```
> restart: with (linalg):with(liesymm):with(difforms):
> setup(x,y,z,t):defform(x=0,y=0,z=0,t=0,Vx=0,Vy=0,Vz=0,D1=0,D2=0,D3=0,Ax=0,Ay=0,Az=0,C=0,Phi=0,phi=0,theta=0,r=0,a=const,b=const,c=const,aa=const,bb=const,cc=const,Lx=0,Ly=0,Lz=0);
Warning, new definition for norm
Warning, new definition for trace
Warning, new definition for close
Warning, new definition for `&^`
Warning, new definition for d
Warning, new definition for mixpar
Warning, new definition for wdegree
```

CARTAN CONNECTIONS, RICCI ROTATION, AFFINE TORSION, METRIC, CHRISTOFFEL CONNECTIONS, CURVATURE and

CARTAN's EQUATIONS of STRUCTURE

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INTRODUCTION

Examples will be given for matrix Frame fields of functions, [F] , acting algebraically on a column vector of coordinate differentials of an initial domain, R = {r,theta,phi}, to produce a column vector of 1-forms |sigma> on a final range.

|sigma> = [F]|dR>

If the 1-forms |sigma> on the final state are exact, then there exist integrals uniquely defining the coordinates {x,y,z} of the final state in terms of functions of the variables of the initial state. On the restricted region of the initial domain where the columns (contravariant vectors) of the Frame field are linearly independent, then the Frame field can be used as a basis for a vector space. On the restricted region, an inverse matrix of functions [G] can be deduced algebraically. The Frame field is then defined as a Basis Frame. All Basis Frames composed of C1 functions admit, via differential and algebraic processes, a (right) Cartan matrix [C] of connection 1-forms, such that the differential of any column vector of the Frame of basis vectors is a linear combination of all of the column vectors of the Frame.

d[F] = [F][C]

The differential process is said to be closed. The right Cartan matrix of connection 1-forms can be computed explicitly from the matrix formulas

[C] = - [dG][F] = + [G][dF]

Application of the connection rule to the exterior differentials of the column vector of 1-forms |sigma> yields

d|sigma> = [F] { [C] ^ |dR>}

If every Pfaffian component of the column vector of 1-forms, |sigma>, is integrable, the exterior derivative of each component must vanish. Hence the column vector of (affine torsion) 2-forms,
[C] ^ |dR>, must vanish in the integrable case. When |sigma> is not uniquely integrable, the associated right Cartan matrix of connection 1-forms [C] is said to admit Torsion. The lack of integrability implies that there does not exist a unique function of the independent variables on the initial state that maps to the associated coordinate on the final state. There may be more than one such function (!), or none. In certain (lucky) instances, the torsion components can be made to vanish by the use of suitable integrating factors, which in effect redefine the matrix elements of the Frame field [F]. In the most general case, such integrating factors do not exist for domains of dimension > 2. A component 1-form of |sigma> which admits affine torsion but does not admit an integrating factor is said to exhibit topological torsion. It is the irreducible topological torsion situations that are the most interesting.

A metric [metricfinal] will (can) be imposed arbitrarily on the final rang**e**, and its compatible preimage [pullbackmetric] on the initial domain will be computed relative to the linear map, [F].

[pullbackmetric] = [F transpose][metricfinal][F]

A special symmetric Christoffel Connection, { Christoffel }, can be defined on the initial state in terms of algebraic processes and exterior differentiations of the pullbackmetric. If the metric on the final state is a set of constants, then it follows that

d[pullbackmetric] = [C transpose][pullbackmetric] + [pullbackmetric][C]

This condition generates Christoffel (Riemannian) Curvatures on the initial state can also be deduced in terms of another exterior differential process applied to the Christoffel connection.

These concepts will be compared to a Cartan Connection deduced from differential and algebraic processes applied to the Frame matrix, not the metric. The right Cartan matrix of Connection 1-forms is also defined on the initial state, and also can be used to compute curvatures.

In general, the Christoffel Connection is not the same as the Cartan connection. They are equivalent only in special situations.

The Christoffel connection is always free of Affine Torsion, the Cartan connection can support anti-symmetries defined as Affine Torsion.

The Cartan Curvatures, based on the Cartan connection, are not always the same as the Riemann curvatures based upon the Christoffel connection.

It is classic to decompose the Cartan Connection coefficients into the sum of the Christoffel connection coefficients, and another piece usually defined as the Ricci rotation coefficients, [T] .

[C] = { Christoffel } + [T]

As the Christoffel connections are symmetric in the lower two indices, any anti-symmetries of the Cartan connection must be associated with the RIcci coefficients [T].

The Ricci Rotation coefficients [T] can have both a symmetric and an anti-symmetric part. The anti-symmetries are equivalent to the anti-symmetries of the Right Cartan connection, and are defined as the Affine Torsion coefficients [Affine Torsion]. Hence, the Cartan connection is decomposed as:

[C] = { Christoffel } + [Tsym] + [Affine Torsion]

The *matrix* of Affine Torsion connection components is a matrix of 1-forms with certain anti-symmetric coefficients.

The same anti-symmetric set of coefficients can appear in a column *vector* of Affine Torsion 2-forms, which is defined by the matrix exterior product of the Right Cartan connection and the differential position vector on the initial state.

|Affine Torsion 2-forms> = [C]^|dR>

Note that the Ricci coefficients can contain symmetric as well as anti-symmetric parts. The symmetric parts can have contributions which depend upon the existence of the Affine Torsion codfficients, and contributions which are independent from the existence of Affine Torsion coefficients.

The examples will start with the classic map of spherical coordinates mapped into the Cartesian coordiantes {x,y,z}. The Jacobian matrix of partial derivatives of the mapping function will serve as the primitive definition of a Basis Frame, [F]. The Right Cartan connection matrix for any basis frame is defined as [C] = - [dG][F] = + [G][dF] , where [G] is the inverse of [F]. Realize that the Right Cartan matrix is defined entirely on the initial state in terms of initial state independent variables. The curvature and the Affine Torsion of the initial state based upon an integrable set of mapping functions are both zero.

Initially it will be assumed that the metric on the final state is a set of constants equal to the identity matrix. Then a perturbation of this finalstate metric will be defined and its effect on curvatures and torsion will be examined. Then a pertubation will be made on the original Frame Field, such that the perturbed Frame Field will admit Affine Torsion. The effect of the combined perturbations on curvature and Torsion will be computed.

The classic map from spherical to Cartesian Coordinates is given by the expressions:

```
> x:=r*sin(theta)*cos(phi);y:=r*sin(theta)*sin(phi);z:=r*cos(theta);
>
```

$$x := r\sin(\theta)\cos(\phi)$$

$$y := r\sin(\theta)\sin(\phi)$$

$$z := r\cos(\theta)$$

The induced 1-forms are exact differentials and have expressions in terms of the variables of the initial state:

```
> sigmax_or_dx:=d(x);
```

$$\mathit{sigmax\_or\_dx} := \sin(\theta)\cos(\phi)\,d(r) + r\cos(\phi)\cos(\theta)\,d(\theta) - r\sin(\theta)\sin(\phi)\,d(\phi)$$

```
> sigmay_or_dy:=d(y);
```

$$\mathit{sigmay\_or\_dy} := \sin(\theta)\sin(\phi)\,d(r) + r\sin(\phi)\cos(\theta)\,d(\theta) + r\sin(\theta)\cos(\phi)\,d(\phi)$$

```
> sigmaz_or_dz:=d(z);
```

$$\mathit{sigmaz\_or\_dz} := \cos(\theta)\,d(r) - r\sin(\theta)\,d(\theta)$$

The three 1-forms induced by the coordinate mapping can be deduced by multiplying a Frame matrix times a vector of differentials on the initial state. The Frame matrix elements are now computed for the assumed coordinate mapping. The frame matrix is equivalent to the JAcobian matrix of the mapping functions relative to the variables of the initial state. The components are as follows:

```
>
> FF11:=getcoeff(d(x)&^d(theta)&^d(phi));FF12:=getcoeff(d(x)&^d(phi)&^d(r));FF13:=getcoeff(d(x)&^d(r)&^d(theta));
```

$$FF11 := \sin(\theta)\cos(\phi)$$
$$FF12 := r\cos(\phi)\cos(\theta)$$
$$FF13 := -r\sin(\theta)\sin(\phi)$$

```
>
> FF21:=getcoeff(d(y)&^d(theta)&^d(phi));FF22:=getcoeff(d(y)&^d(phi)&^d(r));FF23:=getcoeff(d(y)&^d(r)&^d(theta));
```

$$FF21 := \sin(\theta)\sin(\phi)$$
$$FF22 := r\sin(\phi)\cos(\theta)$$
$$FF23 := r\sin(\theta)\cos(\phi)$$

