Frame fields for Spheres in R4

```
> restart: with(linalg):with(difforms):with(liesymm):with(plots):
```

Warning, new definition for norm

```
Warning, new definition for norm
Warning, new definition for trace
Warning, new definition for `&^`
Warning, new definition for close
Warning, new definition for d
Warning, new definition for mixpar
Warning, new definition for wdegree
```

```
>
```

Cartan's Repere Mobile will be used to evaluate structural equations,
curvature 2- forms, and torsion 2-forms for two Frame Fields on the sphere S3 in R4.
One Frame Field will be constructed from the Hopf Map (both chiral left and right versions)
which is a map from R4 to R3. Three of the vecotrs of the Frame field will be deduced from perfect differentials of the three Hopf map functions. The fourth vector of the Frame Field will be found by constructing the adjoint to the three perfect differentials.
In the second case (the Instanton map) the normal field will be generated as a perfect differential, and the three remaining vectors of the Frame Field will be non-holonomic forms.
The Frame field will be scaled by various factors that produce special cases of interest.
and the other Frame will be constructed from the Instanton map (which also has right an left versions).
In the Hopf map the three "tangent" vectors to the Fibers of the projection are well defined holonomic differentials, while the "normal" field is anholonomic.
In the Instanton case, the "normal field" is holonomic and defined as a perfect differential, but the three "tangent" vectors to the fibers are anholonomic.

```
> setup(X,Y,Z,S):
> r2:=(x^2+y^2+z^2);dR:=[d(X),d(Y),d(Z),d(S)];scale:=((X^p+Y^p+Z^p+S^p)^(n/p));
>
```

$$r2 := x^2 + y^2 + z^2$$

$$dR := [\,\mathrm{d}(X), \mathrm{d}(Y), \mathrm{d}(Z), \mathrm{d}(S)\,]$$

$$scale := (X^p + Y^p + Z^p + S^p)^{\left(\frac{n}{p}\right)}$$

```
> p:=2;
```

$$p := 2$$

The Hopf map is defined as as a projection from R4 {X,Y,Z,S} to R3 {x,y,z}, by means of the functions given below
(ch is the chiral factor equal to plus 1 or minus 1.)

```
> x:=2*(S*Y+ch*X*Z);y:=2*(Y*Z-ch*S*X);z:=((X^2+Y^2)-(((S)^2+Z^2)));
```

Hopf Map R4 to R3

$$x := 2\,S\,Y + 2\,ch\,X\,Z$$

$$y := 2\,Y\,Z - 2\,ch\,S\,X$$

$$z := X^2 + Y^2 - S^2 - Z^2$$

Note that a point on a euclidian 2-sphere of radius squared (in R3),
r2 = x^2+y^2+z^2
is related to the points on the euclidean 3 sphere of radius squared (in R4

```
> r2:=factor(subs(ch^2=1,simplify(x^2+y^2+z^2)));
```

$$r2 := (S^2 + Z^2 + X^2 + Y^2)^2$$

The differentials of these functions define three independent tangent fields as perfect differentials

```
> e1:=d(x);e2:=d(y);e3:=d(z);
```

$$e1 := 2\,\mathrm{d}(S)\,Y + 2\,S\,\mathrm{d}(Y) + 2\,ch\,\mathrm{d}(X)\,Z + 2\,ch\,X\,\mathrm{d}(Z)$$

$$e2 := 2\,\mathrm{d}(Y)\,Z + 2\,Y\,\mathrm{d}(Z) - 2\,ch\,\mathrm{d}(S)\,X - 2\,ch\,S\,\mathrm{d}(X)$$

$$e3 := 2\,X\,d(X) + 2\,Y\,d(Y) - 2\,S\,d(S) - 2\,Z\,d(Z)$$

and can be adjoined to a 1-form normal field which is orthogonal to each of the tangent fields.

```
> n1:=innerprod([Y,-X,ch*S,-ch*Z],dR);
```

$$n1 := Y\,d(X) - X\,d(Y) + ch\,S\,d(Z) - ch\,Z\,d(S)$$

These fields, to within a factor, lead to an assignment of a global frame field inverse matrix with components prportional to :

```
> HE1:=[ch*Z,S,ch*X,Y];HE2:=[-ch*S,Z,Y,-ch*X];HE3:=[X,Y,-Z,-S];HN1:=[Y,-X,ch*S,-ch*Z];
```

$$HE1 := [ch\,Z, S, ch\,X, Y]$$

$$HE2 := [-ch\,S, Z, Y, -ch\,X]$$

$$HE3 := [X, Y, -Z, -S]$$

$$HN1 := [Y, -X, ch\,S, -ch\,Z]$$

$$0$$

Arbitrarily , and for algebraic simplification, each direction field will be divided by the factor (X^p+Y^p+Z^p+S^p)^(n/p). It is of some interest to note that all vectors above have zero divergence with respect to [X,Y,Z,S]

```
> HFFINV:=evalm(subs(ch^2=1,(array([HE1,HE2,HE3,HN1])))):
```

Ther are intersting features depending on the scaling. If the index n = 0,1,2,3,4.

The Frame Matrix (Repere Mobile) HFF and its induced metric.

```
> HFF:=simplify(evalm(subs(ch^2=1,n=1,inverse(HFFINV/scale))));DET:=factor(subs(ch^2=1,det(HFF)));Gun:=subs(ch^2=1,innerprod(transpose(HFF),HFF));INV:=subs(ch^2=1,innerprod(HFF,HFF));
```

$$HFF := \begin{bmatrix} \frac{ch\,Z}{\sqrt{S^2+Z^2+X^2+Y^2}} & -\frac{ch\,S}{\sqrt{S^2+Z^2+X^2+Y^2}} & \frac{X}{\sqrt{S^2+Z^2+X^2+Y^2}} & \frac{Y}{\sqrt{S^2+Z^2+X^2+Y^2}} \\ \frac{S}{\sqrt{S^2+Z^2+X^2+Y^2}} & \frac{Z}{\sqrt{S^2+Z^2+X^2+Y^2}} & \frac{Y}{\sqrt{S^2+Z^2+X^2+Y^2}} & -\frac{X}{\sqrt{S^2+Z^2+X^2+Y^2}} \\ \frac{ch\,X}{\sqrt{S^2+Z^2+X^2+Y^2}} & \frac{Y}{\sqrt{S^2+Z^2+X^2+Y^2}} & -\frac{Z}{\sqrt{S^2+Z^2+X^2+Y^2}} & \frac{ch\,S}{\sqrt{S^2+Z^2+X^2+Y^2}} \\ \frac{Y}{\sqrt{S^2+Z^2+X^2+Y^2}} & -\frac{ch\,X}{\sqrt{S^2+Z^2+X^2+Y^2}} & -\frac{S}{\sqrt{S^2+Z^2+X^2+Y^2}} & -\frac{ch\,Z}{\sqrt{S^2+Z^2+X^2+Y^2}} \end{bmatrix}$$

$$DET := 1$$

$$Gun := \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$INV :=$$

$$\left[\begin{matrix} \frac{-ch\,S^2+Z^2+Y^2+X^2\,ch}{S^2+Z^2+X^2+Y^2}, & -\frac{Z\,ch\,S+Z\,S+ch\,X\,Y-Y\,X}{S^2+Z^2+X^2+Y^2}, & -\frac{-ch\,X\,Z+Y\,ch\,S+Z\,X+S\,Y}{S^2+Z^2+X^2+Y^2}, & 2\frac{X\,ch\,S}{S^2+Z^2+X^2+Y^2} \\ \frac{Z\,ch\,S+Z\,S+ch\,X\,Y-Y\,X}{S^2+Z^2+X^2+Y^2}, & \frac{-ch\,S^2+Z^2+Y^2+X^2\,ch}{S^2+Z^2+X^2+Y^2}, & 2\frac{X\,S}{S^2+Z^2+X^2+Y^2}, & \frac{S\,Y-Z\,X+Y\,ch\,S+ch\,X\,Z}{S^2+Z^2+X^2+Y^2} \\ \frac{-ch\,X\,Z+Y\,ch\,S+Z\,X+S\,Y}{S^2+Z^2+X^2+Y^2}, & -2\frac{X\,S}{S^2+Z^2+X^2+Y^2}, & \frac{-ch\,S^2+Z^2+Y^2+X^2\,ch}{S^2+Z^2+X^2+Y^2}, & \frac{-Y\,X+ch\,X\,Y-Z\,S-Z\,ch\,S}{S^2+Z^2+X^2+Y^2} \\ -2\frac{X\,ch\,S}{S^2+Z^2+X^2+Y^2}, & -\frac{S\,Y-Z\,X+Y\,ch\,S+ch\,X\,Z}{S^2+Z^2+X^2+Y^2}, & -\frac{-Y\,X+ch\,X\,Y-Z\,S-Z\,ch\,S}{S^2+Z^2+X^2+Y^2}, & \frac{-ch\,S^2+Z^2+Y^2+X^2\,ch}{S^2+Z^2+X^2+Y^2} \end{matrix}\right]$$

Note that the Determinant never vanishes (except at the origin of R4) and is positive definite, an indication of an orientable structure.

