Spinors, Minimal Surfaces, Torsion, Helicity, Chirality, Spin, Twistors, Orientation, Continuity, Fractals, Point Particles, Polarization, the Light Cone and the Hopf Map.

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Abstract

The correspondence between the Spinor map and the Hopf map is investigated, and the relationship of Spinor maps to Minimal surfaces is discussed. It appears that a point particle could be considered as a minimal surface of zero real radius, but of finite area in 4D. (Draft)

1. Introduction

A re-reading of Cartan's book on Spinors (and Chadrasekhar's book on Black Holes) leads to the thought that there is a connection between all of the ideas in the above title. It is remarkable to me that both Cartan and Chandresekhar do not mention the fact that an isotropic (complex null) vector is related to the generator of a Minimal Surface. [1 -See Osserman p.63]. This is surprising to me, as Cartan was a differential geometer who knew about minimal surfaces. Cartan defined [2 - see Cartan p. 41] the Spinor as a *mapping* of a complex pair to a special 3 component complex vector in such a way that its quadratic form (sum of squares of the three components) is zero. This relationship of Spinor maps to minimal surfaces is ignored by many other authors, as well as Cartan and Chandrasekhar (A recent personal communication with Rindler also indicates that he also was not aware of the connection of these ideas.). It would appear that most physicists are not aware of the connection between spinors and minimal surfaces, which yields an interesting and useful geometric interpretation of spinors. Evidently the idea of connecting Spinors and minimal surfaces was noticed by Dennis Sullivan (the topologist) about 1989. This reference I found (after I had stumbled on the idea) in the recent article by See R. Kusner and N. Schmitt, "The Spinor Representation of Minimal Surfaces" (http://xxx.lanl.gov/abs/dg-ga/9512003 or http://www.gang.umass.edu/schmitt).

Besides the lack of reference to the fact that Cartan Spinor mappings are related to minimal surfaces, it is also apparent that the relationship of Spinor Maps to the Hopf map has been ignored by almost everybody. Recall that the Hopf map is a map from a vector of 4 components to a vector (the Hopf vector) of 3 components, such that the sum of squares of the three components is the square of the sum of squares of the 4 components. The map is ambiguous to within a sign (plus or minus one). If the components of the Hopf vector in 3 space are presumed to be the dimensionless ratios (x/ct,y/ct/,z/ct), then the Hopf map can be viewed as a map from R4 to a projective 3-space. Fixing the value of the sum of squares of the 4 components to a constant (say unity) generates the equation of the light cone in R4. There are three versions of the (real) Hopf vector, all with the same value for the sum of squares, which can be arranged such that they are mutually orthogonal. The implication is that there are at least three distinct constraints that can represent the light cone.

The correspondence between the Spinor map and the Hopf map will be investigated below, where it will be demonstrated that the rudimentary Cartan Spinor map is a complex three dimensional "vector" whose real and imaginary components are both Hopf vectors. Each of the two Hopf vectors that make up the Cartan spinor are mutually orthogonal. As mentioned above, it is possible to construct three linearly independent Hopf vectors that are mutually orthogonal, and when these orthogonal Hopf vectors are combined in complex pairs, it is possible to construct six independent spinors. Integration of a given complex null vector leads to a complex "position" vector. The real and imaginary parts of the "position" vector separately describe a pair of minimal surfaces. In differential geometry, the null spinor is called an isotropic vector, and the pair of minimal surfaces are called conjugate surfaces. Linear combinations of the two conjugate components of the "position" vector also generate a minimal surface. The analog in physics can be described in terms of the optics of polarization. One extreme minimal surface is linear polarization, while the other extreme is circualr polarization. A linear combination of the minimal surfaces is analogous to elliptical polarization. Eachof the polarizations is ambiguous with respect to a sign (right handed vs left handed, horizontal vs. vertical)

The Hopf map also appears embedded in the classical physics literature. It is latent in the classical optics theory of partial polarization [3- O'Neill]; in the classical electromagnetic theory of Bateman and Whittaker [4- Bateman]; in the theory of hydrodynamic wakes [5- Kiehn], in the examples of electric wave singular solutions that give the appearance of breaking time reversal symmetry [6 - Kiehn]. Yet these classic examples, and many others, do not focus attention on the fact the the Hopf vector fields, so constructed in terms of ordered complex pairs, are related to spinors,. It took relativistic quantum theory to focus popularity on spinors, leading to a (false) popular opinion that spinors were something of a "quantum mechanical"origin.

1.1. Hopf Maps and Hopf Vectors

Consider the map from R4(X,Y,Z,S) to R3(u,v,w) given by the formulas

$$\mathbf{H}1 = [u1, v1, w1] = [2(XZ + YS), 2(XS - YZ), (X^2 + Y^2) - (Z^2 + S^2)] \quad (1.1)$$

These formulas define the format of a Hopf map. The Hopf vector $\mathbf{H}1$ is real and has the property that

$$\mathbf{H}1 \cdot \mathbf{H}1 = (u1)^2 + (v1)^2 + (w1)^2 = (X^2 + Y^2 + Z^2 + S^2)^2. \quad (1.2)$$

Hence a real 4 dimensional sphere maps to a real 3 dimensional sphere. If the functions $[u1, v1, w1]$ are defined as $[x/ct, y/ct, z/ct]$, then the 4D sphere $(X^2 + Y^2 + Z^2 + S^2)^2 = 1$, implies that the Hopf map formulas are equivalent to the 4D light cone.

