \ln\sqrt{-g}$ such that $\Gamma^j_{jk} = \frac{\partial \phi}{\partial x^k}$ then $\frac{\partial \Gamma^i_{lj}}{\partial x^k} = \frac{\partial \Gamma^i_{kj}}{\partial x^l}$ and k_{ij} vanishes when the torsion is zero. In this case, where we have parity conservation - equivalent right- and left-handedness, we have the usual formulas used in GRT. In general for a symmetric Riemann space, such as used in GRT, we have [19, p.308]

$$\gamma^m_{ik} = \frac{1}{2} g^{jm} (\partial_k g_{ij} + \partial_i g_{kj} - \partial_j g_{ik}) \tag{212}$$

We note in passing that if the units of g_{ij} are $[g_{ij}] = L^2$ needed to convert unitless coordinates to physical lengths, and noting that ∂_k would also then be unitless, i.e., $\left[\frac{\partial}{\partial x^k}\right] = 1$, then $[\gamma^m_{ik}]$ =1 as well. In greater generality to Eq.(ii), we have [19, p.310]

$$\Gamma^m_{ik} = \frac{1}{2} g^{jm} (e_k(g_{ij}) + e_i(g_{kj}) - e_j(g_{ik})) - \frac{1}{2}(c^m_{ki} + g^{jm} g_{li} c^l_{kj} + g^{jm} g_{ik} c^l_{ij}) \tag{213}$$

The last term can be called a contorsion of the manifold. Here the differential generators of motion, e_i, satisfy

$$[e_i, e_j] = c_{ij}^k e_k. \tag{214}$$

We note that, as $\left[g^{jm}\right] = L^{-2}$, then $[e_k(g_{ij})] = L^2$. Thus, $\left[c_{ij}^k\right] = 1$ and $[e_k] = 1$. In quantum dynamics, one uses dimensioned quantities and $[\Gamma_{ik}^m] = L^{-1}$. The problem is to relate Eq.(213) originating in differential geometry to physics.

We start with the simple non relativistic case of three coordinates and an independent evolution variable. For instance, in transporting a (quantum) moving frame from one point to another, because of manifold evolution, we have, for the quantum symmetry $so(3)$ expressing inherent isotropy,

$$[e_i, e_j] = \ell^{-1} e_k \qquad (\hookrightarrow i \to j \to k \to)(\text{iv}) \tag{215}$$

Here the number ℓ^{-1} has physical dimensions of one of the e_i. If we rescale these quantities by r, then in the limit $r \to \infty$ we have $[e_i, e_j] = \frac{\ell^{-1}}{r} e_k = 0$, This is the case for the natural co-ordinates $e_i \to \frac{\partial}{\partial x^i}$ where $\left[\frac{\partial}{\partial x^j}, \frac{\partial}{\partial x^k}\right]$ This equation is involved with the relation of the classical variables (primal) and dual space variables used in the geometric approach to quantum dynamics. Here

Primal	Dual
idx^k	$\frac{\hbar}{i}\frac{\partial}{\partial x^k}$

The factors of the imaginary unit and $\hbar$ are not necessary and are included merely for convenience in making comparison to first quantization involving dimensioned variables clearer. We have

$$\left[\frac{\hbar}{i}\frac{\partial}{\partial x^j}, \frac{\hbar}{i}\frac{\partial}{\partial x^k}\right] = [p_j, p_k] = 0. \tag{216}$$

The differential geometric duality condition, the relationship between the primal (dx^i) and the dual space$\left(\frac{\partial}{\partial x^k}\right)$, normally used in differential geometry is

$$\frac{dx^i}{\partial x^j} = \delta_j^i \tag{217}$$

The quantities dx^i, defined to be differentials along the co-ordinate plane normals dual to the co-ordinates, x^k (the covariant-contravariant dualism), have their own algebra

$$dx^i \wedge dx^j = -dx^j \wedge dx^i \tag{218}$$

The coordinates commute among themsleves

$$x^i x^j = x^j x^i. \tag{219}$$

The combination of the x^i and the dx^k form an algebra themselves.