The induced metric as defined by the Frame Field is conformal.

```
> subs(ch^2=1,innerprod(HFF,HFFINV));
```

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Now use the Cartan method to evaluate the stuctural equations

```
> HDR:=innerprod(subs(n=1,HFFINV/scale),[d(X),d(Y),d(Z),d(S)]):Hsigma1:=wcollect(factor(wcollect(HDR[1])));Hsigma2:=wcollect(factor(wcollect(HDR[2])));Hsigma3:=factor(wcollect(HDR[3]));Homega:=factor(wcollect(HDR[4]));d(Hsigma1)&^Hsigma1;
```

Note that the scaling has been chosed to produce an orthonormal Frame. Any other scaling would produce a

$$Hsigma1 := \frac{ch\,Z\,\mathrm{d}(X)}{\sqrt{S^2+Z^2+X^2+Y^2}} + \frac{S\,\mathrm{d}(Y)}{\sqrt{S^2+Z^2+X^2+Y^2}} + \frac{ch\,X\,\mathrm{d}(Z)}{\sqrt{S^2+Z^2+X^2+Y^2}} + \frac{Y\,\mathrm{d}(S)}{\sqrt{S^2+Z^2+X^2+Y^2}}$$

$$Hsigma2 := -\frac{ch\,S\,\mathrm{d}(X)}{\sqrt{S^2+Z^2+X^2+Y^2}} + \frac{Z\,\mathrm{d}(Y)}{\sqrt{S^2+Z^2+X^2+Y^2}} + \frac{Y\,\mathrm{d}(Z)}{\sqrt{S^2+Z^2+X^2+Y^2}} - \frac{ch\,X\,\mathrm{d}(S)}{\sqrt{S^2+Z^2+X^2+Y^2}}$$

$$Hsigma3 := \frac{X\,\mathrm{d}(X) + Y\,\mathrm{d}(Y) - Z\,\mathrm{d}(Z) - S\,\mathrm{d}(S)}{\sqrt{S^2+Z^2+X^2+Y^2}}$$

$$Homega := \frac{Y\,\mathrm{d}(X) - X\,\mathrm{d}(Y) + ch\,S\,\mathrm{d}(Z) - ch\,Z\,\mathrm{d}(S)}{\sqrt{S^2+Z^2+X^2+Y^2}}$$

$$\left(\frac{(Y\,ch\,Z - S\,X)\,Y}{(S^2+Z^2+X^2+Y^2)^2} - \frac{(-S^2+Y^2)\,ch\,Z}{(S^2+Z^2+X^2+Y^2)^2} + \frac{(-Z\,ch\,S + Y\,X)\,S}{(S^2+Z^2+X^2+Y^2)^2}\right) \&\hat{}(\mathrm{d}(S), \mathrm{d}(X), \mathrm{d}(Y))$$
$$+ \left(\frac{(-ch\,S\,X + Y\,Z)\,S}{(S^2+Z^2+X^2+Y^2)^2} + \frac{(-Z\,S + ch\,X\,Y)\,Y}{(S^2+Z^2+X^2+Y^2)^2} - \frac{(-S^2+Y^2)\,ch\,X}{(S^2+Z^2+X^2+Y^2)^2}\right) \&\hat{}(\mathrm{d}(S), \mathrm{d}(Z), \mathrm{d}(Y))$$
$$+ \left(-\frac{(-Z\,S + ch\,X\,Y)\,ch\,Z}{(S^2+Z^2+X^2+Y^2)^2} + \frac{(Y\,ch\,Z - S\,X)\,ch\,X}{(S^2+Z^2+X^2+Y^2)^2} - \frac{ch\,(Z^2 - X^2)\,S}{(S^2+Z^2+X^2+Y^2)^2}\right) \&\hat{}(\mathrm{d}(Z), \mathrm{d}(X), \mathrm{d}(Y))$$
$$+ \left(\frac{(-ch\,S\,X + Y\,Z)\,ch\,Z}{(S^2+Z^2+X^2+Y^2)^2} - \frac{(-Z\,ch\,S + Y\,X)\,ch\,X}{(S^2+Z^2+X^2+Y^2)^2} - \frac{ch\,(Z^2 - X^2)\,Y}{(S^2+Z^2+X^2+Y^2)^2}\right) \&\hat{}(\mathrm{d}(S), \mathrm{d}(Z), \mathrm{d}(X))$$

The forms sigma (1,2,3) and omega are the 1-forms relative to the basis frame FF
In the example, the 1-forms sigma1 sigma2 and sigma3 are exact differentials equal to the differentials of the mapping functions defined by the Hopf map giving x,y,z in terms of X,Y,Z,T
The in-exact 1-form of Action is proportional to the 1-form small omega.
For a parametrized surface, small omega vanishes, and such subspaces have Zero Torsion of the Affine type.
From the Frame use the standard methods to compute the Cartan Matrix of connection 1-forms.

See http://www.uh.edu/~rkiehn/pdf/defects2.pdf
for details of the Cartan method for an arbitrary Repere Mobile.

```
> dHFF:=array([[d(HFF[1,1]),d(HFF[1,2]),d(HFF[1,3]),d(HFF[1,4])],[d(HFF[2,1]),d(HFF[2,2]),d(HFF[2,3]),d(HFF[2,4])],[d(HFF[3,1]),d(HFF[3,2]),d(HFF[3,3]),d(HFF[3,4])],[d(HFF[4,1]),d(HFF[4,2]),d(HFF[4,3]),d(HFF[4,4])]]):
```

```
> Hcartan:=(evalm(subs(HFFINV&*dHFF))):
```

Evaluate each component of the connection coefficients on transverse space of E1,E2,E3.

```
> HGamma11:=wcollect(subs(ch^2=1,factor(wcollect(Hcartan[1,1]))));
```

$$HGamma11 := -\frac{(Y^2\,S + S\,Z^2 + S^3 + S\,X^2)\,\mathrm{d}(S)}{(S^2+Z^2+X^2+Y^2)^2} - \frac{(S^2\,Z + Z^3 + Z\,X^2 + Y^2\,Z)\,\mathrm{d}(Z)}{(S^2+Z^2+X^2+Y^2)^2} - \frac{(S^2\,X + X\,Z^2 + X^3 + Y^2\,X)\,\mathrm{d}(X)}{(S^2+Z^2+X^2+Y^2)^2}$$
$$- \frac{(S^2\,Y + Y\,X^2 + Y^3 + Y\,Z^2)\,\mathrm{d}(Y)}{(S^2+Z^2+X^2+Y^2)^2}$$

```
> HGamma12:=wcollect(subs(ch^2=1,factor(wcollect(Hcartan[1,2]))));
```

$$HGamma12 := -\frac{(S^2\,Z + Z^3 + Z\,X^2 + Y^2\,Z)\,\mathrm{d}(S)}{(S^2+Z^2+X^2+Y^2)^2} - \frac{(-S\,Z^2 - S^3 - S\,X^2 - Y^2\,S)\,\mathrm{d}(Z)}{(S^2+Z^2+X^2+Y^2)^2}$$

$$-\frac{(ch\,X^2\,Y+Y\,ch\,S^2+Y\,ch\,Z^2+Y^3\,ch)\,\mathrm{d}(X)}{(S^2+Z^2+X^2+Y^2)^2}-\frac{(-ch\,X\,Z^2-ch\,X^3-ch\,X\,Y^2-ch\,S^2\,X)\,\mathrm{d}(Y)}{(S^2+Z^2+X^2+Y^2)^2}$$

```
> HGamma13:=wcollect(subs(ch^2=1,factor(wcollect(Hcartan[1,3]))));
```

$$HGamma13:=-\frac{Y\,\mathrm{d}(S)}{S^2+Z^2+X^2+Y^2}-\frac{ch\,X\,\mathrm{d}(Z)}{S^2+Z^2+X^2+Y^2}+\frac{ch\,Z\,\mathrm{d}(X)}{S^2+Z^2+X^2+Y^2}+\frac{S\,\mathrm{d}(Y)}{S^2+Z^2+X^2+Y^2}$$

```
> HGamma21:=wcollect(subs(ch^2=1,wcollect(factor(wcollect(Hcartan[2,1])))));
```

$$HGamma21:=\frac{(S^2\,Z+Z^3+Z\,X^2+Y^2\,Z)\,\mathrm{d}(S)}{(S^2+Z^2+X^2+Y^2)^2}+\frac{(-S\,Z^2-S^3-S\,X^2-Y^2\,S)\,\mathrm{d}(Z)}{(S^2+Z^2+X^2+Y^2)^2}+\frac{(ch\,X^2\,Y+Y\,ch\,S^2+Y\,ch\,Z^2+Y^3\,ch)\,\mathrm{d}(X)}{(S^2+Z^2+X^2+Y^2)^2}+\frac{(-ch\,X\,Z^2-ch\,X^3-ch\,X\,Y^2-ch\,S^2\,X)\,\mathrm{d}(Y)}{(S^2+Z^2+X^2+Y^2)^2}$$

```
> HGamma22:=wcollect(subs(ch^2=1,factor(wcollect(Hcartan[2,2]))));
```

$$HGamma22:=-\frac{(Y^2\,S+S\,Z^2+S^3+S\,X^2)\,\mathrm{d}(S)}{(S^2+Z^2+X^2+Y^2)^2}-\frac{(S^2\,Z+Z^3+Z\,X^2+Y^2\,Z)\,\mathrm{d}(Z)}{(S^2+Z^2+X^2+Y^2)^2}-\frac{(S^2\,X+X\,Z^2+X^3+Y^2\,X)\,\mathrm{d}(X)}{(S^2+Z^2+X^2+Y^2)^2}-\frac{(S^2\,Y+Y\,X^2+Y^3+Y\,Z^2)\,\mathrm{d}(Y)}{(S^2+Z^2+X^2+Y^2)^2}$$

```
> HGamma23:=wcollect(subs(ch^2=1,factor(wcollect(Hcartan[2,3]))));
```