The Hopf map can also be represented by a map from C2 to R3, as given by the formulas

$$\begin{aligned} \mathbf{H}1 &= [u1, v1, w1] \quad (1.3) \\ &= [\alpha \cdot \beta^* + \beta \cdot \alpha^*, i(\alpha \cdot \beta^* - \beta \cdot \alpha^*), \alpha \cdot \alpha^* - \beta \cdot \beta^* \end{aligned}$$

The variables α and β also can be viewed as two distinct complex variables defining ordered pairs of the four variables $[X, Y, Z, S]$. For example, the classic format given above for **H**1 can be obtained from the expansion,

$$\alpha = X + iY \qquad\qquad \beta = Z + iS. \tag{1.4}$$

Other selections for the ordered pairs of (X, Y, Z, S) (along with permutations of the 3 vector components) give distinctly different Hopf vectors. For example, the ordered pairs,

$$\alpha = X + iZ\,, \qquad\qquad \beta = Y + iS, \tag{1.5}$$

give

$$\begin{aligned} \mathbf{H}2 &= [u2, v2, w2] \qquad (1.6) \\ &= [\alpha \cdot \beta^* + \beta \cdot \alpha^*, \alpha \cdot \alpha^* - \beta \cdot \beta^*, i(\alpha \cdot \beta^* - \beta \cdot \alpha^*)] \\ &= [2(YX - SZ), X^2 + Z^2 - Y^2 - S^2, -2(ZY + SX)] \end{aligned}$$

which is another Hopf vector, a map from R4 to R3, but with the property that **H**2 is orthogonal to **H**1 :

$$\mathbf{H}2 \cdot \mathbf{H}1 = 0. \tag{1.7}$$

Similarly, a third linearly independent orthogonal Hopf vector **H**3 can be found

$$\begin{aligned} \mathbf{H}3 &= [u3, v3, w3] \qquad (1.8) \\ &= [\alpha \cdot \alpha^* - \beta \cdot \beta^*, -(\alpha \cdot \beta^* + \beta \cdot \alpha^*), -i(\alpha \cdot \beta^* - \beta \cdot \alpha^*)] \\ &= [X^2 + Y^2 - Z^2 - S^2, -2(YX + SZ), 2(-ZX + SY)] \end{aligned}$$

such that

$$\mathbf{H}2 \cdot \mathbf{H}1 = \mathbf{H}3 \cdot \mathbf{H}2 = \mathbf{H}2 \cdot \mathbf{H}3 = 0. \tag{1.9}$$

$$\mathbf{H}1 \cdot \mathbf{H}1 = \mathbf{H}2 \cdot \mathbf{H}2 = \mathbf{H}3 \cdot \mathbf{H}3 = (X^2 + Y^2 + Z^2 + S^2)^2. \tag{1.10}$$

The three linearly independent Hopf vectors can be used as a basis of R3 excluding the origin.

Each Hopf vector can be differentiated with respect to the variables (X, Y, Z, S) forming a gradient field on R4. That is, the mapping functions (u, v, w) can be differentiated with respect to (X, Y, Z, S) to produce a set of three exact 1- forms. The exterior product $d(u1)\hat{}d(v1)\hat{}d(w1)$ produces a 3 form, whose dual which generates the "adjoint" field. The components of the adjoint field can be used to construct a non-integrable 1-form. The three exact 1-forms and the non-integrable 1-form can be used as a basis frame for the space. The exterior derivatives of the basis frame produce the usual Cartan connection which is not torsion free. By this mechanism the differential structure of R4 as induced by the Hopf map is determined.

From another point of view, each of the four functions X, Y, Z, S can be considered as complex variables, so that the Hopf map has a realization from C4 to C3.

1.2. Isotropic Vectors and Minimal surfaces

Consider the two lemmas given in R. Osserman's book " A Survey of Minimal Surfaces" Dover

Theorem 1.1. ***Lemma 8.1 (Osserman)*** *Let D be a domain in the complex z-plane, $g(z)$ an arbitrary meromorphic function in D and $f(z)$ an anlytic function in D having the property that at each point where $g(z)$ has a pole of order m, $f(z)$ has a zero of order at least 2m. Then the functions*

$$z1 = f(1 - g^2)/2 \tag{1.11}$$

$$z2 = i\, f(1 + g^2)/2, \tag{1.12}$$

$$z3 = \mp fg, \tag{1.13}$$

will be analytic in D and satisfy the equation of an "isotropic" null vector:

$$(z1)^2 + (z2)^2 + (z3)^2 = 0. \tag{1.14}$$

The only exception is for $z3 = 0$, $z1 = i\ z2$.

It is unfortunate that the historic word isotropic is used to describe the "null " vector, for in engineering practice, the word isotropic is usually interpreted as meaning the same in all directions. Technically the word isotropic used for the null vector is correct, for no matter what direction the null vector points in C3, its quadratic form, as a sum of squares of the three components, is zero.

An equivalent formulation for an isotropic (null) vector was given by Cartan in terms of $\alpha(z)$ and $\beta(z)$, as follows.

$$z1 = \alpha^2 - \beta^2 \tag{1.15}$$

$$z2 = i\ (\alpha^2 + \beta^2), \tag{1.16}$$

$$z3 = \mp 2\alpha\beta, \tag{1.17}$$

Evidently D. Sullivan noticed that these formulas of Cartan could be related to minimal surfaces in 1989 (hence predates my own recent independent appreciation of this fact). The formulas can also be interpreted in terms of the sequence of maps from the 2D space $\{\varpi = u + iv\}$ to the 8D space $\{\alpha = X(\varpi) + iY(\varpi), \beta = Z(\varpi) + iS(\varpi)\}$ to the 6D space $\{z1(\varpi), z2(\varpi), z3(\varpi)\}$. The quadratic form of an arbitrary vector on C3, $(z1)^2 + (z2)^2 + (z3)^2$, can be complex, real, or zero. However, the spinor construction given above always produces an *isotropic* or null vector: the quadratic form vanishes. The mapping described above is the original defineition of a Cartan Spinor. A Cartan Spinor is in fact, not the pair of functions, $\alpha(\varpi)$ and $\beta(\varpi)$, but the map to the isotropic complex 3 vector, such that

$$(z1)^2 + (z2)^2 + (z3)^2 = 0. \tag{1.18}$$

The isotropic (null) condition imposes two constraints on the 6D space of 3 complex variables reducing the dimension to a 4D space of two complex variables.

