Spinors are the Cause of Topological Fluctuations and Thermodynamic Irreversibility

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ABSTRACT

The topological perspective of thermodynamics defines the 1-form of Work as the interior product of the process direction field V with the antisymmetric 2-form F = dA generated by the exterior derivative of that 1-form of Action, A, used to encode the physical system. The functional coefficients of the 2-form F can be used to construct an antisymmetric matrix whose eigenvectors are either vectors of eigenvalue zero, or isotropic complex Spinors with imaginary eigenvalues.

If the process is composed of those eigenvectors with eigen value zero, then the Work 1-form vanishes, the evolution is Hamiltonian, and conservative. If the Work 1-form is not zero, then the process must contain Spinor components. It follows that topological fluctuations in kinematics, such as temperature and pressure, must be generated by processes that have Spinor components.

1. A TOPOLOGICAL PERSPECTIVE

At its foundations, Thermodynamics is a topological theory independent from the geometric constraints of metric scales or gauge symmetries. The theory of non equilibrium thermodynamics, from the perspective of continuous topological evolution,[19] is based on four axioms.

Axiom 1. *Thermodynamic physical systems can be encoded in terms of a 1-form of covariant Action Potentials,* $A_k(x,y,z,t...)$, *on a* ≥ 4 *dimensional abstract variety of ordered independent variables,* $\{x,y,z,t...\}$. *The variety supports a differential volume element* $\Omega_4 = dx\hat{}dy\hat{}dz\hat{}dt...$

Axiom 2. *Thermodynamic processes are assumed to be encoded, to within a factor,* $\rho(x,y,z,t...)$, *in terms of contravariant vector direction fields,* $V_4(x,y,z,t...)$.

Axiom 3. *Continuous topological evolution*[19] *of the thermodynamic system can be encoded in terms of Cartan's magic formula (see p. 122 in*[9]*). The Lie differential, when applied to an exterior differential 1-form of Action,* $A = A_k dx^k$, *is equivalent, abstractly, to the first law of thermodynamics.*

$$\text{Cartan's Magic Formula } L_{(\rho\mathbf{V}_4)}A = i(\rho\mathbf{V}_4)dA + d(i(\rho\mathbf{V}_4)A) \tag{1}$$

$$\text{First Law} : W + dU = Q, \tag{2}$$

$$\text{Inexact Heat 1-form } Q = W + dU = L_{(\rho\mathbf{V}_4)}A \tag{3}$$

$$\text{Inexact Work 1-form } W = i(\rho\mathbf{V}_4)dA, \tag{4}$$

$$\text{Internal Energy } U = i(\rho\mathbf{V}_4)A. \tag{5}$$

Axiom 4. *Topological equivalence classes of systems and continuous processes can be defined in terms of the Pfaff topological dimension of the 1-forms of Action, Work, and Heat.*

Cartan's magic formula is abstractly equivalent to the First Law of thermodynamics. For more than 20 years, I have taken this correspondence to be both literal as well as abstract. For a given functional 1-form, A, that encodes some equivalence class of physical systems, it is usual for continuous evolutionary processes, V_4, to produce Work, W, and generate Heat, Q.

If the Pfaff topological dimension of W or Q is greater than 2, the the process is irreducibly path dependent, and if greater than 3, thermodynamically irreversible. This important result is impossible in terms of Hamiltonian dynamics, for which the 1-forms of Heat and Work are always less than Pfaff topological dimension 3, hence admit integrating factors or are exact. For Hamiltonian evolution the concepts of Work and Heat are never path dependent nor irreversible, as topological properties are preserved from initial to final state.

In effect, Cartan's methods can be used to formulate precise mathematical definitions for many thermodynamic concepts in terms of topological properties - without the use of statistics, or geometric constraints, such as metric or connections. The fundamental tool is that of continuous topological evolution, which is distinct from the usual perspective of continuous geometric evolution. Moreover, the method of continuous topological evolution is applicable to the study of non equilibrium thermodynamical systems and irreversible processes- again without the explicit use of statistics or metric constraints. A recent advance was made (in early 2005, following hints deduced over the last 20 years) when it was realized that the concepts of (topological) fluctuations and irreversible processes, as well as topological evolution, are to be associated with processes that have Spinor components relative to the 2-form, $F = dA$.

1.1. Eigen direction fields of the 2-form, F=dA

One of the extraordinary results of the Cartan method is that path dependent work, irreversibility, and fluctuations imply that the evolutionary process, V, must have components proportional to the Spinor isotropic "eigenvectors or eigen direction fields" that are associated with the non-zero antisymmetric matrix $[\mathbb{F}]$ generated by functional coefficients of the 2-form, $F = dA$. Classical historic methods ignore the possibilities that processes can have Spinor, as well as vector (tensor), components relative to the 2-form, F. The foundations of classical methods are based upon symmetric matrices and geometric representations, such as the metric, stress and strain matrices, the stress-energy tensor, and orthonormal basis sets. Symmetric matrices do not admit Spinor isotropic eigen direction fields. It is only when the matrices have antisymmetric components that it is possible for matrices to have Spinor isotropic eigen direction fields.

By direct computation on space time, $\{x, y, z, t\}$, the exterior differential 1-form of Work, W, is recognized to be the 4D generalization of the Newtonian concept of Force times Distance:

$$\text{Work:} \quad W = i(\rho \mathbf{V}_4)dA = i(\rho \mathbf{V}_4)F = \omega^\flat = f_m dx^m + Pdt = f_\mu dx^\mu \tag{6}$$

When an EM formulation is used, the spatial coefficients of W, $f_m dx^m$, are recognized as the Lorentz force. When a Fluid formulation is used, the spatial coefficients of the Work 1-form are recognized as the Lagrange-Euler equations for a fluid. The components of the 1-form of Work, W, can be composed in terms of the matrix product of an anti-symmetric matrix, $[\mathbb{F}] = -[\mathbb{F}]^{transpose}$, and the direction field $\rho \mathbf{V}_4 = \rho[V^x, V^y, V^z, 1]$ as a column vector, $|\rho \mathbf{V}_4\rangle$. The antisymmetric matrix is generated by the functional coefficients of the (anti-symmetric) 2-form of field intensities, $F = dA$, The differential 1- form, W, (which is a scalar with respect to diffeomorphisms) can be constructed as a matrix quadratic form:

$$W = \omega^\flat \equiv \langle [dx, dy, dz, dt] | \circ [\mathbb{F}] \circ |\rho \mathbf{V}_4\rangle \,. \tag{7}$$

The Work 1-form, $W = \omega^\flat$ is not identically zero, but the Skew-symmetric self product can be written as,

$$
\begin{aligned}
i(\rho\mathbf{V}_4)W &= i(\rho\mathbf{V}_4)i(\rho\mathbf{V}_4)F = i(\rho\mathbf{V}_4)\omega^\flat && (8)\\
\text{Skew product} &= \langle\rho\mathbf{V}_4|\circ[\mathbb{F}]\circ|\rho\mathbf{V}_4\rangle \Rightarrow 0, && (9)
\end{aligned}
$$

and is always zero by antisymmetry of $[\mathbb{F}]$.

Suppose $|\rho\mathbf{V}_4\rangle$ is an eigenvector $|\mathbf{e}\rangle$ of the matrix $[\mathbb{F}]$ generated by the 2-form dA such that the work generated by a process in the direction of an eigenvector can be written as:

$$W = i(\mathbf{e})dA = i(\mathbf{e})F \simeq [\mathbb{F}]\circ|\mathbf{e}\rangle = \gamma\,|\mathbf{e}\rangle\,. \tag{10}$$

Then, due to antisymmetry of $[\mathbb{F}]$, it follows that

$$\langle\mathbf{e}|\circ[\mathbb{F}]\circ|\mathbf{e}\rangle = \gamma\,\langle\mathbf{e}|\circ|\mathbf{e}\rangle \Rightarrow 0. \tag{11}$$

For division algebras there are two choices: either $\gamma = 0$, or $\langle\mathbf{e}|\circ|\mathbf{e}\rangle = 0$. The implication is that for non zero eigenvalues γ, the algebraic quadratic form must vanish:

$$\langle\mathbf{e}|\circ|\mathbf{e}\rangle = (e^1)^2 + (e^2)^2 + (e^3)^2 + (e^4)^2 = 0. \tag{12}$$

Such a product with value zero is defined to be a skew symmetric product.

The result is that the "eigenvectors", or eigen direction fields, for the antisymmetric matrix $[\mathbb{F}]$ are of two types:

1. Vector direction fields, $|\mathbf{e}\rangle$, with eigenvalue zero, $\gamma = 0$, and $\langle\mathbf{e}|\circ|\mathbf{e}\rangle \neq 0$
2. Spinor direction fields, $|\boldsymbol{\sigma}\rangle$, with eigenvalues which are not zero, $\gamma \neq 0$, but for which the quadratic form vanishes, $\langle\boldsymbol{\sigma}|\circ|\boldsymbol{\sigma}\rangle = 0$. The spinor eigenvalues are pure imaginary.

It is the latter, Spinor, type of eigenvector direction field that is the principle focus of this article. Such direction fields $|\boldsymbol{\sigma}\rangle$ with zero quadratic forms in classic differential geometry are called isotropic complex vectors . These objects were defined by E. Cartan as images of pure Spinors.[2] The map from complex Spinor components to a complex isotropic vector with a skew symmetric product is not unique. On the otherhand, the map from a tensor to a tensor is uniquely defined in terms of a diffeomorphism. A rotation of a spinor by 2π changes its sign, while the rotation of a vector does not change its sign. As will be demonstrated below, the non-uniqueness is associated with properties of polarization and chirality.

The bottom line for purposes herein is that Spinors are normal consequences of antisymmetric matrices, and as topological artifacts, are not restricted to physical microscopic, quantum or relativistic constraints. In the Cartan exterior algebra, every p form over N independent variables that is not closed has an embedding into a vector space of dimension equal to combinations of N things taken p at a time. Hence, if not zero, the exterior derivative of a p-form will be an antisymmetric object that may have Spinor eigenvectors. In classical analysis, the effect of Spinors in the dynamics of physical systems (such as a fluid or a plasma) are ignored.

For a 4 x 4 antisymetric matrix the components can be denoted (in equidimensional $(x, y, z, s = Ct)$ electromagnetic format) as

$$[\mathbb{F}] = \begin{matrix} 0 & B_3 & -B_2 & E_1/C \\ -B_3 & 0 & B_1 & E_2/C \\ B_2 & -B_1 & 0 & E_3/C \\ -E_1/C & -E_2/C & -E_3/C & 0 \end{matrix} \tag{13}$$

The eigenvalues of the antisymmetric matrix $[\mathbb{F}]$ have recognizable features in terms of the electromagnetic format. Consider the functions

$$\begin{aligned}
\text{Magnetic energy density} \quad u &= \langle \mathbf{B}| \circ |\mathbf{B}\rangle /\mu = \langle \mathbf{B}| \circ |\mathbf{H}\rangle, & (14)\\
\text{Electric energy density} \quad v &= \varepsilon \langle \mathbf{E}| \circ |\mathbf{E}\rangle = \langle \mathbf{D}| \circ |\mathbf{E}\rangle, & (15)\\
\text{Hamiltonian Field Energy density } h &= \langle \mathbf{D}| \circ |\mathbf{E}\rangle + \langle \mathbf{B}| \circ |\mathbf{H}\rangle \quad \text{(times 2)} & (16)\\
\text{Bulk Viscosity dissipation} &= 2\langle \mathbf{E}| \circ |\mathbf{B}\rangle & (17)\\
\varepsilon\mu C^2 &= 1 & (18)\\
\text{(Lorentz Space) Impedance} \quad Z &= \sqrt{\mu/\varepsilon} & (19)
\end{aligned}$$

Then the 4 eigenvalues of $[\mathbb{F}]$ are (in general)

$$\text{Eigenvalues: } \lambda = \pm(\sqrt{\mu/2})\sqrt{-h \pm \sqrt{h^2 - (\sqrt{\varepsilon/\mu}2\langle \mathbf{E}| \circ |\mathbf{B}\rangle)^2}}. \tag{20}$$

and are pure imaginary. The 4 associated eigenvectors $|\boldsymbol{\sigma}\rangle = 0$ are complex isotropic direction fields; they are spinors.

$$\langle \boldsymbol{\sigma}| \circ |\boldsymbol{\sigma}\rangle = 0 \tag{21}$$

The characteristic polynomial for $[\mathbb{F}]$ in terms of its eigenvalues λ is

$$\begin{aligned}
CHP \quad : \quad &= \lambda^4 - M\lambda^3 + G\lambda^2 - K\lambda + \ B = 0 & (22)\\
&= \lambda^4 + \mu h\lambda^2 + \ \mu^2(\sqrt{\varepsilon/\mu}\langle \mathbf{E}| \circ |\mathbf{B}\rangle)^2 = 0, & (23)
\end{aligned}$$

from which it is apparent that the coefficient of the cubic term, M, (equal to the trace of the matrix and related to Mean Curvature) is zero. Also the coefficient of linear term, K, (equal to the trace of the matrix adjoint to $[\mathbb{F}]$ and related to the Cubic Curvature) is zero.

Note that the determinant of the matrix is

$$\det[\mathbb{F}] = (\langle \mathbf{E}| \circ |\mathbf{B}\rangle /C)^2 \tag{24}$$

so that if the determinant is not zero, then the "viscosity" (bulk viscosity or rotation-expansion) coefficient is not zero, and the matrix does not have eigen*values* which are zero. Hence, all eigenvector direction fields are spinors if the bulk viscosity coefficient is not zero.

When $\langle \mathbf{E}| \circ |\mathbf{B}\rangle \neq 0$ there exists a unique direction field T_4 defined by

$$i(T_4)dx\hat{}dy\hat{}dz\hat{}dt = i(T_4)\Omega_4 = A\hat{}dA \tag{25}$$

such that evolution in the direction of T_4 is thermodynamically irreversible.[19] T_4 is defined as the Topological Torsion 4 Vector. As $i(T_4)dA = 2(\mathbf{E} \circ \mathbf{B})\Omega_4 \neq 0$, T_4 must have spinor components.

Now consider domains where $\langle \mathbf{E}| \circ |\mathbf{B}\rangle \Rightarrow 0$. In such domains, there exist two (real) eigenvectors with zero eigen values, and two spinor eigen direction fields. The eigenvectors with zero eigenvalues form the two "extremal" solutions of Hamiltonian dynamics. For the electromagetic format, the Work 1-form becomes, relative to a process $[\mathbf{J}, \rho]$

$$W = i(J_4)dA = i(J_4)F = (\rho\mathbf{E} + \mathbf{J} \times \mathbf{B})_m dx^m + (\mathbf{J} \circ \mathbf{E})dt. \tag{26}$$

The spatial components are immediately recognizable as the Lorentz force (density) $(\rho\mathbf{E}+\mathbf{J}\times\mathbf{B})_m$. The timelike component is recognized as the Power (density), $(\mathbf{J}\circ\mathbf{E})$. One of the eigenvectors with zero eigenvalue corresponds to a process where

$$\begin{aligned}
\text{Bulk viscosity} \quad (\mathbf{E}\circ\mathbf{B}) &= 0, && (27)\\
\text{Ohmic power} \quad (\mathbf{J}\circ\mathbf{E}) &= 0 && (28)\\
\text{Free charge density} \quad \rho &= 0, && (29)\\
\text{J} \parallel \text{to B} \quad \mathbf{J}\times\mathbf{B} &= 0. && (30)
\end{aligned}$$

The current $\mathbf{J}$ is in the direction of the $\mathbf{B}$ field and is orthogonal to the $\mathbf{E}$ field.

The second eigenvector of zero eigen value, corresponds to the "force-free" case, where the components of the Lorentz force vanish because the electric term, $\rho\mathbf{E}$, cancels the magnetic term, $\mathbf{J}\times\mathbf{B}$.

$$\begin{aligned}
\text{Bulk viscosity} \quad (\mathbf{E}\circ\mathbf{B}) &= 0 && (31)\\
\text{Ohmic power} \quad (\mathbf{J}\circ\mathbf{E}) &= 0 && (32)\\
\text{Terms cancel} \quad \rho\mathbf{E}+\mathbf{J}\times\mathbf{B} &= 0. && (33)
\end{aligned}$$

The current is not parallel to the $\mathbf{B}$ field.