$$HGamma23:=\frac{ch\,X\,\mathrm{d}(S)}{S^2+Z^2+X^2+Y^2}-\frac{Y\,\mathrm{d}(Z)}{S^2+Z^2+X^2+Y^2}-\frac{ch\,S\,\mathrm{d}(X)}{S^2+Z^2+X^2+Y^2}+\frac{\mathrm{d}(Y)\,Z}{S^2+Z^2+X^2+Y^2}$$

```
> HGamma31:=wcollect(subs(ch^2=1,factor(wcollect(Hcartan[3,1]))));
```

$$HGamma31:=\frac{Y\,\mathrm{d}(S)}{S^2+Z^2+X^2+Y^2}+\frac{ch\,X\,\mathrm{d}(Z)}{S^2+Z^2+X^2+Y^2}-\frac{ch\,Z\,\mathrm{d}(X)}{S^2+Z^2+X^2+Y^2}-\frac{S\,\mathrm{d}(Y)}{S^2+Z^2+X^2+Y^2}$$

```
> HGamma32:=wcollect(subs(ch^2=1,factor(wcollect(Hcartan[3,2]))));
```

$$HGamma32:=-\frac{ch\,X\,\mathrm{d}(S)}{S^2+Z^2+X^2+Y^2}+\frac{Y\,\mathrm{d}(Z)}{S^2+Z^2+X^2+Y^2}+\frac{ch\,S\,\mathrm{d}(X)}{S^2+Z^2+X^2+Y^2}-\frac{\mathrm{d}(Y)\,Z}{S^2+Z^2+X^2+Y^2}$$

```
> HGamma33:=wcollect(subs(ch^2=1,factor(wcollect(Hcartan[3,3]))));
```

$$HGamma33:=-\frac{S\,\mathrm{d}(S)}{S^2+Z^2+X^2+Y^2}-\frac{Z\,\mathrm{d}(Z)}{S^2+Z^2+X^2+Y^2}-\frac{X\,\mathrm{d}(X)}{S^2+Z^2+X^2+Y^2}-\frac{Y\,\mathrm{d}(Y)}{S^2+Z^2+X^2+Y^2}$$

The "Space-S" components are:

```
> Hhh1:=wcollect(subs(ch^2=1,factor(wcollect(Hcartan[4,1]))));
>
```

$$Hhh1:=\frac{(-Y^2\,X-S^2\,X-X\,Z^2-X^3)\,\mathrm{d}(S)}{(S^2+Z^2+X^2+Y^2)^2}+\frac{(ch\,X^2\,Y+Y\,ch\,S^2+Y\,ch\,Z^2+Y^3\,ch)\,\mathrm{d}(Z)}{(S^2+Z^2+X^2+Y^2)^2}+\frac{(Y^2\,S+S\,Z^2+S^3+S\,X^2)\,\mathrm{d}(X)}{(S^2+Z^2+X^2+Y^2)^2}+\frac{(-ch\,Z\,S^2-ch\,Z^3-ch\,X^2\,Z-Y^2\,ch\,Z)\,\mathrm{d}(Y)}{(S^2+Z^2+X^2+Y^2)^2}$$

```
> Hgg1:=wcollect(subs(ch^2=1,factor(wcollect(factor(wcollect(Hcartan[1,4]))))));
```

$$Hgg1:=-\frac{(-Y^2\,X-S^2\,X-X\,Z^2-X^3)\,\mathrm{d}(S)}{(S^2+Z^2+X^2+Y^2)^2}-\frac{(ch\,X^2\,Y+Y\,ch\,S^2+Y\,ch\,Z^2+Y^3\,ch)\,\mathrm{d}(Z)}{(S^2+Z^2+X^2+Y^2)^2}-\frac{(Y^2\,S+S\,Z^2+S^3+S\,X^2)\,\mathrm{d}(X)}{(S^2+Z^2+X^2+Y^2)^2}-\frac{(-ch\,Z\,S^2-ch\,Z^3-ch\,X^2\,Z-Y^2\,ch\,Z)\,\mathrm{d}(Y)}{(S^2+Z^2+X^2+Y^2)^2}$$

```
> Hhh2:=wcollect(subs(ch^2=1,factor(wcollect(Hcartan[4,2]))));
```

$$Hhh2:=-\frac{(ch\,X^2\,Y+Y\,ch\,S^2+Y\,ch\,Z^2+Y^3\,ch)\,\mathrm{d}(S)}{(S^2+Z^2+X^2+Y^2)^2}-\frac{(S^2\,X+X\,Z^2+X^3+Y^2\,X)\,\mathrm{d}(Z)}{(S^2+Z^2+X^2+Y^2)^2}$$

$$-\frac{(-Z X^2 - S^2 Z - Z^3 - Y^2 Z)\, d(X)}{(S^2+Z^2+X^2+Y^2)^2} - \frac{(-ch\, S^3 - ch\, S\, Z^2 - ch\, S\, X^2 - Y^2\, ch\, S)\, d(Y)}{(S^2+Z^2+X^2+Y^2)^2}$$

> **Hgg2:=wcollect(subs(ch^2=1,factor(wcollect(Hcartan[2,4]))));**

$$Hgg2 := \frac{(ch\, X^2\, Y + Y\, ch\, S^2 + Y\, ch\, Z^2 + Y^3\, ch)\, d(S)}{(S^2+Z^2+X^2+Y^2)^2} + \frac{(S^2 X + X Z^2 + X^3 + Y^2 X)\, d(Z)}{(S^2+Z^2+X^2+Y^2)^2} + \frac{(-Z X^2 - S^2 Z - Z^3 - Y^2 Z)\, d(X)}{(S^2+Z^2+X^2+Y^2)^2} + \frac{(-ch\, S^3 - ch\, S\, Z^2 - ch\, S\, X^2 - Y^2\, ch\, S)\, d(Y)}{(S^2+Z^2+X^2+Y^2)^2}$$

> **Hhh3:=wcollect(subs(ch^2=1,factor(wcollect(Hcartan[4,3]))));**

$$Hhh3 := \frac{ch\, Z\, d(S)}{S^2+Z^2+X^2+Y^2} - \frac{ch\, S\, d(Z)}{S^2+Z^2+X^2+Y^2} + \frac{Y\, d(X)}{S^2+Z^2+X^2+Y^2} - \frac{d(Y)\, X}{S^2+Z^2+X^2+Y^2}$$

> **Hgg3:=wcollect(subs(ch^2=1,factor(wcollect(Hcartan[3,4]))));**

$$Hgg3 := -\frac{ch\, Z\, d(S)}{S^2+Z^2+X^2+Y^2} + \frac{ch\, S\, d(Z)}{S^2+Z^2+X^2+Y^2} - \frac{Y\, d(X)}{S^2+Z^2+X^2+Y^2} + \frac{d(Y)\, X}{S^2+Z^2+X^2+Y^2}$$

> **HOmega:=factor(wcollect(subs(ch^2=1,factor(simplify(wcollect(Hcartan[4,4]))))));**

$$HOmega := -\frac{S\, d(S) + Z\, d(Z) + X\, d(X) + Y\, d(Y)}{S^2+Z^2+X^2+Y^2}$$

Note that the Big Omega term is a perfect differential, and is zero only when the arguement R4 is a constant. That is - off the sphere R4 = constant the Omega term does not vanish. Hence if the radius of the two sphere is expanding, then R4 is not constant and one has a dilatation. (The source of dilatons?)

There are in general two sets of torsion two forms.
The affine two forms, big Sigma, which depend upon the product of little omega and the connection components, little gamma.
The second set of torsion 2-forms is related to Big Omega and the connection components, little gamma.

See http://www.uh.edu/~rkiehn/pdf/defects2.pdf

AFFINE TORSION 2-forms   Big Sigma

> **HSigma1:=(wcollect(factor(Homega&^Hgg1)));HSigma2:=wcollect(factor(Homega&^Hgg2));HSigma3:=Homega&^Hgg3;**

$$HSigma1 := -\frac{(ch^2\, S\, Z - ch\, X\, Y)\,(d(Y)\ \&\wedge\ d(Z))}{(S^2+Z^2+X^2+Y^2)^{(3/2)}} - \frac{(-ch^2\, Z^2 + X^2)\,(d(Y)\ \&\wedge\ d(S))}{(S^2+Z^2+X^2+Y^2)^{(3/2)}} - \frac{(ch^2\, Z\, Y - ch\, S\, X)\,(d(Z)\ \&\wedge\ d(S))}{(S^2+Z^2+X^2+Y^2)^{(3/2)}} - \frac{(-Y\, ch\, Z + S\, X)\,(d(X)\ \&\wedge\ d(Y))}{(S^2+Z^2+X^2+Y^2)^{(3/2)}} - \frac{(Y^2\, ch - ch\, S^2)\,(d(X)\ \&\wedge\ d(Z))}{(S^2+Z^2+X^2+Y^2)^{(3/2)}} - \frac{(Z\, ch\, S - Y\, X)\,(d(X)\ \&\wedge\ d(S))}{(S^2+Z^2+X^2+Y^2)^{(3/2)}}$$

$$HSigma2 := \frac{(ch^2\, S^2 - X^2)\,(d(Y)\ \&\wedge\ d(Z))}{(S^2+Z^2+X^2+Y^2)^{(3/2)}} + \frac{(-ch^2\, S\, Z - ch\, X\, Y)\,(d(Y)\ \&\wedge\ d(S))}{(S^2+Z^2+X^2+Y^2)^{(3/2)}} + \frac{(ch\, X\, Z + ch^2\, S\, Y)\,(d(Z)\ \&\wedge\ d(S))}{(S^2+Z^2+X^2+Y^2)^{(3/2)}} + \frac{(-Y\, ch\, S - Z\, X)\,(d(X)\ \&\wedge\ d(Y))}{(S^2+Z^2+X^2+Y^2)^{(3/2)}} + \frac{(Z\, ch\, S + Y\, X)\,(d(X)\ \&\wedge\ d(Z))}{(S^2+Z^2+X^2+Y^2)^{(3/2)}} + \frac{(-Z^2\, ch + Y^2\, ch)\,(d(X)\ \&\wedge\ d(S))}{(S^2+Z^2+X^2+Y^2)^{(3/2)}}$$

