```
> restart:
> with(linalg):with(liesymm):with(difforms):with(plots):with(plottools):
Warning, new definition for norm
Warning, new definition for trace
Warning, new definition for close
Warning, new definition for `&^`
Warning, new definition for d
Warning, new definition for mixpar
Warning, new definition for wdegree
Warning, new definition for translate
>
setup(X,Y,Z,S):defform(X=0,Y=0,Z=0,S=0,ch=const,B=const,u=0,v=0,w=0,s=0,n=const,e=const,p=const,c1=const,c2=const);
```

SPINORS, MINIMAL SURFACES, CHIRALITY, HELICITY, TORSION, SPIN CONTINUITY, ORIENTATION, "POINT PARTICLES" POLARIZATION and the HOPF MAP

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Introduction

A re-reading of Cartan's book on Spinors (and Chadrasekhar's book on Black Holes) leads to the thought that there is a connection between all of the ideas in the above title. It is remarkable to me that both Cartan and Chandresekhar do not mention the fact that an isotropic (complex) vector is related to the generator of a Minimal Surface. Neither do the two authors mention the fact that the expressions they utilize to define 4D spinors are essentially versions of the Hopf Map. Cartan defines the spinor as a complex pair mapped to a 3 component complex vector in such a way that its quadratic form is zero.

The relationship of spinors to minimal surfaces is ignored by many other authors as well as Cartan and Chandrasekhar. It would appear that most physicists are not aware of this relationship which yields an interesting and useful interpretation of spinors. There have been a few mathematicians that have made use recently of spinor representations to discuss minimal surfaces. (See R. Kusner and N. Schmitt, "The Spinor Representation of Minimal Surfaces" http://xxx.lanl.gov/abs/dg-ga/9512003)

"Point Particles as Real and Complex Spheres of zero radii"

A point particle is typically modeled as a 3 dimensional euclidean real ball with vanishingly small radius vector. The length of the radius vector squared is defined by the sum of squares of its real components. The surface area of the real ball tends to zero as the length of the radius shrinks. However if a "point" particle is defined as a (complex) sphere of vanishingly small radius, then complex point particles could be represented by an isotropic null vector. whose length squared is defined, in the same manner, as the sum of the squares of its components. In a Euclidean space (where the signature of the fundamental quadratic form is zero) the isotropic vector is not realized in terms of real variables. In Minkowski space, where the signature of the fundamental form is 1, the isotropic vectors (of null length) can be represented by real vectors. It is suggested that a "point" in real Euclidean space be extended to include complex euclidean space, and/or Minkowski space. The surface area of a real "point" is zero in real euclidean space, but the but the surface area of a complex "point" can be finite, even though its "length" is zero. This result follows from the fact that an isotropic null vector can be used as the generator of a minimal surface (see below). The minimal surface is not of zero area.

Isotropic Vectors and Minimal surfaces

Consider the two lemmas given in R. Osserman's book " A Survey of Minimal Surfaces" Dover

{Lemma 8.1} Let D be a domain in the complex z-plane, g(z) an arbitrary meromorphic function in D and f(z) an anlytic function in D having the property that at each point where g(z) has a pole of order m, f(z) has a zero of order at least 2m. Then the functions

z1=f(1-g^2)/2 <=> alpha ^2-beta ^2
z2=i f(1+g^2)/2 <=>i(alpha ^2+beta ^2)
z3=- f x g <=>- 2alpha x beta

will be analytic in D and satisfy the equation of an "isotropic" null vector:

(z1)^{2}+(z2)^{2}+(z3)^{2}=0.

The only exception is for z3=0,z1=i z2

The equivalent isotropic formulation was given by Cartan in terms of alpha (z) and beta (z)

The quadratic form (z1)^{2}+(z2)^{2}+(z3)^{2} can be complex, real, or zero. A Cartan spinor is in fact, not the pair of functions, alpha (z) and beta (z), but the special map to the isotropic complex 3 vector, such that (z1)^{2}+(z2)^{2}+(z3)^{2}=0. The isotropic condition imposes two constraints on the 6D space of 3 complex variables reducing the dimension to a 4D space of two complex variables.

Every simply connected minimal surface in E3 can be represented in the form of a position vector created by complex integration of z1,z2,z3 as given by the formulas above. If [z1,z2,z3] is considered to be a "Velocity" vector in complex 3 space, parametrized by the variable z, then integration of the Velocity vector determines a complex position vector parameterized by z. The real and the imaginary parts of the poistion vector so found form conjugate minimal surfaces.
Either component of theisotropic position vector may be used to induce a 2D real metric, whose Gaussian curvature is negative and whose mean curvature is zero. Hence it follows that a Cartan spinor (isotropic 3 vector) generates (two) minimal surfaces.

The Hopf Map

The Hopf map is a projection from 4D of variables [X,Y,Z,S] to 3D of variables [V1,V2,V3].
A "standard" version is given by the formulas:

$$V1 := X^2 + Y^2 - Z^2 - S^2$$
$$V2 := 2\,Z\,X - 2\,Y\,S$$
$$V3 := 2\,X\,S + 2\,Y\,Z$$

with $\quad sq\ Magnitude\ Hopf\ V = V1\hat{}2 + V2\hat{}2 + V3\hat{}2 \ := (X^2 + Y^2 + Z^2 + S^2)^2$

Hence if the RHS is equal to a constant, defining a 3 sphere, then the map yields a 2 sphere of the radius equal to the square of the 3 sphere radius. The formulas hold for the map from 4 complex dimensions to 3 complex dimensions. The usual value of the Hopf 3 sphere is plus one, but note that the formulas allow maps of imaginary 3 spheres into real 2 spheres, as well.

If the length of the "radius" of the 3 sphere is chosen to be zero, then the map is from an isotropic vector in 4 space to an isotropic vector in 3 space. Hence the relationship of the Hopf map to Cartan spinors follows. The Cartan spinor, as a complex isotropic 3-vector, consists of two (real and imaginary) parts. The two components, as real 3 vectors, are orthogonal. As will be shown below, each part of the Cartan spinor has the component format of a Hopf map. However, the complex quadratic form of the Cartan spinor vanishes, for the sum of squares of each orthogonal component is identical, and thereby cancel.

The Photon

In a study of the singular solutions to Maxwell's equations (http://www22.pair.com/pdf/timerev.pdf) it was noticed that the expanding 3-sphere, typically associated with a light pulse wave front, was a "strange" sphere in the sense that the implicit formula that generated the spherical surface had a normal field that vanished. The characteristic fourth order polynomial -- in cases without longitudinal anomalies split -- into two degenerate factors. It was as if the light signal consisted of two spheres with different orientations of the normal field. The thought occurred then that the photon could be modeled as a doublet of spherical shells, but each shell had a different orientation for its normal field. The doublet shells of the "photon pulse" might be represented as the coherence length of the photon (typically 3 meters for optical photons of the equilibrium variety.)

