

Euclidean spaces, Euclidean Transformations and Euclidean Geometry

Felix Klein in his Erlanger program defined a geometry in terms of its invariants with respect to a group of transformations. Euclidean geometry is defined by those invariants of a particular sub group of linear transformations commonly called (rigid body) translations (where there are no fixed points) and (rigid body) rotations (where there is a fixed point in a locally planar sub manifold, through which can be defined a rotation axis.) What are the invariants of Euclidean transformations (e.g., rigid body translations and rotations)? Well, it is empirically evident (e.g., experimentally true) that size, shape, parallelism and orthogonality are preserved by Euclidean transformations. It is these (and other) properties that in effect define Euclidean geometry. (Can you give other invariants of Euclidean transformations?)

The concept of a rigid body is related to the idea that the distance between any pair of points in the body stays the same under translations and rotations. The Euclidean distance is defined in terms of a certain quadratic form: Consider a Vector **B** defined as a position vector to a point b from a fixed origin, and another vector **A** to another point a. In three dimensional space, the distance, s, from a to b is defined as the Pythagorean length of the vector **C = B - A**, or

$$s^2 = +1(C_x)^2 + 1(C_y)^2 + 1(C_z)^2$$

It is assumed that the distance is defined by a straight line between the pair of points. Other definitions of distance are possible and useful. This quadratic form given above is the three dimensional extension developed by the Greeks for the hypotenuse of a right triangle. The quadratic form has three plus signs, and usually the numeric coefficient, 1, is not written in the formula. The symbol 1 implies that the scales in the three linear independent different directions are the same, and equal to unity. This equivalence is the basis of the concept of isotropy (same in all directions).

Given a vector array of components, the quadratic form has a realization in terms of a matrix product. A Euclidean format is given by the expression:

$$s^2 = \langle \mathbf{C} | \circ [g] \circ | \mathbf{C} \rangle = \langle \mathbf{C} | \circ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \circ | \mathbf{C} \rangle$$

The matrix [g] is defined as the metric. The column vector |C⟩ is defined in terms of its components, as

$$| \mathbf{C} \rangle = \begin{bmatrix} C_x \\ C_y \\ C_z \end{bmatrix}$$

and the row vector ⟨C| is the transpose of |C⟩.

The utility of the matrix notation is naturally suited to the study of Linear transformations, for the operation of an arbitrary matrix [M] on a vector |C⟩ produces a new vector |C'⟩ defined by the formula

$$\begin{bmatrix} C'_x \\ C'_y \\ C'_z \end{bmatrix} = | \mathbf{C}' \rangle = [M] \circ | \mathbf{C} \rangle = \begin{bmatrix} m_{11} & m_{12} & m_{13} \\ m_{21} & m_{22} & m_{23} \\ m_{31} & m_{32} & m_{33} \end{bmatrix} \circ \begin{bmatrix} C_x \\ C_y \\ C_z \end{bmatrix} = \begin{bmatrix} m_{11}C_x + m_{12}C_y + m_{13}C_z \\ m_{21}C_x + m_{22}C_y + m_{23}C_z \\ m_{31}C_x + m_{32}C_y + m_{33}C_z \end{bmatrix}$$

For constant matrix elements, m_{ij} the formula defines a LINEAR transformation, as the components of

$|\mathbf{C}'\rangle$ are linear combinations of the components of the original vector $|\mathbf{C}\rangle$. The expression is homogeneous of degree 1, for if each component of $|\mathbf{C}\rangle$ is multiplied by a constant λ then the resultant vector $|\mathbf{C}'\rangle$ has each of its components multiplied by the same factor λ . Such transformations depend upon the "scale" assigned to each vector component.

It is also possible to consider transformations where the matrix elements are not constants, but may have a common factor, which is a function of the components of $|\mathbf{C}\rangle$. In other words consider transformations of the form,

$$|\mathbf{C}'\rangle = \gamma(\mathbf{C})[M] \circ |\mathbf{C}\rangle = [M] \circ |\gamma\mathbf{C}\rangle.$$

It is apparent that these represent linear mappings of the vector $|\mathbf{C}\rangle$ scaled by the factor γ . For example, if $\gamma(\mathbf{C}) = 1/(aC_x + bC_y + cC_z)$, then the components of $|\mathbf{C}'\rangle$ are "fractionally" related to the components of $|\mathbf{C}\rangle$. The formula defines what is known as a fractional Moebius projective transformation. Such a formula demonstrates that the components of $|\mathbf{C}'\rangle$ are homogeneous of degree zero in terms of the components of $|\mathbf{C}\rangle$.