$HSigma3 :=$

$$2\frac{Y\,ch\,S\,(d(X)\,\&\wedge\,d(Z))}{(S^2+Z^2+X^2+Y^2)^{(3/2)}}-2\frac{Y\,ch\,Z\,(d(X)\,\&\wedge\,d(S))}{(S^2+Z^2+X^2+Y^2)^{(3/2)}}-2\frac{S\,ch\,X\,(d(Y)\,\&\wedge\,d(Z))}{(S^2+Z^2+X^2+Y^2)^{(3/2)}}+2\frac{ch\,Z\,X\,(d(Y)\,\&\wedge\,d(S))}{(S^2+Z^2+X^2+Y^2)^{(3/2)}}$$

\>

> HPhi1:=wcollect(factor(HOmega&^Hgg1));HPhi2:=wcollect(factor(HOmega&^Hgg2));HPhi3:=wcollect(factor(HOmega&^Hgg3));

$$HPhi1 := \frac{(ch-1)\,(ch+1)\,(-ch\,Z^2\,X^2+ch\,Y^2\,S^2)\,(d(Y)\,\&\wedge\,d(Z))}{(S^2+Z^2+X^2+Y^2)^3}$$
$$+\frac{(ch-1)\,(ch+1)\,(-Z\,ch\,S\,X^2-Z\,ch\,S\,Y^2-Y\,X\,S^2-Y\,X\,Z^2)\,(d(Y)\,\&\wedge\,d(S))}{(S^2+Z^2+X^2+Y^2)^3}$$
$$+\frac{(ch-1)\,(ch+1)\,(X^2\,Y\,ch\,S+X^3\,Z+Y^3\,ch\,S+Y^2\,Z\,X)\,(d(Z)\,\&\wedge\,d(S))}{(S^2+Z^2+X^2+Y^2)^3}$$
$$+\frac{(ch-1)\,(ch+1)\,(-S^2\,ch\,X\,Z-S^3\,Y-Z^3\,ch\,X-Z^2\,S\,Y)\,(d(X)\,\&\wedge\,d(Y))}{(S^2+Z^2+X^2+Y^2)^3}$$
$$+\frac{(ch-1)\,(ch+1)\,(ch\,X\,Y\,S^2+ch\,X\,Y\,Z^2+Z\,S\,X^2+Z\,S\,Y^2)\,(d(X)\,\&\wedge\,d(Z))}{(S^2+Z^2+X^2+Y^2)^3}$$
$$+\frac{(ch-1)\,(ch+1)\,(-Z^2\,X^2+Y^2\,S^2)\,(d(X)\,\&\wedge\,d(S))}{(S^2+Z^2+X^2+Y^2)^3}$$

$$HPhi2 := \frac{(ch-1)\,(ch+1)\,(Z\,ch\,S\,X^2+Z\,ch\,S\,Y^2-Y\,X\,S^2-Y\,X\,Z^2)\,(d(Y)\,\&\wedge\,d(Z))}{(S^2+Z^2+X^2+Y^2)^3}$$
$$+\frac{(ch-1)\,(ch+1)\,(ch\,S^2\,X^2-ch\,Z^2\,Y^2)\,(d(Y)\,\&\wedge\,d(S))}{(S^2+Z^2+X^2+Y^2)^3}$$
$$+\frac{(ch-1)\,(ch+1)\,(-X^3\,S+X^2\,Y\,ch\,Z-Y^2\,S\,X+Y^3\,ch\,Z)\,(d(Z)\,\&\wedge\,d(S))}{(S^2+Z^2+X^2+Y^2)^3}$$
$$+\frac{(ch-1)\,(ch+1)\,(-Z^3\,Y+S^3\,ch\,X-S^2\,Y\,Z+Z^2\,ch\,S\,X)\,(d(X)\,\&\wedge\,d(Y))}{(S^2+Z^2+X^2+Y^2)^3}$$
$$+\frac{(ch-1)\,(ch+1)\,(-S^2\,X^2+Z^2\,Y^2)\,(d(X)\,\&\wedge\,d(Z))}{(S^2+Z^2+X^2+Y^2)^3}$$
$$+\frac{(ch-1)\,(ch+1)\,(Z\,S\,X^2+Z\,S\,Y^2-ch\,X\,Y\,S^2-ch\,X\,Y\,Z^2)\,(d(X)\,\&\wedge\,d(S))}{(S^2+Z^2+X^2+Y^2)^3}$$

$$HPhi3 := \frac{(ch-1)\,(ch+1)\,(-X^3\,Z-Y^2\,Z\,X-Y\,ch\,S^3-Y\,ch\,S\,Z^2)\,(d(Y)\,\&\wedge\,d(Z))}{(S^2+Z^2+X^2+Y^2)^3}$$
$$+\frac{(ch-1)\,(ch+1)\,(-Y^2\,S\,X-X^3\,S+Y\,ch\,Z\,S^2+Y\,ch\,Z^3)\,(d(Y)\,\&\wedge\,d(S))}{(S^2+Z^2+X^2+Y^2)^3}$$
$$+\frac{(ch-1)\,(ch+1)\,(-ch\,S^2\,X^2-ch\,Z^2\,X^2-ch\,Y^2\,S^2-ch\,Z^2\,Y^2)\,(d(Z)\,\&\wedge\,d(S))}{(S^2+Z^2+X^2+Y^2)^3}$$
$$+\frac{(ch-1)\,(ch+1)\,(-S^2\,X^2-Z^2\,X^2-Y^2\,S^2-Z^2\,Y^2)\,(d(X)\,\&\wedge\,d(Y))}{(S^2+Z^2+X^2+Y^2)^3}$$
$$+\frac{(ch-1)\,(ch+1)\,(-S^3\,ch\,X-Z^2\,ch\,S\,X+Y\,Z\,X^2+Y^3\,Z)\,(d(X)\,\&\wedge\,d(Z))}{(S^2+Z^2+X^2+Y^2)^3}$$

$$+\frac{(ch-1)\,(ch+1)\,(S\,Y\,X^2+S\,Y^3+S^2\,ch\,X\,Z+Z^3\,ch\,X)\,(\mathrm{d}(X)\,\&\wedge\,\mathrm{d}(S))}{(S^2+Z^2+X^2+Y^2)^3}$$

```
> factor(HPhi1&^HSigma1);factor(HPhi2&^HSigma2); factor(HPhi3&^HSigma3);
```

$$0$$
$$0$$
$$0$$

```
#
Theta:=([Hgg1&^Hhh1,Hgg1&^Hhh2,Hgg1&^Hhh3],[Hgg2&^Hhh1,Hgg2&^Hhh2,Hgg2&^Hhh3],[Hgg3&^Hhh1,Hgg3&^Hhh2,Hgg3&^Hhh3]):Theta11:=(wcollect(factor(Theta[1,1])));Theta12:=(wcollect(factor(Theta[1,2])));Theta13:=(wcollect(factor(Theta[1,3])));Theta21:=(wcollect(factor(Theta[2,1])));Theta22:=(wcollect(factor(Theta[2,2])));Theta23:=(wcollect(factor(Theta[2,3])));Theta31:=(wcollect(factor(Theta[3,1])));Theta32:=(wcollect(factor(Theta[3,2])));Theta33:=(wcollect(factor(Theta[3,3])));
```

