

Dissipation, Irreversibility and Symplectic Lagrangian Systems on Thermodynamic Space of Dimension $2n+2$

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Abstract

Recent activity in topological classifications of closed symplectic integrable Hamiltonian systems focuses attention on those properties of a Lagrangian formulation for which the fundamental 2-form is exact. The Lagrangian formulation, based on a Cartan-Hilbert Action which has n degrees of freedom, leads to an unconstrained symplectic system which is dissipative and of dimension $2n+2$. Canonical momentum constraints lead to a contact submanifold of dimension $2n+1$ with a unique extremal field. If the $2n+2$ symplectic system is to exist, it is necessary that the momenta are not defined canonically, $\partial L = \partial v$; $p \neq 0$; and that there must exist anholonomic differential fluctuations $\Phi v = dv$; $\int A dt \neq 0$ in the velocity and/or in position, $\Phi x = dx$; $\int V dt \neq 0$. The implication is that (non-extremal) evolution on the $2n+2$ symplectic domain can be dissipative but the process is not described kinematically in terms of a single parameter group. The fluctuations in velocity lead to non-zero temperature gradients and the fluctuations in position lead to non-zero pressure gradients. Both types of fluctuations lead to distinct contributions to a zero point energy. These $2n+2$ domains can act as a source of magnetic dynamo action in a plasma, where velocity fluctuations associated with temperature produce a charge acceleration mechanism in regions where $E \times B \neq 0$: Anholonomic differential fluctuations in position lead to the dissipative terms in the Navier-Stokes equations. Using the fact that Cartan's Lie derivative of the Action with respect to a vector field V is a cohomological equivalent to the First Law of Thermodynamics, it is possible to decide if a given process V is irreversible or not. On the $2n+2$ symplectic domain, defined as Thermodynamic Space, two distinct evolutionary processes may be defined in terms of the Adiabatic Vector and the Torsion Current. The first process is a symplectomorphism, and therefore is reversible; the second process is not a symplectomorphism, and is irreversible in a thermodynamic sense.

I. INTRODUCTION

In 1974 it was suggested that a certain extension to Hamilton's principle [1] could be made such that the evolutionary processes considered would describe dissipative systems. In short, rather than study those extremal vector fields that satisfy the Cartan-Hamilton equation, $i(\mathbf{V})dA = 0$; it was suggested to consider those processes that satisfy the extended equation $i(\mathbf{V})dA = \int \mathbf{A} + d\mu$: It was not appreciated at that time that the topological domain of the extremal conservative systems was a contact manifold, while the topological domain of the dissipative extension was a symplectic manifold. In an attempt to understand more recent developments in "Hamiltonian" symplectic topology, especially the successful classifications of topological defects, or invariants, on compact domains [2], it was decided to investigate the symplectic concepts from a Lagrangian point of view. The Lagrangian point of view has its advantages, for the fundamental 2-form is deduced by construction, $\omega = dA$. The disadvantage is that all symplectic domains so constructed are non-compact. This apparent drawback became an advantage when it was appreciated that such non-compact domains are precisely that which is needed to describe closed but not isolated, or open thermodynamic systems. The results to be described below imply that symplectic processes on symplectic domains of dimension $2n+2$ are applicable to closed thermodynamic systems and reversible processes, while non-symplectic processes on symplectic manifolds of dimension $2n+2$ describe open thermodynamic systems and irreversible processes. These results are to be compared with the historical results of extremal processes on contact domains of dimension $2n+1$ which are used to describe reversible processes on isolated thermodynamic systems. The symplectic space of dimension $2n+2$ is defined as Thermodynamic Space, in order to distinguish it from the classic State Space of dimension $2n+1$.

A. Extensions of the Cartan-Hilbert Action 1-form

This article considers those physical systems that can be described by a Lagrange function $L(q; v; t)$ and a 1-form of Action given by the expression:

$$A = L(q; v; t)dt + p_i(dq_i - v dt); \quad (1)$$

At first glance it appears that the domain of definition is a $3n+1$ dimensional variety of independent variables, $\{q; v; p; t\}$: Do not assume that p is constrained to be a jet; e.g., $p \in \mathcal{L} = \mathcal{L}(v)$: Instead, consider p to be a (set of) Lagrange multiplier(s) to be determined later. Note that the Action 1-form has the format used in the Cartan-Hilbert invariant integral [3], except that $p \in \mathcal{L} = \mathcal{L}(v)$ necessarily. Also, do not assume at this stage that v is a kinematic velocity function, such that $(dq_i - v dt) \neq 0$:

For the given Action, construct the Pfa[®] sequence $fA; dA; A \wedge dA; dA \wedge dA; \dots; g$ in order to determine the Pfa[®] dimension or class of the 1-form [4]. The top (non-zero) Pfa[±]_n of this sequence is given by the formula,

$$(dA)^{n+1} = (n+1)! \int_{S_{k=1}^n} (\partial L = \partial v^k - p_k) \wedge dv^k \wedge dp_1 \wedge \dots \wedge dp_n \wedge dq^1 \wedge \dots \wedge dq^n \wedge dt; \quad (2)$$

which indicates that the Pfa[®] topological dimension is 2n+2 and not the geometrical dimension 3n+1, which might be expected as the 1-form was de⁻ned initially on a space of 3n+1 "independent" variables. The implication is that there exists an irreducible number of independent variables equal to 2n+2 which completely characterize the di[®]erential topology of the system. It follows that the exact two form dA satis⁻es the equations

$$(dA)^{n+1} \notin 0; \text{ but } A \wedge (dA)^{n+1} = 0; \quad (3)$$

On the 2n+2 domain, the components of 2n+1 form $T = A \wedge (dA)^n$ generate what is herein de⁻ned as the Torsion Current, a contravariant vector density, T^m . The components of the "Torsion current" are orthogonal (transversal) to the 2n+2 components of the covector, A_m ; that make up the coe[±]icients of the Action 1-form. In other words,

$$A \wedge T = A \wedge (A \wedge (dA)^n) = 0 \quad i(T)(A) = T^m A_m = 0; \quad (4)$$

This topological result does not depend upon geometric ideas such as metric. The Torsion Current will reappear below when it is demonstrated that evolution along the direction of the Torsion current is irreversible in a thermodynamic sense.

The 2n+2 symplectic domain so constructed can not be compact without boundary for it has a volume element which is exact. For the 2n+2 domain to be symplectic, the top Pfa[±]an can never vanish. The domain is therefore orientable. Examination of the constraint that the symplectic space be of dimension 2n+2 implies that the Lagrange multipliers, p , cannot be used to de⁻ne momenta in the classical "conjugate or canonical" manner; e.g.,

$$(\partial L = \partial v^k - p_k) \wedge dv^k \notin 0 \quad (5)$$

If, however, the constraints of canonical momenta are subsumed, such that $\partial L = \partial v^k - p_k = 0$; then the 2-form dA is not symplectic on its maximal dimension 2n+2, but instead the top Pfa[±]an de⁻nes a contact manifold on a state space of topological dimension 2n+1 with the formula

$$A \wedge (dA)^n = n! \int L(t; q; v; g; dp_1 \wedge \dots \wedge dp_n \wedge dq^1 \wedge \dots \wedge dq^n \wedge dt; \quad (6)$$

The Torsion current reduces to a single component on the contact manifold. It is this 2n+1 dimensional contact manifold that serves as the arena for most of classical mechanics prior to 1965, especially for those theories which were built from the calculus of variations. The 2n+1 dimensional contact manifold, or state space, admits a unique "extremal" evolutionary ⁻eld, that satis⁻es "Hamilton's equations" $i(V)dA = 0$. [5].

For the maximal non-canonical symplectic physical system of Pfaffian dimension $2n+2$, consider evolutionary processes to be representable by vector fields of the form $\dot{W} = \dot{v}V; A; F; 1g$; relative to the independent variables $f; q; v; p; t$; Define the "virtual work" 1-form, W , as $W = i(W)dA$, a 1-form which must vanish for the extremal case, and be non-zero for the symplectic case. For any n , it may be shown by direct computation that the virtual work 1-form consists of three distinct terms,

$$W = \int \left(\frac{\partial L}{\partial v} \delta v + \frac{\partial L}{\partial q} \delta q + \frac{\partial L}{\partial p} \delta p \right) dt \quad (7)$$

where

$$\delta v = dv; \int dt \neq 0; \quad (8)$$

$$\delta q = dq; \int V dt \neq 0; \quad (9)$$

and

$$\delta p = (dp; \int F dt); \quad (10)$$

The first term involves differential (possibly anholonomic) fluctuations δv and the second term involves differential fluctuations, δq : The concept of differential fluctuations represents the error in the assumption that the vector field describes an evolution of a singly parametrized group. The third term contains the factor δp which when zero defines Newton's laws if p is interpreted as the "momentum". When the evolutionary "velocity" V is assumed to be equal to the Lagrangian velocity, v , then the third term (with possible differential fluctuations in momentum) can be ignored. Without constraints of zero differential fluctuations, the virtual work one form is zero when the three bracket factors vanish, independent of any differential fluctuations. This special case is the basic assumption of classical mechanics. However, note that the first bracket cannot vanish if the domain is symplectic of dimension $2n+2$.