At this point the Frame matrix will be modified to make the third 1-form sigmaz not exact and not closed. A constant coefficient aa will scale the perturbation of the FRame. The associated Frame matrix elements are then:

```
> sigmaz:=wcollect(d(z)+aa*(y*d(x)-x*d(y)));dzz:=sigmaz;
> FF31:=getcoeff(dzz&^d(theta)&^d(phi));FF32:=getcoeff(dzz&^d(phi)&^d(r));FF33:=simplify(getcoeff(dzz&^d(r)&^d(theta)));
```

$$sigmaz := \cos(\theta)\,d(r) + aa\,(-r^2\sin(\theta)^2\sin(\phi)^2 - r^2\sin(\theta)^2\cos(\phi)^2)\,d(\phi) - r\sin(\theta)\,d(\theta)$$
$$dzz := \cos(\theta)\,d(r) + aa\,(-r^2\sin(\theta)^2\sin(\phi)^2 - r^2\sin(\theta)^2\cos(\phi)^2)\,d(\phi) - r\sin(\theta)\,d(\theta)$$
$$FF31 := \cos(\theta)$$
$$FF32 := -r\sin(\theta)$$
$$FF33 := -aa\,r^2 + aa\,r^2\cos(\theta)^2$$

```
>
```

These matrix elements will be put into a Frame matrix format as:

```
> R:=[x,y,z]:FF:=array([[FF11,FF12,FF13],[FF21,FF22,FF23],[FF31,FF32,FF33]]);
>
```

$$FF := \begin{bmatrix} \sin(\theta)\cos(\phi) & r\cos(\phi)\cos(\theta) & -r\sin(\theta)\sin(\phi) \\ \sin(\theta)\sin(\phi) & r\sin(\phi)\cos(\theta) & r\sin(\theta)\cos(\phi) \\ \cos(\theta) & -r\sin(\theta) & -aa\,r^2 + aa\,r^2\cos(\theta)^2 \end{bmatrix}$$

Setting aa to zero reproduces the unperturbed JAcobian matrix.
The inverse to the Frame is :

```
> GG:=simplify(evalm(inverse(FF)));DETFF:=simplify(det(FF));
```

$GG :=$

$$[-\sin(\theta)\,(r\sin(\phi)\cos(\theta)\,aa - \cos(\phi)),\ \sin(\theta)\,(r\cos(\phi)\cos(\theta)\,aa + \sin(\phi)),\ \cos(\theta)]$$

$$\left[-\frac{-\sin(\phi)\,aa\,r + \sin(\phi)\,aa\,r\cos(\theta)^2 - \cos(\phi)\cos(\theta)}{r},\ \frac{-\cos(\phi)\,aa\,r + \cos(\phi)\,aa\,r\cos(\theta)^2 + \sin(\phi)\cos(\theta)}{r},\ -\frac{\sin(\theta)}{r}\right]$$

$$\left[-\frac{\sin(\phi)}{r\sin(\theta)},\ \frac{\cos(\phi)}{r\sin(\theta)},\ 0\right]$$

$$DETFF := \sin(\theta)\, r^2$$

Note that the Frame matrix has a singularity at values of theta equal to multiples of pi, and at r=0. The determinant of the Frame matrix does not depend upon the perturbation. It will be shown below that the pertubation influences the components of Affine Torsion.

The next equation checks to see that the specified frame produces the desired differential structures: |V>=[FF]|dR>. The 1-forms computed from the inverse Frame are also evaluated as |A> = [GG]|dR>

```
> sigmaf:=simplify(evalm(innerprod(FF,[d(r),d(theta),d(phi)]))):vx_or_sigmax:=sigmaf[1];vy_or_sigmay:=sigmaf[2];vz_or_sigmaz:=sigmaf[3];sigmag:=simplify(evalm(innerprod(GG,[d(r),d(theta),d(phi)]))):ax:=wcollect(sigmag[1]);ay:=wcollect(sigmag[2]);az:=sigmag[3];
>
```

$$vx\_or\_sigmax := \sin(\theta)\cos(\phi)\, d(r) + r\cos(\phi)\cos(\theta)\, d(\theta) - r\sin(\theta)\sin(\phi)\, d(\phi)$$

$$vy\_or\_sigmay := \sin(\theta)\sin(\phi)\, d(r) + r\sin(\phi)\cos(\theta)\, d(\theta) + r\sin(\theta)\cos(\phi)\, d(\phi)$$

$$vz\_or\_sigmaz := \cos(\theta)\, d(r) - r\sin(\theta)\, d(\theta) - d(\phi)\, aa\, r^2 + d(\phi)\, aa\, r^2\cos(\theta)^2$$

$$ax := (\sin(\theta)\cos(\phi) - \sin(\theta)\, r\sin(\phi)\cos(\theta)\, aa)\, d(r) + \cos(\theta)\, d(\phi) + (\sin(\theta)\, r\cos(\phi)\cos(\theta)\, aa + \sin(\theta)\sin(\phi))\, d(\theta)$$

$$ay := -\frac{(-\sin(\phi)\, aa\, r + \sin(\phi)\, aa\, r\cos(\theta)^2 - \cos(\phi)\cos(\theta))\, d(r)}{r} - \frac{\sin(\theta)\, d(\phi)}{r} - \frac{(-\cos(\phi)\, aa\, r\cos(\theta)^2 - \sin(\phi)\cos(\theta) + \cos(\phi)\, aa\, r)\, d(\theta)}{r}$$

$$az := -\frac{\sin(\phi)\, d(r) - \cos(\phi)\, d(\theta)}{r\sin(\theta)}$$

A test is made to see if the 1-forms |V> are closed. For the perturbed Frame example the construction indicates that the third 1-form sigmaz is not closed unless aa = 0. The third 1-form vz indeed exhibits topological torsion, which is a prerequisite to produce AFFINE TORSION, and also indicates that the system of perturbed 1-forms is not integrable.

```
> closure_vx:=d(vx);closure_vy:=d(vy);closure_vz:=wcollect(d(vz));TopTorsion_vz:=vz&^d(vz);
```

$$closure\_vx := d(vx)$$

$$closure\_vy := d(vy)$$

$$closure\_vz := d(vz)$$

$$TopTorsion\_vz := vz\ \&\hat{}\ d(vz)$$

Now the conventional metric on {x,y,z} will be modified to have non - constant coefficients. The metric on the final state induces a metric on the initial state via the congruent mapping [pullbackmetric] = [Frame_transpose][finalmetric][Frame]. For the example, the coefficient bb gives a measure of the perturbation of the Euclidean metric on the final state.

```
> finalmetric := array([[1+bb/r, 0, 0], [0,1+bb/r, 0], [0, 0, 1+bb/r]]);
```

$$finalmetric := \begin{bmatrix} 1+\frac{bb}{r} & 0 & 0 \\ 0 & 1+\frac{bb}{r} & 0 \\ 0 & 0 & 1+\frac{bb}{r} \end{bmatrix}$$

r

```
> pullbackmetric:=simplify((innerprod(transpose(FF),finalmetric,FF)));m11:=simplify(pullbackmetric[1,2]);
```

$$pullbackmetric := \left[\frac{r+bb}{r}, 0, -\cos(\theta)\, aa\, r^2 - \cos(\theta)\, aa\, r\, bb + \cos(\theta)^3\, aa\, r^2 + \cos(\theta)^3\, aa\, r\, bb\right]$$

$$[0, r^2 + r\, bb, \sin(\theta)\, aa\, r^3 + \sin(\theta)\, aa\, r^2\, bb - \sin(\theta)\, aa\, r^3 \cos(\theta)^2 - \sin(\theta)\, aa\, r^2 \cos(\theta)^2\, bb]$$

$$[-\cos(\theta)\, aa\, r^2 - \cos(\theta)\, aa\, r\, bb + \cos(\theta)^3\, aa\, r^2 + \cos(\theta)^3\, aa\, r\, bb,$$
$$\sin(\theta)\, aa\, r^3 + \sin(\theta)\, aa\, r^2\, bb - \sin(\theta)\, aa\, r^3 \cos(\theta)^2 - \sin(\theta)\, aa\, r^2 \cos(\theta)^2\, bb, r^2 - r^2\cos(\theta)^2 + r\, bb - \cos(\theta)^2\, r\, bb$$
$$+ aa^2\, r^4 - 2\, aa^2\, r^4 \cos(\theta)^2 + aa^2\, r^3\, bb - 2\, aa^2\, r^3\, bb \cos(\theta)^2 + aa^2\, r^4 \cos(\theta)^4 + aa^2\, r^3 \cos(\theta)^4\, bb]$$

$$m11 := 0$$

Now Compute the Right Cartan Matrix [CR]

```
> cartan:=simplify(innerprod(inverse(FF),d(FF))):
```