The interior Curvature 2-forms are:

$$\Theta11 := 0$$

$$\Theta12 := \frac{(ch^2\,S\,Y-ch\,X\,Z)\,(\mathrm{d}(Y)\,\&\wedge\,\mathrm{d}(Z))}{(S^2+Z^2+X^2+Y^2)^2}+\frac{(-ch\,S\,X-ch^2\,Z\,Y)\,(\mathrm{d}(Y)\,\&\wedge\,\mathrm{d}(S))}{(S^2+Z^2+X^2+Y^2)^2}+\frac{(X^2+Y^2\,ch^2)\,(\mathrm{d}(Z)\,\&\wedge\,\mathrm{d}(S))}{(S^2+Z^2+X^2+Y^2)^2}$$
$$+\frac{(-ch\,S^2-Z^2\,ch)\,(\mathrm{d}(X)\,\&\wedge\,\mathrm{d}(Y))}{(S^2+Z^2+X^2+Y^2)^2}+\frac{(Y\,ch\,Z+S\,X)\,(\mathrm{d}(X)\,\&\wedge\,\mathrm{d}(Z))}{(S^2+Z^2+X^2+Y^2)^2}+\frac{(Y\,ch\,S-Z\,X)\,(\mathrm{d}(X)\,\&\wedge\,\mathrm{d}(S))}{(S^2+Z^2+X^2+Y^2)^2}$$

$$\Theta13 := \frac{(-ch^2\,S\,Z-ch\,X\,Y)\,(\mathrm{d}(Y)\,\&\wedge\,\mathrm{d}(Z))}{(S^2+Z^2+X^2+Y^2)^2}+\frac{(X^2+ch^2\,Z^2)\,(\mathrm{d}(Y)\,\&\wedge\,\mathrm{d}(S))}{(S^2+Z^2+X^2+Y^2)^2}+\frac{(ch\,S\,X-ch^2\,Z\,Y)\,(\mathrm{d}(Z)\,\&\wedge\,\mathrm{d}(S))}{(S^2+Z^2+X^2+Y^2)^2}$$
$$+\frac{(-Y\,ch\,Z+S\,X)\,(\mathrm{d}(X)\,\&\wedge\,\mathrm{d}(Y))}{(S^2+Z^2+X^2+Y^2)^2}+\frac{(ch\,S^2+Y^2\,ch)\,(\mathrm{d}(X)\,\&\wedge\,\mathrm{d}(Z))}{(S^2+Z^2+X^2+Y^2)^2}+\frac{(-Z\,ch\,S-Y\,X)\,(\mathrm{d}(X)\,\&\wedge\,\mathrm{d}(S))}{(S^2+Z^2+X^2+Y^2)^2}$$

$$\Theta21 := -\frac{(ch^2\,S\,Y-ch\,X\,Z)\,(\mathrm{d}(Y)\,\&\wedge\,\mathrm{d}(Z))}{(S^2+Z^2+X^2+Y^2)^2}-\frac{(-ch\,S\,X-ch^2\,Z\,Y)\,(\mathrm{d}(Y)\,\&\wedge\,\mathrm{d}(S))}{(S^2+Z^2+X^2+Y^2)^2}$$
$$-\frac{(X^2+Y^2\,ch^2)\,(\mathrm{d}(Z)\,\&\wedge\,\mathrm{d}(S))}{(S^2+Z^2+X^2+Y^2)^2}-\frac{(-ch\,S^2-Z^2\,ch)\,(\mathrm{d}(X)\,\&\wedge\,\mathrm{d}(Y))}{(S^2+Z^2+X^2+Y^2)^2}-\frac{(Y\,ch\,Z+S\,X)\,(\mathrm{d}(X)\,\&\wedge\,\mathrm{d}(Z))}{(S^2+Z^2+X^2+Y^2)^2}$$
$$-\frac{(Y\,ch\,S-Z\,X)\,(\mathrm{d}(X)\,\&\wedge\,\mathrm{d}(S))}{(S^2+Z^2+X^2+Y^2)^2}$$

$$\Theta22 := 0$$

$$\Theta23 := -\frac{(-X^2-ch^2\,S^2)\,(\mathrm{d}(Y)\,\&\wedge\,\mathrm{d}(Z))}{(S^2+Z^2+X^2+Y^2)^2}-\frac{(ch^2\,S\,Z-ch\,X\,Y)\,(\mathrm{d}(Y)\,\&\wedge\,\mathrm{d}(S))}{(S^2+Z^2+X^2+Y^2)^2}$$
$$-\frac{(-ch\,X\,Z-ch^2\,S\,Y)\,(\mathrm{d}(Z)\,\&\wedge\,\mathrm{d}(S))}{(S^2+Z^2+X^2+Y^2)^2}-\frac{(-Y\,ch\,S-Z\,X)\,(\mathrm{d}(X)\,\&\wedge\,\mathrm{d}(Y))}{(S^2+Z^2+X^2+Y^2)^2}-\frac{(-Z\,ch\,S+Y\,X)\,(\mathrm{d}(X)\,\&\wedge\,\mathrm{d}(Z))}{(S^2+Z^2+X^2+Y^2)^2}$$
$$-\frac{(Y^2\,ch+Z^2\,ch)\,(\mathrm{d}(X)\,\&\wedge\,\mathrm{d}(S))}{(S^2+Z^2+X^2+Y^2)^2}$$

$$\Theta31 := -\frac{(-ch^2\,S\,Z-ch\,X\,Y)\,(\mathrm{d}(Y)\,\&\wedge\,\mathrm{d}(Z))}{(S^2+Z^2+X^2+Y^2)^2}-\frac{(X^2+ch^2\,Z^2)\,(\mathrm{d}(Y)\,\&\wedge\,\mathrm{d}(S))}{(S^2+Z^2+X^2+Y^2)^2}$$
$$-\frac{(ch\,S\,X-ch^2\,Z\,Y)\,(\mathrm{d}(Z)\,\&\wedge\,\mathrm{d}(S))}{(S^2+Z^2+X^2+Y^2)^2}-\frac{(-Y\,ch\,Z+S\,X)\,(\mathrm{d}(X)\,\&\wedge\,\mathrm{d}(Y))}{(S^2+Z^2+X^2+Y^2)^2}-\frac{(ch\,S^2+Y^2\,ch)\,(\mathrm{d}(X)\,\&\wedge\,\mathrm{d}(Z))}{(S^2+Z^2+X^2+Y^2)^2}$$
$$-\frac{(-Z\,ch\,S-Y\,X)\,(\mathrm{d}(X)\,\&\wedge\,\mathrm{d}(S))}{(S^2+Z^2+X^2+Y^2)^2}$$

$$\Theta 32 := \frac{(-X^2 - ch^2 S^2)\,(d(Y)\ \&\wedge\ d(Z))}{(S^2+Z^2+X^2+Y^2)^2} + \frac{(ch^2 S Z - ch\, X\, Y)\,(d(Y)\ \&\wedge\ d(S))}{(S^2+Z^2+X^2+Y^2)^2} + \frac{(-ch\, X\, Z - ch^2 S\, Y)\,(d(Z)\ \&\wedge\ d(S))}{(S^2+Z^2+X^2+Y^2)^2} + \frac{(-Y\, ch\, S - Z\, X)\,(d(X)\ \&\wedge\ d(Y))}{(S^2+Z^2+X^2+Y^2)^2} + \frac{(-Z\, ch\, S + Y\, X)\,(d(X)\ \&\wedge\ d(Z))}{(S^2+Z^2+X^2+Y^2)^2} + \frac{(Y^2 ch + Z^2 ch)\,(d(X)\ \&\wedge\ d(S))}{(S^2+Z^2+X^2+Y^2)^2}$$

$$\Theta 33 := 0$$

\>
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NOW RESTART FOR THE INSTANTON CASE

$$r2 := x^2 + y^2 + z^2$$

$$dR := [\, d(X),\ d(Y),\ d(Z),\ d(S)\,]$$

\>

These fields, to within a factor, lead to an assignment of a global frame field inverse matrix with components prportional to :

```
> E3:=[-Y,+X,S,-Z];E2:=[-Z,-S,X,Y];E1:=[S,-Z,Y,-X];N1:=[X,Y,Z,S];
```

$$E3 := [-Y,\ X,\ S,\ -Z]$$

$$E2 := [-Z,\ -S,\ X,\ Y]$$

$$E1 := [S,\ -Z,\ Y,\ -X]$$

$$N1 := [X,\ Y,\ Z,\ S]$$

Arbitrarily , and for algebraic simplification, each direction field will be divided by the factor X^2+Y^2+Z^2+S^2

```
> FFINV:=evalm(subs(ch^2=1,(array([E1,E2,E3,N1])/(X^2+Y^2+Z^2+S^2)^(1/2)))):
```

```
> FF:=simplify(evalm(subs(ch^2=1,inverse(FFINV))));DET:=factor(subs(ch^2=1,det(FF)));Gun:=subs(ch^2=1,innerprod(transpose(FF),FF));INV:=subs(ch^2=1,(innerprod(FF,FF)));
```

$$FF := \begin{bmatrix} \frac{S}{\sqrt{S^2+Z^2+X^2+Y^2}} & -\frac{Z}{\sqrt{S^2+Z^2+X^2+Y^2}} & -\frac{Y}{\sqrt{S^2+Z^2+X^2+Y^2}} & \frac{X}{\sqrt{S^2+Z^2+X^2+Y^2}} \\ -\frac{Z}{\sqrt{S^2+Z^2+X^2+Y^2}} & -\frac{S}{\sqrt{S^2+Z^2+X^2+Y^2}} & \frac{X}{\sqrt{S^2+Z^2+X^2+Y^2}} & \frac{Y}{\sqrt{S^2+Z^2+X^2+Y^2}} \\ \frac{Y}{\sqrt{S^2+Z^2+X^2+Y^2}} & \frac{X}{\sqrt{S^2+Z^2+X^2+Y^2}} & \frac{S}{\sqrt{S^2+Z^2+X^2+Y^2}} & \frac{Z}{\sqrt{S^2+Z^2+X^2+Y^2}} \\ -\frac{X}{\sqrt{S^2+Z^2+X^2+Y^2}} & \frac{Y}{\sqrt{S^2+Z^2+X^2+Y^2}} & -\frac{Z}{\sqrt{S^2+Z^2+X^2+Y^2}} & \frac{S}{\sqrt{S^2+Z^2+X^2+Y^2}} \end{bmatrix}$$

$$DET := -1$$

$$Gun := \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$INV := \begin{bmatrix} -\frac{X^2+Y^2-S^2-Z^2}{S^2+Z^2+X^2+Y^2} & 0 & -2\frac{SY+ZX}{S^2+Z^2+X^2+Y^2} & -2\frac{-SX+YZ}{S^2+Z^2+X^2+Y^2} \\ 0 & 1 & 0 & 0 \\ 2\frac{-ZX+SY}{S^2+Z^2+X^2+Y^2} & 0 & -\frac{Y^2-X^2-S^2+Z^2}{S^2+Z^2+X^2+Y^2} & 2\frac{ZS+YX}{S^2+Z^2+X^2+Y^2} \\ -2\frac{SX+YZ}{S^2+Z^2+X^2+Y^2} & 0 & 2\frac{YX-ZS}{S^2+Z^2+X^2+Y^2} & \frac{-X^2+Y^2-Z^2+S^2}{S^2+Z^2+X^2+Y^2} \end{bmatrix}$$