$$\begin{pmatrix} C'_x \\ C'_y \\ C'_z \end{pmatrix} = \begin{bmatrix} m_{11} & m_{12} & m_{13} \\ m_{21} & m_{22} & m_{23} \\ m_{31} & m_{32} & m_{33} \end{bmatrix} \circ \begin{pmatrix} \gamma(\mathbf{C})C_x \\ \gamma(\mathbf{C})C_y \\ \gamma(\mathbf{C})C_z \end{pmatrix} = \frac{\begin{pmatrix} m_{11}C_x + m_{12}C_y + m_{13}C_z \\ m_{21}C_x + m_{22}C_y + m_{23}C_z \\ m_{31}C_x + m_{32}C_y + m_{33}C_z \end{pmatrix}}{(aC_x + bC_y + cC_z)}$$

Rotations and Lorentz translations

The objective is to study those transformations $[M]$ on the vectors $|\mathbf{C}\rangle$ that preserve the quadratic form as an invariant of the transformation

$$s^2 = \langle \mathbf{C} | \circ [g] \circ |\mathbf{C}\rangle.$$

Substitution leads to the expression (for fixed $[g]$)

$$(s')^2 = \langle \mathbf{C}' | \circ [g] \circ |\mathbf{C}'\rangle = \langle \mathbf{C} | \circ [M]^{transpose} \circ [g] \circ [M] \circ |\mathbf{C}\rangle.$$

It is apparent that if $(s')^2 = (s)^2$ it must be true that

$$[M]^{transpose} \circ [g] \circ [M] = [g].$$

When $[g]$ is the identity (euclidean) metric, it follows that the matrices $[M]$ that satisfy the equation are elements of the orthogonal group. That is

$$[M]^{transpose} = [M]^{inverse}.$$

For electromagnetism, the quadratic form with n-1 minus signs is of interest. The "metric" is called the Minkowski metric and has the matrix representation:

$$[g]_{Minkowski} \Rightarrow [\eta] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

Again distance is defined in the same manner, only now the metric is constrained to be the Minkowski metric.

$$s^2 = \langle \mathbf{C} | \circ [\eta] \circ |\mathbf{C}\rangle = +(C_1)^2 + (C_2)^2 + (C_3)^2 - (C_4)^2.$$

If the components of $|\mathbf{C}\rangle$ are identified with a "position vector" as

$$|\mathbf{C}\rangle = \begin{pmatrix} x \\ y \\ z \\ ct \end{pmatrix}$$

then the distance function becomes the ubiquitous quadratic form

$$s^2 = +(x)^2 + (y)^2 + (z)^2 - (ct)^2.$$

If the components of $|\mathbf{C}\rangle$ are identified with the gradient of a function whose zero set defines an electromagnetic propagating discontinuity, then the distance function become the eikonal expression of Fock:

$$\text{If } s^2 = +(\partial\phi/\partial x)^2 + (\partial\phi/\partial y)^2 + (\partial\phi/\partial z)^2 - (\partial\phi/\partial ct)^2 \Rightarrow 0 \text{ (Eikonal constraint)}$$

It is natural to examine what class of transformations preserve the Minkowski distance function, for then propagating electric field discontinuities are propagating discontinuities for all observers of the equivalence class. Such transformations form not a unique single frame of reference, but an infinite class of equivalent frames of reference define in the physics literature as "Inertial Frames".

What are the equivalence class of transformations that preserve the distance function using the Minkowski metric. Repeating the algebra above for the Minkowski metric leads to the expressions,

$$(s')^2 = \langle \mathbf{C}' | \circ [\eta] \circ | \mathbf{C}' \rangle = \langle \mathbf{C} | \circ [L]^{\text{transpose}} \circ [\eta] \circ [L] \circ | \mathbf{C} \rangle.$$

For invariance of the Minkowski distance function it must be true that

$$[L]^{\text{transpose}} \circ [\eta] \circ [L] = [\eta].$$

What is the format of those matrices (Lorentz transformations) that satisfy this constraint? It turns out that it is possible to construct any Lorentz transformation (to within a factor) in terms of products of six generators (Matrices with a single parameter). Three of the matrix generators are called "spatial rotations". Three of the generators are called "Lorentz boosts". Although the parameters of the Lorentz generators are usually expressed as constants (independent from the "coordinates" {x,y,z,ct}, the constraint is not necessary. As will be demonstrated, the parameters may be written in terms of ratios of "velocities" such as v/c. When the v/c are constants, the inertial frames are not accelerated. However, there are Lorentz transformations where the inertial frames are accelerated. Inertial means that the distance function is preserved and that the eikonal is preserved—even for certain accelerated systems. The accelerated systems have curved and twisted spatial subspaces, where the non-accelerated systems have euclidean flat spatial subspaces.

See <http://www22.pair.com/csdc/pdf/lorentz.pdf>