The matrix elements of the Right Cartan connection matrix (indices and values on the initial non-cartesian state) using the matrix methods:

```
> Gamma11:=wcollect(cartan[1,1]);Gamma12:=wcollect(cartan[1,2]);Gamma13:=wcollect(cartan[1,3]);
```

$$\Gamma 11 := (-r\cos(\theta)^3\, aa + r\cos(\theta)\, aa)\, \mathrm{d}(\phi)$$

$$\Gamma 12 := r^2 \sin(\theta)\cos(\theta)^2\, aa\, \mathrm{d}(\phi) - r\, \mathrm{d}(\theta)$$

$$\Gamma 13 := (-aa\cos(\theta) + \cos(\theta)^3\, aa)\, r\, \mathrm{d}(r) + (-1 + \cos(\theta)^2)\, r\, \mathrm{d}(\phi) - r^2 \sin(\theta)\cos(\theta)^2\, aa\, \mathrm{d}(\theta)$$

```
> Gamma21:=wcollect(cartan[2,1]);Gamma22:=wcollect(cartan[2,2]);Gamma23:=wcollect(cartan[2,3]);
```

$$\Gamma 21 := \frac{(aa\, r\cos(\theta)^2 \sin(\theta) - aa\, r\sin(\theta))\, \mathrm{d}(\phi)}{r} + \frac{\mathrm{d}(\theta)}{r}$$

$$\Gamma 22 := \frac{\mathrm{d}(r)}{r} + \frac{(\cos(\theta)^3\, aa\, r^2 - \cos(\theta)\, aa\, r^2)\, \mathrm{d}(\phi)}{r}$$

$$\Gamma 23 := (-aa\cos(\theta)^2 \sin(\theta) + aa\sin(\theta))\, \mathrm{d}(r) - \cos(\theta)\sin(\theta)\, \mathrm{d}(\phi) + (-r\cos(\theta)^3\, aa + r\cos(\theta)\, aa)\, \mathrm{d}(\theta)$$

```
> Gamma31:=wcollect(cartan[3,1]);Gamma32:=wcollect(cartan[3,2]);Gamma33:=wcollect(cartan[3,3]);
```

$$\Gamma 31 := \frac{\mathrm{d}(\phi)}{r}$$

$$\Gamma 32 := \frac{\cos(\theta)\, \mathrm{d}(\phi)}{\sin(\theta)}$$

$$\Gamma 33 := \frac{\mathrm{d}(r)}{r} + \frac{\mathrm{d}(\theta)\cos(\theta)}{\sin(\theta)}$$

The Trace of the connection will be determined and in this case is equal to a closed 1-form.

```
>
>
> TRACEGAMMA:=simplify(wcollect(Gamma11+Gamma22+Gamma33));TRACECURV:=d(TRACEGAMMA);
```

$$TRACEGAMMA := \frac{2\sin(\theta)\, \mathrm{d}(r) + r\cos(\theta)\, \mathrm{d}(\theta)}{r\sin(\theta)}$$

$$TRACECURV := 0$$

The Affine Torsion coefficients are computed, and depends upon the Frame pertubation coefficient aa.:

```
> AFFINE1:=wcollect(simplify(Gamma11&^d(r)+Gamma12&^d(theta)+Gamma13&^d(phi)));AFFINE2:=wcollect(simplify(Gamma21&^d(r)+Gamma22&^d(theta)+Gamma23&^d(phi)));AFFINE3:=wcollect(simplify(Gamma31&^d(r)+Gamma32&^d(theta)+Gamma33&^d(phi)));
```

$$AFFINE1 := r^2 \sin(\theta)\cos(\theta)^2\, aa\, (d(\phi)\ \&\wedge\ d(\theta)) + (r\cos(\theta)^3\, aa - r\cos(\theta)\, aa)\,(d(r)\ \&\wedge\ d(\phi)) - r^2\sin(\theta)\cos(\theta)^2\, aa\,(d(\theta)\ \&\wedge\ d(\phi)) + (-r\cos(\theta)^3\, aa + r\cos(\theta)\, aa)\,(d(\phi)\ \&\wedge\ d(r))$$

$$AFFINE2 := -\frac{(-\cos(\theta)^3\, aa\, r^2 + \cos(\theta)\, aa\, r^2)\,(d(\phi)\ \&\wedge\ d(\theta))}{r} + \frac{d(r)\ \&\wedge\ d(\theta)}{r} - \frac{(aa\, r\cos(\theta)^2\sin(\theta) - aa\, r\sin(\theta))\,(d(r)\ \&\wedge\ d(\phi))}{r} - \frac{(\cos(\theta)^3\, aa\, r^2 - \cos(\theta)\, aa\, r^2)\,(d(\theta)\ \&\wedge\ d(\phi))}{r} + \frac{d(\theta)\ \&\wedge\ d(r)}{r} - \frac{(-aa\, r\cos(\theta)^2\sin(\theta) + aa\, r\sin(\theta))\,(d(\phi)\ \&\wedge\ d(r))}{r}$$

$$AFFINE3 := \frac{d(\phi)\ \&\wedge\ d(r)}{r} + \frac{\cos(\theta)\,(d(\phi)\ \&\wedge\ d(\theta))}{\sin(\theta)} + \frac{d(r)\ \&\wedge\ d(\phi)}{r} + \frac{\cos(\theta)\,(d(\theta)\ \&\wedge\ d(\phi))}{\sin(\theta)}$$

Now the components of the right Cartan matrix will be computed by the tensor method, as a check

```
> dim:=3;coord:=[r,theta,phi];GG:=simplify(inverse(FF));
```

$$dim := 3$$

$$coord := [r, \theta, \phi]$$

$$GG := \begin{bmatrix} -\sin(\theta)(r\sin(\phi)\cos(\theta)\, aa - \cos(\phi)) & \sin(\theta)(r\cos(\phi)\cos(\theta)\, aa + \sin(\phi)) & \cos(\theta) \\ -\frac{-\sin(\phi)\, aa\, r + \sin(\phi)\, aa\, r\cos(\theta)^2 - \cos(\phi)\cos(\theta)}{r} & \frac{-\cos(\phi)\, aa\, r + \cos(\phi)\, aa\, r\cos(\theta)^2 + \sin(\phi)\cos(\theta)}{r} & -\frac{\sin(\theta)}{r} \\ -\frac{\sin(\phi)}{r\sin(\theta)} & \frac{\cos(\phi)}{r\sin(\theta)} & 0 \end{bmatrix}$$

First compute the differentials of the inverse matrix [GG]

```
> for i from 1 to dim do for j from 1 to dim do for k from 1 to dim do d1GG[i,j,k]:= (diff(GG[i,j],coord[k])) od od od:
```

Compute the elements of the matrix product of - d[G][F]

```
> for b from 1 to dim do for a from 1 to dim do for k from 1 to dim do  ss:=0;for m from 1 to dim do ss := ss+(d1GG[a,m,k]*FF[m,b]); CC[a,b,k]:=simplify(-ss) od od od od ;
> ;
> for b from 1 to dim do for a from 1 to dim do for k from 1 to dim do if CC[a,b,k]=0 then else print(`Cartan_RIGHT`(a,b,k)=factor(CC[a,b,k])) fi od od od ;
```

THE non zero CARTAN RIGHT CONNECTION coefficients.