Moreover, in the non-canonical symplectic $2n+2$ domain, the work 1-form can never vanish, for there do not exist null eigen vectors of the anti-symmetric matrix of functions that make up the components of the exact non-degenerate 2-form dA . In the contact $2n+1$ domain, however, there exists a unique vector field with a null eigen value, such that the virtual work 1-form indeed vanishes: $W = i(W)dA = 0$. This result serves as the basis of the d'Alembert principle. It is of some interest to consider those points upon which the symplectic 2-form has a null eigen value(s) as topological defects in the symplectic domain of dimension $2n+2$. As the eigen values of an anti-symmetric matrix come in pairs, "extremal" vectors representing topological defects of the symplectic domain are not unique, a well known result of the calculus of variations having envelope solutions. Note that the 1-form of virtual work depends on both the system (the Action) and the process (the vector field).

When the symplectic work one form is closed but not zero, such that (locally) $W = i(W)dA = dE \neq 0$, the process represented by W is defined to be a symplectic process. Such processes preserve the symplectic structure for

$$L_{(W)}dA = 0: \quad (11)$$

This requirement for a process to be symplectic is to be recognized as the generalization of the Helmholtz conservation of vorticity law in hydrodynamics. For a symplectic process the functions E are never constant and never without a gradient over the symplectic domain. Vector fields that satisfy this closure condition are elements of Lie groups, and in the technical mathematics literature, the functions E are often called "Casimirs" -or somewhat inappropriately, "Hamiltonians". Although not constants over the domain, these "potential" or "energy" functions E are evolutionary invariants of a symplectic process. Most engineers and applied scientists have a greater appreciation for these functions when it is pointed out that they are equivalent to the Bernoulli invariants in hydrodynamics. The engineer would call E a Bernoulli "constant", a function invariant along a streamline, but of different values for different neighboring streamlines: $E = (P + \frac{1}{2}gh + \frac{1}{2}v^2)Vol$:

To prove that the Bernoulli-Casimirs are always evolutionary invariants with respect to symplectic vector fields, construct the Lie derivative of E with respect to W :

$$L_{(W)}E = i(W)dE + d(i(W)E) = i(W)i(W)dA + d(i(W)E) = 0 + 0: \quad (12)$$

Both the first and second terms vanish algebraically. However, for the classic "Hamiltonian" defined above in terms of the Legendre transformation, $H(t; q; v; p) = \int p_k v^k - L(t; q; v; g)$; a direct computation indicates that the Hamiltonian need not be an invariant of a symplectic process - even if the Hamiltonian is explicitly time independent. For consider the evolutionary equation,

$$L_{(W)}H = i(W)dH = f(\partial H = \partial q) \dot{V} + (\partial H = \partial p) \dot{F} + (\partial H = \partial v) \dot{A} + (\partial H = \partial t)g \quad (13)$$

or equivalently

$$L_{(W)}H = f(\dot{q} - \partial L = \partial q + F) \dot{V} + (p - \partial L = \partial v) \dot{A} - (\partial L = \partial t)g: \quad (14)$$

For the domain to remain symplectic and of dimension $2n+2$, the first factor of the second term cannot vanish. The first factor of the first term, when set to zero, is equivalent to the classical Lagrange-Euler equations. Hence, even in this case, if the accelerations A are such that $(p - \partial L = \partial v) \dot{A} \neq 0$; then the "Hamiltonian energy" H ; is not an evolutionary invariant, even though the Bernoulli-Casimir energies are evolutionary invariants of a symplectic process. A simple example of this situation is where the mechanical (Hamiltonian) energy of a system decays to perhaps some

non-zero value at a singular point of the symplectic domain, but the angular momentum stays constant during the process. Numerical simulations of such evolutionary possibilities in fluids have been studied by Carnevale [6].

Even more interesting correspondences can be made to thermodynamics, for then the first term in the expression for W ; which depends on differential vortuactions, Φv ; suggests a relationship to the thermodynamic Helmholtz free energy (functions of the type TS that involve temperature) and the second term, which depends on differential vortuactions, Φq ; to the thermodynamic Enthalpy (functions of the type PV that involve pressure). The combination of the two types defines the Gibbs free energy (functions of the type $TS + PV$) of closed thermodynamic systems and reversible processes. Hence, these empirical thermodynamic potentials, some 100 years old in concept, are to be recognized as the Bernoulli-Casimirs of the symplectic vector fields on spaces of dimension $2n+2$. The thermodynamic potentials are symplectic evolutionary invariants, but the Hamiltonian energy is not. The need for recognizing the differences between mechanical energy and the thermodynamic energies was discussed by Stuke [7], where, without mention of symplectic evolution, he deduces the need for "acceleration" potentials in certain dissipative systems. These acceleration potentials, which can be shown to be the equivalent of Bernoulli-Casimir functions, were used by Stuke to construct the Enthalpy and Gibbs free energy in certain hydrodynamic examples.