Theorem 1.2. ***Lemma 8.2 (Osserman)*** *Every simply connected minimal surface in E3 can be represented in the form of a position vector*

$$\mathbf{R}_{\mathrm{real}} = [X1(u, v), Y1(u, v), Z1(u, v)], \tag{1.19}$$

where $\varpi = (u + i\,v)$. A conjugate minimal surface can be constructed from the imaginary components of the integral formulation,

$$\mathbf{R}_{imag} = [X2(u, v), Y2(u, v), Z2(u, v)]. \tag{1.20}$$

The position vector is computed from an isotropic complex 3 vector by means of the formulas:

$$X1(u,v) = \mathrm{Re}\int z1(\varpi)d\varpi + constant \qquad X2(u,v) = \mathrm{Im}\int z1(\varpi)d\varpi + constant \tag{1.21}$$

$$Y1(u,v) = \mathrm{Re}\int z2(\varpi)d\varpi + constant \qquad Y2(u,v) = \mathrm{Im}\int z2(\varpi)d\varpi + constant \tag{1.22}$$

$$Z1(u,v) = \mathrm{Re}\int z3(\varpi)d\varpi + constant \qquad Z2(u,v) = \mathrm{Im}\int z3(\varpi)d\varpi + constant \tag{1.23}$$

Either (real) component of the complex position vector, or any linear combination of the components, may be used to induce a 2D real metric, whose Gaussian curvature is negative and whose mean curvatures is zero. Hence it follows that a Cartan Spinor (isotropic 3 vector) generates (two) minimal surfaces. In the examples below, for a simple choice of the functions α and β, the catenoid of revolution occurs as the real part of the integration and the helix is determined from the imaginary part of the integration. A linear combination of the two "conjugate" components is used to form the helicat, which is yet another minimal surface. Each of the functions defined above is ambiguous to a factor of ± 1. The mean curvature vanishes (the minimal surface condition) for all combinations of plus or minus signs.

As mentioned above, the real and imaginary parts of the minimal surface position vector correspond to extremes in "polarization". The interesting fact is that if ψ is a complex "constant" of the type $\psi = A\exp(i\theta)$, then each component of the complex position vector

$$\mathbf{R} = A\exp(i\theta)[\mathbf{R}_{\mathrm{real}} + i\mathbf{R}_{\mathrm{imag}}] \tag{1.24}$$

also generates a minimal surface (of mixed polarization)

1.2.1. Example 1. The catenoid

Define the complex functions as below and deduce the following results:

$$\begin{aligned} \alpha &= 1/\varpi, \;\; \beta = 1, && (1.25)\\ \mathbf{H}1 &= [1/\varpi^2 - 1,\, i(1/\varpi^2 + 1)\,, 2\,/\varpi]\,, && (1.26)\\ \mathbf{R} &= [-1/\varpi - \varpi, i(1/\varpi + \varpi), 2\ln\varpi && (1.27) \end{aligned}$$

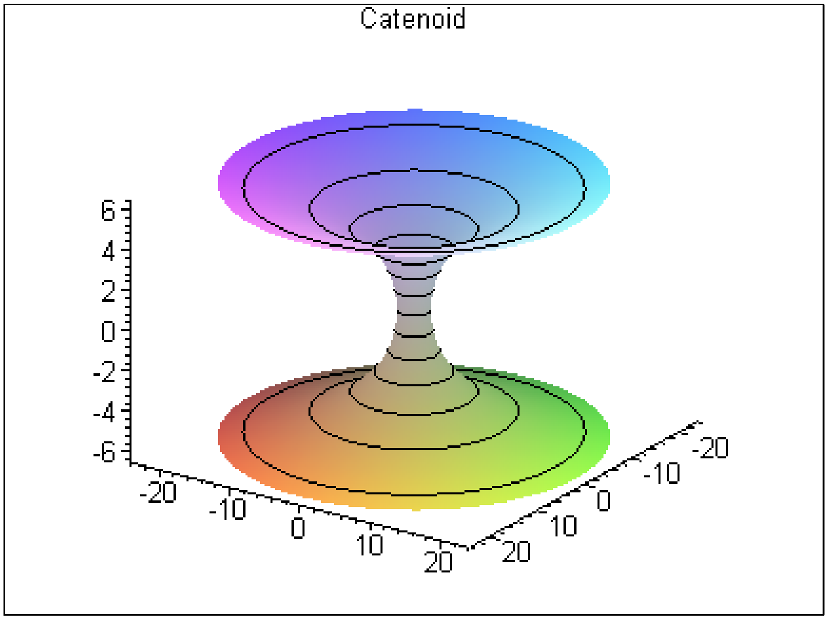

Figure 1.1:

Substitute $\varpi = \exp(u + iv)$ to obtain

$$\mathbf{R}_{\text{real}} = [-cos(v)(exp(-2u) + 1)exp(u), -sin(v)(exp(-2u) + 1)exp(u), 2u]). \quad (1.28)$$

and

$$\mathbf{R}_{\text{imag}} = [sin(v)(exp(-2u) - 1)exp(u), -cos(v)(exp(-2u) - 1)exp(u), 2v]). \quad (1.29)$$

Plotting the Real position vector for u and v varying from -π to +π, yields the catenoid:

1.2.2. Example 2 The helix

Plotting the Imaginary part of the position vector yields the helix: (There are actually 4 helicies. Right handed up, Left handed up, Right handed down, Left handed down.)