Similar arguments can be used to demonstrate that if *any* matrix has an antisymmetric component, then there *may* exist isotropic spinor eigenvectors. It is also possible to extend the algebraic arguments to the generalized eigenvalue problem for an antisymmetric matrix, $[\mathbb{F}]$, relative to a signature matrix, $[\eta]$:

$$\begin{aligned}
\text{Generalized Eigenvalue equation} \quad &: \quad [\mathbb{F}]\circ|\mathbf{e}\rangle = \gamma\,[\eta]\circ|\mathbf{e}\rangle\,, && (34)\\
\langle\mathbf{e}|\circ[\mathbb{F}]\circ|\mathbf{e}\rangle &= \gamma\,\langle\mathbf{e}|\circ[\eta]\circ|\mathbf{e}\rangle \Rightarrow 0. && (35)
\end{aligned}$$

The signature matrix $[\eta]$ is a diagonal matrix of with plus or minus unit entries. Also note that

$$\begin{aligned}
[[\eta]\circ[\mathbb{F}]]\circ|\mathbf{e}\rangle &= \gamma\,|\mathbf{e}\rangle\,, && (36)\\
\text{or, } [\mathbb{F}_{Gottlieb}]\circ|\mathbf{e}\rangle &= \gamma\,|\mathbf{e}\rangle && (37)
\end{aligned}$$

such that the eigenvectors of the matrix $[[\eta]\circ[\mathbb{F}]]=[\mathbb{F}_{Gottlieb}]$ (Cite Gottlieb) are the same as the eigen direction fields of the matrix $[\mathbb{F}]$ relative it a signature matrix $[\eta]$. The same conclusions about the *number* of Spinor eigenvectors hold, even though the Gottlieb matrix is not anti-symmetric. Solutions will be compared below for the 4D Euclidean metric and the 4D Minkowski metric. It will be demonstrated how the constraint of a Euclidean metric or a Minkowski metric influences the Spinor category of the Eigenvectors for an antisymmetric matrix, $[\mathbb{F}]$. The signs and algebraic content of the similarity invariants are usually different for $[\mathbb{F}]$ and $[\mathbb{F}_{Gottlieb}]$.

It is unfortunate that the historic word "isotropic" is used to describe a vector with a zero quadratic form, for in engineering practice, the word isotropic is usually interpreted as meaning a property that is "the same in all directions". Technically the word isotropic used for the "vector of zero quadratric form" is correct, for no matter what direction the Spinor points, its quadratic form, as a sum of squares of the components, is zero, and is the "same" in all directions. The discussion above is valid for all antisymmetric N x N matrices.

The Work 1-form, W, admits an integrating factor if its Pfaff topological dimension is less than 3. The evolutionary process then can be reparameterized by a judicious choice of an integrating factor such that $dW=0$. If the Work is an exact differential 1-form without periods it is not path dependent. If the work is closed but not exact, then there are topological obstructions that make the work path dependent.

If Work is to be path dependent, then the evolutionary process must have Spinor components.

As a corollary, if the Work is path independent, then the process is reversible. Irreversible processes must have Spinor components. To insure that the Skew product is zero, a possible formulation of W is of the form,

$$W = i(V)F = f_k(dx^k - V^k dt) = f_k \Delta^k + dH(x, y, z, t), \tag{38}$$

where $\Delta^k = (dx^k - V^k dt)$ and represents possible kinematic topological fluctuations, or deviations from kinematic perfection. It must be true that $H(x, y, z, t)$ is an evolutionary invariant, or as Cartan says, a "first integral" such that

$$L(V)H = i(V)dH = 0. \tag{39}$$

It follows that $i(V)i(V)F = i(V)W = 0$. If $W = 0$, then the process V does not have spinor components, but is proportional to an eigenvector of $[F]$ with eigen value zero. It follows that if $W = 0$ and on the hypersurface, $H =$ constant, either the fluctuations 1-forms vanish, or the components of the force vanish:

$$\text{If} \quad W = 0 \text{ then} \tag{40}$$

$$(dx^k - V^k dt) = \Delta^k \Rightarrow 0, \tag{41}$$

$$\text{or} \quad f_k \Rightarrow 0 \tag{42}$$

In both cases, $W \Rightarrow f_k \Delta^k = 0$, and the process V does not have spinor components. A more subtle case occurs when the 1-form $f_k \Delta^k$ is exact and equal to $-dH(x, y, z, t)$. On the otherhand, if the work 1-form is non zero, the process must have spinor components. If there are no first integrals, the force components do not vanish identically, and there must be kinematic (topological) fluctuation. Such topological fluctuations will be related to pressure (and temperature on the Tangent bundle).

1.2. The 3 component isotropic spinor representations.

1.2.1. The Cartan Segre construction

There are many constructions to produce a 3 component isotropic-vector or Spinor direction field, $|\mathbf{S}\rangle = [\sigma_1, \sigma_2, \sigma_3]^T$, such that $\langle \mathbf{S}| \circ |\mathbf{S}\rangle = 0$. For example, consider the Cartan construction (see page 41[2]), where the two complex variables, α and β, are complex conjugates:

$$\text{Cartan} \quad : 3\text{D} \quad \text{isotropic Spinor : R2} \Rightarrow \text{C3} \tag{43}$$

$$\alpha = u + \sqrt{-1}v, \quad \beta = u - \sqrt{-1}v \tag{44}$$

$$\sigma_1 = \alpha^2 - \beta^2 = 2(u^2 - v^2) \tag{45}$$

$$\sigma_2 = \pm \sqrt{-1}(\alpha^2 + \beta^2) = \pm \sqrt{-1}(2(u^2 - v^2)), \tag{46}$$

$$\sigma_3 = \mp 2\alpha\beta = \mp 2(u^2 + v^2), \tag{47}$$

$$\langle \mathbf{S}| \circ |\mathbf{S}\rangle = \sigma_1^2 + \sigma_2^2 + \sigma_3^2 = 0. \tag{48}$$

1.2.2. The Hopf Construction

Another construction is related to the Hopf map (see details below), where the two complex variables, α and β, are distinct, and not merely complex conjugates.

$$\text{Hopf} \quad : 3\text{D} \quad \text{isotropic Spinor : R4} \Rightarrow \text{C2} \Rightarrow \text{C3} \tag{49}$$

$$\alpha = X + \sqrt{-1}Y, \quad \beta = Z + \sqrt{-1}S \tag{50}$$

$$\sigma_1 = \alpha^2 - \beta^2 = X^2 - Y^2 - Z^2 + S^2 + \sqrt{-1}(2YX - 2SZ), \tag{51}$$

$$\sigma_2 = \pm \sqrt{-1}(\alpha^2 + \beta^2) = -2YX - 2SZ + \sqrt{-1}(X^2 - Y^2 + Z^2 - S^2), \tag{52}$$

$$\sigma_3 = \mp 2\alpha\beta = 2XZ - 2YS + \sqrt{-1}(2YZ + 2XS), \tag{53}$$

$$\langle \mathbf{S}| \circ |\mathbf{S}\rangle = \sigma_1^2 + \sigma_2^2 + \sigma_3^2 = 0. \tag{54}$$

The 3 component Spinor image $|\mathbf{S}\rangle$ is an isotropic direction field with a zero value for its quadratic form. The ambiguity in signs can be related to the concepts of polarization and chirality.

The formulas above are remarkable for the complex pair, α, β, can have many realizations in terms of maps and their complex conjugates. In each case, the map to the 3D spinor is given by same formula set given for evaluating the components of $|\mathbf{S}\rangle = [\sigma_1, \sigma_2, \sigma_3]^T$, in terms of the complex functions, α and β. It is also true that the formulas for the components can be permuted to give different Spinor direction fields. A 4 component Spinor may be viewed as the set $[\sigma_1, \sigma_2, \sigma_3,\ S]$ subject to the "projective complex hypersurface" constraint that $\langle \mathbf{S}| \circ |\mathbf{S}\rangle = S^2 = \sigma_1^2 + \sigma_2^2 + \sigma_3^2 = 0$.

1.2.3. The Bateman plane wave construction

Similar formulas were used by Bateman in his original studies of Electromagnetic waves,[1] but (apparently) without the realization that these concepts were related to Spinors or surfaces of Zero mean curvature. The formulation has an interesting interpretation (for a charge current free region) in terms of the Poincare invariants and the energy density of the Field. The complex vector field $\mathbf{M} = (\mathbf{B} + \sqrt{-1}\mathbf{E}/c)$ is a spinor when the Poincare functions vanish:

$$\begin{aligned}
\text{Bateman} &: \text{3D isotropic Spinor : R6} \Rightarrow \text{C3} && (55)\\
|\mathbf{M}\rangle &= \left|(\mathbf{B} + \sqrt{-1}\mathbf{E}/c)/\sqrt{\mu}\right\rangle = [\sigma_1, \sigma_2, \sigma_3]^T, && (56)\\
\sigma_k &= \mathbf{B}_k + \sqrt{-1}\mathbf{E}_k/c && (57)\\
\langle \mathbf{M}| \circ |\mathbf{M}\rangle &= (\langle \mathbf{B}| \circ |\mathbf{B}\rangle/\mu - \langle \mathbf{E}| \circ |\mathbf{E}\rangle/\mu c^2) + 2\sqrt{-1}\,\langle \mathbf{E}| \circ |\mathbf{B}\rangle\,(\varepsilon c/\mu) && (58)\\
&= \langle \mathbf{B}| \circ |\mathbf{H}\rangle - \langle \mathbf{D}| \circ |\mathbf{E}\rangle + 2\sqrt{-1}\,\langle \mathbf{E}| \circ |\mathbf{B}\rangle\,c/Z^2 && (59)\\
&= (Poincare\ I) - \sqrt{-1}(2 \cdot Poincare\ II)c/Z^2 && (60)\\
\langle \mathbf{M}| \circ |\mathbf{M}\rangle &= \sigma_1^2 + \sigma_2^2 + \sigma_3^2 = 0,\ \text{if Poincare I =0, and Poincare II=0.} && (61)
\end{aligned}$$

The requirements that the second Poincare invariant vanishes, $\langle \mathbf{E}| \circ |\mathbf{B}\rangle \Rightarrow 0$, implies that the Pfaff topological dimension of the 1-form, A, such that $F = dA$, is less than 4, for the determinant of $[\mathbb{F}]$ is proportional to $dA\hat{}dA \simeq \langle \mathbf{E}| \circ |\mathbf{B}\rangle$. When the deterinant of the matrix is zero, there must exist 1 or more zero eigenvalues, and hence eigenvectors which are not Spinors. Also the magnitude of the field intensities are such that $\mathbf{B} = \mathbf{E}/c$ when the first Poincare invariant $\langle \mathbf{B}| \circ |\mathbf{H}\rangle - \langle \mathbf{D}| \circ |\mathbf{E}\rangle \Rightarrow 0$. Hence the Bateman spinor describes the classic plane wave with the two properties,

$$\begin{aligned}
\text{Classic} &: \text{Plane Wave Properties} && (62)\\
\mathbf{B} &= \mathbf{E}/c && (63)\\
\langle \mathbf{E}| \circ |\mathbf{B}\rangle &= 0. && (64)
\end{aligned}$$

1.2.4. The Weierstrass construction

To demonstrate the connection to Minimal Surface theory, note that the Cartan map can be written in Weierstrass form

$$\begin{aligned}
\text{Weierstrass} &: \text{3D isotropic Spinor : R2} \Rightarrow \text{C3} && (65)\\
f &= \alpha^2, \qquad g = \beta/\alpha && (66)\\
\sigma_1 &= (1 - g^2)f && (67)\\
\sigma_2 &= \pm\sqrt{-1}(1 + g^2)f, && (68)\\
\sigma_3 &= \mp 2fg && (69)\\
\langle \mathbf{S}| \circ |\mathbf{S}\rangle &= \sigma_1^2 + \sigma_2^2 + \sigma_3^2 = 0. && (70)
\end{aligned}$$

The components of the Spinor yield a complex direction field, $|\mathbf{S}(z)\rangle$, if each of the functions g and f are assumed to functions of a complex variable, z. The integral of the Spinor $|\mathbf{S}(z)\rangle$ with respect to the complex variable z yields a complex position vector:

$$|\mathbf{R}(z)\rangle = \left| \begin{array}{c} \int \sigma_1 dz \\ \int \sigma_2 dz \\ \int \sigma 3 dz \end{array} \right\rangle . \tag{71}$$

The real part of the position vector Real($|\mathbf{R}(z)\rangle$) defines a minimal surface, and the imaginary part of the position vector Imag($|\mathbf{R}(z)\rangle$) defines another (conjugate) minimal surface as an immersion from $z = u + iv$ to $\{x, y, z\}$.

1.2.5. The Entropy-Fractal construction

A most remarkable variant on the Weierstrass construction is given by choosing $g = z$ and $f = F(z)$. The spinor formulas become

$$\text{Entropy} \quad : \quad \text{3D isotropic Spinor : R2} \Rightarrow \text{C3} \tag{72}$$
$$f = F(z), \qquad g = z \tag{73}$$
$$\sigma_1 = (1 - z^2)F(z) \tag{74}$$
$$\sigma_2 = \pm\sqrt{-1}(1 + z^2)F(z), \tag{75}$$
$$\sigma_3 = \mp 2zF(z) \tag{76}$$
$$\langle \mathbf{S}| \circ |\mathbf{S}\rangle = \sigma_1^2 + \sigma_2^2 + \sigma_3^2 = 0. \tag{77}$$

Again, the complex position vector can be constructed from the single complex function, $F(z)$

$$|\mathbf{R}(z)\rangle = \left| \begin{array}{c} \int \sigma_1 dz \\ \int \sigma_2 dz \\ \int \sigma_3 dz \end{array} \right\rangle = \left| \begin{array}{c} \int (1 - z^2)F(z)dz \\ \int \pm\sqrt{-1}(1 + z^2)F(z)dz \\ \int \mp 2zF(z)dz \end{array} \right\rangle . \tag{78}$$

For example, $F(z) = -a/z^2$generates the right helicoid and the function $F(z) = \sqrt{-1}a/z^2$ generates the catenoid, which are conjugate minimal surfaces.

The function $F(z)$ can be deduced from the triple differentian of another complex function $s(z)$ (p. 188[17]). Of greater physical interest is to construct the function $s(z)$ with complex constants $A, B, C, D, ...$ as follows:

$$s(z) = A + B \cdot zln(z) + Cz + Dz^2 + Ez^4.... \tag{79}$$

The construction $A + Dz^2$ forms the complex germ of the Mandlebrot set, and it generates a minimal surface, as does every interate (using the theorem of Sophus Lie that any complex analytic function generates a minimal surface). The even powers of z can be used to express every iterate of the Mandelbrot set.

$$\text{Mandelbrot term} \quad : \quad A + Dz^2 \tag{80}$$
$$\text{First iterate} \quad : \quad A + D(A^2 + 2Az^2 + Dz^4) \tag{81}$$
$$\text{Second iterate} \quad : \quad ...... \tag{82}$$

The terms $B \cdot zln(z) + Cz$ also are recognizeable as the Gibbs entropy when $C = -B$:

$$\text{Gibbs Entropy term} = B(zln(z) - z). \tag{83}$$

The function s(z) so constructed can be interpreted in terms an entropy component plus an iterate of the Mandelbrot set.

$$s(z) = \text{Entropy} + (\text{sum of Mandelbrot iterates}) \tag{84}$$

Struik points out that the third derivative of s, or $s\acute{} = F(z)$, can be used for all real minimal surfaces.

$$
\begin{aligned}
s\acute{} &= B + B\ln(z) + C + 2Dz + 4Ez^3.... && (85)\\
s\acute{} &= B/z + 2D + 12Ez^2.... && (86)\\
s\acute{} &= -B/z^2 + 24Ez.... && (87)
\end{aligned}
$$

$$
\begin{aligned}
\text{Entropy Fractal} &: \quad \text{3D isotropic Spinor : R2} \Rightarrow \text{C3} && (88)\\
F(z) &= -B/z^2 + 24Ez...., && (89)\\
\sigma_1 &= (1 - z^2)(-B/z^2 + 24Ez....) && (90)\\
\sigma_2 &= \pm\sqrt{-1}(1 + z^2)(-B/z^2 + 24Ez....), && (91)\\
\sigma_3 &= \mp 2z(-B/z^2 + 24Ez....) && (92)\\
\langle \mathbf{S}| \circ |\mathbf{S}\rangle &= \sigma_1^2 + \sigma_2^2 + \sigma_3^2 = 0. && (93)
\end{aligned}
$$

This result is remarkable for it relates a formula for entropy + fractals to a formula for minimal surfaces and to Spinors. All iterates of the Mandelbrot germ are retained and contribute even powers to the generalized formula for $s(z)$. Also the leading term in the Spinor formulation, $F(z) = -B/z^2....$ generates the conjugate catenoid and helicoid for the entropy contributions which are then modified for the fractal contributions.