Note that the Determinant never vanishes (except at the origin of R4) and is positive definite, an indication of an orientable structure.
The induced metric as defined by the Frame Field is conformal.

```
> subs(ch^2=1,innerprod(FF,FFINV));
```

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Now use the Cartan method to evaluate the stuctural equations

```
> DR:=innerprod(FFINV,[d(X),d(Y),d(Z),d(S)]):sigma1:=wcollect(factor(wcollect(DR[1])));sigma2:=wcollect(factor(wcollect(DR[2])));sigma3:=wcollect(factor(wcollect(DR[3])));omega:=factor(wcollect(DR[4]));hdsigma1:=wcollect(factor(sigma1&^d(sigma1)));hdsigma2:=factor(sigma2&^d(sigma2));hdsigma3:=factor(sigma3&^d(sigma3));domega:=d(omega);
```

The rescaling of all the basis elements by the non-negative number yields a simple algebra:

$$\sigma 1 := \frac{S\,d(X)}{\sqrt{S^2+Z^2+X^2+Y^2}} - \frac{Z\,d(Y)}{\sqrt{S^2+Z^2+X^2+Y^2}} + \frac{Y\,d(Z)}{\sqrt{S^2+Z^2+X^2+Y^2}} - \frac{X\,d(S)}{\sqrt{S^2+Z^2+X^2+Y^2}}$$

$$\sigma 2 := -\frac{Z\,d(X)}{\sqrt{S^2+Z^2+X^2+Y^2}} - \frac{S\,d(Y)}{\sqrt{S^2+Z^2+X^2+Y^2}} + \frac{X\,d(Z)}{\sqrt{S^2+Z^2+X^2+Y^2}} + \frac{Y\,d(S)}{\sqrt{S^2+Z^2+X^2+Y^2}}$$

$$\sigma 3 := -\frac{Y\,d(X)}{\sqrt{S^2+Z^2+X^2+Y^2}} + \frac{X\,d(Y)}{\sqrt{S^2+Z^2+X^2+Y^2}} + \frac{S\,d(Z)}{\sqrt{S^2+Z^2+X^2+Y^2}} - \frac{Z\,d(S)}{\sqrt{S^2+Z^2+X^2+Y^2}}$$

$$\omega := \frac{X\,d(X)+Y\,d(Y)+Z\,d(Z)+S\,d(S)}{\sqrt{S^2+Z^2+X^2+Y^2}}$$

$hdsigma1 :=$

$$2\frac{S\,\&\hat{}(d(X),d(Y),d(Z))}{S^2+Z^2+X^2+Y^2} + 2\frac{Y\,\&\hat{}(d(X),d(Z),d(S))}{S^2+Z^2+X^2+Y^2} - 2\frac{Z\,\&\hat{}(d(X),d(Y),d(S))}{S^2+Z^2+X^2+Y^2} - 2\frac{X\,\&\hat{}(d(Y),d(Z),d(S))}{S^2+Z^2+X^2+Y^2}$$

$hdsigma2 :=$

$$2\frac{S\,\&\hat{}(d(X),d(Y),d(Z)) + Y\,\&\hat{}(d(X),d(Z),d(S)) - Z\,\&\hat{}(d(X),d(Y),d(S)) - X\,\&\hat{}(d(Y),d(Z),d(S))}{S^2+Z^2+X^2+Y^2}$$

$hdsigma3 :=$

$$2\frac{S\,\&\hat{}(d(X),d(Y),d(Z)) + Y\,\&\hat{}(d(X),d(Z),d(S)) - Z\,\&\hat{}(d(X),d(Y),d(S)) - X\,\&\hat{}(d(Y),d(Z),d(S))}{S^2+Z^2+X^2+Y^2}$$

$$domega := 0$$

The forms sigma (1,2,3) and omega are the 1-forms relative to the basis frame FF
In the example, the 1-forms sigma1 sigma2 and sigma3 are exact differentials equal to the differentials of the mapping functions defined by the Hopf map giving x,y,z in terms of X,Y,Z,T
The in-exact 1-form of Action is proportional to the 1-form small omega.
For a parametrized surface, small omega vanishes, and such subspaces have Zero Torsion of the Affine type.
From the Frame use the standard methods to compute the Cartan Matrix of connection 1-forms.

See http://www.uh.edu/~rkiehn/pdf/defects2.pdf
for details of the Cartan method for an arbitrary Repere Mobile.

```
> dFF:=array([[d(FF[1,1]),d(FF[1,2]),d(FF[1,3]),d(FF[1,4])],[d(FF[2,1]),d(FF[2,2]),d(FF[2,3]),d(FF[2,4])],[d(FF[3,1]),d(FF[3,2]),d(FF[3,3]),
```

d(FF[3,4])],[d(FF[4,1]),d(FF[4,2]),d(FF[4,3]),d(FF[4,4])]]):

> cartan:=(evalm(FFINV&*dFF)):

Evaluate each component of the connection coefficients on transverse space of E1,E2,E3.

> Gamma11:=factor(wcollect(cartan[1,1]));

$$\Gamma 11 := 0$$

> Gamma12:=factor(wcollect(cartan[1,2]));

$$\Gamma 12 := \frac{Y\,\mathrm{d}(X) - X\,\mathrm{d}(Y) - S\,\mathrm{d}(Z) + Z\,\mathrm{d}(S)}{S^2 + Z^2 + X^2 + Y^2}$$

> Gamma13:=factor(wcollect(cartan[1,3]));

$$\Gamma 13 := \frac{-Z\,\mathrm{d}(X) - S\,\mathrm{d}(Y) + X\,\mathrm{d}(Z) + \mathrm{d}(S)\,Y}{S^2 + Z^2 + X^2 + Y^2}$$

> Gamma21:=factor(wcollect(cartan[2,1]));

$$\Gamma 21 := -\frac{Y\,\mathrm{d}(X) - X\,\mathrm{d}(Y) - S\,\mathrm{d}(Z) + Z\,\mathrm{d}(S)}{S^2 + Z^2 + X^2 + Y^2}$$

> Gamma22:=factor(wcollect(cartan[2,2]));

$$\Gamma 22 := 0$$

> Gamma23:=factor(wcollect(cartan[2,3]));

$$\Gamma 23 := -\frac{S\,\mathrm{d}(X) - \mathrm{d}(Y)\,Z + Y\,\mathrm{d}(Z) - X\,\mathrm{d}(S)}{S^2 + Z^2 + X^2 + Y^2}$$

> Gamma31:=factor(wcollect(cartan[3,1]));

$$\Gamma 31 := -\frac{-Z\,\mathrm{d}(X) - S\,\mathrm{d}(Y) + X\,\mathrm{d}(Z) + \mathrm{d}(S)\,Y}{S^2 + Z^2 + X^2 + Y^2}$$

> Gamma32:=factor(wcollect(cartan[3,2]));

$$\Gamma 32 := \frac{S\,\mathrm{d}(X) - \mathrm{d}(Y)\,Z + Y\,\mathrm{d}(Z) - X\,\mathrm{d}(S)}{S^2 + Z^2 + X^2 + Y^2}$$

> Gamma33:=factor(wcollect(cartan[3,3]));

$$\Gamma 33 := 0$$

The "Space-S" components are:

> hh1:=factor(wcollect(cartan[4,1]));

>

$$hh1 := -\frac{S\,\mathrm{d}(X) - \mathrm{d}(Y)\,Z + Y\,\mathrm{d}(Z) - X\,\mathrm{d}(S)}{S^2 + Z^2 + X^2 + Y^2}$$

> gg1:=factor(wcollect(factor(wcollect(cartan[1,4]))));

$$gg1 := \frac{S\,\mathrm{d}(X) - \mathrm{d}(Y)\,Z + Y\,\mathrm{d}(Z) - X\,\mathrm{d}(S)}{S^2 + Z^2 + X^2 + Y^2}$$

> hh2:=factor(wcollect(cartan[4,2]));

$$hh2 := -\frac{-Z\,\mathrm{d}(X) - S\,\mathrm{d}(Y) + X\,\mathrm{d}(Z) + \mathrm{d}(S)\,Y}{S^2 + Z^2 + X^2 + Y^2}$$

> gg2:=factor(wcollect(cartan[2,4]));

$$gg2 := \frac{-Z\,\mathrm{d}(X) - S\,\mathrm{d}(Y) + X\,\mathrm{d}(Z) + \mathrm{d}(S)\,Y}{S^2 + Z^2 + X^2 + Y^2}$$

> hh3:=factor(wcollect(cartan[4,3]));

$$hh3 := \frac{Y\,\mathrm{d}(X) - X\,\mathrm{d}(Y) - S\,\mathrm{d}(Z) + Z\,\mathrm{d}(S)}{S^2 + Z^2 + X^2 + Y^2}$$

> gg3:=factor(wcollect(cartan[3,4]));

$$gg3 := -\frac{Y\,\mathrm{d}(X) - X\,\mathrm{d}(Y) - S\,\mathrm{d}(Z) + Z\,\mathrm{d}(S)}{S^2 + Z^2 + X^2 + Y^2}$$

> Omega:=factor(simplify(wcollect(cartan[4,4])));

$$\Omega := 0$$

Note that the Big Omega term is a perfect differential, and is zero only when the arguement R4 is a constant. That is - off the sphere R4 = constant the Omega term does not vanish. Hence if the radius of the two sphere is expanding, then R4 is not constant and one has a dilatation. (The source of dilatons?)