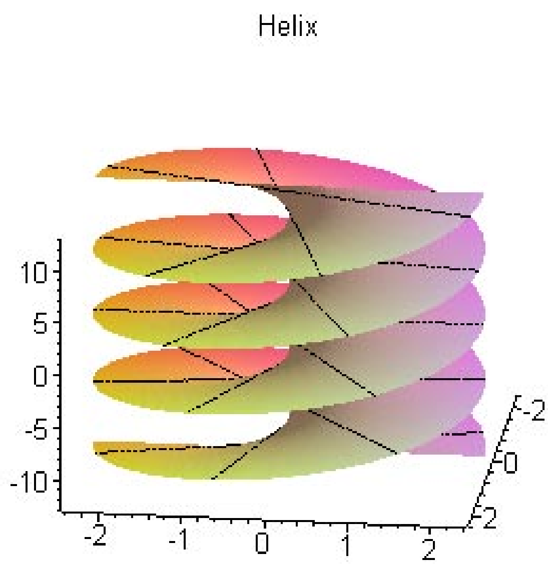

Figure 1.2:

1.2.3. Example 3 The helicat

Combining the two conjugate forms of the minimal surface position vector (real/3-2Imag/3) yields the double layered helicat.Note that there are two distorted minimal surfaces interwoven together. Another view is given below which demonstrates the spiral arms associated with constant z slices.

Note the contour lines form Spiral Arms which converge to circles for the catenoid and to straight lines for the helicoid.

1.3. The Cartan Map as Spinors and the Hopf map

The isotropic Complex position vector, [z1,z2,z3] can be decomposed into a real and imaginary part, such that both have the same sum of squares, and are orthogonal. In otherwords, the Cartan Spinor can be represented as

$$|\boldsymbol{\sigma}_{12}\rangle = |\mathbf{H}1\rangle + i\ |\mathbf{H}2\rangle \qquad with \quad \langle\boldsymbol{\sigma}_{12}| \circ |\boldsymbol{\sigma}_{12}\rangle = 0 \tag{1.30}$$

Two other Cartan spinors are represented by the combinations.

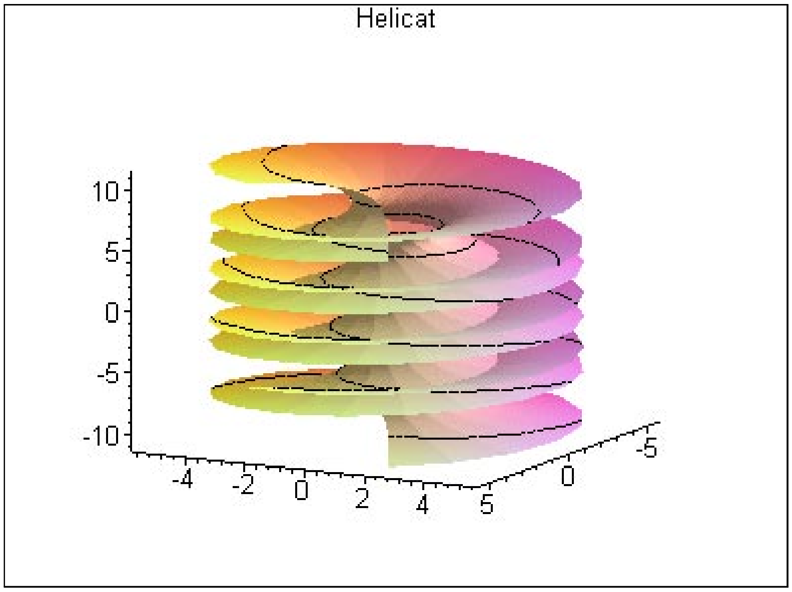

Figure 1.3:

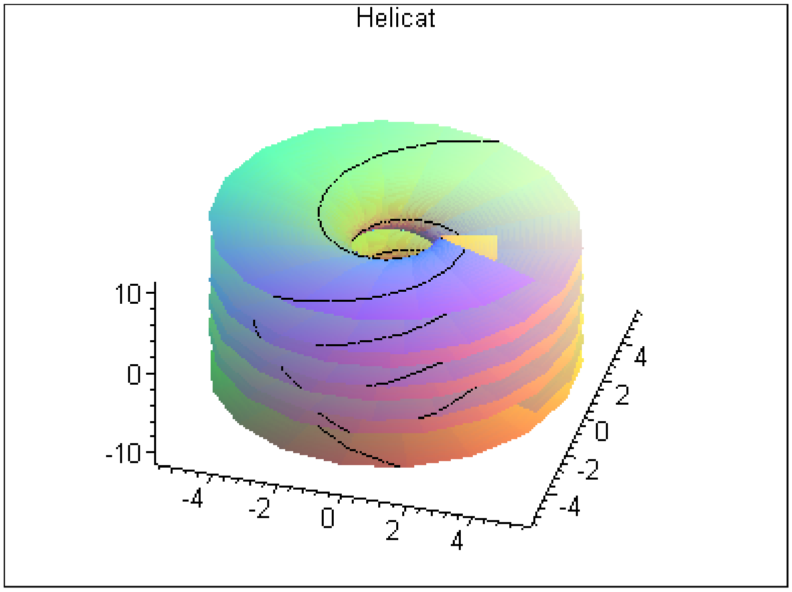

Figure 1.4:

$$|\boldsymbol{\sigma}_{23}\rangle = |\mathbf{H}2\rangle + i\ |\mathbf{H}3\rangle \qquad with \quad \langle\boldsymbol{\sigma}_{23}| \circ |\boldsymbol{\sigma}_{23}\rangle = 0 \tag{1.31}$$

$$|\boldsymbol{\sigma}_{31}\rangle = |\mathbf{H}3\rangle + i\ |\mathbf{H}1\rangle \qquad with \quad \langle\boldsymbol{\sigma}_{31}| \circ |\boldsymbol{\sigma}_{31}\rangle = 0 \tag{1.32}$$