The thermodynamic concepts of pressure and temperature are explicitly absent from that classical mechanics which has focused attention on the extremal contact manifolds of dimension $2n+1$, and which has ignored the concept of differential vortuactions on symplectic spaces of dimension $2n+2$. It is suggested that the occurrence of a pressure gradient, or a temperature gradient should be taken as the signature of a symplectic process.

When the virtual work 1-form is not closed, ($dW \neq 0$ such that the evolutionary processes are not symplectic processes by definition) then the process can become thermodynamically irreversible. These ideas stem from Cartan's definition of an evolutionary process in terms of the equation,

$$L_{(W)}A = i(W)dA + d(i(W)A) = W + dU = Q; \quad (15)$$

and the equation of closure,

$$L_{(W)}dA = di(W)dA = dW = dQ: \quad (16)$$

As mentioned above, when dQ is zero, physicists call this last equation the Helmholtz conservation of vorticity equation, but it is essentially the requirement that the vector field W be a symplectic vector field. Note that Cartan's equation (of topological evolution) in the form, $W + dU = Q$; is precisely the cohomological equivalent of the first law of thermodynamics. This correspondence admits of a useful criteria for connecting dynamical systems and thermodynamics in a non-statistical manner. Note that the dynamical correspondence permits a precise statement to be made about

the differences between work and heat: the 1-form of heat is not transversal to the evolutionary process, but the 1-form of virtual work is always transversal to the process:

$$i(W)W = i(W)i(W)dA = 0 \text{ but } i(W)Q = i(W)dU \neq 0: \quad (17)$$

This idea of transversality is never made clear in most thermodynamic treatments.

The thermodynamic criteria for irreversibility is that the heat 1-form, Q , does not admit an integrating factor [8]. By the Frobenius Theorem, the Pfaff dimension of Q must be greater than 2; e.g. $Q \wedge dQ \neq 0$: To test for irreversibility of the process W , construct the exterior product $Q \wedge dQ$ using the above formulas. By the Frobenius theorem, a given process, W ; acting on a physical system, A , is irreversible when

$$Q \wedge dQ = L_{(W)}A \wedge L_{(W)}dA \neq 0: \quad (18)$$

Before proving the existence of such irreversible processes, note that when the evolutionary vector fields are symplectic (or extremal), such that $dW = dQ = 0$; then such closed processes are reversible in a thermodynamic sense. The Cartan equation for symplectic evolution becomes (for $V = v$)

$$L_{(W)}A = W + dU = f p_i \otimes L_{(W)}v + f F_i \otimes L_{(W)}q + dU = d(TS_i PV + U) = Q; \quad (19)$$

which defines the heat 1-form Q as the "gradient" of the Gibbs free energy, $G = TS_i PV + U$. Compare to Stueckelberg [7]. For symplectic vector fields, the Mechanical Energy of the system need not be an evolutionary invariant, but the "angular momentum" (Casimir) is an evolutionary invariant. By means of a symplectic process the Hamiltonian energy can decay to a singular state where the symplectic condition fails (where the momentum become "canonical"), and then stay in that "equilibrium" state of non-zero energy forever. A special case of symplectic evolution is given by the Adiabatic constraint,

$$L_{(W)}A = i(W)dA + d(i(W)A) \quad (20)$$

$$= i(W)dA + d(p^2(V(q; v; p; t))_i v) + L(q; v; t) = 0 \quad (21)$$

For simplicity, first assume that $p^2(V(q; v; p; t))_i v = E(q; v; p; t) = 0$; then this special vector W is uniquely defined in an algebraic manner as the Adiabatic vector, Z : If $[F_{uv}]$ is the matrix of coefficients of the 2-form, dA , then the unique adiabatic process is given by the vector field,

$$Z = [F_{uv}]^{i-1} \pm \text{grad}(L): \quad (22)$$

The adiabatic process defined by Z is symplectic and reversible. The function $E(q; v; p; t)$ plays a role similar to the Weierstrass excess function in the calculus of variations.

To prove the existence of an irreversible process, note that there always exists a $2n+1$ form $T = (A \wedge (dA)^n)$, whose $2n+2$ coefficients define the Torsion current, T^m , on the $2n+2$ symplectic space. In 4D, the three form $A \wedge (dA)$ has been defined as the Topological Torsion 3-form. The Torsion current depends only on the system (the Action) and not upon a process. The divergence of this Torsion current is proportional to the measure of the $2n+2$ volume, that defines the symplectic space, and cannot be zero on the symplectic domain. The $2n+2$ components of the $2n+1$ form T generate what is called the "subsidiary Pfaffian system" by Forsythe [9].