At the quantum level in physics it has been discovered that there is relationship between the co-ordinates and the dual space which is non orthogonal can be simply specified in terms of physical variables by

$$\left[p_k, ix^i\right] = \hbar\delta_k^j \tag{220}$$

The numerical value of the quantity $\hbar$ is chosen so as to make physical momenta, $p_k = \frac{\hbar}{i}\frac{\partial}{\partial x^k}$, consistent with physical lengths. It has the same status as the speed of light, connecting the standard units of length to the units of time for the maximum speed of evolution of a wave submanifold of the MST. Equation (220) can be viewed as a requirement on the geometry of a combined coordinate-momentum space. The geometrical relationship given in Eq.(v) can be rewritten in terms of new operators defined in the following determinantal nomogram

$$\begin{bmatrix} L_x & L_y & L_z \\ x & y & z \\ p_x = \frac{\hbar}{i}\frac{\partial}{\partial x} & p_y = \frac{\hbar}{i}\frac{\partial}{\partial y} & p_z = \frac{\hbar}{i}\frac{\partial}{\partial z} \end{bmatrix}$$

We observe, as is well known, that

$$[L_x, L_y] = i\hbar L_z, \quad [L_y, L_z] = i\hbar L_x, \quad [L_z, L_x] = i\hbar L_y \tag{221}$$

The algebra defined in Eq.(221) is a representation of the angular momentum Lie algebra $so(3)$. In this form the operators L_i are rotation operators about the i^{th} axis. As a moving frame advances it moves along the $i^{th}-$ axis in a helix motion; we can identify the variables with angular momentum variables at small scale. The inclusion of the imaginary unit with the coordinates introduces minus signs in the spatial part of the line element (the coordinate differentials, which are infinitessimal displacements along the co-ordinate curves, are not the same as the dx^i, which point along coordinate planes), so the Euclidean space-time metric becomes Minkowski. This encourages us to introduce the units of $\hbar$ above. Since torque has units of $[r \times F] = E$, Planck's constant has units of torque-time. The physical picture is that at small enough scale the transport follows helices, so that there is an advance along the axis of the helix during the transport. At large enough scale (from a distance) the helical motion can be ignored, the space assumed to be pseudometric and we proceed as usual in GRT. On the other hand, we have left- and right-handed helices, a fact that cannot be ignored. In the advance along an evolutionary path these are different by a small amount leading to parity non-conservation. It is the usual practice in GRT to ignore these small quantum effects and set $c^k_{ij} = 0$ leading to the so-called *natural coordinate frame* where the transporters of linear motion $e_k \rightarrow \frac{\hbar}{i}\partial_k$. Thus the underlying geometry at small scales, where ultimately the tug of gravity must find its purchase, is quite different from what is assumed in GRT.

The analysis above shows that the natural symmetry of the local quantum system, related to the presence of torsion at a microscopic scale, is basically a rotation group. In the relativistic generalization, the appropriate group is the Poincaré group. It turns out that the discussion above, using $so(3)$, can be neatly generalized to a 4D space as the limit of a 5D one, called the de Sitter group. This enables the introduction of a parameter at whose limit produces the natural coordinates associated with the Poincaré group. The next few paragraphs describes how this is done.

The de Sitter group, used in supergravity theory, can be used to produce the transition to the naturaal coordinates as we now show. On the other hand, if we limit the size of the de Sitter "radius" we can describe the left- right-hand assymetry of a spacetime. This arises from the fact that on a Lie group three connections can be defined with respect to parallel translations. (É. Cartan and J.A. Schouten, Proc. Amsterdam Acad. Sci. **29**, 803 (1926)) If T_a is a fixed element of the group, T_x is a variable element of the group, and we consider the vector space under multiplication formed by the elements $\{T_x\}$, we can form the right-invariant fields $R_a(x) = T_x T_a$ and left-invariant fields $L_a(x) = T_a T_x$. Left connections are the ones compatible with parallel translations made from left-invariant vector fields and similarly for right connections. In general, the translation of a vector v^ν along a path is given by

$$\delta v^\nu = dv^\nu + \Gamma^\nu_{\lambda\mu} v^\lambda d\xi^\mu + T^\nu_{\lambda\mu} d\xi^\mu. \tag{222}$$