These formulas can be obtained from the Cartan representation for the isotropic 3 vector. As an example consider the permuted form,.

$$z1 = \alpha^2 - \beta^2 \tag{1.33}$$

$$z2 = -2\alpha\beta, \tag{1.34}$$

$$z3 = i\ (\alpha^2 + \beta^2), \tag{1.35}$$

Make the substitutions $\{\alpha = X + iZ, \quad \beta = Y - iS\}$ to obtain the equations

$$|\boldsymbol{\sigma}_{31}\rangle = \left| \begin{array}{l} X^2 + S^2 - Y^2 - Z^2 + i\,2(ZX + SY) \\ -2(YX + SZ) + i\,2(-ZY + SX) \\ 2(-ZX + SY) + i\,(X^2 + Y^2 - Z^2 - S^2) \end{array} \right\rangle = |\mathbf{H}3\rangle + i\ |\mathbf{H}1\rangle \tag{1.36}$$

1.4. The Adjoint field to the Hopf Map

The Hopf Map, as characterized by the equations:

$$[u1, v1, w1] = [2(XZ + YS), 2(XS - YZ), (X^2 + Y^2) - (Z^2 + S^2)], \tag{1.37}$$

can be used to generated 3 linear independent 1-forms on R4, by forming the gradient with respect to $[X, Y, Z, S]$ of each of the three functions that define the map. These three covariant 4 component vectors may be used in the construction of a frame matrix on R4. A fourth linearly independent vector is needed, to complete the basis frame. This fourth vector can be constructed from the adjoint operation (on matrices or differential forms) to within an arbitrary scaling factor,$1/\lambda$. The linearly independent 1-forms are therefor,

$$d(u1) = 2Zd(X) + 2Sd(Y) + 2Xd(Z) + 2Yd(S) \tag{1.38}$$

$$d(u2) = 2Sd(X) - 2Zd(Y) - 2Yd(Z) + 2Xd(S) \tag{1.39}$$

$$d(u3) = 2Xd(X) + 2Yd(Y) - 2Zd(Z) - Sd(S) \tag{1.40}$$

$$A = \{-Yd(X) + Xd(Y) - Sd(Z) + Zd(S)\}/\lambda \tag{1.41}$$

The Frame Matrix so generated is given by the expression:

$$F = \begin{bmatrix} Z & S & X & Y \\ S & -Z & -Y & X \\ X & Y & -Z & -S \\ -Y/\lambda & X/\lambda & -S/\lambda & Z/\lambda \end{bmatrix}, \quad Det[F] = \left(Z^2 + S^2 + Y^2 + X^2\right)^2/\lambda \tag{1.42}$$

It is some interest to examine the properties of the adjoint 1-form, A,defined hereafter as the Hopf 1-form. For $\lambda = 1$, it follows that the Hopf 1-form is of Pfaff dimension 4.

It is also of interest to consider factors λ that are of the format of the Holder norm, where n and p are integers, and (a,b,k,m) are arbitrary constants.

$$\lambda = (aX^p + bY^p + kZ^p + mS^p)^{n/p} \tag{1.43}$$

The exponents n and p determine the homogeneity of the resulting 1-form, which is given below an ambiguous format (the plus of minus sign)

$$A_\pm = A_\pm/\lambda = \{\pm(Yd(X) - Xd(Y)) - Sd(Z) + Zd(S)\}/\lambda. \tag{1.44}$$

For example, for n=p=2, the scaling factor becomes related to the classic quadratic form. The scaled Hopf 1-form, A, is then homogeneous of degree zero.

For arbitrary n and p, the 3-form of topological (Hopf) torsion

$$\text{Topological Torsion } = (A_\pm)\hat{}d(A_\pm) = i(\pm\mathbf{T}_4)d(X)\hat{}d(Y)\hat{}d(Z)\hat{}d(S) \tag{1.45}$$

generates a direction field defined as the 4 component Torsion vector, $\mathbf{T}_4$.

$$\mathbf{T}_4 = \pm[X, Y, Z, S]/\beta. \tag{1.46}$$

The factor β depends upon the integers n and p as well as the constants (a,b,k,m).

The Topological Parity 4-form, whose coefficient is the 4 divergence of the Torsion vector, $\mathbf{T}_4$, becomes

$$\begin{aligned} \text{Topological Parity} &= d(A_\pm)\hat{}(d(A_\pm) && (1.47) \\ &= -4(\pm\lambda)^{(-2n/p)})(n-2)d(X)\hat{}d(Y)\hat{}d(Z)\hat{}d(S) && (1.48) \end{aligned}$$

It is most remarkable that for n=2, any p and any (a,b,k,m), the topological parity vanishes and the scaled Hopf 1-form is of Pfaff dimension 3, not 4. In such cases the ratios of the integrals of the topological torsion 3 form over various closed manifiolds are rational, and the closed integrals of the 3-form are topological deformation invariants. (coherent structures).