There are in general two sets of torsion two forms.
The affine two forms, big Sigma, which depend upon the product of little omega and the connection components, little gamma.
The second set of torsion 2-forms is related to Big Omega and the connection components, little gamma.

See http://www.uh.edu/~rkiehn/pdf/defects2.pdf

AFFINE TORSION 2-forms   Big Sigma

```
> Sigma1:=(wcollect(factor(omega&^gg1)));
  ds3:=wcollect(factor(d(sigma3)));Sigma2:=wcollect(factor(omega&^gg2));Sigma3:=omega&^gg3;
```

$$\Sigma 1 := \frac{(Z^2+Y^2)\,(\mathrm{d}(Y)\ \&\wedge\ \mathrm{d}(Z))}{(S^2+Z^2+X^2+Y^2)^{(3/2)}} + \frac{(Z\,S-Y\,X)\,(\mathrm{d}(Y)\ \&\wedge\ \mathrm{d}(S))}{(S^2+Z^2+X^2+Y^2)^{(3/2)}} + \frac{(-S\,Y-Z\,X)\,(\mathrm{d}(Z)\ \&\wedge\ \mathrm{d}(S))}{(S^2+Z^2+X^2+Y^2)^{(3/2)}} + \frac{(-S\,Y-Z\,X)\,(\mathrm{d}(X)\ \&\wedge\ \mathrm{d}(Y))}{(S^2+Z^2+X^2+Y^2)^{(3/2)}} + \frac{(Y\,X-Z\,S)\,(\mathrm{d}(X)\ \&\wedge\ \mathrm{d}(Z))}{(S^2+Z^2+X^2+Y^2)^{(3/2)}} + \frac{(-S^2-X^2)\,(\mathrm{d}(X)\ \&\wedge\ \mathrm{d}(S))}{(S^2+Z^2+X^2+Y^2)^{(3/2)}}$$

$$ds3 := \frac{(Z\,X-S\,Y)\,(\mathrm{d}(Y)\ \&\wedge\ \mathrm{d}(Z))}{(S^2+Z^2+X^2+Y^2)^{(3/2)}} + \frac{(S\,X+Y\,Z)\,(\mathrm{d}(Y)\ \&\wedge\ \mathrm{d}(S))}{(S^2+Z^2+X^2+Y^2)^{(3/2)}} + \frac{(-2\,X^2-2\,Y^2-Z^2-S^2)\,(\mathrm{d}(Z)\ \&\wedge\ \mathrm{d}(S))}{(S^2+Z^2+X^2+Y^2)^{(3/2)}} + \frac{(2\,S^2+2\,Z^2+Y^2+X^2)\,(\mathrm{d}(X)\ \&\wedge\ \mathrm{d}(Y))}{(S^2+Z^2+X^2+Y^2)^{(3/2)}} + \frac{(-Y\,Z-S\,X)\,(\mathrm{d}(X)\ \&\wedge\ \mathrm{d}(Z))}{(S^2+Z^2+X^2+Y^2)^{(3/2)}} + \frac{(Z\,X-S\,Y)\,(\mathrm{d}(X)\ \&\wedge\ \mathrm{d}(S))}{(S^2+Z^2+X^2+Y^2)^{(3/2)}}$$

$$\Sigma 2 := \frac{(Z\,S+Y\,X)\,(\mathrm{d}(Y)\ \&\wedge\ \mathrm{d}(Z))}{(S^2+Z^2+X^2+Y^2)^{(3/2)}} + \frac{(S^2+Y^2)\,(\mathrm{d}(Y)\ \&\wedge\ \mathrm{d}(S))}{(S^2+Z^2+X^2+Y^2)^{(3/2)}} + \frac{(-S\,X+Y\,Z)\,(\mathrm{d}(Z)\ \&\wedge\ \mathrm{d}(S))}{(S^2+Z^2+X^2+Y^2)^{(3/2)}} + \frac{(-S\,X+Y\,Z)\,(\mathrm{d}(X)\ \&\wedge\ \mathrm{d}(Y))}{(S^2+Z^2+X^2+Y^2)^{(3/2)}} + \frac{(Z^2+X^2)\,(\mathrm{d}(X)\ \&\wedge\ \mathrm{d}(Z))}{(S^2+Z^2+X^2+Y^2)^{(3/2)}} + \frac{(Z\,S+Y\,X)\,(\mathrm{d}(X)\ \&\wedge\ \mathrm{d}(S))}{(S^2+Z^2+X^2+Y^2)^{(3/2)}}$$

$$\Sigma 3 := \left(-\frac{X\,Z}{(S^2+Z^2+X^2+Y^2)^{(3/2)}} + \frac{S\,Y}{(S^2+Z^2+X^2+Y^2)^{(3/2)}}\right)(\mathrm{d}(Y)\ \&\wedge\ \mathrm{d}(Z)) + \left(-\frac{X\,S}{(S^2+Z^2+X^2+Y^2)^{(3/2)}} - \frac{Z\,Y}{(S^2+Z^2+X^2+Y^2)^{(3/2)}}\right)(\mathrm{d}(Y)\ \&\wedge\ \mathrm{d}(S)) + \left(-\frac{S^2}{(S^2+Z^2+X^2+Y^2)^{(3/2)}} - \frac{Z^2}{(S^2+Z^2+X^2+Y^2)^{(3/2)}}\right)(\mathrm{d}(Z)\ \&\wedge\ \mathrm{d}(S)) + \left(\frac{X^2}{(S^2+Z^2+X^2+Y^2)^{(3/2)}} + \frac{Y^2}{(S^2+Z^2+X^2+Y^2)^{(3/2)}}\right)(\mathrm{d}(X)\ \&\wedge\ \mathrm{d}(Y)) + \left(\frac{X\,S}{(S^2+Z^2+X^2+Y^2)^{(3/2)}} + \frac{Z\,Y}{(S^2+Z^2+X^2+Y^2)^{(3/2)}}\right)(\mathrm{d}(X)\ \&\wedge\ \mathrm{d}(Z)) + \left(-\frac{X\,Z}{(S^2+Z^2+X^2+Y^2)^{(3/2)}} + \frac{S\,Y}{(S^2+Z^2+X^2+Y^2)^{(3/2)}}\right)(\mathrm{d}(X)\ \&\wedge\ \mathrm{d}(S))$$

```
>
```

```
> Phi1:=wcollect(factor(Omega&^gg1));Phi2:=wcollect(factor(Omega&^gg2));Phi3:=wcollect(factor(Omega&^gg3));
```

$$\Phi 1 := 0$$

$$\Phi 2 := 0$$

$$\Phi3 := 0$$

> **factor(Phi1&^Sigma1+Phi2&^Sigma2+Phi3&^Sigma3);**

$$0$$

>

>

The two vector two forms are not independent

> **Theta:=([gg1&^hh1,gg1&^hh2,gg1&^hh3],[gg2&^hh1,gg2&^hh2,gg2&^hh3],[gg3&^hh1,gg3&^hh2,gg3&^hh3]);**