If the Torsion current, to within a factor, is used as a candidate for an evolutionary process, then the Lie derivative of the Action with respect to the Torsion current satisfies the "conformal" or similarity equation

$$L_{(T)}A = \lambda A \quad (23)$$

The existence of the Torsion vector implies

$$L_{(T)}dA = d\lambda \wedge A + \lambda dA \quad (24)$$

such that the Pfaffian dimension of the heat 1-form is greater than 2:

$$Q \wedge dQ = L_{(T)}A \wedge L_{(T)}dA = \lambda^2 A \wedge dA \neq 0 \quad (25)$$

Hence the existence of an irreversible process on the symplectic space has been demonstrated, by construction, in the form of the Torsion current. With respect to evolution in the direction of the torsion current, the symplectic volume is contracting or expanding exponentially unless $\lambda = 0$; and therefore such vector fields cannot represent a symplectic process (which preserves the volume element). The factor, λ ; is a Liapunov function and defines the stability of the process (depending on the sign of λ): When $\lambda = 1$, the Torsion vector has been called the "Liouville vector" [10]. Note that herein a constructive process has been displayed for the Torsion current for any given Lagrangian.

Further note that the evolution of the 3-form $A \wedge dA$ is given by the expression,

$$L_{(T)}A \wedge dA = (L_{(T)}A)dA + A \wedge L_{(T)}dA = 2\lambda A \wedge dA \quad (26)$$

II. An Electromagnetic Example

The best way to exemplify the techniques described above is apply them to an electromagnetic situation. Most everyone has had some experience with electrodynamics on a four dimensional space-time. On the four dimensional space-time of independent variables, $(x; y; z; t)$ the 1-form of Action can be written in the form

$$A = \int_{k=1}^3 A_k(x; y; z; t) dx^k + \dot{A}(x; y; z; t) dt \quad (27)$$

The first step is to construct the Pfa[®] Sequence, $fA; dA; A^{\wedge}dA; dA^{\wedge}dA$. Following the usual constructions, the components of the 2-form dA become

$$dA = f@A_k=@x^j ; @A_j=@x^k g dx^j \wedge dx^k = F_{jk} dx^j \wedge dx^k = B_z dx^{\wedge} dy^{\wedge} ; E_x dx^{\wedge} dt ; \quad (28)$$

where in usual notation,

$$E = j @A=@t ; grad A ; \quad B = curl A ; \quad (29)$$

The topological torsion 3-form, $A^{\wedge}dA$, induces the torsion current

$$T = f(E \wedge A + B \wedge A) ; A^{\wedge} B g ; fS ; hg ; \quad (30)$$

such that

$$A^{\wedge}dA = i(T) dx^{\wedge} dy^{\wedge} dz^{\wedge} dt = S^x dy^{\wedge} dz^{\wedge} dt ; ; j h dx^{\wedge} dy^{\wedge} dz \quad (31)$$

The 4-form of topological parity becomes

$$dA^{\wedge}dA = j 2(E^{\wedge} B) dx^{\wedge} dy^{\wedge} dz^{\wedge} dt = (div S + @h=@t) dx^{\wedge} dy^{\wedge} dz^{\wedge} dt \quad (32)$$

It is apparent that a 4D system cannot be symplectic unless $(E^{\wedge} B)$ never vanishes. Moreover, in the symplectic case the torsion current does not satisfy a conservation law (or the "equation" of continuity")

Note that the Poincare lemma always leads to the first Maxwell pair of (Faraday Induction) equations:

$$ddA = fcurl E + @B=@t g dx^{\wedge} dy^{\wedge} dz^{\wedge} dt ; ; + ; ; j div B dx^{\wedge} dy^{\wedge} dz g) 0 ; \quad (33)$$

or

$$fcurl E + @B=@t = 0 ; \quad div B = 0g ; \quad (34)$$

The result is actually true for a variety of any dimension ≥ 4 and for any set of covariant symbols: The concept of Faraday induction is universal.

Consider an arbitrary process defined by the 4 vector field $W = \frac{1}{2}fV ; 1g$: Then the Work 1-form becomes the Lorentz force law:

$$W = (i(W)dA) = (\frac{1}{2}fE + V \wedge B g dx^k ; f\frac{1}{2}V^{\wedge} E g dt) ; \quad (35)$$

With these constructions now apply the constraints that produce the Hamiltonian extremal and the Helmholtz symplectic equivalence classes.