The quantity $T^\nu_{\lambda\mu}$ is called the torsion tensor and is what makes the total connection asymmetric. The torsion does not affect the motion of point particles as

$$\frac{d^2x^i}{dt^2} + (\tilde{\Gamma}^i_{jk} + \alpha T^i_{jk})\frac{dx^j}{dt}\frac{dx^k}{dt} = \frac{d^2x^i}{dt^2} + \tilde{\Gamma}^i_{jk}\frac{dx^j}{dt}\frac{dx^k}{dt}. \tag{223}$$

From which we derive the fact that point particle geodesics move along the symmetric path. If $\grave{\Gamma}^i_{jk}$ is a left-connection and $\acute{\Gamma}^i_{jk}$ is a right-connection then the bi-invariant, two-sided connection is given by $\tilde{\Gamma}^i_{jk} = \frac{1}{2}(\grave{\Gamma}^i_{jk} + \acute{\Gamma}^i_{jk})$. If T^i_{jk} is a torsion tensor, then $\acute{T}^i_{jk} = -\grave{T}^i_{jk}$ and consequently $\tilde{T}^i_{jk} = 0$. Under the symmetry reversing the directions of lines (parity reversal) clearly $\acute{T}^i_{jk} \longleftrightarrow \grave{T}^i_{jk}$. Furthermore, under the parity reversal, changing left $\longleftrightarrow$ right, then $\tilde{\Gamma}^i_{jk} \longleftrightarrow \tilde{\Gamma}^i_{jk}$, i.e., the symmetric connection is parity invariant. For a Lie group $[T_i, T_j] = c^k_{ij} T_k$, the curvatures are, $\grave{R}^i_{j\alpha\beta} = \acute{R}^i_{j\alpha\beta}$, and $\tilde{R}^i_{j\alpha\beta} = -\frac{1}{4} c^i_{kj} c^k_{\alpha\beta}$. In the symmetric case one can define the Killing metric $g_{ij} = c^k_{im} c^m_{jk}$. We can choose $\grave{\Gamma}^i_{jk} = \mathring{\Gamma}^i_{jk} + c^i_{jk}$, in which case $\acute{\Gamma}^i_{jk} = \mathring{\Gamma}^i_{jk} - c^i_{jk}$ so, consequently, $\tilde{\Gamma}^i_{jk} = \frac{1}{2}(\grave{\Gamma}^i_{jk} + \acute{\Gamma}^i_{jk})$. As the c^k_{im} form the adjoint representation of the group we can say the connections induce the adjoint representation of the group. We can define the left- and right- parts of the group, $\grave{G}$ and $\acute{G}$ respectively, as those transforming left-invariant vector fields into left-invariant fields and the same for right-invariant fields, then the left- and right-connections are invariants on $\grave{G}$ and $\acute{G}$ respectively. With the difference between left- and right-handed connections defined, we can apply it to the orthogonal Lie groups $O(n, m)$, and in particular to the 5D Lie roup $O(4, 1)$ called the de Sitter group. By using this group we can provide left- and right-handed connections from the group and control the scale. The proportion of left- and right-handedness is obtained from the quantum dynamics of the MST and presumably represents a dynamically determined physical constant. Since, abstract group theoretic methods have to be instantiated as particular representations for any quandum dynamical system, the left- and rignt-handed subsystems can belong to different representations.