>

$$\Theta := \Bigg[0, \left(\frac{S\,X}{(Z^2+S^2+X^2+Y^2)^2}+\frac{Y\,Z}{(Z^2+S^2+X^2+Y^2)^2}\right)(\mathrm{d}(Z)\,\&\wedge\,\mathrm{d}(X))$$
$$+\left(\frac{S\,Y}{(Z^2+S^2+X^2+Y^2)^2}-\frac{Z\,X}{(Z^2+S^2+X^2+Y^2)^2}\right)(\mathrm{d}(S)\,\&\wedge\,\mathrm{d}(X))$$
$$+\left(-\frac{Y\,Z}{(Z^2+S^2+X^2+Y^2)^2}-\frac{S\,X}{(Z^2+S^2+X^2+Y^2)^2}\right)(\mathrm{d}(S)\,\&\wedge\,\mathrm{d}(Y))$$
$$+\left(-\frac{Z^2}{(Z^2+S^2+X^2+Y^2)^2}-\frac{S^2}{(Z^2+S^2+X^2+Y^2)^2}\right)(\mathrm{d}(Y)\,\&\wedge\,\mathrm{d}(X))$$
$$+\left(\frac{Y^2}{(Z^2+S^2+X^2+Y^2)^2}+\frac{X^2}{(Z^2+S^2+X^2+Y^2)^2}\right)(\mathrm{d}(S)\,\&\wedge\,\mathrm{d}(Z))$$
$$+\left(-\frac{S\,Y}{(Z^2+S^2+X^2+Y^2)^2}+\frac{Z\,X}{(Z^2+S^2+X^2+Y^2)^2}\right)(\mathrm{d}(Y)\,\&\wedge\,\mathrm{d}(Z)),$$
$$\left(\frac{S^2}{(Z^2+S^2+X^2+Y^2)^2}+\frac{Y^2}{(Z^2+S^2+X^2+Y^2)^2}\right)(\mathrm{d}(Z)\,\&\wedge\,\mathrm{d}(X))$$
$$+\left(-\frac{S\,Z}{(Z^2+S^2+X^2+Y^2)^2}-\frac{Y\,X}{(Z^2+S^2+X^2+Y^2)^2}\right)(\mathrm{d}(S)\,\&\wedge\,\mathrm{d}(X))$$
$$+\left(\frac{Z^2}{(Z^2+S^2+X^2+Y^2)^2}+\frac{X^2}{(Z^2+S^2+X^2+Y^2)^2}\right)(\mathrm{d}(S)\,\&\wedge\,\mathrm{d}(Y))$$
$$+\left(-\frac{Y\,Z}{(Z^2+S^2+X^2+Y^2)^2}+\frac{S\,X}{(Z^2+S^2+X^2+Y^2)^2}\right)(\mathrm{d}(Y)\,\&\wedge\,\mathrm{d}(X))$$
$$+\left(-\frac{Y\,Z}{(Z^2+S^2+X^2+Y^2)^2}+\frac{S\,X}{(Z^2+S^2+X^2+Y^2)^2}\right)(\mathrm{d}(S)\,\&\wedge\,\mathrm{d}(Z))$$

$$+\left(\frac{Y X}{(Z^2+S^2+X^2+Y^2)^2}+\frac{S Z}{(Z^2+S^2+X^2+Y^2)^2}\right)(d(Y) \&\wedge d(Z))\Big]\Big],\Big[\Big[$$

$$\left(-\frac{Y Z}{(Z^2+S^2+X^2+Y^2)^2}-\frac{S X}{(Z^2+S^2+X^2+Y^2)^2}\right)(d(Z) \&\wedge d(X))$$

$$+\left(-\frac{S Y}{(Z^2+S^2+X^2+Y^2)^2}+\frac{Z X}{(Z^2+S^2+X^2+Y^2)^2}\right)(d(S) \&\wedge d(X))$$

$$+\left(\frac{S X}{(Z^2+S^2+X^2+Y^2)^2}+\frac{Y Z}{(Z^2+S^2+X^2+Y^2)^2}\right)(d(S) \&\wedge d(Y))$$

$$+\left(\frac{S^2}{(Z^2+S^2+X^2+Y^2)^2}+\frac{Z^2}{(Z^2+S^2+X^2+Y^2)^2}\right)(d(Y) \&\wedge d(X))$$

$$+\left(-\frac{X^2}{(Z^2+S^2+X^2+Y^2)^2}-\frac{Y^2}{(Z^2+S^2+X^2+Y^2)^2}\right)(d(S) \&\wedge d(Z))$$

$$+\left(\frac{S Y}{(Z^2+S^2+X^2+Y^2)^2}-\frac{Z X}{(Z^2+S^2+X^2+Y^2)^2}\right)(d(Y) \&\wedge d(Z)), 0,$$

$$\left(-\frac{S Z}{(Z^2+S^2+X^2+Y^2)^2}+\frac{Y X}{(Z^2+S^2+X^2+Y^2)^2}\right)(d(Z) \&\wedge d(X))$$

$$+\left(\frac{Z^2}{(Z^2+S^2+X^2+Y^2)^2}+\frac{Y^2}{(Z^2+S^2+X^2+Y^2)^2}\right)(d(S) \&\wedge d(X))$$

$$+\left(\frac{S Z}{(Z^2+S^2+X^2+Y^2)^2}-\frac{Y X}{(Z^2+S^2+X^2+Y^2)^2}\right)(d(S) \&\wedge d(Y))$$

$$+\left(-\frac{S Y}{(Z^2+S^2+X^2+Y^2)^2}-\frac{Z X}{(Z^2+S^2+X^2+Y^2)^2}\right)(d(Y) \&\wedge d(X))$$

$$+\left(-\frac{S Y}{(Z^2+S^2+X^2+Y^2)^2}-\frac{Z X}{(Z^2+S^2+X^2+Y^2)^2}\right)(d(S) \&\wedge d(Z))$$

$$+\left(\frac{X^2}{(Z^2+S^2+X^2+Y^2)^2}+\frac{S^2}{(Z^2+S^2+X^2+Y^2)^2}\right)(d(Y) \&\wedge d(Z))\Big]\Big],\Big[\Big[$$

$$\left(-\frac{Y^2}{(Z^2+S^2+X^2+Y^2)^2}-\frac{S^2}{(Z^2+S^2+X^2+Y^2)^2}\right)(d(Z) \&\wedge d(X))$$

$$+\left(\frac{Y X}{(Z^2+S^2+X^2+Y^2)^2}+\frac{S Z}{(Z^2+S^2+X^2+Y^2)^2}\right)(d(S) \&\wedge d(X))$$

$$+\left(-\frac{X^2}{(Z^2+S^2+X^2+Y^2)^2}-\frac{Z^2}{(Z^2+S^2+X^2+Y^2)^2}\right)(d(S) \&\wedge d(Y))$$

$$+\left(-\frac{S X}{(Z^2+S^2+X^2+Y^2)^2}+\frac{Y Z}{(Z^2+S^2+X^2+Y^2)^2}\right)(d(Y) \&\wedge d(X))$$

$$+\left(-\frac{S X}{(Z^2+S^2+X^2+Y^2)^2}+\frac{Y Z}{(Z^2+S^2+X^2+Y^2)^2}\right)(d(S) \&\wedge d(Z))$$

$$+\left(-\frac{S Z}{(Z^2+S^2+X^2+Y^2)^2}-\frac{Y X}{(Z^2+S^2+X^2+Y^2)^2}\right)(d(Y) \&\wedge d(Z)),$$

$$\left(\frac{S Z}{(Z^2+S^2+X^2+Y^2)^2}-\frac{Y X}{(Z^2+S^2+X^2+Y^2)^2}\right)(d(Z) \&\wedge d(X))$$

$$+\left(-\frac{Y^2}{(Z^2+S^2+X^2+Y^2)^2}-\frac{Z^2}{(Z^2+S^2+X^2+Y^2)^2}\right)(d(S) \&\wedge d(X))$$

$$+\left(-\frac{S Z}{(Z^2+S^2+X^2+Y^2)^2}+\frac{Y X}{(Z^2+S^2+X^2+Y^2)^2}\right)(d(S) \&\wedge d(Y))$$

$$+\left(\frac{Z X}{(Z^2+S^2+X^2+Y^2)^2}+\frac{S Y}{(Z^2+S^2+X^2+Y^2)^2}\right)(d(Y) \&\wedge d(X))$$

$$+\left(\frac{Z X}{(Z^2+S^2+X^2+Y^2)^2}+\frac{S Y}{(Z^2+S^2+X^2+Y^2)^2}\right)(d(S) \&\wedge d(Z))$$

$$+\left(-\frac{S^2}{(Z^2+S^2+X^2+Y^2)^2}-\frac{X^2}{(Z^2+S^2+X^2+Y^2)^2}\right)(d(Y) \&\wedge d(Z)), 0\Big]$$

```
> factor(subs(ch=1,HSigma1&^Sigma1));factor(subs(ch=1,HSigma2&^Sigma2));factor(subs(ch=1,HSigma3&^Sigma3));
```

$$0$$

$$0$$

$$0$$

Torsions are proportional!!!

```
>
> wcollect(factor(wcollect((subs(ch=1,HSigma1)+Sigma1))));
```

$$\frac{(Z S-X Y+Y^2-S^2)(d(Z) \&\wedge d(X))}{(Z^2+S^2+X^2+Y^2)^{(3/2)}}+\frac{(X^2-X Y+S^2+Z S)(d(S) \&\wedge d(X))}{(Z^2+S^2+X^2+Y^2)^{(3/2)}}$$

$$+\frac{(X^2+X Y-Z S-Z^2)(d(S) \&\wedge d(Y))}{(Z^2+S^2+X^2+Y^2)^{(3/2)}}+\frac{(-Y Z+S Y+S X+Z X)(d(Y) \&\wedge d(X))}{(Z^2+S^2+X^2+Y^2)^{(3/2)}}$$

$$+\frac{(-S X+Y Z+S Y+Z X)\,(\mathrm{d}(S)\ \&\wedge\ \mathrm{d}(Z))}{(Z^2+S^2+X^2+Y^2)^{(3/2)}}+\frac{(X Y-Z S+Y^2+Z^2)\,(\mathrm{d}(Y)\ \&\wedge\ \mathrm{d}(Z))}{(Z^2+S^2+X^2+Y^2)^{(3/2)}}$$

>

> **factor(wcollect((subs(ch=1,HSigma1)-Sigma1)&^d(S)&^d(Y)));**

$$\frac{(-Z S+X Y+Y^2-S^2)\ \&\wedge(\mathrm{d}(S), \mathrm{d}(Y), \mathrm{d}(Z), \mathrm{d}(X))}{(Z^2+S^2+X^2+Y^2)^{(3/2)}}$$

>