The idea that this "surface" is an isotropic plane in the sense of a Cartan 3 spinor is intriguing, and will be studied soon.

```
>
> Minsurf:=proc(R)
    local Yu,Yv,NN,magn,NNU,FFF,DET,EE,FF,GG,Yuu,Yvv,Yuv,b11,b12,b22:

    global GUN,Q,H,K,gun:
     GUN:=array([[1,0,0],[0,1,0],[0,0,1]]);
     Yu:=diff(R,u):
     Yv:=diff(R,v):
     NN:=evalm(simplify(crossprod(Yu,Yv))):
     magn:=(factor(simplify(innerprod(NN,GUN,NN))^(1/2))):
     NNU:=simplify(evalm(NN/magn)):
     FFF:=transpose(array([Yu,Yv])):
     gun:=simplify(innerprod(transpose(FFF),GUN,(FFF))):DET:=det(gun):
     gun:=evalm(gun):
     EE:=gun[1,1]:
     FF:=gun[1,2]:
     GG:=gun[2,2]:
     Q:=EE*GG-FF*FF:
     Yuu:=diff(Yu,u):b11:=simplify(innerprod(Yuu,GUN,NNU)):
      Yuv:=diff(Yu,v):b12:=simplify(innerprod(Yuv,GUN,NNU)):
      Yvv:=diff(Yv,v):b22:=simplify(innerprod(Yvv,GUN,NNU)):
      H:=factor(simplify(gun[2,2]*b11+gun[1,1]*b22)-2*gun[1,2]*b12) /(2*det(gun)):
    K:=simplify((b11*b22-b12*b12))/det(gun):
  print(`Immersion R`=R);print(`Mean Curvature`=H);print(`Gauss Curvature`=simplify(K));print(`Metric
  Det Q`=simplify(Q)); end:
```

Consider a three component "Velocity" vector with components V1,V2,V3 defined in terms of a pairs of complex functions { a(u+iv), b(u+iv) }. In certain circumstances this pair of complex functions will be defined as spinor pair. The construction is essentially a projection from 4D to 3D. The properties of the quadratic form created from the sum of squares of the three components can be used to define categories of vectors. The quadratic form can be real, complex, or null. It is the null set that motivated Cartan's development of spinors. Some special cases lead to:

Real 3D Hopf Vectors

(with a real non-zero quadratic form)
First consider the "real" Hopf vector defined in terms of a and b by the equations:

```
> V01:=(b*conjugate(b)-a*conjugate(a));V02:=a*conjugate(b)+b*conjugate(a);V03:=-I*(a*conjugate(b)-b*conjugate(a));
```

$$V01 := b\,\bar{b} - a\,\bar{a}$$

$$V02 := a\,\bar{b} + b\,\bar{a}$$

$$V03 := -I\,(a\,\bar{b} - b\,\bar{a})$$

This choice generates the standard real Hopf Map from {X,Y,Z,S} to {V1,V2,V3},
by substituting b = X+ch1*iY and a = Z+ch2*iS,
where ch is a "chirality" factor with values plus or minus 1:

```
> V1:=subs(c1^2=1,c2^2=1,evalc(subs(b=X+c1*I*Y,a=Z+c2*I*S,V01)));V2:=subs(c1^2=1,c2^2=1,evalc(subs(b=X+c1*I*Y,a=Z+c2*I*S,V02)));V3:=subs(c1^2=1,c2^2=1,evalc(subs(b=X+c1*I*Y,a=Z+c2*I*S,V03)));
```

$$V1 := X^2 + Y^2 - Z^2 - S^2$$

$$V2 := 2\,Z\,X + 2\,c2\,S\,c1\,Y$$

$$V3 := 2\,c2\,S\,X - 2\,Z\,c1\,Y$$

The sq magnitude of the (Real) 3D "Hopf vector" is equal to the 4th power of the magnitude of the radius of the 4D sphere.

```
> `sq Magnitude Hopf V =V1^2+V2^2+V3^2
`:=factor(subs(c1^2=1,c2^2=1,(factor(V1*V1+V2*V2+V3*V3))));
```

$$sq\ Magnitude\ Hopf\ V = V1\text{^}2+V2\text{^}2+V3\text{^}2 \ := (X^2 + Y^2 + Z^2 + S^2)^2$$

```
> ### WARNING: `Adjoint` might conflict with Maple's meaning of that name
Adjoint:=subs(c1^2=1,c2^2=1,wcollect(d(V1)&^d(V2)&^d(V3)));
```

$$\begin{aligned} Adjoint := &(-8\,Z\,c1\,X^2 - 8\,Z^3\,c1 - 8\,c1\,Y^2\,Z - 8\,S^2\,Z\,c1)\,\&\text{^}(d(X), d(Z), d(Y)) \\ &+ (8\,c2\,X^3 + 8\,c2\,X\,Z^2 + 8\,c2\,S^2\,X + 8\,Y^2\,X\,c2)\,\&\text{^}(d(X), d(Z), d(S)) \\ &+ (8\,X^2\,S\,c1 + 8\,S\,Y^2\,c1 + 8\,S^3\,c1 + 8\,Z^2\,c1\,S)\,\&\text{^}(d(X), d(Y), d(S)) \\ &+ (8\,Z^2\,c2\,Y + 8\,c2\,X^2\,Y + 8\,Y^3\,c2 + 8\,Y\,S^2\,c2)\,\&\text{^}(d(Z), d(S), d(Y)) \end{aligned}$$

```
> A1:=factor(getcoeff(Adjoint&^d(X))/(8*(X^2+Y^2+Z^2+S^2)));A2:=-factor(getcoeff(Adjoint&^d(Y))/(8*(X^2+Y^2+Z^2+S^2)));A3:=factor(getcoeff(Adjoint&^d(Z)/(8*(X^2+Y^2+Z^2+S^2))));A4:=-factor(getcoeff(Adjoint&^d(S))/(8*(X^2+Y^2+Z^2+S^2)));
```

$$A1 := c2\,Y$$

$$A2 := -c2\,X$$

$$A3 := S\,c1$$

$$A4 := Z\,c1$$

```
> Action:=A1*d(X)+A2*d(Y)+A3*d(Z)+A4*d(S);
```

$$Action := c2\,Y\,d(X) - c2\,X\,d(Y) + S\,c1\,d(Z) + Z\,c1\,d(S)$$

```
> F:=d(Action);H:=factor(Action&^F);K:=F&^F;
```

$$F := 2\,c2\,(d(Y)\,\&\text{^}\,d(X))$$

$$H := 2\,c2\,c1\,(\&\text{^}(d(Z), d(Y), d(X))\,S + Z\,\&\text{^}(d(S), d(Y), d(X)))$$

$$K := 0$$

The map is from the square of a 3 sphere to a 2 sphere. The sdjoint field has a Topological Parity (orientation) which depends upon the product of the signs of c1 and c2
****

Next consider another, complex, mapping which yields

Complex but Isotropic 3D Hopf Vectors:

(with real but NULL quadratic form)

```
> cV01:=(b*b-a*a);cV02:=(I*(a*a+b*b));cV03:=2*a*b;
```

$$cV01 := b^2 - a^2$$

$$cV02 := I\,(a^2 + b^2)$$

$$cV03 := 2\,a\,b$$

```
> cV1:=subs(c1^2=1,c2^2=1,evalc(subs(b=X-c1*I*Y,a=Z+c2*I*S,cV01)));cV2:=subs(c1^2=1,c2^2=1,evalc(subs(b=X-c1*I*Y,a=Z+c2*I*S,cV02)));cV3:=subs(c1^2=1,c2^2=1,evalc(subs(b=X-c1*I*Y,a=Z+c2*I*S,cV03)));
```

$$cV1 := X^2 - Y^2 - Z^2 + S^2 + I\,(-2\,X\,c1\,Y - 2\,Z\,c2\,S)$$

$$cV2 := -2\,Z\,c2\,S + 2\,X\,c1\,Y + I\,(Z^2 - S^2 + X^2 - Y^2)$$

$$cV3 := 2\,Z\,X + 2\,c2\,S\,c1\,Y + I\,(2\,c2\,S\,X - 2\,Z\,c1\,Y)$$

Note that the complex Hopf vector consists of two distinct real Hopf vectors which are orthogonal. The two orthogonal Hopf vectors define a Hopf plane.