A. The Hamiltonian Extremal Class

In the extremal Hamiltonian case, $dA \wedge dA = 0 \Rightarrow E \cdot B = 0$; and the work 1-form must vanish: $W = i(W)dA = 0$: Therefore the extremal field constraint requires that

$$\int (E + \mathbf{V} \times \mathbf{B})_k dx^k - \int \mathbf{V} \cdot \mathbf{E} dt = 0; \quad (36)$$

an equation that is to be satisfied for any value of the "normalization" factor $\frac{1}{2}$: It is apparent that the extremal constraint forces the Lorentz force to vanish, $(E + \mathbf{V} \times \mathbf{B}) \cdot \mathbf{V} = 0$; and the dissipative power to vanish, $\mathbf{V} \cdot \mathbf{E} = 0$: The first condition is the classic constraint for a charge particle moving in crossed magnetic and electric fields.

B. The Helmholtz Symplectic Class

The Helmholtz (symplectic) closure requirement, $d(i(W)dA) = 0$; implies that the Lorentz force need not be zero, but it should have zero curl: $d(\frac{1}{2}\int (E + \mathbf{V} \times \mathbf{B})_k dx^k - \int \mathbf{V} \cdot \mathbf{E} dt) = 0$: First consider that case where $\frac{1}{2}$ is a constant. Then, the necessary condition to satisfy the closure condition, for arbitrary displacements of the independent variables, is that

$$\text{curl} \int (E + \mathbf{V} \times \mathbf{B})_k dx^k = 0 \quad (37)$$

and similarly

$$\text{div} \int (E + \mathbf{V} \times \mathbf{B})_k dx^k + \text{grad} \int \mathbf{V} \cdot \mathbf{E} dt = 0: \quad (38)$$

Substituting the Maxwell result, $\text{curl} \mathbf{E} = -\text{grad} \phi$; leads to the Master equation of the Imperfect Plasma:

$$\text{grad} \phi + \text{curl} \int \mathbf{V} \times \mathbf{B} = 0: \quad (39)$$

In the symplectic case which is equivalent to a Fomenko system the Lorentz Force cannot vanish, and the symplectic evolutionary process satisfies the equation

$$\frac{1}{2}(\mathbf{E} + \mathbf{V} \times \mathbf{B}) = \text{grad}(\phi): \quad (40)$$

The Master equation is modified slightly to account for a non-constant form of the "scaling" function, $\frac{1}{2}$:

$$\text{grad} \phi + \text{curl} \int \mathbf{V} \times \mathbf{B} = \text{grad} \ln \frac{1}{2} \int (\mathbf{E} + \mathbf{V} \times \mathbf{B}): \quad (41)$$

In elementary physics the scaling function $\frac{1}{2}$ is to be recognized as the charge distribution. In elementary mathematics, the scaling function $\frac{1}{2}$ is to be recognized as the integrating factor.

The symplectic manifold condition of maximal rank over the 4 dimensional domain requires that the second Poincare invariant is not zero:

$$dA \wedge dA = 2(\mathbf{E} \cdot \mathbf{B}) dx \wedge dy \wedge dz \wedge dt \neq 0: \quad (42)$$

The symplectic evolutionary process condition requires that

$$\frac{1}{2}(\mathbf{E} \cdot \mathbf{B}) = \mathbf{B} \cdot \text{grad } \mathcal{E} \neq 0 \quad (43)$$

There must exist a gradient (of pressure or temperature) in the direction of the \mathbf{B} field lines. Similarly, there is a dissipation if the motion is in the direction of the gradient, for then

$$\frac{1}{2}(\mathbf{E} \cdot \mathbf{V}) = \mathbf{V} \cdot \text{grad } \mathcal{E} \quad (44)$$

Note here is no "ohmic" dissipation for evolution \mathbf{V} in the direction orthogonal to $\text{grad } \mathcal{E}$.

Physically, then, in a symplectic system there must exist a component of electric force that accelerates charged particles along the magnetic field lines, and that component of force, as an artifact of the symplectic constraints, is the ultimate source of the magnetic dynamo. A similar situation holds in hydrodynamics where fluid mass can be accelerated along the lines of vorticity. For the extremal, non-symplectic case, the Lorentz force must vanish, and there is no magnetic dynamo action.