2. THE 3 COMPONENT HOPF VECTOR

This construction for the isotropic spinor $\boldsymbol{\sigma} = [\sigma_1, \sigma_2, \sigma_3]^T$ should be compared to the construction of the 3 component Hopf vector, $|\mathbf{H1}\rangle = [h_1, h_2, h_2]^T$, which also has topological significance. The Hopf vector can be represented by a map from a set of complex variables. A particular example is given by the formulas

$$
\begin{aligned}
\text{3D Hopf} &: \quad \text{R4} \Rightarrow \text{C2} \Rightarrow \text{R3} && (94)\\
\alpha &= X + \sqrt{-1}Y, \quad \beta = Z + \sqrt{-1}S && (95)\\
h_1 &= \alpha \cdot \beta^* + \beta \cdot \alpha^* = 2XZ + 2YS, && (96)\\
h_2 &= \sqrt{-1}(\alpha \cdot \beta^* - \beta \cdot \alpha^*) = -2YZ + 2XS, && (97)\\
h_3 &= \alpha \cdot \alpha^* - \beta \cdot \beta^* = X^2 + Y^2 - Z^2 - S^2, && (98)\\
\langle \mathbf{H1}| \circ |\mathbf{H1}\rangle &= h_1^2 + h_2^2 + h_3^2 = (X^2 + Y^2 + Z^2 + S^2)^2 && (99)
\end{aligned}
$$

The Hopf vector is real and of finite quadratic form. For a Hypersphere $(X^2 + Y^2 + Z^2 + S^2)^2 = \pm 1$ in R4, the map produces a Spherical Surface in R3, of unit radius. The variables α and β also can be viewed as two distinct complex variables defining ordered pairs of the four variables $[X, Y, Z, S]$. For example, the classic format given above for $|\mathbf{H1}\rangle$ was obtained from the expansion, $\alpha = X + \sqrt{-1}Y$, $\beta = Z + \sqrt{-1}S$. Other selections for the ordered pairs of (X, Y, Z, S) (along with permutations of the 3 vector components) give distinctly different Hopf vectors. For example, the ordered pairs, $\alpha = X + \sqrt{-1}Z$, $\beta = Y + \sqrt{-1}S$, give

$$|\mathbf{H}2\rangle = [2(YX - SZ), X^2 + Z^2 - Y^2 - S^2, -2(ZY + SX)]^T \tag{100}$$

which is another Hopf vector, a map from R4 to R3, but with the property that $\mathbf{H}2$ is orthogonal to $\mathbf{H}1$:

$$\langle \mathbf{H}2| \circ |\mathbf{H}1\rangle = 0. \tag{101}$$

Similarly, a third linearly independent orthogonal Hopf vector $\mathbf{H}3$ can be found

$$|\mathbf{H}3\rangle = [X^2 + Y^2 - Z^2 - S^2, -2(YX + SZ), 2(-ZX + SY)]^T \tag{102}$$

such that

$$\langle \mathbf{H}2| \circ |\mathbf{H}1\rangle = \langle \mathbf{H}3| \circ |\mathbf{H}2\rangle = \langle \mathbf{H}1| \circ |\mathbf{H}3\rangle = 0 \tag{103}$$

$$\langle \mathbf{H}1| \circ |\mathbf{H}1\rangle = \langle \mathbf{H}2| \circ |\mathbf{H}2\rangle = \langle \mathbf{H}3| \circ |\mathbf{H}3\rangle = (X^2 + Y^2 + Z^2 + S^2)^2.. \tag{104}$$

By comparing the two tables for the 3D Hopf vector and the 3D Spinor it is apparent that the real and the imaginary components of the Spinor are composed of two (perhaps permuted) orthogonal Hopf vectors added together in a complex fashion.

$$Spinor = \mathbf{H1} \pm \sqrt{-1}\mathbf{H2} \ . \tag{105}$$

Spinors can be associated with complex linear combinations of Hopf Maps

The 3D isotropic (null) complex position vector, $[z1, z2, z3]$ can be decomposed into a real and an imaginary part, such that both parts have the same magnitude and are orthogonal. In short, the Cartan Spinor, [16] can be represented as the complex sum of two Hopf vectors. The spinors come in two triples of the form

$$|\boldsymbol{\sigma}_{12}\rangle = |\mathbf{H}1\rangle \pm \sqrt{-1}\,|\mathbf{H}2\rangle \quad with \quad \langle \boldsymbol{\sigma}_{12}| \circ |\boldsymbol{\sigma}_{12}\rangle = 0 \tag{106}$$

$$|\boldsymbol{\sigma}_{23}\rangle = |\mathbf{H}2\rangle \pm \sqrt{-1}\,|\mathbf{H}3\rangle \quad with \quad \langle \boldsymbol{\sigma}_{23}| \circ |\boldsymbol{\sigma}_{23}\rangle = 0 \tag{107}$$

$$|\boldsymbol{\sigma}_{31}\rangle = |\mathbf{H}3\rangle \pm \sqrt{-1}\,|\mathbf{H}1\rangle \quad with \quad \langle \boldsymbol{\sigma}_{31}| \circ |\boldsymbol{\sigma}_{31}\rangle = 0 \tag{108}$$

The 3D complex Spinors constructed as above can be used to describe conjugate pairs of surfaces of zero mean curvature (see page 63 in[8]).

Another realization of the Hopf map is given by the formulas

$$\text{3D Hopf} \ : \ \text{R4} \Rightarrow \text{C1} \Rightarrow \text{R3} \tag{109}$$

$$\alpha = u + \sqrt{-1}v, \quad \beta = u - \sqrt{-1}v \tag{110}$$

$$h_1 = \alpha \cdot \beta^* + \beta \cdot \alpha^* = 2(u^2 - v^2), \tag{111}$$

$$h_2 = \sqrt{-1}(\alpha \cdot \beta^* - \beta \cdot \alpha^*) = -4uv, \tag{112}$$

$$h_3 = \alpha \cdot \alpha^* - \beta \cdot \beta^* = 0, \tag{113}$$

$$\langle \mathbf{H1}| \circ |\mathbf{H1}\rangle = h_1^2 + h_2^2 + h_3^2 = +(2(u^2 + v^2))^2 \tag{114}$$

In terms of the EM format

$$\text{3D Hopf} \ : \ \text{R6} \Rightarrow \text{C1} \Rightarrow \text{R3} \tag{115}$$

$$\alpha = u + \sqrt{-1}v = c^2 \langle \mathbf{B}| \circ |\mathbf{B}\rangle + \sqrt{-1}\,\langle \mathbf{E}| \circ |\mathbf{E}\rangle\,, \tag{116}$$

$$\beta = u - \sqrt{-1}v = c^2 \langle \mathbf{B}| \circ |\mathbf{B}\rangle - \sqrt{-1}\,\langle \mathbf{E}| \circ |\mathbf{E}\rangle \tag{117}$$

$$h_1 = \alpha \cdot \beta^* + \beta \cdot \alpha^* = 2(c^2 \langle \mathbf{B}| \circ |\mathbf{B}\rangle - \langle \mathbf{E}| \circ |\mathbf{E}\rangle), \tag{118}$$

$$h_2 = \sqrt{-1}(\alpha \cdot \beta^* - \beta \cdot \alpha^*) = -4c\,\langle \mathbf{E}| \circ |\mathbf{B}\rangle\,, \tag{119}$$

$$h_3 = \alpha \cdot \alpha^* - \beta \cdot \beta^* = 0, \tag{120}$$

$$\langle \mathbf{H1}| \circ |\mathbf{H1}\rangle = h_1^2 + h_2^2 + h_3^2 = (2c^2 \langle \mathbf{B}| \circ |\mathbf{B}\rangle + \langle \mathbf{E}| \circ |\mathbf{E}\rangle)^2 \tag{121}$$

2.1. Spinors Eigen direction fields of vectors F=dA.

The construction of the eigenvectors for a 4x4 antisymmetric matrix can present a formidable algebraic problem. In that which follows, specific choices for the matrix elements of eq. **??** will be made to simplify the algebra. The choices will correspond to the case $\langle \mathbf{E}| \circ |\mathbf{B}\rangle \neq 0$ (the Pfaff dimension 4 case) and the case $\langle \mathbf{E}| \circ |\mathbf{B}\rangle = 0$ (the Pfaff dimension 2 or 3 case).

$$\text{Example 1 :} \qquad \langle \mathbf{E}| \circ |\mathbf{B}\rangle \neq 0 \qquad \text{(Euclidean)} \tag{122}$$

$$[\mathbb{F}]_{PTD=4} = \begin{bmatrix} 0 & B_3 & 0 & 0 \\ -B_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & E_3/c \\ -0 & 0 & -E_3/c & 0 \end{bmatrix}, \tag{123}$$

$$M = 0,\ G = \{E_3^2 + c^2 B_3^2\}/C^2,\ K = 0,\ B = (E_3 B_3/c)^2 \tag{124}$$

$$[\mathbb{E}igenvectors] = \begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & i & -i \\ -i & i & 0 & 0 \\ 1 & 1 & 0 & 0 \end{bmatrix} \tag{125}$$

$$\text{Eigenvalues} = [iE_3/c,\ -iE_3/c,\ iB_{3,}\ -iB_3]. \tag{126}$$

Note that for the PTD=4 case, all 4 eigenvectors are Spinors. Also note that although the mean curvature, M, is zero, the Gauss curvature is positive, and the surface is not an embedding of a minimal surface, where the Gauss curvature of the 2 surface is negative.

$$\text{Example 2} \quad : \quad \mathbf{E} \neq 0,\ \mathbf{B} \neq 0, \langle \mathbf{E}| \circ |\mathbf{B}\rangle = 0 \ \text{(Euclidean)} \tag{127}$$

$$[\mathbb{F}]_{PTD=2} = \begin{bmatrix} 0 & 0 & -B_2 & E_1/c \\ B_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -E_1/c & 0 & 0 & 0 \end{bmatrix} \tag{128}$$

$$M = 0,\ G = \{+c^2 B_2^2 + E_1^2\}/c^2,\ K = 0,\ B = 0 \tag{129}$$

$$[\mathbb{E}igenvectors] = \begin{bmatrix} \sqrt{-G}/B_2 & -\sqrt{-G}/B_2 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 \\ -E_1/B_2 c & -E_1/B_2 c & (B_2 c)/E_1 & 0 \end{bmatrix} \tag{130}$$

$$\text{Eigenvalues} = [+\sqrt{-G}/c, -\sqrt{-G}/c, 0, 0]. \tag{131}$$

The (Euclidean) quadratic forms (or norm) of the first two column (Spinor) eigenvectors are always zero. Note that the Spinor eigenvalues are pure imaginary, and are complex conjugates. The imaginary eigenvalue implies the existence of a rotation is to be associated with the Spinor eigenvectors. The norm of the 4th eigenvector column is 1, but the norm of the third eigenvector column is a number depending upon the ratio of (magnetic plus electric energy) to (electric energy).

$$\text{Example 3} \quad : \quad \mathbf{E}=0,\ \mathbf{B}\neq 0, \langle\mathbf{E}|\circ|\mathbf{B}\rangle = 0 \text{ (Euclidean)} \tag{132}$$

$$[\mathbb{F}]_{PTD=2} = \begin{bmatrix} 0 & B_3 & -B_2 & 0 \\ -B_3 & 0 & B_1 & 0 \\ B_2 & -B_1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{133}$$

$$M = 0,\ G=\{\mathbf{B}\circ\mathbf{B}\},\ K=0,\ B=0 \tag{134}$$

$$[\mathbb{E}igenvectors] = \begin{bmatrix} (\sqrt{-G}B_3 - B_2B_1)/(B_1^2+B_3^2) & (\sqrt{-G}B_3 - B_2B_1)/(B_1^2+B_3^2) & 1 & 0 \\ 1 & 1 & B_2/B_1 & 0 \\ (-\sqrt{-G}B_1 - B_3B_2)/(B_1^2+B_3^2) & -\sqrt{-G}B_1 - B_2B_3)/(B_1^2+B_3^2) & B_3/B_1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{135}$$

$$\text{Eigenvalues} = [+\sqrt{-G}, -\sqrt{-G}, 0, 0]. \tag{136}$$

$$\text{Example 4} \quad : \quad \mathbf{E}\neq 0,\ \mathbf{B}=0, \langle\mathbf{E}|\circ|\mathbf{B}\rangle = 0 \text{ (Euclidean)} \tag{137}$$

$$[\mathbb{F}]_{PTD=2} = \begin{bmatrix} 0 & 0 & 0 & E_1 \\ 0 & 0 & 0 & E_1 \\ 0 & 0 & 0 & E_1 \\ -E_1 & -E_2 & -E_3 & 0 \end{bmatrix} \tag{138}$$

$$M = 0,\ G=\{-\mathbf{E}\circ\mathbf{E}/c^2\},\ K=0,\ B=0 \tag{139}$$

$$[\mathbb{E}igenvectors] = \begin{bmatrix} -i\sqrt{G}/B_3 & +i\sqrt{G}/B_3 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 0 & 0 & -E_1/E_3 & -E_1/E_3 \\ E_1/(B_3c) & E_1/(B_3c) & 0 & 0 \end{bmatrix} \tag{140}$$

$$\text{Eigenvalues} = [+\sqrt{G}, -\sqrt{G}, 0, 0]. \tag{141}$$

2.1.1. Special cases of the Minkowski metric, {plus,plus,plus,minus}

Consider four examples with specific choices for the matrix elements of eq. **??** that correspond to $\langle\mathbf{E}|\circ|\mathbf{B}\rangle \neq 0$ (the Pfaff dimension 4 case) and $\langle\mathbf{E}|\circ|\mathbf{B}\rangle = 0$ (the Pfaff dimension 3 case).

$$\text{Example 1}: \quad \langle\mathbf{E}|\circ|\mathbf{B}\rangle \neq 0 \quad \text{(Minkowski)} \tag{142}$$

$$[F]_{PTD=4} = \begin{bmatrix} 0 & B_3 & 0 & 0 \\ -B_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & E_3/c \\ 0 & 0 & -E_3/c & 0 \end{bmatrix}, \tag{143}$$

$$M = 0,\ G=\{-E_3^2+c^2B_3^2\}/C^2,\ K=0,\ B=-(E_3B_3/c)^2 \tag{144}$$

$$[\mathbb{E}igenvectors] = \begin{bmatrix} 0 & 0 & -i & i \\ 0 & 0 & 1 & 1 \\ 1 & -1 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{bmatrix} \tag{145}$$

$$\text{Eigenvalues} = [E_3/c,\ -iE_3/c,\ iB_3,\ -iB_3]. \tag{146}$$

Note that for the PTD=4 case, all 4 eigenvectors are Spinors in the sense that $\langle\mathbf{e}| \circ [\eta]\circ|\mathbf{e}\rangle = 0$, and $\gamma \neq 0$. Also note that although the mean curvature, M, is zero, the Gauss curvature is negative or positive, and the surface is not an embedding of a minimal surface, where the Gauss curvature of the 2 surface is always negative.

$$\begin{aligned}
\text{Example 2} &: \quad \mathbf{E} \neq 0,\ \mathbf{B} \neq 0, \langle \mathbf{E}| \circ |\mathbf{B}\rangle = 0 \ \ \text{(Minkowski)} && (147)\\
[\mathbb{F}]_{PTD=2} &= \begin{bmatrix} 0 & 0 & -B_2 & E_1/c \\ 0 & 0 & 0 & 0 \\ B_2 & 0 & 0 & 0 \\ -E_1/c & 0 & 0 & 0 \end{bmatrix} && (148)\\
M &= 0,\ G = \{+c^2 B_2^2 - E_1^2\}/c^2,\ K = 0,\ B = 0 && (149)\\
[\mathbb{E}igenvectors] &= \begin{bmatrix} \sqrt{-G}/B_2 & -\sqrt{-G}/B_2 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 \\ E_1/B_2 c & E_1/B_2 c & (B_2 c)/E_1 & 0 \end{bmatrix} && (150)\\
\text{Eigenvalues} &= [+\sqrt{-G}, -\sqrt{-G}, 0, 0]. && (151)
\end{aligned}$$

The (Euclidean) quadratic forms (or norm) of the first two column (Spinor) eigenvectors are always zero. Note that the Spinor eigenvalues are pure imaginary, and are complex conjugates. The imaginary eigenvalue implies the existence of a rotation is to be associated with the Spinor eigenvectors. The norm of the 4th eigenvector column is 1, but the norm of the third eigenvector column is a number depending upon the ratio of (magnetic plus electric energy) to (electric energy).

