When is a Dynamical System irreversible?

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Cartan's Magic Formula has many useful applications.

$$\mathrm{L}_{(\mathbf{V})} \int_a^b A = \int_a^b \mathrm{L}_{(\mathbf{V})} A = \int_a^b Q$$

For those physical systems that can be represented by a 1- form of Action, A, and those processes that can be represented by a dynamical system, V

$$\int_a^b \mathrm{L}_{(\mathbf{V})} A = \int_b^a i(\mathbf{V})dA + d(i(\mathbf{V})A) = \int_a^b Q$$

a.) Those solutions, V, such that $\mathrm{L}_{(\mathbf{V})} \int_a^b A \Rightarrow 0$ **are equivalent to those paths in the calculus of variations that leave the Action integral stationary.**

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b.) The equation may be interpreted as an equation of motion describing continuous topological evolution.

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c.) When the term $i(\mathbf{V})dA$ **is identified with the 1-form of virtual work,** W, **and the term** $d(i(\mathbf{V})A)$ **is identified with the Internal energy,** $d(U)$, **then Cartan's Magic formula is recognized as the first law of Thermodynamics:**

$$\mathrm{L}_{(\mathbf{V})} \int_a^b A = \int_b^a W + d(U) = \int_a^b Q$$

Cartan's Magic formula may be utilized to test if a dynamical system V represents an irreversible process.

a.) From thermodynamics, a process is irreversible when the Heat 1-form, Q, **does NOT admit an integrating factor (the Temperature).**

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b.) From Frobenius, the 1-form Q **does not admit an integrating factor iff**

$$Q\hat{}dQ \neq 0.$$

c.) Hence use Cartan's Magic formula to compute

$$Q\hat{}dQ = \mathrm{L}_{(\mathbf{V})} A\hat{}\mathrm{L}_{(\mathbf{V})} dA$$

for a given physical system, A**, and a given process, V.**

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If $\mathbf{L}_{(V)}A\wedge\mathbf{L}_{(V)}dA \neq 0$, then the process represented by the dynamical system V is irreversible in a thermodynamic sense.

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Example: The Sliding Bowling Ball

The Observation

Consider a bowling ball given an initial amount of translational energy and rotational energy. Assume the angular momentum and the linear momentum are orthogonal to themselves and also to the ambient graviatational field. Then place the bowling ball, subject to these initial conditions, in contact with the bowling alley. Initially, it is observed that the ball slips or skids, dissipating its linear and angular momentum, until the No-Slip condition is achieved. Note that it is possible for the angular momentum or the linear momentum to change sign during the irreversible phase of the evolution. The dynamical system representing the evolutionary process is irreversible until the No-Slip condition is reached. Thereafter, the dynamical system is reversible, and momentum is conserved.

The Analysis

Assume that the physical system may be represented by a 1-form of Action constructed from a Lagrange function:

$$L = L(x,\theta,v,\omega,t) = \{\beta m(\lambda\omega)^2/2 - mv^2/2\}$$

Let the topological constraints be defined anholonomically by the Pfaffian system:

$$\{dx - vdt\} \Rightarrow 0, \quad \{d\theta - \omega dt\} \Rightarrow 0, \ \{dx - \lambda d\theta\} \Rightarrow 0$$

Define the constrained 1-form of Action as

$$A = L(x,\theta,v,\omega,t)dt + p\{dx - vdt\} + l\{d\theta - \omega dt\} + s\{\lambda d\theta - dx\}$$

where $\{p,l,s\}$ **are Lagrange multipliers. Rearrange the variables to give (in the language of optimal control theory) a pre-Hamiltonian action:**

$$A = (-p-s)dx + (l+\lambda s)d\theta - \{-pv + l\omega - L\}dt.$$

It is apparent that the Pfaff dimension of this Action 1-form is 2n+2 = 6. The Action defines a symplectic manifold of dimension 6.

For simplicity, assume initially that the Lagrange multipliers (momenta) are defined canonically; e.g.,

$$p = \partial L/\partial v \Rightarrow -mv, \quad l = \partial L/\partial \omega \Rightarrow \beta m\lambda^2\omega$$

which implies that

$$A = (-mv - s)dx + (\beta m\lambda^2\omega + \lambda s)d\theta - \{-mv^2/2 + \beta m(\lambda\omega)^2/2\}dt.$$

The volume element of the symplectic manifold is given by the expression

$$6Vol = 6m^2\beta\lambda^2\{v - \lambda\omega\}dx\hat{}d\theta\hat{}dv\hat{}d\omega\hat{}ds\hat{}dt = dA\hat{}dA\hat{}dA$$

The symplectic manifold has a singular subset upon which the Pfaff dimension of the Action 1-form is 2n+1 = 5. The constraint for such a contact manifold is precisely the no-slip condition:

$$\{v - \lambda\omega\} \Rightarrow 0$$

On the 5 dimensional contact manifold there exists a unique extremal (Hamiltonian) field which (to within a projective factor) defines the conservative reversible part of the evolutionary process. As this unique extremal vector satisfies the equation

$$i(\mathbf{V})dA = 0,$$

it is easy to show that dynamical systems defined by such vector fields must be reversible in the thermodynamic sense. (**As** $dQ = d(i(\mathbf{V})dA) = 0$ **for all Hamiltonian or symplectic processes**, $Q\hat{}dQ = 0$)

However, **on the 6 dimensional symplectic manifold**, **there does not exist a unique extremal field**, **nor a unique stationary field**, **that can be used to define the dynamical system**. **The symplectic manifold does support vector fields**, **S**, **that leave the Action integral invariant, but these vector fields are not unique in the sense that that they depend on an arbitrary gauge addition to the 1-form of Action that may be required to satisfy initial conditions**.

There does exist a unique torsion field (**or current**) **defined** (**to within a projective factor**, σ) **by the 6 components of the 5 form**,

$$Torsion = A\hat{}dA\hat{}dA$$

This unique vector, **T**, **independent of gauge additions**, **has the properties that**

$$\mathrm{L}_{(\mathbf{T})}A = \Gamma \bullet A \quad and \quad i(\mathbf{T})A = 0.$$

This "Torsion" vector field statisfies the equation

$$\mathrm{L}_{(\mathbf{T})}A\hat{}\mathrm{L}_{(\mathbf{T})}dA = Q\hat{}dQ \neq 0.$$

Hence a dynamical system having a component constructed from this unique Torsion vector field becomes a candidate to describe the initial irreversible decay of angular momentum and kinetic energy.

Solving for the components of the Torsion vector for the bowling ball problem leads to the (**unique**) **decaying dynamical system**:

$$dv/dt = -\sigma/2\{\beta\lambda^2\omega^2 - 2\lambda v\omega + v^2\}/(v - \lambda\omega)$$

$$d\omega/dt = -\sigma/2\{-\beta\lambda^2\omega^2 + 2\beta\lambda v\omega - v^2\}/\lambda\beta(v - \lambda\omega)$$

$$ds/dt - \sigma s = \sigma/2\{-m\beta\lambda^2\omega^2 + mv^2\}/(v - \lambda\omega)$$

$$dx/dt = v$$

$$d\theta/dt = \omega$$

It is to be noted that the non-canonical "symplectic momentum" variables, defined by inspection from the constrained 1-form of Action as

$$P_x \doteq -(mv + s), \qquad P_\theta \doteq (m\beta\lambda^2\omega + s\lambda),$$

both decay at the same (unit σ) rate, until the NoSlip condition is satisfied. (See rollball.pdf)