CC(abk) index (1,-1,-1))

$$\mathrm{Cartan\_RIGHT}(1, 1, 3) = -r\cos(\theta)\, aa\,(\cos(\theta) - 1)(\cos(\theta) + 1)$$

$$\mathrm{Cartan\_RIGHT}(2, 1, 2) = \frac{1}{r}$$

$$\mathrm{Cartan\_RIGHT}(2, 1, 3) = \sin(\theta)\, aa\,(\cos(\theta) - 1)(\cos(\theta) + 1)$$

$$\text{Cartan\_RIGHT}(3, 1, 3) = \frac{1}{r}$$

$$\text{Cartan\_RIGHT}(1, 2, 2) = -r$$

$$\text{Cartan\_RIGHT}(1, 2, 3) = r^2 \sin(\theta) \cos(\theta)^2\, aa$$

$$\text{Cartan\_RIGHT}(2, 2, 1) = \frac{1}{r}$$

$$\text{Cartan\_RIGHT}(2, 2, 3) = r \cos(\theta)\, aa\, (\cos(\theta) - 1)\, (\cos(\theta) + 1)$$

$$\text{Cartan\_RIGHT}(3, 2, 3) = \frac{\cos(\theta)}{\sin(\theta)}$$

$$\text{Cartan\_RIGHT}(1, 3, 1) = r \cos(\theta)\, aa\, (\cos(\theta) - 1)\, (\cos(\theta) + 1)$$

$$\text{Cartan\_RIGHT}(1, 3, 2) = -r^2 \sin(\theta) \cos(\theta)^2\, aa$$

$$\text{Cartan\_RIGHT}(1, 3, 3) = r\, (\cos(\theta) - 1)\, (\cos(\theta) + 1)$$

$$\text{Cartan\_RIGHT}(2, 3, 1) = -\sin(\theta)\, aa\, (\cos(\theta) - 1)\, (\cos(\theta) + 1)$$

$$\text{Cartan\_RIGHT}(2, 3, 2) = -r \cos(\theta)\, aa\, (\cos(\theta) - 1)\, (\cos(\theta) + 1)$$

$$\text{Cartan\_RIGHT}(2, 3, 3) = -\cos(\theta) \sin(\theta)$$

$$\text{Cartan\_RIGHT}(3, 3, 1) = \frac{1}{r}$$

$$\text{Cartan\_RIGHT}(3, 3, 2) = \frac{\cos(\theta)}{\sin(\theta)}$$

These results agree with matrix method.
Next check for Affine Torsion using the tensor methods:

```
> for j from 1 to dim do for i from 1 to dim do for k from 1 to dim do  ss :=
  (CC[i,j,k]-CC[i,k,j])/2; CCTTS[i,j,k]:=ss od od od  ;
>
> for i from 1 to dim do for j from 1 to dim do for k from 1 to dim do if
  CCTTS[i,j,k]=0 then else print(`RIGHT_AffineTorsion`(i,k,j)=CCTTS[i,k,j]) fi od
  od od ;
```

$$\text{RIGHT\_AffineTorsion}(1, 3, 1) = r \cos(\theta)^3\, aa - r \cos(\theta)\, aa$$

$$\text{RIGHT\_AffineTorsion}(1, 3, 2) = -r^2 \sin(\theta) \cos(\theta)^2\, aa$$

$$\text{RIGHT\_AffineTorsion}(1, 1, 3) = -r \cos(\theta)^3\, aa + r \cos(\theta)\, aa$$

$$\text{RIGHT\_AffineTorsion}(1, 2, 3) = r^2 \sin(\theta) \cos(\theta)^2\, aa$$

$$\text{RIGHT\_AffineTorsion}(2, 3, 1) = -aa \cos(\theta)^2 \sin(\theta) + aa \sin(\theta)$$

$$\text{RIGHT\_AffineTorsion}(2, 3, 2) = -r \cos(\theta)^3\, aa + r \cos(\theta)\, aa$$

$$\text{RIGHT\_AffineTorsion}(2, 1, 3) = -aa \sin(\theta) + aa \cos(\theta)^2 \sin(\theta)$$

$$\text{RIGHT\_AffineTorsion}(2, 2, 3) = r \cos(\theta)^3\, aa - r \cos(\theta)\, aa$$

IF NO ENTRIES APPEAR ABOVE, THE AFFINE TORSION IS ZERO

Note for the torsion example, there is no affine torsion of the connection coefficients if aa = 0. Torsion does not depend upon bb.

Christoffel Connection coefficients from the induced metric

```
>
> metric:= simplify(subs(evalm(pullbackmetric)));
```

$metric :=$

$$\left[\frac{r+bb}{r}, 0, -\cos(\theta)\, aa\, r^2 - \cos(\theta)\, aa\, r\, bb + \cos(\theta)^3\, aa\, r^2 + \cos(\theta)^3\, aa\, r\, bb\right]$$

$$[0, r^2 + r\, bb, \sin(\theta)\, aa\, r^3 + \sin(\theta)\, aa\, r^2\, bb - \sin(\theta)\, aa\, r^3 \cos(\theta)^2 - \sin(\theta)\, aa\, r^2 \cos(\theta)^2\, bb]$$

$$[-\cos(\theta)\, aa\, r^2 - \cos(\theta)\, aa\, r\, bb + \cos(\theta)^3\, aa\, r^2 + \cos(\theta)^3\, aa\, r\, bb,$$
$$\sin(\theta)\, aa\, r^3 + \sin(\theta)\, aa\, r^2\, bb - \sin(\theta)\, aa\, r^3 \cos(\theta)^2 - \sin(\theta)\, aa\, r^2 \cos(\theta)^2\, bb, r^2 - r^2 \cos(\theta)^2 + r\, bb - \cos(\theta)^2\, r\, bb$$
$$+ aa^2 r^4 - 2\, aa^2 r^4 \cos(\theta)^2 + aa^2 r^3\, bb - 2\, aa^2 r^3\, bb \cos(\theta)^2 + aa^2 r^4 \cos(\theta)^4 + aa^2 r^3 \cos(\theta)^4\, bb]$$

```
> metricinverse:=inverse(metric):
> for i from 1 to dim do for j from 1 to dim do for k from 1 to dim do
  d1gun[i,j,k] := (diff(metric[i,j],coord[k])) od od od:
> #for i from 1 to dim do for j from 1 to dim do for k from 1 to dim do if
  d1gun[i,j,k]=0 then  else print(`dgun`(i,j,k)=d1gun[i,j,k]) fi od od od;
> for i from 1 to dim do for j from i to dim do for k from 1 to dim do C1S[i,j,k]
  := 0  od od od; for i from 1 to dim do for j from 1 to dim do for k from 1 to
  dim do C1S[i,j,k] := 1/2*d1gun[i,k,j]+1/2*d1gun[j,k,i]-1/2*d1gun[i,j,k]  od od
  od;
>  for k from 1 to dim do for i from 1 to dim do for j from 1 to dim do ss := 0;
  for m to dim do ss := ss+metricinverse[k,m]*C1S[i,j,m] od; C2S[k,i,j] :=
  simplify(factor(ss),trig) od od od;
> for i from 1 to dim do for j from 1 to dim do for k from 1 to dim do if
  C2S[i,j,k]=0 then  else print(`Christoffel_Gamma2`(i,j,k)=simplify(C2S[i,j,k]))
  fi od od od;
```

The non zero Christoffel Connection coefficients 2nd kind on the initial space (domain)

Gamma2(i,j,k) index (1,-1,-1)

Christoffel_Gamma2(1, 1, 1) =

$$\frac{1}{2}\,\frac{-3\, aa^2 r^2\, bb \cos(\theta)^2 + 3\, aa^2 r^2 \cos(\theta)^4\, bb - bb - 4\, aa^2 r^3 \cos(\theta)^2 + 4\, aa^2 r^3 \cos(\theta)^4}{r\,(r+bb)}$$

$$\text{Christoffel\_Gamma2}(1, 1, 2) = -2\, r^2\, aa^2 \sin(\theta) \cos(\theta)^3 + r^2\, aa^2 \sin(\theta) \cos(\theta)$$

$$\text{Christoffel\_Gamma2}(1, 1, 3) = -\frac{1}{2}\, r \cos(\theta)\, aa\, (-2\, aa^2 r^3 + 4\, aa^2 r^3 \cos(\theta)^2 - 2\, aa^2 r^2\, bb + 5\, aa^2 r^2\, bb \cos(\theta)^2$$
$$- 2\, aa^2 r^3 \cos(\theta)^4 - 4\, aa^2 r^2 \cos(\theta)^4\, bb + r^2 \cos(\theta)^6\, aa^2\, bb - 2\, r + 2\, r \cos(\theta)^2 - bb + \cos(\theta)^2\, bb) / (r + bb)$$

$$\text{Christoffel\_Gamma2}(1, 2, 1) = -2\, r^2\, aa^2 \sin(\theta) \cos(\theta)^3 + r^2\, aa^2 \sin(\theta) \cos(\theta)$$