From the argument developed above for symplectic systems, the Bernoulli-Casimir energy function \mathcal{E} is either of the type TS and/or of the type PV. For a solid, assume the former representation dominates. Then the "Lorentz force" must have the form of a spatial gradient of the temperature, $\frac{1}{2}(\mathbf{E} + \mathbf{V} \times \mathbf{B}) = \text{grad}(kT)$: For motion that is along the magnetic field lines, the term $\mathbf{V} \times \mathbf{B} = 0$: Then, incorporating the empirical Ohmic relation, $\mathbf{j} = \frac{3}{4}\mathbf{E}$; it is apparent that the symplectic case leads to a derivation of flux equations in the Thompson format for thermal power.

$$\mathbf{j} = (1 - \frac{1}{2}\frac{3}{4})\text{grad}(kT) \quad (45)$$

The suggestion is that the source of magnetic dynamo forces is to be associated with a temperature gradient and the existence of differential velocity fluctuations in a symplectic system. The theory predicts that not only should there exist a Bernoulli-Casimir Pressure gradient, but also there should also exist a Bernoulli-Casimir Temperature gradient that will be exacerbated by domains where $\mathbf{E} \cdot \mathbf{B} \neq 0$:

The Bernoulli-Casimir Pressure gradient is associated with those velocity functions which induce differential fluctuations in position, Φq . The (novel and differential) Bernoulli-Casimir Temperature gradient is associated with those acceleration functions which induce differential fluctuations in velocity, Φv . These ideas are to be compared with the Davies-Unruh concept that uniform acceleration in the vacuum field induces a temperature. [14].

C. The Torsion Current and Irreversible Processes (Pfa[®] dimension 4)

Assume that the Pfa[®] dimension of the domain of interest is 4, hence the space is symplectic. However, consider evolutionary fields that are not constrained to be

symplectic such that $dW \neq 0$. Direct evaluation of the virtual work 1-form, $W = i(W)dA$ yields (the Lorentz force)

$$W = i(W)dA = (f\frac{1}{2}E + J \int Bg_k dx^k - \int J^2 E g dt) \quad (46)$$

The obvious first choice for the evolutionary vector field has been based on the classic assumption that $W = fJ; \frac{1}{2}g - \frac{1}{2}fV; 1g$. The expression for virtual work becomes

$$W = \frac{1}{2}(fE + V \int Bg_k dx^k - \int fV^2 E g dt): \quad (47)$$

However, another perhaps not so obvious a candidate for a solution vector field is the expression for the Torsion current. That is, examine the evolution along the four dimensional vector field,

$$T = f(E \int A + B\dot{A}); A^2 Bg: \quad (48)$$

The expression for virtual work becomes

$$W = i(T)dA = (f(A^2 B)E + (E \int A) \int Bg_k dx^k - \int fE^2 B\dot{A}g dt) = (E^2 B)A: \quad (49)$$

The torsion current is an associated field relative to the 1-form of Action, in the sense that

$$i(T)A = 0: \quad (50)$$

It follows that the Lie derivative of the Action along the direction of the Torsion current is a conformal process in the sense that

$$L_{(T)}A = \int A = (E^2 B)A = Q: \quad (51)$$

By direct computation,

$$L_{(T)}dA = d\int A + \int dA = dQ \quad (52)$$

from which it follows that

$$Q \wedge dQ = \int A \wedge dA: \quad (53)$$

If the topological parity $\int A = (E^2 B)$ does not vanish, then the Torsion current T represents an irreversible non-conservative process, as for such cases the heat 1-form, Q , does not admit an integrating factor. When $\int A = (E^2 B) = 0$; the process is stable in a Liapunov sense.

The formula $L_{(w)}A = \int A$ was the fundamental principle used by the present author in 1974 to describe "An Extension of Hamilton's Principle to Include Dissipative Systems". It was not known at that time the such processes implied the existence of a symplectic structure, nor the fact that the irreversible processes were not symplecto-morphisms.

D. The Torsion Current and Reversible Processes (Pfa[®] dimension 3)

If the non-zero Torsion current has a zero divergence everywhere, then the function $\dot{\mu} = 0$ defines a holonomic constraint of projection on the 4D space, upon which the Pfa[®] domain generated by the Action 1-form is no longer of dimension 4. The space does not support a symplectic structure of Pfa[®] dimension 4. Instead the 2-form defines a contact manifold of Pfa[®] dimension 3. On the 4 dimensional domain of initial independent variables, the contact 2-form is of rank 2, not 4, which implies that on the 4D space there exist TWO extremal fields with null eigenvalues. From the analysis above, the Torsion current is one of these two extremal vector fields. Evolution along the divergence free paths of the torsion current is reversible. It is of particular interest when this hypersurface is minimal.

If the divergence free condition is not global, then evolution along the divergence free paths of the torsion current preserves the contact structure, or singularity in the symplectic domain, for when $\dot{\mu} > 0$;

$$L_{(T)}(A \wedge dA) = (L_{(T)}(A)) \wedge dA + A \wedge L_{(T)}(dA) = 0 + 0: \quad (54)$$

This divergence free process path is reversible, and stable if $\dot{\mu} < 0$: Such closed process paths form limit cycles.