The two distinct orthogonal Hopf vectors compose the real and imaginary parts of the Complex Hopf vector, but note that the sum of squares of components for the complex Hopf vector vanishes.!!

The Null complex Hopf vector is an

isotropic vector = CARTAN SPINOR.

```
> `sq Magnitude cHV`:=(subs(c1^2=1,c2^2=1,evalc(cV1*cV1+cV2*cV2+cV3*cV3)));
```

$$sq\ Magnitude\ cHV := 0$$

The real and imaginary Hopf vectors have the same squared magnitude, and moreover are orthogonal.

```
> RealHV:=evalc([Re(cV1),Re(cV2),Re(cV3)]);ImagHV:=evalc([Im(cV1),Im(cV2),Im(cV3)]);
```

$$RealHV := [X^2 - Y^2 - Z^2 + S^2, -2\,Z\,c2\,S + 2\,X\,c1\,Y, 2\,Z\,X + 2\,c2\,S\,c1\,Y]$$

$$ImagHV := [-2\,X\,c1\,Y - 2\,Z\,c2\,S, Z^2 - S^2 + X^2 - Y^2, 2\,c2\,S\,X - 2\,Z\,c1\,Y]$$

```
> `sq Magnitude RealHV`:=factor(subs(c1^2=1,c2^2=1,innerprod(RealHV,RealHV)));`sq Magnitude ImagHV`=factor(subs(c1^2=1,c2^2=1,innerprod(ImagHV,ImagHV)));`Dot product RealHV and ImagHV`=factor(subs(c1^2=1,c2^2=1,innerprod(RealHV,ImagHV)));NHV:=crossprod(RealHV,ImagHV):`NormHV=Cross product RealHV and ImagHV`:=crossprod(RealHV,ImagHV);`sq Magnitude NormHV `:=factor(subs(c1=1,c2=1,innerprod(NHV,NHV)));
```

$$sq\ Magnitude\ RealHV := (X^2 + Y^2 + Z^2 + S^2)^2$$

$$sq\ Magnitude\ ImagHV = (X^2 + Y^2 + Z^2 + S^2)^2$$

$$Dot\ product\ RealHV\ and\ ImagHV = 0$$

$$NormHV{=}Cross\ product\ RealHV\ and\ ImagHV := [\\
(-2\,Z\,c2\,S + 2\,X\,c1\,Y)\,(2\,c2\,S\,X - 2\,Z\,c1\,Y) - (2\,Z\,X + 2\,c2\,S\,c1\,Y)\,(Z^2 - S^2 + X^2 - Y^2),\\
(2\,Z\,X + 2\,c2\,S\,c1\,Y)\,(-2\,X\,c1\,Y - 2\,Z\,c2\,S) - (X^2 - Y^2 - Z^2 + S^2)\,(2\,c2\,S\,X - 2\,Z\,c1\,Y),\\
(X^2 - Y^2 - Z^2 + S^2)\,(Z^2 - S^2 + X^2 - Y^2) - (-2\,Z\,c2\,S + 2\,X\,c1\,Y)\,(-2\,X\,c1\,Y - 2\,Z\,c2\,S)]$$

$$sq\ Magnitude\ NormHV\ := (X^2 + Y^2 + Z^2 + S^2)^4$$

Hence the three vectors, RealHV,ImagHV, and NHV are all 3D orthogonal. If Real HV and Imag HV are divided by the same sq Magnitude, And if NHV (the normal to the Hopf plane) is divided by the sq of the sq magnitude, then all three vectors form an orthonormal basis for 3D, mod the origin.

****

In another notation define the real part of the Complex Hopf vector by the symbol **E**, and the imaginary part of the Hopf vector by the symbol c**B**. Then the sq Magnitude consists of a real part (**E.E** - c^2 **B.B**) and a complex part (2c **E.B**). Both must vanish if the Vector is to be istropic. For a Lorentz type of consitutive constraint, it is apparent that both the first and the second Poincare invariants are zero for a Cartan Spinor => isotropoic vector.

```
> VV:=[Ex,Ey,Ez]+c*I*[Bx,By,Bz];evalc(Re(innerprod(VV,VV)));evalc(Im(innerprod(VV,VV)));
```

$$VV := [Ex, Ey, Ez] + I\,c\,[Bx, By, Bz]$$

$$Ex^2 - c^2\,Bx^2 + Ey^2 - c^2\,By^2 + Ez^2 - c^2\,Bz^2$$

$$2\,Ex\,c\,Bx + 2\,Ey\,c\,By + 2\,Ez\,c\,Bz$$

This construction was used by Bateman in 1914, but I do not believe he was aware of the connection to the Hopf Map, or to Minimal surfaces

****

Those 3D vectors which are **ISOTROPIC** can be used to generate a pair of **MINIMAL SURFACES.**
Consider those 3D isotropic null "Velocity" vectors whose components are functions of the complex variable, z = u + iv.
Then integrate the isotropic Velocity vector with respect to z, to yield a complex position vector, R. Consider the two vectors which are the real and imaginary parts of the complex 3D vector R.
Each component generates a Minimal Surface.
EXAMPLE:

```
> VX:=subs(a=1,b=1/z,cV01);VY:=subs(a=1,b=1/z,cV02);VZ:=subs(a=1,b=1/z,cV03);
> X:=(int(VX,z));Y:=(int(VY,z));Z:=(int(VZ,z));
>
```

$$VX := \frac{1}{z^2} - 1$$

$$VY := I\left(1 + \frac{1}{z^2}\right)$$

$$VZ := 2\,\frac{1}{z}$$

$$X := -\frac{1}{z} - z$$

$$Y := I\left(z - \frac{1}{z}\right)$$

$$Z := 2\ln(z)$$

```
> XX:=factor(subs(z=exp(u+I*v),X));YY:=factor(subs(z=exp(u+I*v),Y));ZZ:=factor(subs(z=exp(u+I*v),Z));
```

$$XX := -\frac{1 + \left(\mathbf{e}^{(-I\,(I\,u - v))}\right)^2}{\mathbf{e}^{(-I\,(I\,u - v))}}$$