Christoffel_Gamma2(1, 2, 2) =

$$-\frac{1}{2}\,\frac{r\,(-4\, aa^2 r^3 \cos(\theta)^2 + 4\, aa^2 r^3 \cos(\theta)^4 - 5\, aa^2 r^2\, bb \cos(\theta)^2 + 5\, aa^2 r^2 \cos(\theta)^4\, bb + 2\, r + bb)}{r + bb}$$

$$\text{Christoffel\_Gamma2}(1, 2, 3) = \frac{1}{2} \sin(\theta)\, aa\, r^2\, (2\, aa^2 r^3 \cos(\theta)^2 - 2\, aa^2 r^3 \cos(\theta)^4 - 4\, aa^2 r^2 \cos(\theta)^4\, bb$$
$$+ r^2 \cos(\theta)^6\, aa^2\, bb + 3\, aa^2 r^2\, bb \cos(\theta)^2 - 2\, r - bb + \cos(\theta)^2\, bb + 2\, r \cos(\theta)^2) / (r + bb)$$

$$\text{Christoffel\_Gamma2}(1, 3, 1) = -\frac{1}{2}\, r \cos(\theta)\, aa\, (-2\, aa^2 r^3 + 4\, aa^2 r^3 \cos(\theta)^2 - 2\, aa^2 r^2\, bb + 5\, aa^2 r^2\, bb \cos(\theta)^2$$

$- 2\, aa^2\, r^3 \cos(\theta)^4 - 4\, aa^2\, r^2 \cos(\theta)^4\, bb + r^2 \cos(\theta)^6\, aa^2\, bb - 2\, r + 2\, r \cos(\theta)^2 - bb + \cos(\theta)^2\, bb) / (r + bb)$

$$\text{Christoffel\_Gamma2}(1, 3, 2) = \frac{1}{2} \sin(\theta)\, aa\, r^2\, (2\, aa^2\, r^3 \cos(\theta)^2 - 2\, aa^2\, r^3 \cos(\theta)^4 - 4\, aa^2\, r^2 \cos(\theta)^4\, bb + r^2 \cos(\theta)^6\, aa^2\, bb + 3\, aa^2\, r^2\, bb \cos(\theta)^2 - 2\, r - bb + \cos(\theta)^2\, bb + 2\, r \cos(\theta)^2) / (r + bb)$$

$$\text{Christoffel\_Gamma2}(1, 3, 3) = -\frac{1}{2} r\, (-\cos(\theta)^2\, bb + 3\, aa^2\, r^2\, bb - 8\, aa^2\, r^3 \cos(\theta)^2 + 4\, aa^2\, r^3 \cos(\theta)^4 - 7\, aa^2\, r^2\, bb \cos(\theta)^2 + 5\, aa^2\, r^2 \cos(\theta)^4\, bb + 4\, aa^2\, r^3 + bb - 2\, r \cos(\theta)^2 - r^2 \cos(\theta)^6\, aa^2\, bb + 2\, r - r^4 \cos(\theta)^2\, aa^4\, bb + 3\, r^4 \cos(\theta)^4\, aa^4\, bb - 3\, r^4 \cos(\theta)^6\, aa^4\, bb + r^4 \cos(\theta)^8\, aa^4\, bb) / (r + bb)$$

$$\text{Christoffel\_Gamma2}(2, 1, 1) = -\frac{1}{2} \frac{\sin(\theta)\, aa^2 \cos(\theta)\, (-3\, bb + 3 \cos(\theta)^2\, bb - 4\, r + 4\, r \cos(\theta)^2)}{r + bb}$$

$$\text{Christoffel\_Gamma2}(2, 1, 2) = -\frac{1}{2} \frac{4\, aa^2\, r^3 \cos(\theta)^4 + 4\, aa^2\, r^2 \cos(\theta)^4\, bb - 2\, r - bb - 6\, aa^2\, r^3 \cos(\theta)^2 - 6\, aa^2\, r^2\, bb \cos(\theta)^2 + 2\, aa^2\, r^3 + 2\, aa^2\, r^2\, bb}{r\, (r + bb)}$$

$$\text{Christoffel\_Gamma2}(2, 1, 3) = \frac{1}{2} \sin(\theta)\, aa\, (-2\, aa^2\, r^3 \cos(\theta)^4 - 4\, aa^2\, r^2 \cos(\theta)^4\, bb + r^2 \cos(\theta)^6\, aa^2\, bb + 2 \cos(\theta)^2\, bb + 4\, aa^2\, r^3 \cos(\theta)^2 + 5\, aa^2\, r^2\, bb \cos(\theta)^2 - 2\, aa^2\, r^3 - 2\, aa^2\, r^2\, bb + 2\, r \cos(\theta)^2) / (r + bb)$$

$$\text{Christoffel\_Gamma2}(2, 2, 1) = -\frac{1}{2} \frac{4\, aa^2\, r^3 \cos(\theta)^4 + 4\, aa^2\, r^2 \cos(\theta)^4\, bb - 2\, r - bb - 6\, aa^2\, r^3 \cos(\theta)^2 - 6\, aa^2\, r^2\, bb \cos(\theta)^2 + 2\, aa^2\, r^3 + 2\, aa^2\, r^2\, bb}{r\, (r + bb)}$$

$$\text{Christoffel\_Gamma2}(2, 2, 2) = \frac{1}{2} \frac{r^2\, aa^2 \sin(\theta) \cos(\theta)\, (5 \cos(\theta)^2\, bb - 5\, bb - 4\, r + 4\, r \cos(\theta)^2)}{r + bb}$$

$$\text{Christoffel\_Gamma2}(2, 2, 3) = \frac{1}{2} r \cos(\theta)\, aa\, (4\, aa^2\, r^3 \cos(\theta)^2 - 2\, aa^2\, r^3 \cos(\theta)^4 - 5\, aa^2\, r^2 \cos(\theta)^4\, bb + r^2 \cos(\theta)^6\, aa^2\, bb - 2\, aa^2\, r^3 - 3\, aa^2\, r^2\, bb + 7\, aa^2\, r^2\, bb \cos(\theta)^2 - 2\, r + 2\, r \cos(\theta)^2 - 2\, bb + 2 \cos(\theta)^2\, bb) / (r + bb)$$

$$\text{Christoffel\_Gamma2}(2, 3, 1) = \frac{1}{2} \sin(\theta)\, aa\, (-2\, aa^2\, r^3 \cos(\theta)^4 - 4\, aa^2\, r^2 \cos(\theta)^4\, bb + r^2 \cos(\theta)^6\, aa^2\, bb + 2 \cos(\theta)^2\, bb + 4\, aa^2\, r^3 \cos(\theta)^2 + 5\, aa^2\, r^2\, bb \cos(\theta)^2 - 2\, aa^2\, r^3 - 2\, aa^2\, r^2\, bb + 2\, r \cos(\theta)^2) / (r + bb)$$

$$\text{Christoffel\_Gamma2}(2, 3, 2) = \frac{1}{2} r \cos(\theta)\, aa\, (4\, aa^2\, r^3 \cos(\theta)^2 - 2\, aa^2\, r^3 \cos(\theta)^4 - 5\, aa^2\, r^2 \cos(\theta)^4\, bb + r^2 \cos(\theta)^6\, aa^2\, bb - 2\, aa^2\, r^3 - 3\, aa^2\, r^2\, bb + 7\, aa^2\, r^2\, bb \cos(\theta)^2 - 2\, r + 2\, r \cos(\theta)^2 - 2\, bb + 2 \cos(\theta)^2\, bb) / (r + bb)$$