We begin by defining a Clifford algebra [14, p.514] in terms of an anti-commutator

$$\{T_a, T_b\} = 2\delta_{ab} \tag{224}$$

Here δ_{ab} is the Kroneker delta. a representation of the (Lie) algebra can be had by defining

$$[T_a, T_b]_{ij} = 4i\epsilon_{abij} = 4i(\delta_{ai}\delta_{bj} - \delta_{aj}\delta_{bi}). \tag{225}$$

Here $\epsilon_{abij} = \epsilon^{ab}_{ij}$ is the completely antisymmetric Levi-Civita symbol. In general the $(T_a)_{ij}$ can be represented by $2n \times 2n$ matrices for $O(2n)$ - the spinor representations. for vector representations, $O(2n+1)$ one more element is needed, defined by

$$T^{2n+1} = T^1 T^2 \cdots T^{2n}. \tag{226}$$

Now the orthogonal groups $O(n,m)$ are defined by the relation $O^T \eta O = \eta$ where $O \in O(n,m)$ and $\eta = diag(1, ..., -1, ..., 1...)$ contains $n + 1s$ and $m - 1s$ along its diagonal. Corresponding to (226) we define for $O(4,1)$

$$P_a = M^5_a. \tag{227}$$

The algebra can then be written

$$\begin{aligned} [P_a, P_b] &= M_{ab}, \\ [P_a, M_{bc}] &= P_b \eta_{ac} - P_c \eta_{ab}, \\ [M_{ab}, M_{cd}] &= \eta_{ac} M_{bc} - \cdots \end{aligned} \tag{228}$$

We now employ the rescaling trick introduced above by letting $P_a \to r P_a$. It is noted that the only commutator which changes is

$$[P_a, P_b] = \frac{1}{r^2} M_{ab}. \tag{229}$$

Thus we have a group structure, the de Sitter group, in which the dual components can be chosen to satisfy

$$[\frac{\partial}{\partial x^a}, \frac{\partial}{\partial x^b}] = \frac{1}{r^2} M^{ab}. \tag{230}$$

Therefore, we have shown we can produce natural, commuting co-ordinates, as $r \to \infty$. These co-ordinates yield the usual tensor analytic results for the symmetric Riemann connection and provide in the static limit concordance with the Newtonian theory as the W-field then also vanishes.. However, as discussed above, even without taking the limit, merely making r large, by symmetrizing the connection, we can have have a Riemannian space.

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Figures

Figure 1. A 2D static manifold viewed as a hypersurface in a 3D space. A moving frame, or triad is seen being pushed over the manifold to obtain the geometry of the manifold. The moving frame and its dual completely determine the geometry of the manifold. The evolving manifolds used in this paper are 3D and are subsequently embedded in a spacetime.

Figure 2. Slices of the G-vector field causing anomalous motion of stars, for z-values and xy-planes, starting from the south pole ($z = -0.075, -0.8, -0.5, -0.3, 0.0, 0.3, -0.5, 0.8, 0.975$).

Figure 3. Slices of the G-vector field causing anomalous motion of stars, for y-values and xz-planes, starting from the back of galaxy moving to front ($z = -0.075, -0.8, -0.5, -0.3, 0.0, 0.3, -0.5, 0.8, 0.975$).

Figure 4. Total CDM distribution. This is conserved as it includes density from both sheets

Figure 5. First sheet, $0 \leq \chi < \pi/2$ evolution of FLRW model G-fields of CDM, For galactic scale $c_r = 1.5$ and epoch $\eta = 0.02.z$-slices: (x, y)-plane for $z = (-.45, 0.0, +4.5)$,

Figure 6. First sheet, $0 \leq \chi < \pi/2$ evolution of FLRW model G-fields of CDM, For galactic scale $c_r = 1.5$ and epoch $\eta = 0.02$. y-slices (x, z)-plane for $y = (-.45, 0.0, +4.5)$

Figure 7. CDM evolution. For increasing epoch of the first sheet. Note the scale change on going down and across the figure to the right and the inversion. The orientation of the galaxy's CDM distribution changes on going through a big crunch.

Figure 8. Structure of the Hopf map $S^3 \rightarrow S^2 \times S^1$ - a simple model of the galactic structure. a) The lines of G-field flux. b) A 3D pictorial of the same.