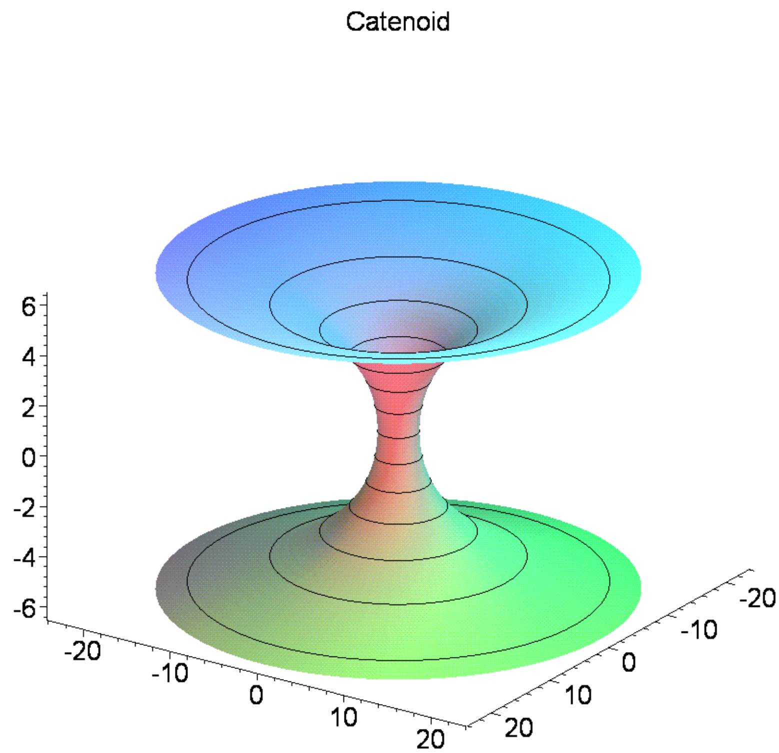
$$YY := \frac{I\left(\mathbf{e}^{(-I\,(I\,u - v))} - 1\right)\left(\mathbf{e}^{(-I\,(I\,u - v))} + 1\right)}{\mathbf{e}^{(-I\,(I\,u - v))}}$$

$$ZZ := 2\ln\left(\mathbf{e}^{(-I\,(I\,u - v))}\right)$$

```
> Rreal:=evalm(simplify(evalm([evalc(Re(XX)),evalc(Re(YY)),evalc(Re(ZZ))])));
```

$$Rreal := [-\cos(v)\left(\mathbf{e}^{(-2\,u)} + 1\right)\mathbf{e}^{u}, -\sin(v)\left(\mathbf{e}^{(-2\,u)} + 1\right)\mathbf{e}^{u}, 2\,u]$$

```
> plot3d(Rreal,u=-Pi..Pi,v=-Pi..Pi,shading=XYZ,lightmodel=light3,axes=framed,style=PATCHCONTOUR,numpoints=5000,orientation=[34,60],title=`Catenoid`);
```




```
> Rimag:=evalm(simplify(evalm([evalc(Im(XX)),evalc(Im(YY)),evalc(Im(ZZ))])));
>
```

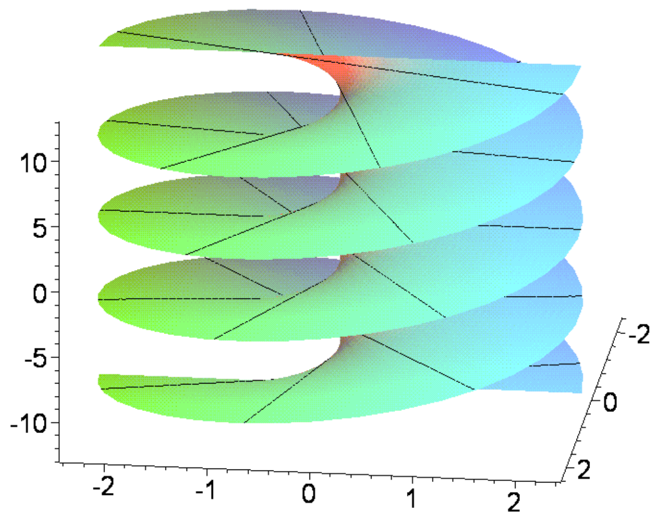
$$Rimag := [\sin(v)(\mathbf{e}^{(-2u)} - 1)\,\mathbf{e}^{u}, -\cos(v)(\mathbf{e}^{(-2u)} - 1)\,\mathbf{e}^{u}, 2v]$$

```
> RR2:=evalm(Rimag);
```

$$RR2 := [\sin(v)(\mathbf{e}^{(-2u)} - 1)\,\mathbf{e}^{u}, -\cos(v)(\mathbf{e}^{(-2u)} - 1)\,\mathbf{e}^{u}, 2v]$$

```
>
> plot3d(RR2,u=-1..1,v=-2*Pi..2*Pi,shading=XY,lightmodel=light3,axes=framed,style=PATCHCONTOUR,numpoints=5000,orientation=[7,64],title=`Helix`);
>
```

Helix



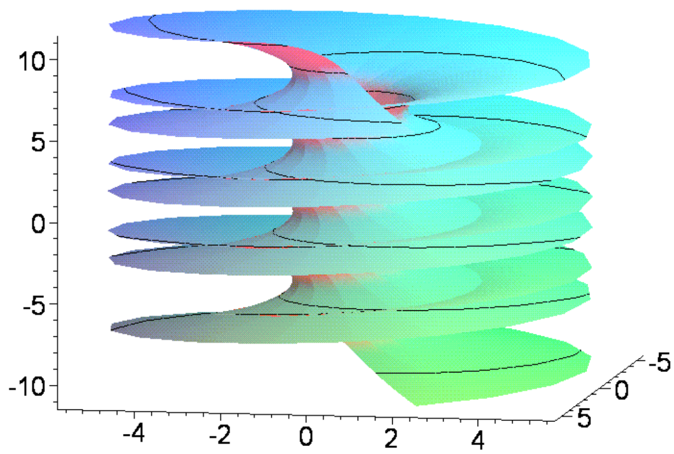

```
> RR2:=evalm(Rimag/3-2*Rreal/3);
```

$RR2 :=$

$$\left[\frac{1}{3}\sin(v)\,(\mathbf{e}^{(-2\,u)}-1)\,\mathbf{e}^{u}+\frac{2}{3}\cos(v)\,(\mathbf{e}^{(-2\,u)}+1)\,\mathbf{e}^{u},\,-\frac{1}{3}\cos(v)\,(\mathbf{e}^{(-2\,u)}-1)\,\mathbf{e}^{u}+\frac{2}{3}\sin(v)\,(\mathbf{e}^{(-2\,u)}+1)\,\mathbf{e}^{u},\,\frac{2}{3}\,v-\frac{4}{3}\,u\right]$$

```
>
>
> plot3d(RR2,u=-2..2,v=-4*Pi..4*Pi,shading=XYZ,lightmodel=light3,axes=framed,style=PATCHCONTOUR,
numpoints=5000,orientation=[10,80],title=`Helicat`);
```

Helicat



Note that the general helicat consists of a pair of surfaces (torsion waves), where the helicoid is one unique limit set (circular polarization ?) and the catenoid is the other limit set (linear polarization ?)

The concept of a sphere with "zero" radius (the isotropic Hopf vector) yields new meaning to the concept of a "point" partical. Could it be that a possible model for point particles is that they are essentially istropic spheres of null real radius, but composed of two conjugate minmal surfaces with finite area.