$$\begin{aligned}
\text{Example 3} &: \quad \mathbf{E} = 0,\ \mathbf{B} \neq 0, \langle \mathbf{E}| \circ |\mathbf{B}\rangle = 0 \ \text{(Minkowski)} && (152)\\
[\mathbb{F}]_{PTD=2} &= \begin{bmatrix} 0 & B_3 & -B_2 & 0 \\ -B_3 & 0 & B_1 & 0 \\ B_2 & -B_1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} && (153)\\
M &= 0,\ G = \{\mathbf{B} \circ \mathbf{B}\},\ K = 0,\ B = 0 && (154)\\
[\mathbb{E}igenvectors] &= \begin{bmatrix} (\sqrt{-G}B_3 - B_2 B_1)/(B_1^2 + B_3^2) & (\sqrt{-G}B_3 - B_2 B_1)/(B_1^2 + B_3^2) & 1 & 0 \\ 1 & 1 & B_2/B_1 & 0 \\ (-\sqrt{-G}B_1 - B_3 B_2)/(B_1^2 + B_3^2) & -\sqrt{-G}B_1 - B_2 B_3)/(B_1^2 + B_3^2) & B_3/B_1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} && (155)\\
\text{Eigenvalues} &= [+\sqrt{-G}, -\sqrt{-G}, 0, 0]. && (156)
\end{aligned}$$

$$\begin{aligned}
\text{Example 4} &: \quad \mathbf{E} \neq 0,\ \mathbf{B} = 0, \langle \mathbf{E}| \circ |\mathbf{B}\rangle = 0 \ \text{(Minkowski)} && (157)\\
[\mathbb{F}]_{PTD=2} &= \begin{bmatrix} 0 & 0 & 0 & E_1 \\ 0 & 0 & 0 & E_2 \\ 0 & 0 & 0 & E_3 \\ -E_1 & -E_2 & -E_3 & 0 \end{bmatrix} && (158)\\
M &= 0,\ G = \{\mathbf{E} \circ \mathbf{E}/c^2\},\ K = 0,\ B = 0 && (159)\\
[\mathbb{E}igenvectors] &= \begin{bmatrix} E_1 c/\sqrt{G} & -E_1 c/\sqrt{G} & -E_2/E_1 & -E_3/E_1 \\ E_2 c/\sqrt{G} & -E_2 c/\sqrt{G} & 1 & 0 \\ E_3 c/\sqrt{G} & -E_3 c/\sqrt{G} & 0 & 1 \\ 1 & 1 & 0 & 0 \end{bmatrix} && (160)\\
\text{Eigenvalues} &= [+\sqrt{G}, -\sqrt{G}, 0, 0]. && (161)
\end{aligned}$$

****

3. NON EQUILIBRIUM SYSTEMS AND SPINORS

It is important to remember that geometric notions of scale and metric are to be avoided in favor of topological properties, some of which are invariants of continuous topological evolution, and some of which are not. Those classical thermodynamic features which are diffeomorphic invariants (useful to many equilibrium applications) are, on the most part, ignored herein, while topological features which are invariants of continuous but non-diffeomorphic transformations (and therefore useful to non equilibrium applications) are emphasized. Topological evolution is understood to occur when topological features (not geometrical features of size and shape) change. The motivation for this perspective was based upon the goal of developing analytical methods which could decide if a given physical system was an equilibrium system or a non equilibrium system, and, also, if a specific analytic process was applied to the physical system, was that process reversible or irreversible. The bottom line is that classical equilibrium systems emphasize vector concepts, while non-equilibrium systems must involve Spinors.

The definition of continuous topological evolution in terms of the Lie differential is to be interpreted algebraically, using the properties of the exterior differential and the inner product associated with exterior differential forms. Many derivations of the Cartan-Lie differential formula presume a dynamic constraint, such that the vector field $\mathbf{V}_4(x,y,z,t)$ be the generator of a single parameter group; if true, then the topological constraint of kinematic perfection is satisfied:

$$\textbf{Kinematic perfection}: \; dx^k - V^k dt \Rightarrow 0. \tag{162}$$

The topological constraint in effect defines (or presumes) a limit process, and the Lie *differential* then can be considered to be a Lie *derivative* of the form A representing an infinitesimal propagation of the 1-form, A, down the flow lines generated by the 1-parameter group. However, such a kinematic constraint is *not* necessarily imposed in the presentation herein; the vector field may have multiple parameters, which leads to the important concept of topological fluctuations,

$$\textbf{Topological Fluctuations}: \; \Delta x^k = dx^k - V^k dt \neq 0. \tag{163}$$

A the title states, major conjecture of this article is that Topological Fluctuations and irreversible processes are related to the Spinor components of the antisymmetric matrix, $F = dA$. These Spinor components have no explicit connection to microphysics, and are topological features that do not depend upon scales. They are different from vectors in the sense that a 2π rotation changes the sign of a spinor, while 2π rotation does not change the sign of a vector.

It is interesting to note that in Felix Klein's discussions of the development of calculus, he says "The primary thing for him (Leibniz) was not the differential quotient (the derivative) thought of as a limit." It is important to remember that the concept of a differential form is different from the concept of a derivative, where a (topological) limit has been pre-defined.

It is remarkable that although the symbols are often different, the same basic constructions and conclusions expressed in the above axioms apply to most classical physical systems. Examples will demonstrate this feature of topological universality. The correspondence so established between the Cartan magic formula acting on a 1-form of Action, and the first law of thermodynamics is taken both literally and seriously in this article. The Cartan methods are not limited to applications in electromagnetic theory. The thermodynamic phenomena and the associated topological results describe herein have universal qualities, and applicability to all physical theories. In fact, the topological methods permit the long sought for integration of mechanical and thermodynamic concepts, without the constraints of equilibrium systems, and statistical analysis. The methods yield explicit constructions for testing when a process acting on a physical system is irreversible. The methods permit irreversible adiabatic processes to be distinguished from reversible adiabatic processes analytically. Indeed, adiabatic processes need not be "slow" or quasi-static.

Given any 1-form, A, W, and Q, the concept of Pfaff topological dimension permits separation of processes and systems into topological equivalence classes. For example, dynamical process can be classified in terms of the topological Pfaff dimension of the Work 1-form. All Hamiltonian systems have a Work 1-form of topological Pfaff dimension of 1, ($dW = 0$) and therefor cannot describe irreversible processes for which the topological Pfaff dimension is 4. It is remarkable that if $dW = 0$, then the process that generates the Work 1-form must consist of eigenvectors of $F = dA$ that are of eigen value zero. In contrast, if $W\hat{}dW$ is not zero, then the process that generates the Work 1-form must contain Spinor eigen components of the 2-form, F. If $dW\hat{}dW \neq 0$ (Pfaff dimension 4), then all eigenvectors of the 2-form, F, are spinors. A discussion of the concept of Pfaff topological dimension and its application to physical systems and physical processes appears in[19] ,,[20] ,[21] and.[22]

3.1. *Physical Systems: Equilibrium, Isolated, Closed and Open*

Physical systems and processes are elements of topological categories determined by the Pfaff topological dimension (or class) of the 1-forms of Action, A, Work, W, and Heat, Q. For example, the Pfaff topological dimension of the exterior differential 1-form of Action, A, determines the various species of thermodynamic systems in terms of distinct topological categories:

$$\textbf{Thermodynamic Systems} \text{ defined by} \tag{164}$$

$$\text{the Pfaff Topological Dimension of } A \tag{165}$$

$$dA = 0 \quad \textbf{Equilibrium} \text{ - Pfaff dimension 1} \tag{166}$$

$$A\hat{}dA = 0 \quad \textbf{Isolated} \text{ - Pfaff dimension 2} \tag{167}$$

$$d(A\hat{}dA) = 0 \quad \textbf{Closed} \text{ - Pfaff dimension 3} \tag{168}$$

$$dA\hat{}dA \neq 0. \quad \textbf{Open} \text{ - Pfaff dimension 4.} \tag{169}$$

There are two topological thermodynamic categories, one is determined by the closure (or differential ideal) of the 1-form of Action, $A \cup dA$, and the other is determined by the closure of the 3-form of topological torsion, $A\hat{}dA \cup dA\hat{}dA$. The first category is represented by a connected Cartan topology, while the second category is represented by a disconnected Cartan topology. The Cartan topology is discussed in detail in Chapter **??**.

3.1.1. Connected Topology $A\hat{}F = 0$

1. Equilibrium physical systems are elements such that the Pfaff topological dimension of the 1-form of Action, A, is 1.

2. Isolated physical systems are elements such that the Pfaff topological dimension of the 1-form of Action, A, is 2, or less. Isolated systems of Pfaff dimension 2 need not be in equilibrium, but (in historic language) do not exchange radiation or mass with the environment.

3.1.2. Disconnected Topology $A\hat{}F \neq 0$

1. Closed physical systems are elements such that the Pfaff topological dimension of the 1-form of Action, A, is 3. Closed systems can exchange radiation, but not mass, with the environment.

2. Open physical systems are such that the Pfaff topological dimension of the 1-form of Action, A, is 4. Open physical systems can exchange both radiation and mass* with the environment.

Note that these topological specifications as given above are determined entirely from the functional properties of the physical system encoded as a 1-form of Action, A. The system topological categories do not involve a process, which is encoded (to within a factor) by some vector direction field, $\mathbf{V}_4$. However, the process $\mathbf{V}_4$ does influence the topological properties of the work 1-form W and the Heat 1-form Q. Compare these topological definitions, whereby Equilibrium or Isolated systems are determined in terms of two independent variables at most, and Duhem's theorem

*The use of the word mass to distinguish between closed and open systems is a legacy that ought to be changed to "mole or particle" number, as it is now known that mass energy can be converted to radiation, and radiation can produce massive pairs.

"Whatever the number of phases, components and chemical reactions, if the initial mole numbers N_k of all components are specified, the equilibrium state of a closed system is completely specified by two independent variables. (p.182?)"

3.2. *Equilibrium versus non Equilibrium Systems*

The intuitive idea for an equilibrium system comes from the experimental recognition that the intensive variables of Pressure and Temperature (conjugate to volume and entropy) become domain constants in an equilibrium state: $dP \Rightarrow 0,\ dT \Rightarrow 0$. A definition made herein is that the Pfaff topological dimension in the interior of a physical system which is in the equilibrium state is at most 1.? Formally, the idea is restated such that the equilibrium state is a Lagrangian submanifold of a 4 dimensional symplectic manifold, and upon this Lagrangian submanifold, the 2-form dA, that generates the symplectic structure, vanishes. Hence the equilibrium state is of Pfaff topological dimension 1:

$$\textbf{Equilibrium}\ \ \{A \neq 0,\ dA = 0,\ A\hat{}dA = 0,\ dA\hat{}dA = 0\}. \tag{170}$$

The isolated physical system is of Pfaff dimension 2,

$$\textbf{Isolated}\ \ \{A \neq 0,\ dA \neq 0,\ A\hat{}dA = 0,\ dA\hat{}dA = 0\}. \tag{171}$$

For both the isolated or equilibrium system, the Cartan topology generated by the elements of the Pfaff sequence for A is then a connected topology of one component, as $A\hat{}dA = 0$.

Although the Pfaff topological dimension of A is at most 2 in the isolated state, processes in the isolated state are such that the Work 1-form and the Heat 1-form must be of Pfaff dimension 1. For suppose $W = PdV$, then $dW = dP\hat{}dV \Rightarrow 0$ if the pressure is a domain constant. Similarly, suppose $Q = TdS$, then $dQ = dT\hat{}dS \Rightarrow 0$ if the temperature is a domain constant. Hence both W and Q are of Pfaff dimension 1 for this isolated example. If the Pfaff dimension of the 1-form of Action is 1, then $dA \Rightarrow 0$. It follows in this more stringent case that $W \Rightarrow 0$. Hence for elementary systems the Pressure must vanish or the Volume is constant, and the Heat 1-form is a perfect differential, $Q = d(U)$.

Of particular interest herein are those regions of base variables for open, non equilibrium, turbulent physical systems, formed by the closure† of the 3-forms $A\hat{}dA$, $W\hat{}dW$, and $Q\hat{}dQ$. For such regions, the Pfaff topological dimension of the 1-forms, A, W, and Q, are all initially of Pfaff topological dimension 4,

$$dA\hat{}dA \neq 0, \quad dW\hat{}dW \neq 0, \quad dQ\hat{}dQ \neq 0, \tag{172}$$

save for defect regions that are of Pfaff dimension 3 (or less). It is remarkable that evolutionary dissipative irreversible processes in such open systems can describe evolution to regions of base variables where the Pfaff topological dimension of the 1-form of Action, A, changes from 4 to 3. Such processes describe topological change in the physical system. For a given 1-form of Action, A, those regions of Pfaff topological dimension 3, once created, form topological "defect structures" in the closure of the 3-form, $A\hat{}F$. The defect structures of the 1-form of Action, A, (of Pfaff dimension 3) can behave as long lived (excited) states of the initial physical system, but they are far from equilibrium and are not isolated, for they are not of Pfaff topological dimension equal to 2 or less. Such excited states (of odd topological dimension) can admit extremal processes of kinematic perfection, and can have a Hamiltonian generator for the kinematics represented as a system of first order ordinary differential equations. The Hamiltonian evolution remains contained in the defect structure, unless topological fluctuations destroy the kinematic perfection.

Such concepts can be applied to a model of cosmology (where the stars are the defect structures), to turbulent plasmas and fluids (where wakes are the defect structures), and to a better understanding of the arrow of time. Although the defects in the turbulent non equilibrium regime are not necessarily equilibrium structures, once formed and self organized as coherent topological structures of Pfaff dimension 3, they can evolve along extremal trajectories that are not dissipative, Indeed such extremal processes have a Hamiltonian representation. These

†The closure of the p-form Σ is the union of Σ and $d\Sigma$, which Cartan has called a differential ideal.

"stationary", or long lived (excited), states of Pfaff dimension 3, indeed are states "far" from the equilibrium state, which requires a Pfaff dimension of 1. Note that the word "far" does not imply a "distance". The Pfaff dimension 3 and 4 sets are not even "connected" to the equilibrium states in a topological sense. The non equilibrium but isolated states of a physical system that are "near-by" to the equilibrium state, are "connected" to the equilibrium state, and are of Pfaff dimension 2.

The descriptive words of self-organized states far from equilibrium have been abstracted from the intuition and conjectures of I. Prigogine.[?] However the topological theory presented herein presents for the first time a solid, formal, mathematical justification (with examples) for the Prigogine conjectures. Precise definitions of equilibrium and non equilibrium systems, as well as reversible and irreversible processes can be made in terms of the topological features of Cartan's exterior calculus. Using Cartan's methods of exterior differential systems, thermodynamic irreversibility and the arrow of time can be well defined in a topological sense,[?] a technique that goes beyond (and without) statistical analysis. Thermodynamic irreversibility and the arrow of time requires that the evolutionary process produce topological change.

3.3. *Change of Pfaff Topological Dimension*

It should be noted that the closed components of the 1-form of Action do not effect the components of the 2-form of intensities, $F = dA = d(A_c + A_0) = 0 + d(A_0) = F_0$. However, these "gauge" additions of closed forms, A_c, do influence the topological dimension of the 1-form of Action. For example, let A_0 be of Pfaff Topological dimension 2, representing an isolated system where $A_0\hat{}dA_0 = 0$. Then by addition of a closed component to the original action, the new 1-form of Action, $A = A_c + A_0$ could have a topological dimension of 3:

$$A\hat{}dA = (A_c + A_0)\hat{}dA_0 = A_c\hat{}dA_0 \neq 0, \tag{173}$$

So the addition of a closed component to the 1-form of Action could change the system from an isolated system of Pfaff dimension 2 to a closed system of Pfaff dimension 3. The 4-form $dA\hat{}dA$ is not influenced by the (gauge) addition to the original 1-form of Action.

$$dA\hat{}dA = dA_0\hat{}dA_0. \tag{174}$$

In higher dimension, such gauge additions imply that the Pfaff dimension can change according to the rule, $2n \Rightarrow 2n + 1$.

It is also possible to change the Pfaff dimension of a 1-form by "renormalization", or better said, by "rescaling" with a multiplying function, often in the form of an integrating factor. For example, consider the 1-form A_0 of Pfaff dimension 4, such that $d(A_0\hat{}dA_0) \neq 0$. Next rescale the 1-form such that $A = \beta A_0$. Then

$$d(A\hat{}dA) = d(\beta^2 A_0\hat{}dA_0) \Rightarrow 0, \tag{175}$$

subject to the constraint that β^2 is an integrating factor for the 3-form $A_0\hat{}dA_0$. In 4 dimensions there exists an infinite number of such functions that serve as integrating factors for the 3-form of Topological Torsion, $A_0\hat{}dA_0$. The integrating factors (which can be formulated from Holder norms) can be interpreted as distributions of "density" which change the Pfaff topological dimension from 4 to 3, or, in general, from $2n+2 \Rightarrow 2n+1$. Such distributions can be put into correspondence with "stationary" states far from equilibrium.