$$\text{Christoffel\_Gamma2}(2, 3, 3) = \frac{1}{2} \cos(\theta) \sin(\theta)\, (-aa^4\, r^4\, bb - 4\, aa^2\, r^3 + 3\, r^4 \cos(\theta)^2\, aa^4\, bb - 3\, r^4 \cos(\theta)^4\, aa^4\, bb + r^4 \cos(\theta)^6\, aa^4\, bb + 6\, aa^2\, r^2\, bb \cos(\theta)^2 - aa^2\, r^2 \cos(\theta)^4\, bb - 2\, r + 4\, aa^2\, r^3 \cos(\theta)^2 - 5\, aa^2\, r^2\, bb - 2\, bb) / (r + bb)$$

$$\text{Christoffel\_Gamma2}(3, 1, 1) = -\frac{1}{2} \frac{aa \cos(\theta)\, (3\, bb + 4\, r)}{r\, (r + bb)}$$

$$\text{Christoffel\_Gamma2}(3, 1, 2) = \frac{aa \sin(\theta)\, (2 \cos(\theta)^2 - 1)}{-1 + \cos(\theta)^2}$$

$$\text{Christoffel\_Gamma2}(3, 1, 3) = \frac{1}{2} \frac{-2\, aa^2\, r^3 \cos(\theta)^2 - 3\, aa^2\, r^2\, bb \cos(\theta)^2 + aa^2\, r^2 \cos(\theta)^4\, bb + 2\, aa^2\, r^3 + 2\, aa^2\, r^2\, bb + 2\, r + bb}{r\, (r + bb)}$$

$$\text{Christoffel\_Gamma2}(3, 2, 1) = \frac{aa\,\sin(\theta)\,(2\cos(\theta)^2 - 1)}{-1 + \cos(\theta)^2}$$

$$\text{Christoffel\_Gamma2}(3, 2, 2) = \frac{1}{2}\,\frac{(5\,bb + 4\,r)\,r\cos(\theta)\,aa}{r + bb}$$

$\text{Christoffel\_Gamma2}(3, 2, 3) =$

$$-\frac{1}{2}\,\frac{(aa^2\,r^2\cos(\theta)^4\,bb - 2\,aa^2\,r^3\cos(\theta)^2 - 4\,aa^2\,r^2\,bb\cos(\theta)^2 + 2\,aa^2\,r^3 + 3\,aa^2\,r^2\,bb + 2\,r + 2\,bb)\cos(\theta)\sin(\theta)}{\cos(\theta)^2\,bb - r + r\cos(\theta)^2 - bb}$$

$\text{Christoffel\_Gamma2}(3, 3, 1) =$

$$\frac{1}{2}\,\frac{-2\,aa^2\,r^3\cos(\theta)^2 - 3\,aa^2\,r^2\,bb\cos(\theta)^2 + aa^2\,r^2\cos(\theta)^4\,bb + 2\,aa^2\,r^3 + 2\,aa^2\,r^2\,bb + 2\,r + bb}{r\,(r + bb)}$$

$\text{Christoffel\_Gamma2}(3, 3, 2) =$

$$-\frac{1}{2}\,\frac{(aa^2\,r^2\cos(\theta)^4\,bb - 2\,aa^2\,r^3\cos(\theta)^2 - 4\,aa^2\,r^2\,bb\cos(\theta)^2 + 2\,aa^2\,r^3 + 3\,aa^2\,r^2\,bb + 2\,r + 2\,bb)\cos(\theta)\sin(\theta)}{\cos(\theta)^2\,bb - r + r\cos(\theta)^2 - bb}$$

$$\text{Christoffel\_Gamma2}(3, 3, 3) = \frac{1}{2}\,\frac{aa\cos(\theta)\,r\,bb\,(1 - \cos(\theta)^2 + aa^2\,r^2 - 2\,r^2\cos(\theta)^2\,aa^2 + aa^2\,r^2\cos(\theta)^4)}{r + bb}$$

If no entries appear above the Christoffel symbols on the domain space vanish

However, for the distorted example it is evident that there are contributions to Christoffel symbols (induced metric) which depend both on the torsion aa and the distortion bb.

The Right Cartan matrix is often defined as the sum of Christoffel Symbols and Rotation coefficients, T(i,j,k)

CartanRight(ijk) = ChristoffelGamma(ijk) + T(ijk)

Compute the T(i,j,k): These coefficients if no-zero indicate the effects of the pertubations on the metric and Basis. If there is no difference between the Christoffel symbols and the Cartan connection symbols, then the T(i,j,k) are zero.

```
> for i from 1 to dim do for j from 1 to dim do for k from 1 to dim do ss:=0; ss
  := (CC[i,j,k]-C2S[i,j,k]); SHIPTR[i,j,k]:=simplify(ss)  od od od   ;
>
>
> for i from 1 to dim do for j from 1 to dim do for k from 1 to dim do if
  C2S[i,j,k]=0 and CC[i,j,k]=0 then else print(`T`(i,j,k)=simplify(SHIPTR[i,j,k]))
  fi od od od ;
```

T(ijk) index (1,-1,-1)

$$T(1, 1, 1) = -\frac{1}{2}\,\frac{-3\,aa^2\,r^2\,bb\cos(\theta)^2 + 3\,aa^2\,r^2\cos(\theta)^4\,bb - bb - 4\,aa^2\,r^3\cos(\theta)^2 + 4\,aa^2\,r^3\cos(\theta)^4}{r\,(r + bb)}$$

$$T(1, 1, 2) = 2\,r^2\,aa^2\sin(\theta)\cos(\theta)^3 - r^2\,aa^2\sin(\theta)\cos(\theta)$$

$$T(1,1,3)=\frac{1}{2}\,aa\cos(\theta)\,r\,(-\cos(\theta)^2 bb+bb-2\,aa^2 r^3+4\,aa^2 r^3\cos(\theta)^2-2\,aa^2 r^2 bb+5\,aa^2 r^2 bb\cos(\theta)^2$$
$$-2\,aa^2 r^3\cos(\theta)^4-4\,aa^2 r^2\cos(\theta)^4 bb+r^2\cos(\theta)^6 aa^2 bb)\,/\,(r+bb)$$

$$T(1,2,1)=2\,r^2 aa^2\sin(\theta)\cos(\theta)^3-r^2 aa^2\sin(\theta)\cos(\theta)$$

$$T(1,2,2)=\frac{1}{2}\,\frac{r\,(-bb-4\,aa^2 r^3\cos(\theta)^2+4\,aa^2 r^3\cos(\theta)^4-5\,aa^2 r^2 bb\cos(\theta)^2+5\,aa^2 r^2\cos(\theta)^4 bb)}{r+bb}$$

$$T(1,2,3)=-\frac{1}{2}\,r^2\sin(\theta)\,aa\,(-\cos(\theta)^2 bb+2\,aa^2 r^3\cos(\theta)^2-2\,aa^2 r^3\cos(\theta)^4-4\,aa^2 r^2\cos(\theta)^4 bb$$
$$+r^2\cos(\theta)^6 aa^2 bb+3\,aa^2 r^2 bb\cos(\theta)^2-2\,r-bb)\,/\,(r+bb)$$

$$T(1,3,1)=\frac{1}{2}\,aa\cos(\theta)\,r\,(4\,r\cos(\theta)^2+3\cos(\theta)^2 bb-4\,r-3\,bb-2\,aa^2 r^3+4\,aa^2 r^3\cos(\theta)^2-2\,aa^2 r^2 bb$$
$$+5\,aa^2 r^2 bb\cos(\theta)^2-2\,aa^2 r^3\cos(\theta)^4-4\,aa^2 r^2\cos(\theta)^4 bb+r^2\cos(\theta)^6 aa^2 bb)\,/\,(r+bb)$$

$$T(1,3,2)=-\frac{1}{2}\,r^2\sin(\theta)\,aa\,(4\,r\cos(\theta)^2+3\cos(\theta)^2 bb+2\,aa^2 r^3\cos(\theta)^2-2\,aa^2 r^3\cos(\theta)^4-4\,aa^2 r^2\cos(\theta)^4 bb$$
$$+r^2\cos(\theta)^6 aa^2 bb+3\,aa^2 r^2 bb\cos(\theta)^2-2\,r-bb)\,/\,(r+bb)$$