Also note that if the scaling factor is restricted to values such that $n = 4,\ p = 2,\ a = b = k = m = 1,$ then the Frame matrix is unimodular, and the scaled Hopf 1-form is homogeneous of degree -2, relative to the substitution $X \Rightarrow \gamma X,\ etc.$ (A somewhat different definition of homogeneity relative to the volume element will be given below.) For this constraint, the 2-form, $F = dA$ has two components in analog to the $\mathbf{E}$ and $\mathbf{B}$ fields of electromagnetism. The two 3 component "blades" are identical only when all of the coefficients are equal to unity. A finite value for the quadratic form leads to a sphere in 3D of coordinates u1,u2,u3.

1.4.1. Electromagnetism of Index zero Hopf 1-forms

Guided by prior investigations, it is of interest to use the scaled Hopf 1-form as the generator of electromagnetic field intensities. The coefficients of the scaled Hopf 1-form can be put into correspondence with the classic vector and scalar potentials, $[\mathbf{A}, \phi]$ (using $S = CT$). The Action for the first examples is then of the format,

$$A_{\pm,0} = A_\pm/\lambda_0 = \{\pm(+Yd(X) - Xd(Y)) - CTd(Z) + CZd(T)\}/\lambda_0 \qquad (1.49)$$

When the number of minus signs in the quadratric form is zero (index 0), and the exponents are n=4, p=2, such that

$$\lambda_0 = (X^2 + Y^2 + Z^2 + S^2)^2, \qquad (1.50)$$

then it is remarkable that the derived 2-form has coefficients ($\mathbf{E}$ and $\mathbf{B}$) that are proportional to the same Hopf Map with the the classic result that $\mathbf{E}^2 = C^2\mathbf{B}^2$, Using the minus ambiguity (parity) sign, the **E** field is anti-parallel to the **B** field. If the positive ambiguity (parity) sign is used, the **E** and **B** fields are parallel:

$$
\begin{aligned}
F &= dA, &(1.51)\\
\mathbf{B} &= curl\mathbf{A} = &(1.52)\\
&[2(CTY + XZ), -2(-YZ + CTX), (-X^2 - Y^2 + Z^2 + (CT)^2)](2/(\lambda_0)^{3/2}) &\\
\mathbf{E} &= -grad\phi - \partial\mathbf{A}/\partial T = &(1.53)\\
&[-2(CTY + XZ), 2(-YZ + CTX), -(-X^2 - Y^2 + Z^2 + (CT)^2)](2C/(\lambda_0)^{3/2}) &
\end{aligned}
$$

It is natural to ask if these **E** and **B** fields admit a Lorentz symmetry constitutive constraint such that vacuum state is charge current free. Recall that a constitutive constraint is a relationship between a 2-form, $F,$ and a 2-form density G, such that the coefficients of $G(\mathbf{D}, \mathbf{H})$ are related to the coefficients of $F(\mathbf{E}, \mathbf{B})$. A Lorenz vacuum condition implies that the fields are solutions of the vector wave equation. The question becomes, "If is presumed that $\mathbf{D} = \varepsilon\mathbf{E}$ and $\mathbf{B} =\mu\mathbf{H}$, do the Maxwell Ampere equations generate a zero 3 form of charge current? ". Direct computation of the index zero Hopf 1-form indicates that $dG = J \neq 0$, unless $\varepsilon\mu C^2 + 1 = 0$. Hence the scaled Hopf Action, where the scaling is of signature zero, does **not** describe a charge current free vacuum, for real positive values of ε, $\mu,$ and C. On the other hand, if it is presumed that the domain is such that say μ,or ε, is negative, then the Hopf Map, scaled as above, does generate charge-current free wave solutions. Negative ε appears to hold in metals and the Earth's ionosphere. Recent announcements indicate constructions that yield negative μ. (Physics Today, p17 May 2000). However, for situations where ε or μ are negative, the Hopf wave solutions imply that the Spin angular momentum $A\hat{}G$ is not a deformation invariant (hence Spin angular momentum of the field is not conserved.)

1.4.2. Electromagnetism of Index one Hopf 1-forms

When the number of minus signs in the quadratric form is one (index 1), and the exponents are n=4, p=2, such that (using lower case letters for Index one Hopf 1-forms)

$$\lambda_1 = (x^2 + y^2 + z^2 - c^2t^2)^2, \tag{1.54}$$

then it is remarkable that the derived 2-form has coefficients (**E** and **B**) that are proportional to different Hopf Maps. The Action 1-form is the same as above, but with a different denominator.

$$A_{\pm,1} = A_{(\pm)}/\lambda_1 = \{\pm(yd(x) - xd(y)) - Ctd(z) + zCd(t)\}/\lambda_1 \tag{1.55}$$

The fact leads to the classic result that $\mathbf{E}^2 = C^2\mathbf{B}^2$, but now the $\mathbf{E}$ field is not collinear with the $\mathbf{B}$ field. Using the negative ambiguity (parity) sign leads to the fields:

$$\begin{aligned}
F &= dA && (1.56)\\
\mathbf{B} &= curl\mathbf{A} = && (1.57)\\
&[2(Cty + xz), -2(-yz + Ctx), (-x^2 - y^2 + z^2 + (Ct)^2)](2/(\lambda_1)^{3/2}) &&\\
\mathbf{E} &= -grad\phi - \partial\mathbf{A}/\partial t = && (1.58)\\
&[2(Cty - xz), 2(-yz - Ctx), -(-x^2 - y^2 + z^2 + (Ct)^2)](2C/(\lambda_1)^{3/2}) &&
\end{aligned}$$

Independent from any other constraints, it is possible to construct the 3-form of Topological Torsion, and its exterior derivative defined as Topological Parity. The Topological parity can be either positive, zero, or negative. For the example Hopf 1-form given above (using the negative ambiguity sign), the Topological Torsion is represented to within a factor by a position vector$[-x, -y, -z, -t]$ inbound in 4 dimensions, and having a negative divergence or parity. If the positive sign of the ambiguity factor is changed, then the parity of the form changes sign. For example, for the 1-form,

$$A1 = A1_+/\lambda_1 = \{+yd(x) - xd(y) - Ctd(z) + zCd(t)\}/\lambda_1, \tag{1.59}$$

the 4 -form of topological parity is positive, and the topological torsion is represented by an outbound position vector (to within a factor).