As an example of how the Pfaff dimension of a 1-form can be modified by a gauge addition, consider a 1-form representing a Bohm-Aharanov-Abrikosov singular "vortex" string,

$$\gamma = b(ydx - xdy)/(x^2 + y^2), \tag{176}$$

to which is added a 1/r potential for a point source.[22] The bare m/r "Coulomb" potential, $A_0 = m/\sqrt{(x^2 + y^2 + z^2)}dt$ exhibits no Topological Torsion but does exhibit Topological Spin. The 1/r potential term implies that $A_0 \neq 0$. Hence the 1-form of Action representing a bare "coulomb" potential, is not in equilibrium, but does represent a connected "isolated" topology of Pfaff dimension 2. The combined 1-form of Action,

$$A = b(ydx - xdy)/(x^2 + y^2) + m/\sqrt{(x^2 + y^2 + z^2)}dt, \tag{177}$$

even though $d\gamma = 0$, is of Pfaff dimension 3, not 2. The Topological Torsion 3-form $A\hat{}F$ depends on both b and m, and is zero if $b = 0$, or if $m = 0$, reducing the Pfaff dimension of the modified 1-form back to 2.

3.4. *Systems with Multiple Components*

One of the most remarkable properties of the Cartan topology generated by a Pfaff sequence is due to the fact that when $A\hat{}dA = 0$, (Pfaff dimension 2 or less) the physical system is reducible to a single connected topological component. This single connected topological component need not be simply connected. The Topological Torsion field vanishes on equilibrium domains.

On the other hand when $A\hat{}dA \neq 0$, (the Pfaff topological dimension of the 1-form, A, is 3 or more) the physical system admits more than one topological component (and the topology is a disconnected topology). The bottom line is that when the Pfaff dimension is 3 or greater (such that conditions of the Frobenius unique integrability theorem are not satisfied), solution uniqueness to the Pfaffian differential equation, $A = 0$, is lost. If solutions exists, there is more than one solution. Such concepts lead to propagating discontinuities (signals), envelope solutions ‡ (Huygen wavelets), an edge of regression (the Spinodal line of phase transitions) a lack of time reversal invariance, and the existence of irreducible affine torsion in the theory of connections. It is the opinion of this author that a dogmatic insistence that a viable physical theory must give a unique prediction from a set of given initial conditions historically has hindered the understanding of irreversibility and non equilibrium systems. Irreversibility and non equilibrium are concepts that require non uniqueness, and demand that the dogma of predictive uniqueness, mentioned above, has to be rejected.

4. THERMODYNAMIC PROCESSES

4.1. *Continuous Processes*

All continuous processes may be put into equivalence classes as determined by the vector direction fields, V, that locally generate a flow. For example on a domain of geometric dimension, n, and for the 1-form, A, those n-1 vector fields, $\mathbf{V}_{associated}$, that satisfy the transversal equation,

$$\textbf{Associated Class:} \quad i(\rho\mathbf{V}_{associated})A = 0, \tag{178}$$

are said to be elements of the associated class of vector fields relative to the form A. If the direction field of the 1-form of Action is considered to be a fiber, then the associated vectors are also said to be "horizontal". The associated vectors will form a distribution orthogonal to the 1-form, A, but the distribution need not be a smooth foliation. That is, the fiber direction field is not necessarily the normal field to an implicit hypersurface. The requirement for a smooth foliation is that the associated 1-form be of Pfaff topological dimension 2 or less. For such associated processes acting on a 1-form of Action, A, the "internal interaction energy" vanishes! As shown below, processes generated by associated vectors relative to the 1-form of Action, A, are also included in the set of thermodynamic locally adiabatic processes. Other locally adiabatic processes are generated by those processes which are associated vectors of the exterior derivative of the internal energy, U. In both cases, locally adiabatic processes are null vectors of the Heat 1-form, in the sense that $i(\rho\mathbf{V}_{adiabatic})Q = 0$.

As defined in the previous chapter those vector fields, $\mathbf{V}_E$, that satisfy the equations,

$$\textbf{Extremal Class:} \quad i(\rho\mathbf{V}_E)dA = 0, \tag{179}$$

are said to be elements of the extremal class of vector fields. As the matrix of functions that define the 2-form dA is antisymmetric, the extremal vector is proportional to that eigen vector of the antisymmetric matrix that has a zero eigen value. If the matrix dA is of maximum rank, then there can be only one (unique) eigen vector with zero eigen value, and that null eigen vector exists only if the Pfaff topological dimension of the 1-form A is odd $(2n + 1)$. In other words, the 2-form dA defines a Contact manifold. The extremal direction field is completely determined (to within a factor) by the component functions of the 1-form A utilized in its definition. Note that the work 1-form $W = i(\rho\mathbf{V}_E)dA \Rightarrow 0$ vanishes for extremal evolutionary processes.

As an example, consider the 1-form of Action and its associated Pfaff sequence given by the expressions

‡See section 7.8

$$
\begin{aligned}
A &= ydx - xdy + sdz - zds, && (180)\\
F &= dA = 2dy\hat{}dx + 2ds\hat{}dz, && (181)\\
A\hat{}F &= 2\{xdy\hat{}dz\hat{}ds - ydx\hat{}dz\hat{}ds + zdx\hat{}dy\hat{}ds - sdx\hat{}dt\hat{}dz, && (182)\\
F\hat{}F &= 8dx\hat{}dy\hat{}dz\hat{}ds, && (183)
\end{aligned}
$$

Note that the 4x4 antisymmetric matrix is of the form

$$
[\mathbb{F}] = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix}, \tag{184}
$$

with eigenvalues and eigen vectors,

$$
\begin{aligned}
\text{EigenVectors} &= [0,0,1,i],[1,i,0,0] \text{ with eigenvalue } = i, && (185)\\
\text{EigenVectors} &= [0,0,1,-i],[1,-i,0,0] \text{ with eigenvalue } = -i, && (186)
\end{aligned}
$$

Each eigenvector is null isotropic such that the sum of squares of the coefficients is zero. This example is a simple example generated from the 1-form, A, which is the adjoint of the three exact differentials generated by the Hopf map (See section 3.5). Note that the matrix $[\mathbb{F}]$ acting on an eigenvector causes the eigenvector to be "rotated" by $\pi/2$ in the complex plane, and the operation is not reflexive of degree 2, but is reflexive of degree 4. That is $[\mathbb{F}]^2 \circ |Spinor\rangle = -|Spinor\rangle$ and $[\mathbb{F}]^4 \circ |Spinor\rangle = +|Spinor\rangle$. The third power of $[\mathbb{F}]$ is the Inverse of $[\mathbb{F}]$.

If the Pfaff topological dimension of the 1-form A is even, as in the example above, then a unique extremal vector (with eigenvalue zero) does not exist. The reduced topological domain (not necessarily the entire geometric domain) is a symplectic manifold of even dimensions, $(2n+2)$. However, on a symplectic manifold of 4 geometric dimensions and 4 topological dimensions, it follows that there does exist a unique vector direction field, the Topological Torsion vector, $\mathbf{V}_T$, (or sometimes written as $\mathbf{T}_4$), completely determined (to within a factor) in terms of the functions which define the physical system.

$$
\begin{aligned}
\textbf{Topological Torsion Class} &: \quad i(\rho\mathbf{V}_T)dA = \rho\sigma A, && (187)\\
&: \quad i(\rho\mathbf{V}_T)A = 0, && (188)
\end{aligned}
$$

Note that the internal energy generated by a process of the Torsion class is zero. For the example above, the Topological Torsion vector is proportional to the 4 vector, $\mathbf{V}_T = 2[x,y,z,s]$, which indeed has a non zero divergence. Moreover, $i(\mathbf{V}_T)A = xy - yx + zs - sz = 0$.

A related vector field is defined as the Iso-Vector, which is not associated, but satisfies the constraints :

$$
\begin{aligned}
\textbf{Iso-Vector Class} &: \quad i(\rho\mathbf{V}_{iso})dA = \rho\sigma A, && (189)\\
\textbf{but} &: \quad d\{i(\rho\mathbf{V}_{iso})A\} = 0. && (190)
\end{aligned}
$$

The internal energy of such Iso-vector processes is a constant.

In the next section it will be shown that evolution with a component in the direction of the "Topological Torsion" vector will produce an irreversible process on the physical system (as encoded by the Action 1-form), if the divergence of the "Topological Torsion" vector is not zero. This "Topological Torsion" vector, equivalent to the 3-form $A\hat{}dA = A\hat{}F$, is always an associated vector, but it is not necessarily an extremal vector, relative to the Action 1-form, A. The Torsion vector is identically zero on domains of Pfaff topological dimension 2. Hence non zero values of the Torsion vector are an indication that the physical system, A, is not an equilibrium system. The Topological Torsion vector exists only on domains of Pfaff topological dimension 3 or greater, in the same sense that Frenet-Serret torsion exists only on domains of geometric dimension 3 or greater. With respect to evolution in the direction of the Torsion Current, the symplectic 4D volume is contracting or expanding exponentially unless $\sigma = 0$. If the divergence of $\mathbf{V}_{Torsion}$ vanishes, $\sigma \Rightarrow 0$, and therefore such vector fields cannot represent a symplectic process (which preserves the volume element, $dA\hat{}dA$). The factor, σ, is a Liapunov function and defines the stability of the process (depending on the sign of σ). When $\sigma = 1$, the Torsion vector has been called the "Liouville vector".[?]

Vector fields which are both extremal and associated are said to be elements of the characteristic class, $\mathbf{V}_{characteristic}$, of vector fields.[?]

$$\textbf{Characteristic Class} \quad : \quad i(\rho \mathbf{V}_{characteristic})A = 0, \tag{191}$$
$$\textbf{and} \quad : \quad i(\rho \mathbf{V}_{characteristic})dA = 0. \tag{192}$$

Note that characteristic flow lines generated by $\mathbf{V}_{characteristic}$ of the Characteristic class preserve the Cartan topology, for each form of the Cartan topological base is invariant with respect to the action of the Lie differential relative to characteristic flows. Characteristics are often associated with wave phenomena, and propagating discontinuities. They are locally adiabatic. The Topological Torsion vector mentioned above may have zero divergence on certain geometric subsets of space-time, but these domains are of Pfaff topological dimension 3 (although of geometric dimension 4). In such cases, the Topological Torsion vector will be a characteristic vector for the 1-form of Action, A. These and other properties of the "Topological Torsion" vector will be described in detail by examples presented below.

4.2. *Reversible and Irreversible Processes*

The Pfaff topological dimension of the exterior differential 1-form of Heat, Q, determines important topological categories of processes. From classical thermodynamics "The quantity of heat in a reversible process always has an integrating factor"[?].[?] Hence, from the Frobenius unique integrability theorem, which requires $Q\hat{}dQ = 0$, all reversible processes are such that the Pfaff dimension of Q is less than or equal to 2. Irreversible processes are such that the Pfaff dimension of Q is greater than 2, and an integrating factor does not exist. A dissipative irreversible topologically *turbulent* process is defined when the Pfaff dimension of Q is 4.

$$\textbf{Processes} \quad : \quad \textbf{as defined by the Pfaff dimension of } Q$$
$$Q\hat{}dQ = 0 \qquad \textbf{Reversible} \text{ - Pfaff dimension 2} \tag{193}$$
$$d(Q\hat{}dQ) \neq 0. \qquad \textbf{Turbulent} \text{ - Pfaff dimension 4.} \tag{194}$$

Note that the Pfaff dimension of Q depends on both the choice of a process, $\mathbf{V}_4$, and the system, A, upon which it acts. As reversible thermodynamic processes are such that $Q\hat{}dQ = 0$, and irreversible thermodynamic processes are such that $Q\hat{}dQ \neq 0$, Cartan's formula of continuous topological evolution can be used to determine if a given process, $\mathbf{V}_4$, acting on a physical system, A, is thermodynamically reversible or not:

$$\textbf{Processes defined by} \quad : \quad \textbf{the Lie differential of } \mathrm{A}$$
$$L_{(\rho \mathbf{V}_4)}A = Q \tag{195}$$
$$\textbf{Reversible Processes } \rho \mathbf{V}_4 \quad : \quad Q\hat{}dQ = 0, \tag{196}$$
$$L_{(\rho \mathbf{V}_4)}A\hat{}L_{(\rho \mathbf{V}_4)}dA = 0, \tag{197}$$
$$\textbf{Irreversible Processes } \rho \mathbf{V}_4 \quad : \quad Q\hat{}dQ \neq 0, \tag{198}$$
$$L_{(\rho \mathbf{V}_4)}A\hat{}L_{(\rho \mathbf{V}_4)}dA \neq 0. \tag{199}$$

Remarkably, Cartan's magic formula can be used to describe the continuous dynamic possibilities of both reversible and irreversible processes, acting on equilibrium or non equilibrium systems, even when the evolution induces topological change, transitions between excited states, or changes of phase, such as condensations.

The idea that irreversible processes must satisfy $Q\hat{}dQ \neq 0$, and the idea that the Pfaff topological dimension of such irreversible domains must be related to symplectic (2n+2) domains $dA\hat{}dA \neq 0$, is opposed to the presentations of Chen, who, in a set of articles of almost identical content, ,? proposed that non-equilibrium thermodynamics and irreversible processes should be studied on contact manifolds (2n+1) constrained such that $Q\hat{}dQ = 0$. According to the criteria that an irreversible process implies that an integrating factor does not exist for the heat 1-form, Q, Chen's theory is patently false.

It is important to note that the direction field, $\mathbf{V}_4$, need not be topologically constrained such that it is singularly parameterized. That is, the evolutionary processes described by Cartan's magic formula are not necessarily restricted to vector fields that satisfy the topological constraints of kinematic perfection, $dx^k - V^k dt = 0$. A discussion of topological fluctuations, where $dx^k - V^k dt = \Delta^k \neq 0$, and an example fluctuation process is described in Section 2.6.

In the next section it will be demonstrated that evolution in the direction of the Topological Torsion vector (or Current), $\mathbf{T}_4$, defined from the components of the 3-form of topological torsion,

$$i(\mathbf{T}_4)dx\hat{}dy\hat{}dz\hat{}dt = A\hat{}dA, \tag{200}$$

induces a process which satisfies the equations of a conformal evolutionary process

$$L_{(\mathbf{T}_4)}A = \sigma A \quad \text{ and } \quad i(\mathbf{T}_4)A = 0, \quad \sigma \neq 0, \tag{201}$$

such that

$$L_{(\mathbf{T}_4)}A\hat{}L_{(\mathbf{T}_4)}dA = Q\hat{}dQ = \sigma^2 A\hat{}dA \neq 0. \tag{202}$$

Conclusion *Evolution in the direction of the Topological Torsion vector, T_4, relative to a physical system encoded by the 1-form A, is thermodynamically irreversible.*

A crucial idea is to recognize that irreversible processes must be on domains of Pfaff topological dimension which support Topological Torsion, $A\hat{}dA \neq 0$, with its attendant properties of non uniqueness, envelopes, regressions, and projectivized tangent bundles Such domains are of Pfaff dimension 3 or greater. Moreover, as described below, it would appear that thermodynamic irreversibility must support a non zero Topological Parity 4-form, $dA\hat{}dA \neq 0$. Such domains are of Pfaff dimension 4 or greater.

4.3. *Adiabatic Processes - Reversible and Irreversible*

The topological formulation of thermodynamics in terms of exterior differential forms permits a precise definition to be made for both reversible and irreversible adiabatic processes in terms of the topological properties of Q. On a geometrical space of N dimensions, a 1-form, Q, will admit N-1 associated vector fields, $V_{Associated}$, such that $i(V_{Associated})Q = 0$. Processes defined by associated vector fields, $V_{Associated}$, relative to Q are defined as (locally) adiabatic processes (or sometimes as null curves), $V_{adiabatic}$.?

$$\textbf{Locally Adiabatic Processes:} \quad i(V_{adiabatic})Q = 0. \tag{203}$$

The N-1 associated vectors will form a distribution of adiabatic processes orthogonal to the 1-form Q. The distribution of adiabatic processes will not form a smooth hypersurface, unless the Pfaff dimension of Q is 2 or less. In other words the null curves (adiabats) form a smooth hypersurface only in the equilibrium or isolated state. Note that all adiabatic processes are defined by vector direction fields, to within an arbitrary factor, $\beta(x, y, z, t)$. That is, if $i(V_A)Q = 0$, then it is also true that $i(\beta V_A)Q = 0$. The adiabatic direction fields and the 1-form of Action can be used to construct an interesting basis frame related to projective connections.