$$T(1,3,3)=\frac{1}{2}\,r\,(-bb+\cos(\theta)^2 bb+3\,aa^2 r^2 bb-8\,aa^2 r^3\cos(\theta)^2+4\,aa^2 r^3\cos(\theta)^4-7\,aa^2 r^2 bb\cos(\theta)^2$$
$$+5\,aa^2 r^2\cos(\theta)^4 bb+4\,aa^2 r^3-r^2\cos(\theta)^6 aa^2 bb-r^4\cos(\theta)^2 aa^4 bb+3\,r^4\cos(\theta)^4 aa^4 bb-3\,r^4\cos(\theta)^6 aa^4 bb$$
$$+r^4\cos(\theta)^8 aa^4 bb)\,/\,(r+bb)$$

$$T(2,1,1)=\frac{1}{2}\,\frac{\sin(\theta)\,aa^2\cos(\theta)\,(-3\,bb+3\cos(\theta)^2 bb-4\,r+4\,r\cos(\theta)^2)}{r+bb}$$

$$T(2,1,2)=$$
$$\frac{1}{2}\,\frac{bb+4\,aa^2 r^3\cos(\theta)^4+4\,aa^2 r^2\cos(\theta)^4 bb-6\,aa^2 r^3\cos(\theta)^2-6\,aa^2 r^2 bb\cos(\theta)^2+2\,aa^2 r^3+2\,aa^2 r^2 bb}{r\,(r+bb)}$$

$$T(2,1,3)=-\frac{1}{2}\,aa\sin(\theta)\,(2\,r+2\,bb-2\,aa^2 r^3\cos(\theta)^4-4\,aa^2 r^2\cos(\theta)^4 bb+r^2\cos(\theta)^6 aa^2 bb+4\,aa^2 r^3\cos(\theta)^2$$
$$+5\,aa^2 r^2 bb\cos(\theta)^2-2\,aa^2 r^3-2\,aa^2 r^2 bb)\,/\,(r+bb)$$

$$T(2,2,1)=$$
$$\frac{1}{2}\,\frac{bb+4\,aa^2 r^3\cos(\theta)^4+4\,aa^2 r^2\cos(\theta)^4 bb-6\,aa^2 r^3\cos(\theta)^2-6\,aa^2 r^2 bb\cos(\theta)^2+2\,aa^2 r^3+2\,aa^2 r^2 bb}{r\,(r+bb)}$$

$$T(2,2,2)=-\frac{1}{2}\,\frac{r^2 aa^2\sin(\theta)\cos(\theta)\,(5\cos(\theta)^2 bb-5\,bb-4\,r+4\,r\cos(\theta)^2)}{r+bb}$$

$$T(2,2,3)=-\frac{1}{2}\,\frac{r^3\cos(\theta)\,aa^3\,(4\,r\cos(\theta)^2-2\,r\cos(\theta)^4-5\cos(\theta)^4 bb+\cos(\theta)^6 bb-2\,r-3\,bb+7\cos(\theta)^2 bb)}{r+bb}$$

$$T(2,3,1)=-\frac{1}{2}\,aa\sin(\theta)\,(4\,r\cos(\theta)^2+4\cos(\theta)^2 bb-2\,r-2\,bb-2\,aa^2 r^3\cos(\theta)^4-4\,aa^2 r^2\cos(\theta)^4 bb$$
$$+r^2\cos(\theta)^6 aa^2 bb+4\,aa^2 r^3\cos(\theta)^2+5\,aa^2 r^2 bb\cos(\theta)^2-2\,aa^2 r^3-2\,aa^2 r^2 bb)\,/\,(r+bb)$$

$$T(2,3,2)=-\frac{1}{2}\,r\cos(\theta)\,aa\,(4\,r\cos(\theta)^2+4\cos(\theta)^2 bb-4\,r-4\,bb+4\,aa^2 r^3\cos(\theta)^2-2\,aa^2 r^3\cos(\theta)^4$$
$$-5\,aa^2 r^2\cos(\theta)^4 bb+r^2\cos(\theta)^6 aa^2 bb-2\,aa^2 r^3-3\,aa^2 r^2 bb+7\,aa^2 r^2 bb\cos(\theta)^2)\,/\,(r+bb)$$

$$T(2,3,3)=-\frac{1}{2}\cos(\theta)\sin(\theta)\,aa^2 r^2\,(-aa^2 r^2 bb-4\,r+3\,aa^2 r^2 bb\cos(\theta)^2-3\,aa^2 r^2\cos(\theta)^4 bb+r^2\cos(\theta)^6 aa^2 bb$$

$+ 6 \cos(\theta)^2 bb - \cos(\theta)^4 bb + 4 r \cos(\theta)^2 - 5 bb) / (r + bb)$

$$T(3, 1, 1) = \frac{1}{2} \frac{aa \cos(\theta) (3 bb + 4 r)}{r (r + bb)}$$

$$T(3, 1, 2) = -\frac{aa \sin(\theta) (2 \cos(\theta)^2 - 1)}{-1 + \cos(\theta)^2}$$

$$T(3, 1, 3) = -\frac{1}{2} \frac{-bb - 2 aa^2 r^3 \cos(\theta)^2 - 3 aa^2 r^2 bb \cos(\theta)^2 + aa^2 r^2 \cos(\theta)^4 bb + 2 aa^2 r^3 + 2 aa^2 r^2 bb}{r (r + bb)}$$

$$T(3, 2, 1) = -\frac{aa \sin(\theta) (2 \cos(\theta)^2 - 1)}{-1 + \cos(\theta)^2}$$

$$T(3, 2, 2) = -\frac{1}{2} \frac{(5 bb + 4 r) r \cos(\theta) aa}{r + bb}$$

$$T(3, 2, 3) = \frac{1}{2} \frac{(\cos(\theta)^2 bb - 2 r - 3 bb) aa^2 r^2 \cos(\theta) \sin(\theta)}{r + bb}$$

$$T(3, 3, 1) = -\frac{1}{2} \frac{-bb - 2 aa^2 r^3 \cos(\theta)^2 - 3 aa^2 r^2 bb \cos(\theta)^2 + aa^2 r^2 \cos(\theta)^4 bb + 2 aa^2 r^3 + 2 aa^2 r^2 bb}{r (r + bb)}$$

$$T(3, 3, 2) = \frac{1}{2} \frac{(\cos(\theta)^2 bb - 2 r - 3 bb) aa^2 r^2 \cos(\theta) \sin(\theta)}{r + bb}$$

$$T(3, 3, 3) = -\frac{1}{2} \frac{aa \cos(\theta) r bb (1 - \cos(\theta)^2 + aa^2 r^2 - 2 r^2 \cos(\theta)^2 aa^2 + aa^2 r^2 \cos(\theta)^4)}{r + bb}$$

```
>
>
>
>
```

Right Cartan(ijk) = ChristoffelGamma(ijk) + T(ijk)

In this example above the rotation coefficients on the domain space depend upon both the metric perturbation bb and the Frame perturbation aa. When both are zero, the Ricci rotation coefficients vanish, and the Right Cartan matrix is exactly equal to the Christoffel symbols based upon the pullbackmetric.

NOW EXAMINE THE EFFECTS OF A METRIC PERTURBATION WITHOUT A TORSION PERTUBATION

```
> for i from 1 to dim do for j from 1 to dim do for k from 1 to dim do ss:=0; ss
  := (CC[i,j,k]-C2S[i,j,k]); SHIPTR[i,j,k]:=simplify(ss)  od od od   ;
>
>
> for i from 1 to dim do for j from 1 to dim do for k from 1 to dim do if
  C2S[i,j,k]=0 and CC[i,j,k]=0 then else
  print(`T`(i,j,k)=simplify(subs(aa=0,SHIPTR[i,j,k]))) fi od od od ;
```

$$T(1, 1, 1) = \frac{1}{2} \frac{bb}{r (r + bb)}$$

$$T(1, 1, 2) = 0$$

$$T(1,1,3)=0$$
$$T(1,2,1)=0$$
$$T(1,2,2)=-\frac{1}{2}\frac{r\,bb}{r+bb}$$
$$T(1,2,3)=0$$
$$T(1,3,1)=0$$
$$T(1,3,2)=0$$
$$T(1,3,3)=\frac{1}{2}\frac{r\,bb\,(-1+\cos(\theta)^2)}{r+bb}$$
$$T(2,1,1)=0$$
$$T(2,1,2)=\frac{1}{2}\frac{bb}{r\,(r+bb)}$$
$$T(2,1,3)=0$$
$$T(2,2,1)=\frac{1}{2}\frac{bb}{r\,(r+bb)}$$
$$T(2,2,2)=0$$
$$T(2,2,3)=0$$
$$T(2,3,1)=0$$
$$T(2,3,2)=0$$
$$T(2,3,3)=0$$
$$T(3,1,1)=0$$
$$T(3,1,2)=0$$
$$T(3,1,3)=\frac{1}{2}\frac{bb}{r\,(r+bb)}$$
$$T(3,2,1)=0$$
$$T(3,2,2)=0$$
$$T(3,2,3)=0$$
$$T(3,3,1)=\frac{1}{2}\frac{bb}{r\,(r+bb)}$$
$$T(3,3,2)=0$$
$$T(3,3,3)=0$$