Similar to the investigation described above, it is natural to ask if these $\mathbf{E}$ and $\mathbf{B}$ fields admit a Lorentz symmetry constitutive constraint such that vacuum state is charge current free. Again, such a condition implies that the fields are solutions of the vector wave equation. Direct computation of the Maxwell Ampere equations indicates that $dG = J = 0$ if the phase velocity constraint vanishes, $\varepsilon\mu C^2 - 1 = 0$. Hence the scaled Hopf Action, where the scaling is of index one, **does** describe a charge current free vacuum, for real positive values of ε, μ, and C.

It is some interest to give the charge current solutions to show how the "phase factor " $(\varepsilon\mu C^2 - 1) \Rightarrow 0$ establishes the vacuum charge free conditions.

$$J^x = -(yx^2 + yz^2 + 5yC^2t^2 - 6zCtx + y^3)(\varepsilon\mu C^2 - 1)4/\lambda^2 \tag{1.60}$$

$$J^y = (x^3 + xy^2 + xz^2 + 5xC^2t^2 + 6zCty)(\varepsilon\mu C^2 - 1)4/\lambda^2 \tag{1.61}$$

$$J^z = -(2x^2 + 2y^2 - z^2 + C^2t^2)(\varepsilon\mu C^2 - 1)8Ct/\lambda^2 \tag{1.62}$$

$$\rho = 0 \tag{1.63}$$

Note that there are possible charge current free (wave solutions) that are governed by curves in space time generated by the intersection of the three surfaces created by setting the coefficents of the current density equal to zero. These solutions are valid for any phase velocity.

The given solution above is not free of Topological Torsion, $A\hat{}F$, and there is a non-zero value of the second Poincare invariant, $\mathbf{E}\cdot\mathbf{B} \neq 0$. However, the Spin 3-form $A\hat{}G$ is also non-zero, but it has, subject to the phase constraint, a zero 4-divergence. (The first Poincare invariant is zero.) The divergence of the Spin 3-form, has 2 parts. The first part is twice the conventional Lagrange density of the fields, $(\mathbf{B}\cdot\mathbf{H} - \mathbf{D}\cdot\mathbf{E})$. The second part is the interaction between the potentials and the charge currents, $(\mathbf{A}\cdot\mathbf{J} - \rho\phi)$. When the divergence of the 3-form is zero, then the closed integrals of Topological Spin are deformation invariants, and have closed integrals with rational (quantized) ratios. That is, with regard to any singly parametrized vector field, V, describing an evolutionary process,

$$\begin{aligned} L_{(\beta V)} \int_{z3} (A\hat{}G) &= \int_{z3} i(\beta V)d(A\hat{}G) + \int_{z3} d(i(\beta V)A\hat{}G) \\ &= 0 + 0 \supset \textit{evolutionary invariance}. \end{aligned} \tag{1.64}$$

The function β is an arbitrary deformation parameter.

1.4.3. Twistors composed by superposing two index 1 Hopf 1-forms

By superposing (adding or subtracting) two different, index 1, Hopf 1-forms (which will be shown below to be equivalent to a Penrose twistor solution) it is possible to construct a vacuum (charge current free wave) solution to the Maxwell system, subject to the constraint that the phase speed satisfies the phase velocity equation, $(\varepsilon\mu C^2 - 1) = 0$.

As an example consider another Hopf 1-form of index 1 formulated as

$$A2 = A2_+/\lambda_1 = \{Ctd(x) + zd(y) - yd(z) - xCd(t)\}/\lambda_1 \tag{1.65}$$

Similar formulas for the field intensities can be determined as above. Note that the parity of the Hopf forms to be superposed can be the same or different. If the parity of the two superposed Hopf 1-forms are opposite, then without consideration of the phase constraint, the Topological Torsion of the "twistor" 1-form vanishes, $A\hat{}F = 0$. Yet the quantized topological spin3-form $A\hat{}G$ does not vanish, and moreover, subject to the phase constraint, the closed integrals of the Spin 3 form are conserved. This result implies that such a construction yields "quantized" values for the Spin integrals.

In this "twistor" case, note the vector represented by the vector $R = [x, y, z, t]$ in R4, is orthogonal to the 1-form of Action. It follows that for a twistor Action,

$$A = A1_- + A2_+ \tag{1.66}$$

$$i(R)A = 0, \; and \; L_{(R)}A = 2A \tag{1.67}$$

$$i(R)dA = 2A, \; and \; L_{(R)}dA = 2dA \tag{1.68}$$

Note that the Hopf 1 form, A and the derived 2-form, $F = dA$, are both homogeneous of degree 2, with respect to R.

The "twistor" Action created by superposing Hopf 1-forms of different parity (but not the general Hopf action) is integrable in the sense of Frobenius,

$$\text{Topological Torsion } H = A\hat{}F = 0. \tag{1.69}$$

The implication is that the 4 forms of Topological Parity, or the second Poincare invariant, (which does not depend upon constitutive properties) is also zero for the twistor 1-form:

$$Second\,Poincare\,invariant\,P2 := d(A\hat{}F) = F\hat{}F = 2\mathbf{E}\cdot\mathbf{B}dx\hat{}dy\hat{}dz\hat{}dt \Rightarrow 0 \tag{1.70}$$

Classically, one would say that the second Poincare invariant vanishes for this twistor Action.