The differences between the inexact 1-forms of Work and Heat become obvious in terms of the topological format. Both 1-forms, W and Q, depend on the process, $\mathbf{V}_4$, and on the physical system, A. However, Work is always transversal to the process, but Heat is not:

$$\textbf{Work is transversal } i(\mathbf{V}_4)W = i(\mathbf{V}_4)i(\mathbf{V}_4)dA = 0, \tag{204}$$

$$\textbf{Heat is NOT transversal } i(\mathbf{V}_4)Q = i(\mathbf{V}_4)dU \nRightarrow 0, \tag{205}$$

unless the process is adiabatic. It is this fundamental difference between Heat, Q, and Work, W, that leads to the Carnot-like statements that it is possible to convert work into heat with 100% efficiency, but it is not possible to convert heat into work with 100% efficiency.

Locally adiabatic direction fields, so defined as null curves of Q, do not imply that the Pfaff dimension of Q must be 2. That is, it is not obvious that Q can be written in the form, $Q = TdS$, as is possible on the manifold of equilibrium, or isolated, states. From the Cartan formulation it is apparent that if Q is not zero, then

$$L_{(\mathbf{V}_A)}A = Q \neq 0, \tag{206}$$

$$i(\mathbf{V}_A)L_{(\mathbf{V}_A)}A = i(\mathbf{V}_A)i(\mathbf{V}_A)dA + i(\mathbf{V}_A)d(i(\mathbf{V}_A)A) \tag{207}$$

$$= 0(transversality) + i(\mathbf{V}_A)d(i(\mathbf{V}_A)A)$$

$$= i(\mathbf{V}_A)Q \tag{208}$$

The necessary condition for a process to be adiabatic is given by the statement that the process is an "associated" vector relative to the exact exterior differential of the internal energy, dU. A locally adiabatic process requires that

$$i(\mathbf{V}_A)Q = i(\mathbf{V}_A)d(i(\mathbf{V}_A)A)) \Rightarrow 0. \tag{209}$$

If $Q \neq 0$, a necessary adiabatic condition is given by the equation,

$$\text{necessary condition } : i(\mathbf{V}_A)dU \Rightarrow 0, \tag{210}$$

while a sufficient condition is given by the equation

$$\text{sufficient condition } : d(i(\mathbf{V}_A)A) = dU \Rightarrow 0. \tag{211}$$

Note that the Topological Torsion vector is an associated vector relative to the Action 1-form, A, and therefore defines a locally adiabatic (but irreversible) process on domains of Pfaff topological dimension 4.

If the heat 1-form is zero, then the process is a globally reversible adiabatic process of a special type. A reversible process is defined such that the Pfaff dimension of Q is less than 3; or, $Q\hat{}dQ = 0$. Hence $i(\mathbf{V}_A)(Q\hat{}dQ) = 0$ for reversible processes. However,

$$i(\mathbf{V}_A)(Q\hat{}dQ) = (i(\mathbf{V}_A)Q)\hat{}dQ - Q\hat{}i(\mathbf{V}_A)dQ, \tag{212}$$

which permits reversible and irreversible adiabatic processes to be distinguished [§] when $Q \neq 0$:

$$\textbf{Reversible Adiabatic Process} = -Q\hat{}i(\mathbf{V}_A)dQ \Rightarrow 0, \tag{213}$$

$$i(\mathbf{V}_A)Q \Rightarrow 0, \tag{214}$$

$$\textbf{Irreversible Adiabatic Process} = -Q\hat{}i(\mathbf{V}_A)dQ \neq 0, \tag{215}$$

$$i(\mathbf{V}_A)Q \Rightarrow 0. \tag{216}$$

It is certainly true that if $L_{(\mathbf{V})}A = Q = 0$, *identically*, then all such processes are (globally) adiabatic, and reversible. (In the next section, it will be demonstrated how these thermodynamic ideas can be associated with the tensor processes of covariant differentiation and parallel transport.) In such special adiabatic cases, the Cartan formalism implies that $W + dU = 0$. Such systems are elements of the Hamiltonian-Bernoulli class of processes, where $W = -dB$.

[§] It is apparent that i(V)Q = 0 defines an adiabatic process, but not necessarily a reversible adiabatic process. This topological point clears up certain misconceptions that appear in the literature.

4.4. *Processes classified by connected topological constraints on the Work 1-form.*

Cartan has shown that all Hamiltonian processes (systems with a generator of ordinary differential equations), $\rho\mathbf{V}_H$, satisfy the following equations of topological constraint on the work 1-form, W :

$$\begin{aligned}
& \textbf{A Hamiltonian process } \mathbf{V}_H \text{ is either } \mathbf{V}_E \text{ or } \mathbf{V}_B \\
& \textbf{Extremal Hamiltonian } \mathbf{V}_E \\
W_E &= i(\rho\mathbf{V}_E)dA = 0 \qquad \text{Pfaff dimension of W} = 0 \qquad (217) \\
& \textbf{Bernoulli-Casimir Hamiltonian } \mathbf{V}_B \\
W_B &= i(\rho\mathbf{V}_B)dA = -dB \quad \text{Pfaff dimension of W} = 1 \qquad (218)
\end{aligned}$$

More details about Cartan's development of Hamiltonian systems appears in section 4.5. A special case occurs when the Bernoulli function is equal to the negative of the internal energy, for then the heat 1-form produced by this special Hamiltonian process vanishes.

For Helmholtz processes (which are not strictly Hamiltonian) the situation is a bit more intricate, but in all cases the Pfaff dimension of the Work 1-form is at most 1. Hamiltonian processes are subsets of Helmholtz processes.

$$\begin{aligned}
& \textbf{Helmholtz (Symplectic) Process } \mathbf{V}_S \\
W_S &= i(\rho\mathbf{V}_S)dA = -dB + \gamma \;\; \text{Pfaff dimension of W} = 1 \qquad (219) \\
dW_S &= 0 \text{ as } \gamma \text{ is closed but not exact.} \qquad (220) \\
& \rho\mathbf{V}_S \text{ is Symplectic when} \\
dA\hat{}dA &\neq 0, \;\; W_S \neq 0, \; dW_S = 0. \qquad (221)
\end{aligned}$$

Helmholtz-Symplectic processes satisfy the following equation which is known as the Helmholtz conservation of vorticity theorem:

$$\textbf{Helmholtz} \quad : \quad \textbf{Conservation of Vorticity} \qquad (222)$$

$$L_{(\rho\mathbf{V}_S)}dA = dW_S + ddU = 0 + 0 = dQ \Rightarrow 0. \qquad (223)$$

Note that a symplectic process preserves the 2-form dA but does not generate a symplectic manifold relative to the Action 1-form, A. The physical system is different from the physical process.

However, the closed but not exact component of work can have finite period integrals, so the evolutionary Helmholtz process can involve changing topology. The closed integrals of Action are not invariant with respect to $\rho\mathbf{V}_S$ unless $\gamma = 0$.

$$L_{(\rho\mathbf{V}_S)}\int_{z1d} A = \int_{z1d}\gamma = \int_{z1d} Q \neq 0 \qquad (224)$$

The Helmholtz class of processes 224 can be split into two types:

Type H_A. Those processes for which the connectivity of the domain of support for the 1-form A is invariant.

$$\text{Helmholtz type A}: \; L_{(\rho\mathbf{V})}\int_{z1} A \Rightarrow 0, \text{ any } \;\rho \neq 0, \; \int_{z1} W = \int_{z1} Q = 0. \qquad (225)$$

Type H_B. Those processes for which the connectivity of the domain of support for the 1-form A can change (the number of holes and handles can change),

$$\text{Helmholtz type B}: L_{(\rho\mathbf{V})}\int_{z1} A \neq 0, \;\; \text{any } \rho \neq 0, \; \int_{z1} W = \int_{z1} Q \neq 0. \qquad (226)$$

Cartan proved**?** that if the 1-form of Action is taken to be of the classic "Hamiltonian" format,

$$A = p_k dq^k - \mathsf{H}(p_k, q^k, t)dt \qquad (227)$$

on a 2n+1 dimensional domain of variables $\{p_k, q^k, t\}$, there exists a *unique* extremal vector field, ρV_E, that satisfies the conditions of Helmholtz type A processes. The closed but not exact 1- forms, γ, introduce non uniqueness into the definition of the work 1-form for Helmholtz type B processes. As $dQ = d(-dB+\gamma+dU) = 0$ for all three processes defined above, all three processes are thermodynamically reversible (see equation (193)).

Conclusion *Helmholtz Type B processes demonstrate that topological change is necessary but not sufficient to produce thermodynamic irreversibility.*

4.5. *Planck's Harmonic Oscillator and Type B processes - How does energy get quantized ?*

Consider a symplectic Harmonic Oscillator system with a Lagrange function

$$\begin{aligned} \text{Lagrange function, } \mathsf{L}(t,x,v) &= -1/2kx^2 + 1/2mv^2 + m_0c^2, && (228)\\ \text{and a 1-form of Action, } A &= pdx - (pv - \mathsf{L}(t,x,v))dt, && (229)\\ \text{where } \hbar k &\doteq p - \partial\mathsf{L}/\partial v = p - mv \neq 0. && (230)\end{aligned}$$

Then search for evolutionary vector fields such that the symplectic non zero virtual work is of the form:

$$\begin{aligned} W &= i(\mathbf{W})dA \\ &= [-(\hbar k)(dv - adt) + F^{diss}(dx - vdt)] \\ &= (F^{diss})dx - (\hbar k)dv + \{(\hbar k)a - F^{diss}v\}dt. && (231)\end{aligned}$$

Consider those cases where

$$\begin{aligned} F^{diss}v &= \beta\Gamma\omega v^2 && (232)\\ \hbar ka &= \beta\Gamma\omega xa && (233)\end{aligned}$$

constrained to yield the Virial Equation

$$\{(\hbar k)a - F^{diss}v) \Rightarrow \beta\Gamma\omega(xa - v^2) \Rightarrow 0. \qquad (234)$$

The Work 1-form then becomes

$$W = \Gamma\beta\{vd(\omega x) - (\omega x)dv\}, \qquad (235)$$

and if β is chosen to be a polynomial distribution of Holder norms, where each term is of the form

$$\beta(p) = 1/\{(\omega x)^p \pm (v)^p\}^{2/p}, \qquad (236)$$

then each term contributes an integer to the integral

$$\oint W = \Gamma 2\pi = \sum(\text{integers}). \qquad (237)$$

In other words, the Virial constraint causes the 1-form of work to be of Pfaff dimension 1 ($dW = 0$). However, the 1-form of Work, W, is closed, but not exact.

An "Open Question" remains: Does a Planck Distribution have a relationship to the polynomial of Holder norms?

Such processes on a thermodynamic system are examples of Helmholtz type B processes on symplectic manifolds. Topological fluctuations (see section 2.6) in both kinematic position and velocity are permitted, but are tamed by the constraint of the Virial condition to yield energy quantization.

Conclusion *The radiation pressure (fluctuation in* $(dx - vdt)$*) or temperature (fluctuation in* $(dv - adt)$*) prevents change in the orbit. Angular momentum is constant but interaction with the environment gives a Bohr-like picture. It would appear that the application of the Virial theorem can have both statistical and topological significance. From a statistical average point of view, the Virial theorem leads to Boyle's ideal gas Law,* $PV = nRT$*. From a topological point of view the Virial theorem appears to be related to the discrete oscillation frequencies associated with quantum mechanics.*

4.6. ***Locally Adiabatic Processes***

Each of the reversible processes must satisfy an additional topological constraint if the process is to be locally adiabatic:

Locally Adiabatic Processes

$$\begin{aligned} \text{Adiabatic process } i(\mathbf{V}_A)Q &= i(\mathbf{V}_A)d(i(\mathbf{V}_A)A)) \Rightarrow 0, \quad Q \neq 0 & (238) \\ \text{with a sufficient condition} &= i(\mathbf{V}_A)A \Rightarrow 0. & (239) \end{aligned}$$

If $-dB = 0$, then $\rho\mathbf{V}_E$ is a characteristic process relative to the 2-form F. If the work 1-form is of Pfaff topological dimension 0, then the process is an extremal process relative to A (see equation 191).

Extremal processes cannot exist on a non singular symplectic domain, because a non degenerate anti-symmetric matrix (formed by the coefficients of the 2-form dA) does not have eigenvectors with zero eigenvalues on spaces of even dimensions. Although unique extremal stationary states do not exist on the domain of Pfaff topological dimension 4, there can exist evolutionary invariant Bernoulli-Casimir functions, B, that generate non extremal, "stationary"states. Such Bernoulli processes can correspond to energy dissipative Helmholtz processes, but they, as well as all Helmholtz processes, are reversible in the thermodynamic sense described in section 2.3. The mechanical energy need not be constant, but the Bernoulli-Casimir function(s), B, are evolutionary invariant(s), and may be used to describe non unique stationary state(s).

The equations given above define several familiar categories of processes, which are in effect constraints the Work 1-form, W. The Work 1-form is generated by a process describing the topological evolution of any physical system encoded by an Action 1-form, A. The Pfaff dimension of the 1-form of virtual work, $W = i(\mathbf{V})dA$ is 1 or less for all three sub categories of Helmholtz processes described above. The Extremal constraint of equation (217) can be used to generate the Euler equations of hydrodynamics for a incompressible fluid. The Bernoulli-Casimir constraint of equation (218) can be used to generate the equations for a barotropic compressible fluid. The Helmholtz constraint of equation (220) can be used to generate the equations for a Stokes flow. All such processes are thermodynamically reversible as $dQ = 0$. None of these constraints on the Work 1-form, W, above will generate the dissipative Navier-Stokes equations, which require that the topological dimension of the 1-form of virtual work must be greater than 2.

Note that for a given 1-form of Action, A, it is possible to construct a matrix of N-1 null (associated) vectors, and then to compute the adjoint matrix of cofactors transposed to create the unique direction field (to within a factor), $\mathbf{V}_{NullAdjoint}$. Evolution in the direction of $\mathbf{V}_{NullAdjoint}$ does not represent an adiabatic process path, as $i(\mathbf{V}_{NullAdjoint})A \neq 0$. However, for a given A, the N-1 null (associated) vectors represent locally adiabatic processes, but they need not span a smooth hypersurface whose surface normal is proportional to a gradient field. In fact, the components of the 1-form of Action, A, may be viewed as the normal vector to an implicit hypersurface, but the implicit hypersurface is not necessarily defined as the zero set of some smooth function.

4.7. ***Reversible processes when the Pfaff topological dimension of Work is 2 or 3***

Before studying irreversible processes, it is of some importance to study those reversible processes for which the Pfaff dimension is 2 or 3. In the process examples above, the work 1-form, W, was of Pfaff dimension 1 at most. As such, the Helmholtz conservation of vorticity theorem is valid, and the differential 1-form of heat is closed, $dQ = 0$. It follows that all such processes are thermodynamically reversible as $Q\hat{}dQ = 0$. However, there are processes where the work 1-form W is of Pfaff dimension >1, and yet the process involved is reversible. First consider Stokes processes where the Pfaff dimension of W is 2:

$$\begin{aligned} \textbf{Stokes Processes} &: \quad \text{If } W = -\beta dU = d(\beta U) + U d\beta, & (240) \\ dW &= -d\beta\hat{}dU & (241) \\ Q &= (1-\beta)dU, & (242) \\ dQ &= -d\beta dU & (243) \\ Q\hat{}dQ &= -(1-\beta)dU\hat{}d\beta\hat{}dU \Rightarrow 0 & (244) \end{aligned}$$

Although the Pfaff topological dimension of the work 1-form is 2, as $Q\hat{}dQ = 0$, the Stokes process is a reversible process.

Next consider Chaotic reversible processes where the work 1-form is of Pfaff dimension 3. The topology induced by the work 1-form is a disconnected topology. The functions ϕ and χ are completely arbitrary in this example, and can be associated with the classical thermodynamic potentials. The contact structure (as the Pfaff topological dimension of Work, $W = 3$) can be of two types: Tight and Overtwisted: Tight contact structures have a global Pfaff dimension equal to 3, Overtwisted contact structures also have a 3-form which is not zero, except at certain singular subsets. The 3-form is not global.

Tight Contact Structures

$$\begin{aligned} \text{If } W &= \phi d\chi - dU = d(\phi\chi - U) - \chi d\phi, && (245) \\ dW &= d\phi\hat{}d\chi && (246) \\ Q &= (W + dU) && (247) \\ dQ &= dW && (248) \\ Q\hat{}dQ &= (W + dU)\hat{}dW && (249) \\ &= -dU\hat{}dW + dU\hat{}dW \Rightarrow 0 && (250) \end{aligned}$$

As $Q\hat{}dQ \Rightarrow 0$, these specialized processes which induce a work 1-form of Pfaff topological dimension 3 are thermodynamically reversible. The Work 1-form generates a contact 3 manifold which has no limit cycles.? It will be shown below (see section 2.6.3) how such processes are related to the classical thermodynamic potentials, for specific choices of the function $(\phi\chi - U)$.