>

NOW EXAMINE THE EFFECTS OF A TORSION PERTURBATION WITHOUT A METRIC PERTUBATION

```
> for i from 1 to dim do for j from 1 to dim do for k from 1 to dim do ss:=0; ss
  := (CC[i,j,k]-C2S[i,j,k]); SHIPTR[i,j,k]:=simplify(ss)  od od od   ;
>
>
> for i from 1 to dim do for j from 1 to dim do for k from 1 to dim do if
  C2S[i,j,k]=0 and CC[i,j,k]=0 then else
  print(`T`(i,j,k)=simplify(subs(bb=0,SHIPTR[i,j,k]))) fi od od od ;
```

$$T(1,1,1)=2\,aa^2\,r\cos(\theta)^2-2\,aa^2\,r\cos(\theta)^4$$

$$T(1, 1, 2) = 2\, r^2\, aa^2 \sin(\theta) \cos(\theta)^3 - r^2\, aa^2 \sin(\theta) \cos(\theta)$$
$$T(1, 1, 3) = 2\, aa^3 \cos(\theta)^3\, r^3 - aa^3 \cos(\theta)^5\, r^3 - aa^3 \cos(\theta)\, r^3$$
$$T(1, 2, 1) = 2\, r^2\, aa^2 \sin(\theta) \cos(\theta)^3 - r^2\, aa^2 \sin(\theta) \cos(\theta)$$
$$T(1, 2, 2) = -2\, aa^2\, r^3 \cos(\theta)^2 + 2\, aa^2\, r^3 \cos(\theta)^4$$
$$T(1, 2, 3) = -r^4 \sin(\theta)\, aa^3 \cos(\theta)^2 + r^4 \sin(\theta)\, aa^3 \cos(\theta)^4 + \sin(\theta)\, aa\, r^2$$
$$T(1, 3, 1) = 2\, r \cos(\theta)^3\, aa - 2\, r \cos(\theta)\, aa - aa^3 \cos(\theta)\, r^3 + 2\, aa^3 \cos(\theta)^3\, r^3 - aa^3 \cos(\theta)^5\, r^3$$
$$T(1, 3, 2) = -2\, r^2 \sin(\theta) \cos(\theta)^2\, aa - r^4 \sin(\theta)\, aa^3 \cos(\theta)^2 + r^4 \sin(\theta)\, aa^3 \cos(\theta)^4 + \sin(\theta)\, aa\, r^2$$
$$T(1, 3, 3) = -4\, aa^2\, r^3 \cos(\theta)^2 + 2\, aa^2\, r^3 \cos(\theta)^4 + 2\, aa^2\, r^3$$
$$T(2, 1, 1) = -2 \sin(\theta)\, aa^2 \cos(\theta) + 2 \sin(\theta)\, aa^2 \cos(\theta)^3$$
$$T(2, 1, 2) = 2\, aa^2\, r \cos(\theta)^4 - 3\, aa^2\, r \cos(\theta)^2 + aa^2\, r$$
$$T(2, 1, 3) = -aa \sin(\theta) + aa^3 \sin(\theta)\, r^2 \cos(\theta)^4 - 2\, aa^3 \sin(\theta)\, r^2 \cos(\theta)^2 + aa^3 \sin(\theta)\, r^2$$
$$T(2, 2, 1) = 2\, aa^2\, r \cos(\theta)^4 - 3\, aa^2\, r \cos(\theta)^2 + aa^2\, r$$
$$T(2, 2, 2) = 2\, r^2\, aa^2 \sin(\theta) \cos(\theta) - 2\, r^2\, aa^2 \sin(\theta) \cos(\theta)^3$$
$$T(2, 2, 3) = -2\, aa^3 \cos(\theta)^3\, r^3 + aa^3 \cos(\theta)^5\, r^3 + aa^3 \cos(\theta)\, r^3$$
$$T(2, 3, 1) = -2\, aa \cos(\theta)^2 \sin(\theta) + aa \sin(\theta) + aa^3 \sin(\theta)\, r^2 \cos(\theta)^4 - 2\, aa^3 \sin(\theta)\, r^2 \cos(\theta)^2 + aa^3 \sin(\theta)\, r^2$$
$$T(2, 3, 2) = -2\, r \cos(\theta)^3\, aa + 2\, r \cos(\theta)\, aa + aa^3 \cos(\theta)\, r^3 - 2\, aa^3 \cos(\theta)^3\, r^3 + aa^3 \cos(\theta)^5\, r^3$$
$$T(2, 3, 3) = 2\, r^2\, aa^2 \sin(\theta) \cos(\theta) - 2\, r^2\, aa^2 \sin(\theta) \cos(\theta)^3$$
$$T(3, 1, 1) = 2\, \frac{aa \cos(\theta)}{r}$$
$$T(3, 1, 2) = -\frac{aa \sin(\theta)\, (2 \cos(\theta)^2 - 1)}{-1 + \cos(\theta)^2}$$
$$T(3, 1, 3) = -aa^2\, r + aa^2\, r \cos(\theta)^2$$
$$T(3, 2, 1) = -\frac{aa \sin(\theta)\, (2 \cos(\theta)^2 - 1)}{-1 + \cos(\theta)^2}$$
$$T(3, 2, 2) = -2\, r \cos(\theta)\, aa$$
$$T(3, 2, 3) = -r^2\, aa^2 \sin(\theta) \cos(\theta)$$
$$T(3, 3, 1) = -aa^2\, r + aa^2\, r \cos(\theta)^2$$
$$T(3, 3, 2) = -r^2\, aa^2 \sin(\theta) \cos(\theta)$$
$$T(3, 3, 3) = 0$$

```
>
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NOW EXAMINE THE EFFECTS OF WITHOUT METRIC PERTURBATIONS AND WITHOUT TORSION PERTUBATIONS

```
> for i from 1 to dim do for j from 1 to dim do for k from 1 to dim do ss:=0; ss
  := (CC[i,j,k]-C2S[i,j,k]); SHIPTR[i,j,k]:=simplify(ss)  od od od   ;
>
>
> for i from 1 to dim do for j from 1 to dim do for k from 1 to dim do if
  C2S[i,j,k]=0 and CC[i,j,k]=0 then else
  print(`T`(i,j,k)=simplify(subs(aa=0,bb=0,SHIPTR[i,j,k]))) fi od od od ;
```

$$T(1, 1, 1) = 0$$
$$T(1, 1, 2) = 0$$

$$T(1, 1, 3) = 0$$
$$T(1, 2, 1) = 0$$
$$T(1, 2, 2) = 0$$
$$T(1, 2, 3) = 0$$
$$T(1, 3, 1) = 0$$
$$T(1, 3, 2) = 0$$
$$T(1, 3, 3) = 0$$
$$T(2, 1, 1) = 0$$
$$T(2, 1, 2) = 0$$
$$T(2, 1, 3) = 0$$
$$T(2, 2, 1) = 0$$
$$T(2, 2, 2) = 0$$
$$T(2, 2, 3) = 0$$
$$T(2, 3, 1) = 0$$
$$T(2, 3, 2) = 0$$
$$T(2, 3, 3) = 0$$
$$T(3, 1, 1) = 0$$
$$T(3, 1, 2) = 0$$
$$T(3, 1, 3) = 0$$
$$T(3, 2, 1) = 0$$
$$T(3, 2, 2) = 0$$
$$T(3, 2, 3) = 0$$
$$T(3, 3, 1) = 0$$
$$T(3, 3, 2) = 0$$
$$T(3, 3, 3) = 0$$

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