Part 2 Torsion, Hopf Maps, Instanton Maps

```
> restart:with(linalg):with(liesymm):with(difforms):
Warning, new definition for norm
Warning, new definition for trace
Warning, new definition for close
Warning, new definition for `&^`
Warning, new definition for d
Warning, new definition for mixpar
Warning, new definition for wdegree
> setup(X,Y,Z,S):defform(X=0,Y=0,Z=0,S=0,ch=const,B=const,u=0,v=0,w=0,s=0,n=const,e=const,p=const);
> Mink:=array([[1,0,0,0],[0,1,0,0],[0,0,1,0],[0,0,0,-1]]);
```

$$Mink := \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

Consider the two complex numbers alpha and beta: (ch is plus or minus 1 and defines a handedness)

```
> alpha:=(ch*X+I*Y);beta:=(Z+I*S);
```

$$\alpha := ch\,X + I\,Y$$
$$\beta := Z + I\,S$$

```
> a0:=u;a1:=v;
>
```

$$a0 := u$$
$$a1 := v$$

```
> x1:=evalc(I*(a0^2-a1^2));x2:=evalc((a0^2+a1^2));x3:=-evalc(I*2*a0*a1);rr:=factor(x1^2+x2^2+x3^2);
```

$$x1 := I\,(u^2 - v^2)$$
$$x2 := u^2 + v^2$$
$$x3 := -2\,I\,u\,v$$
$$rr := 0$$

```
> d(x1);d(x2);d(x3);
```

$$2\,I\,u\,\mathrm{d}(u) - 2\,I\,v\,\mathrm{d}(v)$$
$$2\,u\,\mathrm{d}(u) + 2\,v\,\mathrm{d}(v)$$
$$-2\,I\,v\,\mathrm{d}(u) - 2\,I\,u\,\mathrm{d}(v)$$

```
>
>
>
```

and the map between {X,Y,Z,S}=>{ x,y,z,s=ct} given by the expressions:

```
> ct:=subs(ch^2=1,evalc(B*(conjugate(alpha)*alpha+conjugate(beta)*beta)))
  ;z:=subs(ch^2=1,evalc(conjugate(alpha)*alpha-conjugate(beta)*beta));y:=evalc(I*(conjugate(alpha)*bet
  a-conjugate(beta)*alpha));x:=evalc((conjugate(alpha)*beta+conjugate(beta)*alpha));
>
```

$$ct := B\,(X^2 + Y^2 + Z^2 + S^2)$$
$$z := X^2 + Y^2 - Z^2 - S^2$$
$$y := 2\,Y\,Z - 2\,ch\,X\,S$$
$$x := 2\,ch\,X\,Z + 2\,Y\,S$$

```
> d(z);
```

$$2\,X\,\mathrm{d}(X) + 2\,Y\,\mathrm{d}(Y) - 2\,Z\,\mathrm{d}(Z) - 2\,S\,\mathrm{d}(S)$$

```
> d(x);
```

$$2\,ch\,Z\,\mathrm{d}(X) + 2\,ch\,X\,\mathrm{d}(Z) + 2\,S\,\mathrm{d}(Y) + 2\,Y\,\mathrm{d}(S)$$

```
> d(y);
```

$$2\,Z\,\mathrm{d}(Y) + 2\,Y\,\mathrm{d}(Z) - 2\,ch\,S\,\mathrm{d}(X) - 2\,ch\,X\,\mathrm{d}(S)$$

```
> d(ct);
```

$$2\,B\,X\,\mathrm{d}(X) + 2\,B\,Y\,\mathrm{d}(Y) + 2\,B\,Z\,\mathrm{d}(Z) + 2\,B\,S\,\mathrm{d}(S)$$

```
> Vol4:=factor(d(x)&^d(y)&^d(z)&^d(ct));
```

$$Vol4 := -32\,B\,X\,Y\,(ch-1)\,(ch+1)\,(Z^2+S^2)\ \&\hat{}(\mathrm{d}(X), \mathrm{d}(Y), \mathrm{d}(Z), \mathrm{d}(S))$$

S0 for a either chioce of chirality ch the 4 volume element is zero. The space is 3 dimensional.

```
> r2:=factor(subs(ch^2=1,factor(x^2+y^2+z^2)));`ratio r2/ct^2`:=r2/ct^2;R4:=X^2+Y^2+Z^2+S^2;
>
>
```

$$r2 := (X^2 + Y^2 + Z^2 + S^2)^2$$
$$ratio\ r2/ct\hat{}2 := \frac{1}{B^2}$$
$$R4 := X^2 + Y^2 + Z^2 + S^2$$

```
> J:=factor(wcollect(subs(ch^2=1,d(x)&^d(y)&^d(z))));dJ:=d(J);JJ:=J/(8*R4);
>
```

$J := 8\,(X^2+Y^2+Z^2+S^2)$

$(\&\text{^}(d(Z), d(Y), d(S))\,Y - \&\text{^}(d(X), d(Y), d(S))\,ch\,S - \&\text{^}(d(X), d(Y), d(Z))\,ch\,Z + \&\text{^}(d(X), d(S), d(Z))\,X)$

$$dJ := 0$$

$JJ :=$

$\&\text{^}(d(Z), d(Y), d(S))\,Y - \&\text{^}(d(X), d(Y), d(S))\,ch\,S - \&\text{^}(d(X), d(Y), d(Z))\,ch\,Z + \&\text{^}(d(X), d(S), d(Z))\,X$

```
> N1:=-getcoeff((JJ&^d(X)));N2:=-getcoeff((JJ&^d(Y)));N3:=getcoeff((JJ&^d(Z)));N4:=-getcoeff((JJ&^d(S
)));
```

$$N1 := -Y$$

$$N2 := -X$$

$$N3 := -ch\,S$$

$$N4 := ch\,Z$$

The 3 current J depends upon the chirality ch and is divergence free. J may or may not be exact.

```
>
> NN:=(N1*d(Y)&^d(Z)&^d(S)-N2*d(X)&^d(Z)&^d(S) +N3*d(X)&^d(Y)&^d(S) -N4*d(X)&^d(Y)&^d(Z));
```

$NN :=$

$-Y\,\&\text{^}(d(Y), d(Z), d(S)) + X\,\&\text{^}(d(X), d(Z), d(S)) - \&\text{^}(d(X), d(Y), d(S))\,ch\,S - \&\text{^}(d(X), d(Y), d(Z))\,ch\,Z$

```
> ZQ:=factor((JJ-NN));
```

$ZQ := \&\text{^}(d(Z), d(Y), d(S))\,Y + \&\text{^}(d(X), d(S), d(Z))\,X + Y\,\&\text{^}(d(Y), d(Z), d(S)) - X\,\&\text{^}(d(X), d(Z), d(S))$

```
>
```

The 4D volume element is zero (computed from the map given above) implying a functional relation, or constraint, exists between the functional forms assumed for x,y,z,ct in terms of {X,Y,Z,S}. The result is independent from the chirality factor ch.