Neither of these last 2 processes conserve vorticity (think angular momentum). Yet they are candidates for investigating reconnection processes.?

Of particular interest are those processes for which the work 1-form generates an "Over-twisted Contact structure". Such structures are important for they are the domain of limit cycles. As an example define the Holder function as a quadratic form in terms of two independent functions, ϕ and χ, as:

Definition *The "Holder" variable is defined as* $h^2 = \phi^2 \pm \chi^2$

A constant value for the square Holder norm is elliptic or hyperbolic depending upon the $\pm$ sign. Next define the closed 1-form of Pfaff dimension 1, as

Definition *The closed but not exact 1-form, γ, is defined as*

$$\begin{aligned} \gamma &= (\phi d\chi - \chi d\phi)/(\phi^2 \pm \chi^2) = (\phi d\chi - \chi d\phi)/h^2, && (251) \\ d\gamma &\Rightarrow 0. && (252) \end{aligned}$$

The closed form γ plays the role of a "differential angle variable $\delta\theta$" in the elliptic case.

Now study those processes where the work 1-form is of Pfaff dimension 3, but not globally.

Over-twisted Contact Structures

(Limit cycles)

$$\begin{aligned} \text{If } W &= f(h)\gamma - dU, && (253) \\ dW &= \partial f/\partial h dh\hat{}\gamma && (254) \\ W\hat{}dW &= \{-\partial f/\partial h\ \}dU\hat{}dh\hat{}\gamma && (255) \\ Q &= (W + dU), dQ = dW, && (256) \\ Q\hat{}dQ &= -dU\hat{}dW + dU\hat{}dW \Rightarrow 0 && (257) \end{aligned}$$

Although the Work 3-form $W\hat{}dW$ is not zero almost everywhere, the heat 3-form $Q\hat{}dQ = 0$ is zero globally, Hence the process is thermodynamically reversible. However, the 3-form volume element created by the work 1-form is not global and will admit defect structures. In the example above, the work $3-$ form, $W\hat{}dW$, considered as a 3D - volume element, has singularities which occur at the zeros of the function $-\partial f/\partial h$. If, for example,

$$f(h) = (b + h - h^3/3a^2), \tag{258}$$

then the circle, $-\partial f/\partial h = 0$, defines a limit cycle in the elliptic case, in the two dimensional plane defined by ϕ and χ :

$$\begin{aligned} -\partial f/\partial h &= -\left(1 - h^2/a^2\right) \Rightarrow 0 & (259) \\ h^2 &= \phi^2 + \chi^2 = a^2. & (260) \end{aligned}$$

The limit cycle is stabile (attracting) if the volume element has a negative orientation (contracting), and is unstable otherwise. To cement the ideas, rewrite the work 1-form in terms of a more suggestive set of symbols, and observe that the rotational term has the format of the component of angular momentum orthogonal to the plane of rotation.

$$\begin{aligned} \text{If } W &= i(\rho \mathbf{V}_4)dA = m\Gamma(h)(xdy - ydx) - dU & (261) \\ &\Rightarrow \Gamma(h)\{m(xV^y - yV^x)\}dt - dU, & (262) \\ &= \Gamma(h)\mathbf{L}_z dt - dU. & (263) \end{aligned}$$

As $d\gamma = 0$ except at the fixed point of the "rotation", the Pfaff dimension of W has evolved from Pfaff dimension 3 to Pfaff dimension 1, as $\partial\Gamma/\partial h \Rightarrow 0$. In the Pfaff dimension 1 state, Helmholtz theorem becomes valid and "vorticity" is preserved. In the Pfaff dimension 3 mode, the process does not conserve vorticity. When the system decays (or is attracted) to the Pfaff dimension 1 state, the subsequent work done by a cyclic process is not necessarily zero. The closed but not exact 1-form γ can contribute to a period integral. Upon reflection, what has been described is the approach (Pfaff dimension 3) to a limit cycle (Pfaff dimension 1). The entire process has been done reversibly. Other forms of both the tight and the overtwisted contact structures defined by the work 1-form, can occur and such C2 processes can be thermodynamically irreversible. However, it can be demonstrated below that sequential C1 processes exist for all contact structures that are thermodynamically reversible.

4.8. Spinors carry polarization and chiral properties

If the rank of the N x N matrix $[\mathbb{F}]$ is less than N, then the matrix can have real vectors with eigen value zero. These vectors of eigen value zero will be defined as "null" vectors, and are not the same as Spinors.

As an example, consider the 1-form of Action, $A(x, y, z, s, dx, dy, dz, ds)$ and its associated Pfaff sequence given by the expressions

$$\begin{aligned} A &= -ydx + xdy - sdz + zds, & (264) \\ F &= dA = 2dx\hat{}dy + 2dz\hat{}ds \Rightarrow B_z dx\hat{}dy + E_z dz\hat{}dt, & (265) \\ A\hat{}F &= 2\{xdy\hat{}dz\hat{}ds - ydx\hat{}dz\hat{}ds + zdx\hat{}dy\hat{}ds - sdx\hat{}dy\hat{}dz, & (266) \\ F\hat{}F &= 8dx\hat{}dy\hat{}dz\hat{}ds \Rightarrow 2\left(\mathbf{E}\cdot\mathbf{B}\right)dx\hat{}dy\hat{}dz\hat{}ds. & (267) \end{aligned}$$

If the 1-form is interpreted as a representation of a Maxwell system, then there exists both an $\mathbf{E}$ and a $\mathbf{B}$ field, both equal to the constant value 2 (times a factor to adjust the physical dimensions), and both parallel to one another. That is, $\mathbf{E} \circ \mathbf{B} \neq \mathbf{0}$. The 3-form of topological torsion, $A\hat{}F$, is not zero, and has components proportional to a 4 dimensional position vector, $\mathbf{R}^T = [x, y, z, t]$ representing a 4D expansion.

Note that the 4x4 antisymmetric matrix representation of the 2-form F is of the form

$$[\mathbb{F}] = \begin{bmatrix} 0 & 2 & 0 & 0 \\ -2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 \\ 0 & 0 & -2 & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 0 & B_z & 0 & 0 \\ -B_z & 0 & 0 & 0 \\ 0 & 0 & 0 & E_z \\ 0 & 0 & -E_z & 0 \end{bmatrix}. \tag{268}$$

The matrix has 4 eigenvalues and 4 eigenvectors given by the formulas,

$$\text{Isotropic Eigenvector 1} = [0,0,-i,1] \text{ with eigenvalue} = iE_z, \tag{269}$$
$$\text{Isotropic Eigenvector 2} = [0,0,i,1] \text{ with eigenvalue} = -iE_z, \tag{270}$$
$$\text{Isotropic Eigenvector 3} = [1,i,0,0] \text{ with eigenvalue} = iB_z, \tag{271}$$
$$\text{Isotropic Eigenvector 4} = [1,-i,0,0] \text{ with eigenvalue} = -iB_z. \tag{272}$$

Each eigenvector is null isotropic such that the sum of squares (with euclidean signature) of the coefficients is zero. This example is a simple case generated by the 1-form of Action (potentials), A, whose coefficients form the adjoint field to the three exact differentials generated by the Hopf map (a submersion from 4D to 3D).[22] All eigenvectors of a 2-form F with non-zero determinant have "zero" (euclidean) length. The idea can be extended to Minkowski or anti-Minkowski signatures. The fundamental result is that Spinors are the natural format of propagating singularities generated from the Eikonal equation. It is the objective of this article to exploit the connection between propagating singularities, spinors and the photon.

It should be noted, in the example above, that, as $F\hat{}F \neq 0$, the Pfaff topological dimension of the 1-form, A, and the rank of the 2-form $F = dA$ is 4. It is also true that the topological torsion 3-form is not zero, $A\hat{}dA \neq 0$. It follows that evolution in the direction of $A\hat{}F$ is thermodynamically irreversible, and the topology induced by the 1-form of Action is not an evolutionary invariant. Electrodynamic expansion of the 4D universe is thermodynamically irreversible. Note that the associated $\mathbf{E}$ and $\mathbf{B}$ fields have parallel components; the value $\mathbf{E} \cdot \mathbf{B} \neq 0$ is a measure of the dissipation.[19] This situation implies that the determinant of the 4D matrix, $[\mathbb{F}]$, is non-zero, and there are no eigenvectors with zero eigenvalues.

The existence of Spinors is a topological property related to the Pfaff topological dimension[19] of the 1-form, A, that generates the 2-form of EM field intensities, F. If the Pfaff topological dimension is 4, all four eigenvectors of the matrix $[\mathbb{F}]$ are spinors. If the Pfaff topological dimension of A is 3 or 2, then two of the eigenvectors of the matrix $[\mathbb{F}]$ are spinors with $\gamma \neq 0$, and the other eigenvectors have $\gamma = 0$. If the Pfaff topological dimension of A is 1, then the matrix $[\mathbb{F}]$ is null (as $F = dA \Rightarrow 0$), and there are no spinor eigenvalues. Of particular interest are those cases where the Pfaff topological dimension of A is 3 or 4, for then in each case the 3-form of "topological torsion exists; $A\hat{}F \neq 0$. Explicit formulas in engineering format are given below. Recall that for any 4x4 matrix, the characteristic polynomial (in terms of eigenvalues) has either 4 real solutions, two pairs of complex conjugate solutions, or 1-pair of complex conjugate solutions along with 2 real solutions.

A search of the more recent mathematics literature[6] indicates that Spinors can be related to harmonic forms, and also to conjugate pairs of minimal surfaces. Yet little application of this correspondence has been made in the engineering physical sciences. For purposes herein, the conclusion reached is that Spinors are normal consequences of antisymmetric matrices, and, as topological artifacts, are not restricted to physical microscopic or quantum constraints of scale. According to the topological thermodynamic arguments, they should appear at all scales of Pfaff topological dimension 2 or greater. The fundamental result is that Spinors are the natural format of propagating singularities generated from the Eikonal equation.

Note that the 2-form F can be considered to be a "vector" in a 6 dimensional space. If the components of this 6-vector are such that $\mathbf{E} \cdot \mathbf{B} = 0$, then the determinant of matrix $[\mathbb{F}]$ is zero. This induced topological constraint has been has called the Klein quadric; the projection hypersurface reduces the 6 dimensional space to a 5 dimensional projective subspace. This result forms an interesting connection between modern Kaluza-Klein theory and the embedding of a curved 4D manifold into a Ricci flat space of 5D.[23] The important point is that when $\mathbf{E} \cdot \mathbf{B} = 0$ is zero, the Pfaff topological dimension is at most 3, and there exist on a 4D space two eigenvectors of the matrix $[\mathbb{F}]$ which have zero eigenvalues, and two eigenvectors which are spinors. These results will be used below to formulate a model of a photon deduced from a topological perspective of Electromagnetic theory.

4.9. The concept of Photon Quantization is a topological idea that does not depend upon microphysical scales.

From the topological formulation given above, in terms of exterior differential forms, $\{\ A,\ F,\ G,\ J\ \}$ the question arises as to how discrete (quantum) features of the photon enter into the topological theory. From thermodynamic arguments, if the Maxwell equations are uniquely integrable, then the maximum topological dimension of the 1-form of Action is 2. That is, there exist two functions on the geometrical domain of 4D which generate all of the differential topology associated with the field intensities. Such is the domain of an isolated, or equilibrium, thermodynamic system. Exterior differential 3-forms, such as $A\hat{}F$, do not exist on domains of an isolated topology of Pfaff dimension 2 (or less); the topological structure consists of a single connected component. On non-equilibrium domains, the topological dimension of the 1-form of Action potentials, A, can be 3 or 4. Such domains support exterior differential 3-forms and 4-forms on multiple components of the topological structure. The question is how do you formulate the possible multiple component topological structures of non-equilibrium electrodynamic system? The answer is in terms of closed, but not exact, exterior differential 3-forms which are homogeneous of degree zero.

By inspection, from the set of exterior forms $\{\ A,\ F,\ G,\ J\}$, it is possible to construct two important 3-forms that are related to 4 component "currents" on a 4D domain of $\{x, y, z, t\}$. The 3-forms are written in terms of engineering variables, representing the coefficients of the 3-forms, in the following equations. The objects are zero in isolated-equilibrium systems. They are (topological) artifacts of non-equilibrium electromagnetic systems:

$$\text{Topological Torsion} = A\hat{}F = i(\mathbf{T}_4)dx\hat{}dy\hat{}dz\hat{}dt, \quad \text{units h/e} \tag{273}$$

$$\mathbf{T}_4 = [\mathbf{E}\times\mathbf{A}+\mathbf{B}\phi,\ \mathbf{A}\circ\mathbf{B}] \tag{274}$$

$$\text{Topological Spin} = A\hat{}G = i(\mathbf{S}_4)dx\hat{}dy\hat{}dz\hat{}dt, \ \text{units h} \tag{275}$$

$$\mathbf{S}_4 = [\mathbf{A}\times\mathbf{H}+\mathbf{D}\phi,\ \mathbf{A}\circ\mathbf{D}], \tag{276}$$

These topological objects are universally defined for non equilibrium electromagnetic systems, yet their dynamics and properties have been little utilized. These 3-forms can have non-zero exterior differentials (which are exact exterior differential 4-forms) related to the historical Poincare invariants of the electromagnetic field:

$$\text{Poincare II} \ \ d(A\hat{}F) = F\hat{}F = 2(\mathbf{E}\circ\mathbf{B})dx\hat{}dy\hat{}dz\hat{}dt \tag{277}$$

$$\text{Poincare I} \ \ d(A\hat{}G) = F\hat{}G - A\hat{}J = \{\mathbf{B}\circ\mathbf{H}-\mathbf{D}\circ\mathbf{E}\} - \{\mathbf{A}\circ\mathbf{J}-\rho\phi\}. \tag{278}$$

When and where the exterior derivatives of each 3-form vanish, then by deRham's topological theorems, the closed cyclic integrals of each 3-form will have values that have rational integer ratios; i.e., the closed cyclic integrals are integers times some universal constant. The cyclic integrals therefor are "quantized" relative to the physical constant, h/e, for topological torsion, and to the physical constant, h, for topological spin. These concepts have not made any use of geometric ideas of size and shape, yet yield "quantum" numbers. The do not depend upon geometric scales, nor any explicit use of quantum theory.

$$\iiint_{closed} (A\hat{}F) = m\ h/e \quad \text{if Poincare II} = 0 \tag{279}$$

$$\iiint_{closed} (A\hat{}G) = n\ h \quad \text{if Poincare I} = 0 \tag{280}$$

Further note that the ratios of these two topological quantum numbers yields the Hall impedance, $Z_{Hall} = h/e^2$ (to within a rational fraction).[11]

In terms of topological thermodynamics, the manifolds upon which $d(A\hat{}F) = F\hat{}F = 0$ are non-equilibrium domains of Pfaff topological dimension 3. These submanifolds of space time can emerge (as if by a droplet condensation process) from dissipative thermodynamic systems of Pfaff topological dimension 4 ($d(A\hat{}F) = F\hat{}F \neq 0$). The Cartan topology of Pfaff dimension 3 (or more) is a topology of disconnected multiple components. Each component has a connected topological coherence.

The fifth somewhat heretical claim is:

5. Long lived propagating coherent states can occur in non equilibrium electrodynamic systems,

and the photon is an example of such a soliton.

The non-equilibrium electrodynamic system consists of disconnected components where the Pfaff topological dimension is greater than 2. For a 4D space time set of independent variables, the possibilities are that the domain of interest is of Pfaff dimension 3 or Pfaff dimension 4. Pfaff dimension 3 domains can emerge from Pfaff topological dimension 4 domains by means of continuous thermodynamic irreversible processes in the direction of the topological torsion vector, generated by the components of the 3-form $A\hat{}F$ with $F\hat{}F \neq 0$. What is remarkable is that, on thermodynamic domains of Pfaff topological dimension 3 ($F\hat{}F = 0,\ A\hat{}F \neq 0$) a continuous evolutionary process in the direction of the topological torsion vector becomes equivalent to the unique extremal Hamiltonian field that generates "stationary, or excited, states". Such submanifold domains then can evolve as Soliton structures ("stationary" excited states) maintaining a topological coherence and a long life time. The submanifold structures of Pfaff topological dimension 3 do not depend upon geometric scales, yet they are precisely the domains required such that the 3-forms of topological torsion and topological spin have zero divergence. They are sets that have the properties required for the "quantized" topological properties of spin quanta and flux quanta.