Any other extremal field of the format $W = fJ; \frac{1}{2}g$ must satisfy the equations $J^2 E = 0$; and $\frac{1}{2}E + J \times B = 0$: Assuming that $fJ; \frac{1}{2}g = \frac{1}{2}fV; 1g$; this second extremal evolutionary process implies that the motion of charges is orthogonal to both the E and B fields (the Hall effect).

If it is assumed that the second extremal field is orthogonal to the Torsion vector, then it must be true that

$$fJ; \frac{1}{2}g = \frac{1}{2}fA; \dot{A}g; \quad (55)$$

which is in the format of the London current.:

When $\dot{\mu} \neq 0$; any evolutionary field that is to be reversible must be orthogonal to the Torsion vector current. Conversely, in the symplectic case such that $E^2 B \neq 0$, any evolutionary vector field that has a component in the direction of the Torsion vector must be irreversible. Note that the sign of $\dot{\mu} = E^2 B$ determines the "stability" of the process defined by the Torsion vector.

E. Submanifolds of the Symplectic Case

The criteria that the Torsion current produce a symplectic manifold implies that the Torsion current does not satisfy an "equation of continuity"; that is, the Torsion current is "not conserved". On the other hand, if the Torsion current satisfies an equation of continuity (has zero divergence), the domain is NOT symplectic. In this case, the exterior derivative of the 3-form vanishes over the 4D domain which implies

that there exists a $N-2 = 2$ form, G , such that $dG = J$. The electromagnetic system is now complete, for both the Fields, $F = dA$, and the excitations, G , have been defined without a metric. The sources of the electromagnetic field are topological defects in the symplectic structure of space-time. The Poincare lemma $ddA = dF = 0$ establishes the first Maxwell pair, and the equations $dG = J$ form the second Maxwell pair. No metric constraints, nor connections have been subsumed. This result leads to the concept of quantized flux as the 1-D period integrals of the harmonic components of the Action, $\int_H A$, quantized charge as the 2-D period integrals of $\int_{RR} G$, and quantized spin as 3-D period integrals of the form, $\int_{RRR} A \wedge G$. [11] Now it is apparent that the existence of such results and the entire Maxwell theory follows from the topological concept that certain physical systems can be described by 1-form of Action which is of Pfa[®] dimension 3; that is, from the study of non-symplectic systems for which $A \wedge F \notin 0$; but for which $d(A \wedge dF) = dA \wedge dA = F \wedge F = 2(E \wedge B)dx \wedge dy \wedge dz \wedge dt = 0$: It has been demonstrated that ratio of the 3-D period integrals of spin and torsion form a set of rational fractions defining the topological fractional Hall Impedance. [12]

As G is a non-exact 2-form, it could also define a symplectic structure on the 4-D domain, when $dG = 0$: In such a case $G \wedge G = 2(D \wedge H)dx \wedge dy \wedge dz \wedge dt \notin 0$: The ratio of the integrals of the two symplectic structures then gives the impedance of free space:

$$Z_0 = \frac{\int_{RRR} A \wedge G}{\int_{RR} G} = \frac{\int_{RRR} A \wedge G}{\int_{RR} G} = \frac{q}{\dots} = \text{the impedance of freespace:} \quad (56)$$

The ratio of the period integrals gives the Hall impedance

$$Z_{\text{Hall}} = \frac{\int_{\text{closed}} A \wedge F}{\int_{\text{closed}} A \wedge G} \gg h = e^2 \quad (57)$$

such that

$$\frac{2Z_{\text{Hall}}}{Z_0} = \text{®} = 137:063041 \quad (58)$$

It is not uncommon for a variety to support many topologies. For the symplectic topology generated by the exact 2-form, $F = dA$; the topological domain is not compact, while the symplectic topology induced by G can be compact. As the 2-form G is associated with the fields D and H and sources (charge and currents) it appears that such "quantized" features are to be associated with compact manifolds. However, as the two form F is exact its symplectic topology is non-compact. The associated E and B fields are empirically related to forces, and therefore to mass. The argument seems to justify the Mach idea that mass is an artifact of a non-compact topology, while the fundamentally different concept of charge is a compact artifact.

Similar examples as applied to hydrodynamic systems will be reported elsewhere. In particular, solutions to the Navier-Stokes equations that satisfy the condition that $\text{curl} v \wedge \text{curl}(\text{curl} v) \notin 0$ imply that Action 1-form defines a symplectic domain of Pfa[®] dimension 4. The condition is therefore a necessary condition for the existence of thermodynamically irreversible turbulent evolution.[13]

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