****

The 3-form J = dx^dy^dz has two signs depending on ch, but it is also closed, independent from sign, as it is a monomial constructed from perfect differentials: dJ = 0.
in fact, J =d(X^2+Y^2+Z^2+S^2)^{dZ^dS + ch * dX^dY) .
**
The two complex functions, alpha and beta, define a 4 vector {x,y,z,ct} whose Minkowski sum of squared coefficients vanishes, if B^2= +1 and whose euclidean sum of squared coefficients vanishes if B^2 = -1.
**
For B = 1, ch = 1, such is the definition of the 4 spinor as given by Chandrasekhar "The Mathematical Theory of Black Holes".Oxford (1983) p531.
***
However, the map to x,y,z from X,Y,Z,S (given above) is precisely the Hopf Map
(either left and right handed depending on the sign of ch).
The three 4 component gradient fields defined by dx, dy, dz are linearly independent. In order to form a 4D basis frame it is necessary to find on {X,Y,Z,S} a fourth linearly independent field. This field can be computed by the adjoint method.
In terms of forms, the 3- form J =dx^dy^dz forms a current (with sign dependent upon the factor ch),
and the objective is to find a vector **N**, such that i(**N**) dX^dY^dZ^dS = J.
Then the functions that define the three 4-vectors constructed from the gradient fields, dx,dy,and dz, along with functions that compose the adjoint field, **N**, form a basis frame for the space. These concepts are valid to with a factor.

The orientation of the adjoint field is arbitrary, so multiply **N** by the orientation factor B which can take on value plus or minus one. These fields, to within a factor, lead to an assignment of a global frame field matrix with components proportional to :

```
> HE1:=[ch*Z,S,ch*X,Y];HE2:=[-ch*S,Z,Y,-ch*X];HE3:=[X,Y,-Z,-S];HN1:=evalm(-B*[-Y,X,-ch*S,ch*Z]);
```

$$HE1 := [ch\,Z, S, ch\,X, Y]$$
$$HE2 := [-ch\,S, Z, Y, -ch\,X]$$
$$HE3 := [X, Y, -Z, -S]$$
$$HN1 := [B\,Y, -B\,X, B\,ch\,S, -B\,ch\,Z]$$

All four vector fields have zero divergence.
The first three have zero curl and are integrable (as they are exact gradient fields)
The HN1 adjoint field is NOT integrable as a Pfaffian expression. The associated 1-form has non-zero topological torsion.
Arbitrarily , and for algebraic simplification, each direction field can be divided by the factor Holder factor (X^p+Y^p+Z^p+S^p)^(n/p). It is of some interest to note that all vectors above have zero divergence with respect to [X,Y,Z,S]. This will not be done in that whch follows.

```
> diverge(HE1,[X,Y,Z,S]);diverge(HE2,[X,Y,Z,S]);diverge(HE3,[X,Y,Z,S]);diverge(evalm(HN1),[X,Y,Z,S]);
```

$$0$$
$$0$$
$$0$$
$$0$$

The Frame matrix is defined as:

```
> FF:=array([[HE1[1],HE1[2],HE1[3],HE1[4]],[HE2[1],HE2[2],HE2[3],HE2[4]],[HE3[1],HE3[2],HE3[3],HE3[4]],[HN1[1],HN1[2],HN1[3],HN1[4]]]);DET:=factor(subs(ch^2=1,det(FF)));GG:=simplify(subs(ch^2=1,inverse(FF)));
```

FRAME MATRIX FF, INVERSE FRAME GG, ch = chirality factor, B=orientation

```
>
```

$$FF := \begin{bmatrix} ch\,Z & S & ch\,X & Y \\ -ch\,S & Z & Y & -ch\,X \\ X & Y & -Z & -S \\ B\,Y & -B\,X & B\,ch\,S & -B\,ch\,Z \end{bmatrix}$$

$$DET := B\,(X^2+Y^2+Z^2+S^2)^2$$

$$GG := \begin{bmatrix} \frac{ch\,Z}{X^2+Y^2+Z^2+S^2} & -\frac{ch\,S}{X^2+Y^2+Z^2+S^2} & \frac{X}{X^2+Y^2+Z^2+S^2} & \frac{Y}{B\,(X^2+Y^2+Z^2+S^2)} \\ \frac{S}{X^2+Y^2+Z^2+S^2} & \frac{Z}{X^2+Y^2+Z^2+S^2} & \frac{Y}{X^2+Y^2+Z^2+S^2} & -\frac{X}{B\,(X^2+Y^2+Z^2+S^2)} \\ \frac{ch\,X}{X^2+Y^2+Z^2+S^2} & \frac{Y}{X^2+Y^2+Z^2+S^2} & -\frac{Z}{X^2+Y^2+Z^2+S^2} & \frac{ch\,S}{B\,(X^2+Y^2+Z^2+S^2)} \\ \frac{Y}{X^2+Y^2+Z^2+S^2} & -\frac{ch\,X}{X^2+Y^2+Z^2+S^2} & -\frac{S}{X^2+Y^2+Z^2+S^2} & -\frac{ch\,Z}{B\,(X^2+Y^2+Z^2+S^2)} \end{bmatrix}$$

```
> CHPOLYPP:=(collect((subs(ch=1,B=1,charpoly(FF,lambda))),lambda));CHPOLYMP:=collect((subs(ch=-1,B=1,charpoly(FF,lambda))),lambda);CHPOLYPM:=(collect((subs(ch=1,B=-1,charpoly(FF,lambda))),lambda));CHPOLYMM:=collect((subs(ch=-1,B=-1,charpoly(FF,lambda))),lambda);`Diff between left and right CHPOLYchP`:=factor(subs(CHPOLYPP-CHPOLYMP));`Diff between left and right CHPOLYchM`:=factor(subs(CHPOLYPM-CHPOLYMM));`Pullback of euclidean Metric on x,y,z,ct for B^2=1`:=subs(ch^2=1,B^2=1,innerprod(transpose(FF),FF));`Pullback of Minkowski Metric on x,y,z,ct for B^2=-1`:=subs(ch^2=1,B^2=-1,innerprod(transpose(FF),Mink,FF));
```

CHPOLYPP :=

$$\lambda^4 + (-2\,Y^2 - 2\,Z^2 - 2\,X^2 + 2\,S^2)\,\lambda^2 + 2\,X^2 Z^2 + 2\,Y^2 S^2 + 2\,Y^2 Z^2 + 2\,X^2 S^2 + X^4 + 2\,X^2 Y^2 + Y^4 + Z^4 + 2\,Z^2 S^2 + S^4$$

CHPOLYMP :=

$$\lambda^4 + (-2Y^2 - 2Z^2 + 2X^2 - 2S^2)\lambda^2 + 2X^2Z^2 + 2Y^2S^2 + 2Y^2Z^2 + 2X^2S^2 + X^4 + 2X^2Y^2 + Y^4 + Z^4 + 2Z^2S^2 + S^4$$

$$CHPOLYPM := \lambda^4 - 2Z\lambda^3 + (2S^2Z + 2ZY^2 + 2ZX^2 + 2Z^3)\lambda - 2Y^2S^2 - 2Y^2Z^2 - 2X^2Y^2 - 2X^2Z^2 - S^4 - Y^4 - Z^4 - X^4 - 2Z^2S^2 - 2X^2S^2$$

$$CHPOLYMM := \lambda^4 + 2Z\lambda^3 + (-2S^2Z - 2ZY^2 - 2ZX^2 - 2Z^3)\lambda - 2Y^2S^2 - 2Y^2Z^2 - 2X^2Y^2 - 2X^2Z^2 - S^4 - Y^4 - Z^4 - X^4 - 2Z^2S^2 - 2X^2S^2$$

$$\textit{Diff between left and right CHPOLYchP} := 4\lambda^2(S-X)(S+X)$$

$$\textit{Diff between left and right CHPOLYchM} := 4Z\lambda(S^2 + Y^2 + X^2 + Z^2 - \lambda^2)$$