From the constitutive relations, there exists a 3-form (density) S, (Kiehn 1976) defined as the Spin 3-form,

$$S := A\hat{}G \text{ such that } A\hat{}S = 0. \tag{1.71}$$

The Action of the Lie Derivative on the Spin 1-form, S, is such that

$$L_{(R)}S = (L_{(R)}A)\hat{}G + A\hat{}L_{(R)}G = 2A\hat{}G + A\hat{}(2G) = 4S \tag{1.72}$$

and

$$L_{(R)}dS = 4dS. \tag{1.73}$$

The Spin 3-form, S, and its divergence 4 form, dS are homogeneous of degree 4 relative to the vector R. Subject to the phase constraint, the divergence of the Spin 3-form vanishes, which indicates that the closed integrals of the spin 3-form are conserved as period integrals.

These results are to be compared with the Penrose twistor definitions in terms of differential forms (R. Penrose, "The Central Programme of Twistor Theory ", Chaos,Solitons, and Fractals, **10**, 2-3, p581-611, 1999)

The energy flow $\mathbf{E} \times \mathbf{H}$ of such a solution is collinear with the spatial components of the Spin, $\mathbf{S}$.

1.5. Point Particles as Real and Complex Spheres of "zero radii"

A point particle is typically modeled as a 3 dimensional euclidean real ball with a vanishingly small radius vector. The length of the radius vector squared is defined by the sum of squares of its real components. The surface area of the real ball tends to zero as the length of the radius shrinks. However if a "point" particle is defined as a (complex) sphere of vanishingly small radius, then complex point particles could be represented by an isotropic null vector, whose length squared is defined, in the same euclidean manner as for real vectors, as the sum of the squares of its components. In a Euclidean space (where the signature of the fundamental quadratic form is zero) the isotropic vector is not realized in terms of real variables. In Minkowski space, where the signature of the fundamental form is 1, the isotropic vectors (of null length) can be represented by real vectors. relative to the pseudometric. It is suggested herein that a physical "point" in real Euclidean space be extended to include complex euclidean space, and/or Minkowski space. The surface area of a real "point" is zero in real euclidean space, but the surface area of a complex "point" can be finite, even though its "length" is zero. This result follows from the fact that an isotropic null vector can be used as the generator of a minimal surface (see below). The minimal surface is not of zero area. The idea is to study a "point" volume of "zero" real radius that is bounded by two minimal (perhaps conjugate) minimal surfaces.

2. More on Minimal Surfaces

It is extraordinary that the Hopf Adjoint vector, when suitably normalized to have coefficients homogeneous of degree zero, can be used to define a minimal surface in 4D, where the Gauss curvature (sums of product pairs of curvatures) is real and positive. Real minimal surfaces in 3D have a Gauss curvature (sums of product pairs of curvatures) which is negative.

Consider the Hopf Adjoint vector is of the form

$$A_0 = b(ydx - xdy) + a(tdz - zdt). \tag{2.1}$$

The 1-form of Potentials depends on the coefficients a and b which are presumed to take on values ± 1. There are two cases corresponding to left and right handed "polarizations": $a = b$ or $a = -b$. (There actually are 6 cases to consider, by cyclically permuting the variables, and these can be combined to represent spinor solutions.)

Next normalize the 1-form by dividing through by a Holder norm such that the coefficients of the renormalized 1-form are homogeneous of degree zero. Then construct the similarity invariants of the Jacobian matrix determined from the coefficients of the renormalized 1-form. What is remarkable for this example, is that both the Mean curvature (sum of curvatures), the Adjoint (Kubic curvature = sum of all curvature triples), and quadratic curvature (determinant of the Jacobian matrix = product of all curvatures) of the implicit hypersurface in 4D vanish, for any choice of a or b. The Gauss curvature (sum of all pairs of curvatures) is non-zero, positive, real and is equal to the square of the radius of a 4D euclidean sphere. The cubic interaction energy density is zero.

$$Mean = 0, \tag{2.2}$$
$$Gauss > 0 \tag{2.3}$$
$$Kubic = 0 \tag{2.4}$$
$$Top\_Torsion \neq 0 \tag{2.5}$$
$$J_s \neq 0 \tag{2.6}$$

This situation occurs when the three curvatures of the implicit 3-surface are $\{0, +i\omega, -i\omega)$.. This Hopf surface is therefore a 3D imaginary *minimal* two dimensional hyper surface in 4D and has two non-zero pure imaginary curvatures! Strangely enough the charge-current density is not zero, but it is proportional to the topological Torsion vector that generates the 3 form $A\hat{}F$. The topological Parity 4 form is not zero, and depends on the sign of the coefficients a and b. In other words the 'handedness' of the different 1-forms determines the orientation of the normal field with respect to the implicit surface.

It is also possible to deduce a closed 3-form of "Charge-Current density", J_s, for such 3D hypersurfaces. The coefficients of $A\hat{}J_s$ are exactly equal to the "Kubic" curvature similarity invariant. The spatial scalar product of A and J is balance by $\rho\phi$. It is known that a process described by a vector proportional to the topological torsion vector in a domain where the topological parity $(4ba)$ is non-zero is thermodynamically irreversible.

3. References

[1] Osserman

[2] Cartan

[3] O'Neill

[4] Bateman

[5] Kiehn

[6]

[7] Penrose