Such unique Hamiltonian fields exist for all odd Pfaff topological dimensional systems greater than 2. Such manifolds belong to the class of Contact manifolds. All even Pfaff topological manifolds belong to the class of Symplectic manifolds, and do NOT admit such extremal Hamiltonian processes. In fact, it appears that the class of thermodynamically irreversible processes is an artifact of Pfaff topological dimension 4. The important idea is that non equilibrium electromagnetic systems involve the 3-forms of topological torsion, $A\hat{}F$, and topological Spin, $A\hat{}G$, whose closed homogeneous forms furnish the quantum numbers associated with photons[16].[13]

The subspaces of Pfaff topological dimension 3 may be viewed as topological defects or discontinuities in domains of Pfaff topological dimension 4. For example the generation of the light cone as propagating discontinuities in the 2-form F is directly related to the Klein quadratic implicit surface function, $F\hat{}F = 0$, which constrains the 4D domain of Pfaff topological dimension 4 to subspaces of Pfaff topological dimension 3. The eigenvectors of the matrix $[\mathbb{F}]$ are of two types: they are either eigenvectors of zero eigenvalue, or complex isotropic direction fields commonly called Spinors. Recall that the isotropic spinor eigenvectors are generators of 2D surfaces of zero mean curvature, within the 3D domains. It is these defect structures that are to be associated with the photon. On spaces with a euclidean metric, these 2D defect structures of zero mean curvature are called minimal surfaces, but on spaces with a Minkowski metric these defect structures of zero mean curvature are called maximal surfaces. Both defect surface types are amenable to isotropic spinor formulations.

4.10. The Analysis

The original analysis was conducted without considering the possibility of 3D space having Minkowski signature properties. Several exact solutions to the Navier-Stokes equations, *in a rotating frame of reference but with a Euclidean metric assumption*, had been used to demonstrate bifurcations to catenoidal defect structures that in appearance are close to Falaco Solitons. The Navier-Stokes solutions were locally catenoidal with "open throat". However, the Navier-Stokes solutions (based upon euclidean vector methods) found so far, were locally catenoidal with an "open throat". They do not replicate the formation of the conical singularity that is observed experimentally. These results suggest that the catenoidal open throat structures are minimal surface realizations of topological defects of zero mean curvature in Euclidean 3 space, but the closed throat, string connected, Falaco Solitons are maximal surface realizations of topological defects of zero mean curvature in Minkowski 3 space. The difference leads to the definition of Euclidean Falaco Solitons (equivalent to Wheeler's wormholes) and Minkowski Falaco Solitons (with the appearance of string connected Hadrons, or of strings connecting "branes"). The fundamental idea that is missing in the classic Navier-Stokes analysis is that the dynamics could have Spinor (not vector) solutions, which are ignored classically.

In fact, in the original attempts to analyze the Falaco Soliton experiments, it was thought that the Falaco Solitons might be macroscopic realizations of the Wheeler wormhole (open vortex throat). This structure was presented early on by Wheeler (1955), but was considered to be unattainable in a practical sense. To quote Zeldovich p. 126[25]

> "The throat or "wormhole" (in a Kruskal metric) as Wheeler calls it, connects regions of the same physical space which are extremely remote from each other. (Zeldovich then gives a sketch that topologically is equivalent to the Falaco Soliton in a Euclidean metric). Such a topology implies the existence of 'truly geometrodynamic objects' which are unknown to physics. Wheeler suggests that such objects have a bearing on the nature of elementary particles and anti particles and the relationships between them. However, this idea has not yet borne fruit; and there are no macroscopic "geometrodynamic objects" in nature that we know of. Thus we shall not consider such a possibility."

This quotation dates back to the period 1967-1971. Now the experimental evidence justifies (again) Wheeler's intuition. However, the concept of a wormhole as a catenoidal minimal surface of zero mean curvature in a Euclidean space (with an open throat) is transformed into a Falaco Soliton as a catenoidal maximal surface of zero mean curvature in a Minkowski 3 space. The catenoidal surface of zero mean curvature, and negative Gauss curvature, in a 3D Euclidean space is a Wheeler Wormhole, while the catenoidal surface of zero mean curvature, and positive Gauss curvature, and its conical singular point in a 3D Minkowski space is a part of the fluid Falaco Soliton.

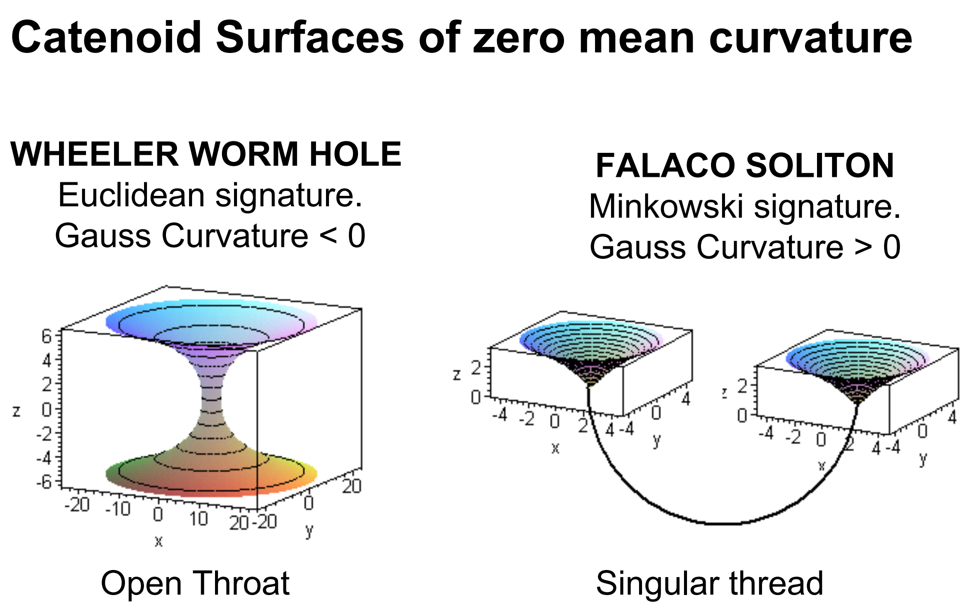
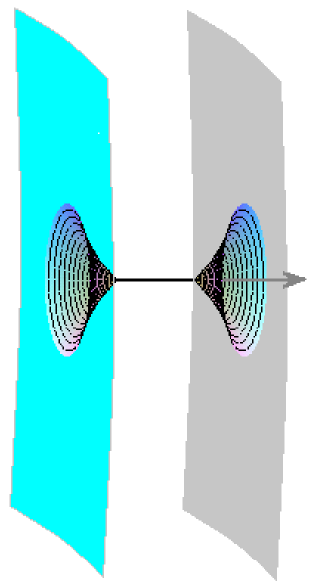


Fig 6. Surfaces of Zero mean Curvature

If the Maximal surfaces appear as deformations in disconnected hypersurfaces of discontinuity, the topological structure has the appearance of "strings connecting branes", a concept touted by the string theorists. The new feature is that the "brane" surface of discontinuity is deformed by the Maximal surface dimple. This structure motivates the next section in which the idea is used to model the photon.

Falaco Solitons



2D Maximal surface deformations in a 3D discontinuity surface, connected with a 1D string

The key experimental features of the Falaco Soliton are that the topological structure deforms a local discontinuity surface in a catenoidal-cone-like manner of zero mean curvature, and that the deformation appears to be stabilized by a connecting (elastic?) string or thread between a pair of deformation structures. In the next section, it is assumed that a thermodynamic (electromagentic) system can be encoded by a 1-form of Action potentials, A, which leads by exterior differentiation to a 2-form of field intensities, $F = dA$. The null eigenvectors of the antisymmetric matrix representation of F will form 3D expanding spherical surfaces of propagating field discontinuities (related to the spatial portions of the Minkowski lightcone where $F\hat{}F = 0$). In addition, the isotropic Spinor eigenvectors of F will form surfaces of zero mean curvature as defect structures on the spherical spatial portions of the lightcone. The result is a Falaco Soliton pair (with $A\hat{}F \neq 0$) between the two bounding cycles of a spherical shell. The claim is that this concept serves as a model for the Photon.

5. A TOPOLOGICAL MODEL FOR A PHOTON.

The idea is to combine the topological features of the Minkowski signature, the possibilities of coherent states of "stationary" topology (solitons) for non-equilibrium, but thermodynamically closed, systems of Pfaff topological dimension 3 (with $A\hat{}F \neq 0$), and the fact that for such systems the electromagnetic 2-form, $F = dA$, has one pair of eigenvectors of eigenvalue zero, and one pair of complex conjugate isotropic null eigenvector arrays with imaginary eigenvalues. The eigenvectors with zero eigenvalues form Minkowski lightcones with $F\hat{}F = 0$. Consider two causal expanding spheres (two light cones) representing the "on" and "off" propagating discontinuity defects (as expanding concentric spheres in 3D). The concentric spherical surfaces of field discontinuity bound an interior region of finite electromagnetic field intensities, **E** and **B**. The conjugate pair of Spinor eigenvectors define 2D surfaces of zero mean curvature as conical topological deformation defects on the light cones.

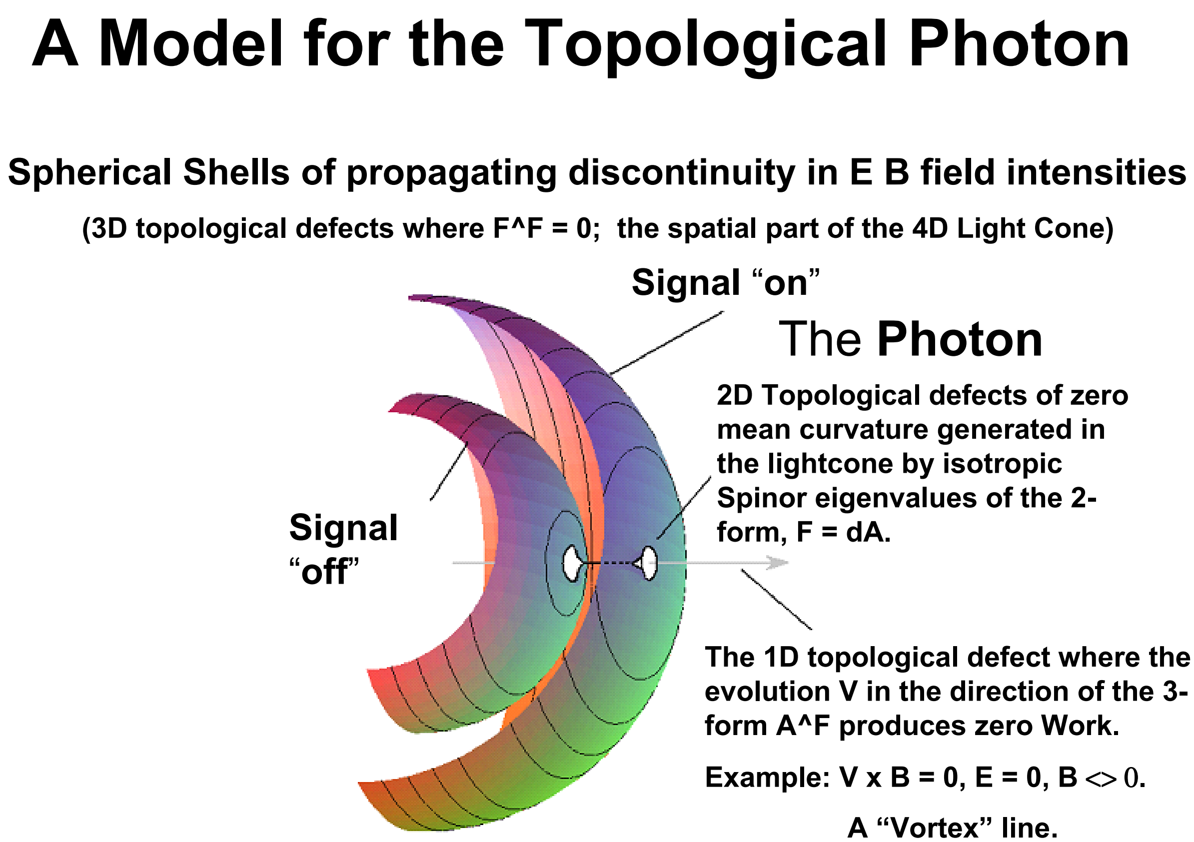


Fig. 7. The Photon as a Falaco Soliton between lightcone shells, which bound a closed non equilibrium thermodynamic state, $\mathbf{A}\hat{}\mathbf{F} \neq 0$.

The conical defects on each light cone are connected by a 1D "string", or "vortex tube", of zero radius, determined by the condition that evolution, V, in the direction of the components of the 3-form, $A\hat{}F$, of topological torsion, are extremal. That is, the thermodynamic work vanishes: $W = i(V)dA = i(V)F \Rightarrow 0$.

As an example, consider the 1-form of Action given by,

$$A = (m_e/e)\{\omega(xdy - ydx) - c^2 dt\}, \tag{281}$$

where the constancts $(m_{electron}/e = h/(ec\lambda_{Compton})), \omega$ and c have been chosen on grounds of dimensional analysis. The Pfaff sequence demonstrates the the Pfaff topological dimension relative to the 1-form A is 3:

$$F = dA = (m_e/e)2\omega \; dx\hat{}dy = B_z dx\hat{}dy, \tag{282}$$
$$A\hat{}F = (m_e c^2/e)B_z dx\hat{}dy\hat{}dt, \tag{283}$$
$$F\hat{}F = 0. \tag{284}$$

There is no **E** field, but there is a **B** field component along the z axis. Hence the example has the properties

$$\mathbf{E} = 0, \; \mathbf{B} \neq 0, \; \mathbf{A} \neq 0, \quad \mathbf{E} \circ \mathbf{B} = 0, \quad \mathbf{A} \circ \mathbf{B} = 0 \tag{285}$$

Note that the scalar and vector potentials are given by the expressions

$$\phi = (m_e c^2/e), \tag{286}$$
$$\mathbf{A} = (m_e/e)\omega[-y, x, 0]. \tag{287}$$

The vector potential is tangent to a circle about the origin in the z = 0 plane. The direction field generated by the toplogical torsion vector is

$$\mathbf{T}_4 = [\mathbf{E} \times \mathbf{A} + \mathbf{B}\phi,\ \mathbf{A} \circ \mathbf{B}] \tag{288}$$
$$= [0, 0, B_z(m_e c^2/e), 0]. \tag{289}$$

For evolutionary processes $\mathbf{V}_4$ in the direction of $\mathbf{T}_4$, it follows that the Work 1-form vanishes.

$$W = i(\mathbf{V}_4)F = -\{\mathbf{E} + \mathbf{V} \times \mathbf{B}\} \circ d\mathbf{r} + (\mathbf{E} \circ \mathbf{V})dt, \tag{290}$$
$$W = i(\mathbf{T}_4)F = -\{\mathbf{0} + \mathbf{B}\phi \times \mathbf{B}\} \circ d\mathbf{r} + (0)dt \Rightarrow 0. \tag{291}$$

The evolutionary velocity field $\mathbf{V}$ in the direction of $\mathbf{T}_4$ is proportional to the $\mathbf{B}$ field.

This result gains credence from the observations of similar topological defects in fluid systems, called Falaco Solitons.[20] Thermodynamic systems of Pfaff topological dimension 3 (based on the 1-form, A) are non equilibrium, thermodynamically closed systems that can exchange energy (radiation) but not mass with their environments. When the Photon is "created" the Pfaff topological dimension is presumed to be 4, with evolution along a space time direction field given by the Topological Torsion vector, $\mathbf{T}_4$. The processes is thermodynamically irreversible, and $(\mathbf{E} \circ \mathbf{B}) \neq 0$. The process evolves continuously to domains of Pfaff topological dimension 3, forming the "condensations" - or Photons - of topological coherence as stationary, but excited, states of a Hamiltonian process. It is conjectured that the conical topological defects are not constrained by a limiting speed C, but can move (transversely on the light cone) with speeds given by the projective Moebius transformations, which can vary from zero to infinity.

6. SUMMARY

From a topological and thermodynamic perspective of the electromagnetic field, there appears to be a common thread between Eikonal solutions, Spinors, propagating topological discontinuities or defects, minimal surfaces, and "topological quantization". All of these properties suggest that the common topological thread is that which is usually perceived as the Photon. A topological perspective of Electromagnetism not only includes features attributed to the Photon, but also points out that non equilibrium thermodynamic concepts can be formulated to produce interesting experiments and practical devices. For example, the fact the irreversible dissipation occurs when the field intensities have a collinear component $(\mathbf{E} \circ \mathbf{B} \neq 0)$ could be used to influence condensation. Stable long lived states in a plasma should be designed about the (Klein quadric) constraint that $(\mathbf{E} \circ \mathbf{B} = 0)$ which yields "stationary" non equilibrium dynamical systems, or excited states described by Hamiltonian processes. Each of these ideas involve the concepts of topological torsion and topological spin, and hence the quantal properties of the photon.

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