Pullback of euclidean Metric on x,y,z,ct for B^2=1 :=

$$\begin{bmatrix} X^2+Y^2+Z^2+S^2 & 0 & 0 & 0 \\ 0 & X^2+Y^2+Z^2+S^2 & 0 & 0 \\ 0 & 0 & X^2+Y^2+Z^2+S^2 & 0 \\ 0 & 0 & 0 & X^2+Y^2+Z^2+S^2 \end{bmatrix}$$

Pullback of Minkowski Metric on x,y,z,ct for B^2=-1 :=

$$\begin{bmatrix} X^2+Y^2+Z^2+S^2 & 0 & 0 & 0 \\ 0 & X^2+Y^2+Z^2+S^2 & 0 & 0 \\ 0 & 0 & X^2+Y^2+Z^2+S^2 & 0 \\ 0 & 0 & 0 & X^2+Y^2+Z^2+S^2 \end{bmatrix}$$

The characteristic polynomial depends upon both the chirality factor, ch, and the sign of the determinant B (orientation).
For B=minus one the Frame has a negative determinant (the discrete reflective cases and global abnormalities or catastrophies). For B =plus one, the Frame has a positive determinant which implies perturbations about the identity are possible (local abnormalities) . The orientation is independent from the chirality
RELATIVE TO THE HOPF MAP:
FOR B^2= plus 1, the induced pullback metric on X,Y,Z,S is euclidean diagonal to within a conformal factor when the metric is presumed to be euclidean on (x,y,z,ct} .
For B^2 = -1, the induced pullback metric on X,Y,Z,S is euclidean diagonal to within a conformal factor when the metric is presumed to be Minkowski on (x,y,z,ct} .

The next equation checks to see that the specified frame produces the desired differential structures:
[FF]|dR>=|sigma>

```
> sigma:=(innerprod(FF,[d(X),d(Y),d(Z),d(S)]));
```

$$\sigma := [ch\,Z\,d(X) + S\,d(Y) + ch\,X\,d(Z) + Y\,d(S), -ch\,S\,d(X) + Z\,d(Y) + Y\,d(Z) - ch\,X\,d(S), X\,d(X) + Y\,d(Y) - Z\,d(Z) - S\,d(S), B\,Y\,d(X) - B\,X\,d(Y) + B\,ch\,S\,d(Z) - B\,ch\,Z\,d(S)]$$

```
> sigma1:=sigma[1];
> sigma2:=sigma[2];
> sigma3:=sigma[3];
> omega:=factor(sigma[4]);curlomega:=d(omega);sigma12:=sigma1&^sigma2;
```

$$\sigma 1 := ch\,Z\,d(X) + S\,d(Y) + ch\,X\,d(Z) + Y\,d(S)$$

$$\sigma 2 := -ch\,S\,d(X) + Z\,d(Y) + Y\,d(Z) - ch\,X\,d(S)$$

$$\sigma 3 := X\,d(X) + Y\,d(Y) - Z\,d(Z) - S\,d(S)$$

$$\omega := B\,(Y\,d(X) - d(Y)\,X + d(Z)\,S\,ch - d(S)\,Z\,ch)$$

$$curlomega := 2B\,(d(Y)\,\&\hat{}\,d(X)) + 2B\,ch\,(d(S)\,\&\hat{}\,d(Z))$$

$$\sigma 12 := (ch\,Z^2 + S^2\,ch)\,(d(X)\,\&\hat{}\,d(Y)) + (-ch^2\,Z\,X + Y\,ch\,S)\,(d(X)\,\&\hat{}\,d(S)) + (ch\,Z\,Y + ch^2\,X\,S)\,(d(X)\,\&\hat{}\,d(Z)) + (-ch^2\,X^2 - Y^2)\,(d(Z)\,\&\hat{}\,d(S)) + (-ch\,X\,S - Y\,Z)\,(d(Y)\,\&\hat{}\,d(S)) + (Y\,S - ch\,X\,Z)\,(d(Y)\,\&\hat{}\,d(Z))$$

The vector of induced 1-forms [FF]|dR>=|sigma> can written as the column vector of components [sigma1,sigma2,sigma3,omega] .

The induced 1-forms created by the Frame acting on the differentials on the domain are tested for integrablility. As the Hopf map was defined by a triple of functions for the first three tangent vectors, it is no surprise that the first three induced 1-forms are exact. It is the fourth component of the Frame Field (the normal or adjoint field) that is of interest. The fourth induced 1-form, omega, is not closed and does not obey the Frobenius integrability theorem. Note that the normal field depends upon both the choice of the chirality, ch, and the orientation.

```
> `dsigma1`:=d(sigma[1]);`dsigma2`:=d(sigma[2]);`dsigma3`:=d(sigma[3]);`omega`:=omega;`vorticy of omega`:=factor(d(omega));`topological torsion of omega`:=factor(subs(B^2=1,omega&^d(omega)));`topological parity of omega`:=subs(B^2=1,d(omega)&^d(omega));
```

$$dsigma1 := 0$$

$$dsigma2 := 0$$

$$dsigma3 := 0$$

$$\omega := B\,(Y\,\mathrm{d}(X) - \mathrm{d}(Y)\,X + \mathrm{d}(Z)\,S\,ch - \mathrm{d}(S)\,Z\,ch)$$

$$vorticy\ of\ \ omega := 2\,B\,((\mathrm{d}(Y)\ \&\hat{}\ \mathrm{d}(X)) + ch\,(\mathrm{d}(S)\ \&\hat{}\ \mathrm{d}(Z)))$$

$$topological\ torsion\ of\ omega :=$$
$$2\,ch\,(Y\,\&\hat{}(\mathrm{d}(X), \mathrm{d}(S), \mathrm{d}(Z)) - \&\hat{}(\mathrm{d}(Y), \mathrm{d}(S), \mathrm{d}(Z))\,X + \&\hat{}(\mathrm{d}(Z), \mathrm{d}(Y), \mathrm{d}(X))\,S - \&\hat{}(\mathrm{d}(S), \mathrm{d}(Y), \mathrm{d}(X))\,Z)$$

$$topological\ parity\ of\ omega := 8\,ch\,\&\hat{}(\mathrm{d}(Y), \mathrm{d}(X), \mathrm{d}(S), \mathrm{d}(Z))$$

It is remarkable that the Topological Torsion and the Topological parity of omega, the non - integrable induced 1-form, does not depend upon the orientation, B, but does depend upon the chirality, ch.

A possible candidate for a metric on X,Y,Z,S would be the symmetric form (which is conformal to the Euclidean metric on R4):

```
>
> pullbackmetric:=subs(ch^2=1,B^2=1,innerprod(transpose(FF),FF));push:=subs(ch^2=1,B^2=-1,innerprod(transpose(GG),R4*GG));
```

$$pullbackmetric := \begin{bmatrix} X^2+Y^2+Z^2+S^2 & 0 & 0 & 0 \\ 0 & X^2+Y^2+Z^2+S^2 & 0 & 0 \\ 0 & 0 & X^2+Y^2+Z^2+S^2 & 0 \\ 0 & 0 & 0 & X^2+Y^2+Z^2+S^2 \end{bmatrix}$$

$$push := \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \frac{1}{B^2} \end{bmatrix}$$

Note that the Frame matrix FF is orthogonal but not orthonormal. It is also conformal in that the normalization is the same for all basis elements. It is independent of the orientation, B and the chirality ch. !!!!

Now Compute the Right Cartan Matrix [CR] and the Left Cartan Matrix. These matrices are negative similarity transforms to within a sign.

```
> cartan:=(subs(ch^2=1,innerprod(GG,d(FF)))):cartanL:=evalm((subs(ch^2=1,innerprod(-d(FF),GG)))):
> CLcong:=(subs(ch^2=1,innerprod(-FF,cartan,GG))):zz:=evalm(FF);inverse(zz):
```

$$zz := \begin{bmatrix} ch\,Z & S & ch\,X & Y \\ -ch\,S & Z & Y & -ch\,X \\ X & Y & -Z & -S \\ B\,Y & -B\,X & B\,ch\,S & -B\,ch\,Z \end{bmatrix}$$

```
> `Should be zero if similarity equivalents`:=wcollect(factor(CLcong[1,2]-cartanL[1,2]));
```

$$\mathit{Should\ be\ zero\ if\ similarity\ equivalents} := -\frac{ch\,X\,(ch-1)\,(ch+1)\,Y\,S\,\mathrm{d}(S)}{(X^2+Y^2+Z^2+S^2)^2} - \frac{ch\,X\,(ch-1)\,(ch+1)\,Y\,Z\,\mathrm{d}(Z)}{(X^2+Y^2+Z^2+S^2)^2} - \frac{ch\,X^2\,(ch-1)\,(ch+1)\,Y\,\mathrm{d}(X)}{(X^2+Y^2+Z^2+S^2)^2} - \frac{ch\,X\,(ch-1)\,(ch+1)\,(-X^2-Z^2-S^2)\,\mathrm{d}(Y)}{(X^2+Y^2+Z^2+S^2)^2}$$

>

The matrix elements of the Right Cartan connection matrix using the matrix methods:

Note that the Right Cartan connection does not depend upon B, but does depend upon ch.

>

```
> Gamma11:=wcollect(cartan[1,1]);Gamma12:=wcollect(cartan[1,2]);Gamma13:=wcollect(cartan[1,3]);Gamma14:=wcollect(cartan[1,4]);
```

$$\Gamma 11 := \frac{S\,\mathrm{d}(S)}{X^2+Y^2+Z^2+S^2} + \frac{Z\,\mathrm{d}(Z)}{X^2+Y^2+Z^2+S^2} + \frac{X\,\mathrm{d}(X)}{X^2+Y^2+Z^2+S^2} + \frac{Y\,\mathrm{d}(Y)}{X^2+Y^2+Z^2+S^2}$$

$$\Gamma 12 := \frac{ch\,Z\,\mathrm{d}(S)}{X^2+Y^2+Z^2+S^2} - \frac{ch\,S\,\mathrm{d}(Z)}{X^2+Y^2+Z^2+S^2} - \frac{Y\,\mathrm{d}(X)}{X^2+Y^2+Z^2+S^2} + \frac{X\,\mathrm{d}(Y)}{X^2+Y^2+Z^2+S^2}$$

$$\Gamma 13 := \frac{Y\,ch\,\mathrm{d}(S)}{X^2+Y^2+Z^2+S^2} - \frac{X\,\mathrm{d}(Z)}{X^2+Y^2+Z^2+S^2} + \frac{Z\,\mathrm{d}(X)}{X^2+Y^2+Z^2+S^2} - \frac{ch\,S\,\mathrm{d}(Y)}{X^2+Y^2+Z^2+S^2}$$

$$\Gamma 14 := -\frac{X\,\mathrm{d}(S)}{X^2+Y^2+Z^2+S^2} - \frac{Y\,ch\,\mathrm{d}(Z)}{X^2+Y^2+Z^2+S^2} + \frac{S\,\mathrm{d}(X)}{X^2+Y^2+Z^2+S^2} + \frac{ch\,Z\,\mathrm{d}(Y)}{X^2+Y^2+Z^2+S^2}$$

****

The Left Cartan matix elements can also be computed by the matrix method (for example)

```
> Gamma11L:=wcollect(cartanL[1,1]);Gamma12L:=wcollect(cartanL[1,2]);Gamma13L:=wcollect(cartanL[1,3]);Gamma14L:=wcollect(cartanL[1,4]);
```

$$\mathit{Gamma11L} := -\frac{S\,\mathrm{d}(S)}{X^2+Y^2+Z^2+S^2} - \frac{Z\,\mathrm{d}(Z)}{X^2+Y^2+Z^2+S^2} - \frac{X\,\mathrm{d}(X)}{X^2+Y^2+Z^2+S^2} - \frac{Y\,\mathrm{d}(Y)}{X^2+Y^2+Z^2+S^2}$$

$$\mathit{Gamma12L} := -\frac{Z\,\mathrm{d}(S)}{X^2+Y^2+Z^2+S^2} + \frac{S\,\mathrm{d}(Z)}{X^2+Y^2+Z^2+S^2} - \frac{Y\,ch\,\mathrm{d}(X)}{X^2+Y^2+Z^2+S^2} + \frac{ch\,X\,\mathrm{d}(Y)}{X^2+Y^2+Z^2+S^2}$$

$$\mathit{Gamma13L} := -\frac{Y\,\mathrm{d}(S)}{X^2+Y^2+Z^2+S^2} - \frac{ch\,X\,\mathrm{d}(Z)}{X^2+Y^2+Z^2+S^2} + \frac{ch\,Z\,\mathrm{d}(X)}{X^2+Y^2+Z^2+S^2} + \frac{S\,\mathrm{d}(Y)}{X^2+Y^2+Z^2+S^2}$$

$$\mathit{Gamma14L} := \frac{X\,\mathrm{d}(S)}{B\,(X^2+Y^2+Z^2+S^2)} - \frac{Y\,ch\,\mathrm{d}(Z)}{B\,(X^2+Y^2+Z^2+S^2)} - \frac{S\,\mathrm{d}(X)}{B\,(X^2+Y^2+Z^2+S^2)} + \frac{ch\,Z\,\mathrm{d}(Y)}{B\,(X^2+Y^2+Z^2+S^2)}$$

Note that the Left Cartan connection is sensitive to both B and ch.

****

```
> Gamma21:=wcollect(cartan[2,1]);Gamma22:=wcollect(cartan[2,2]);Gamma23:=wcollect(cartan[2,3]);Gamma24:=wcollect(cartan[2,4]);
```

$$\Gamma 21 := -\frac{ch\,Z\,\mathrm{d}(S)}{X^2+Y^2+Z^2+S^2} + \frac{ch\,S\,\mathrm{d}(Z)}{X^2+Y^2+Z^2+S^2} + \frac{Y\,\mathrm{d}(X)}{X^2+Y^2+Z^2+S^2} - \frac{X\,\mathrm{d}(Y)}{X^2+Y^2+Z^2+S^2}$$

$$\Gamma 22 := \frac{S\,\mathrm{d}(S)}{X^2+Y^2+Z^2+S^2} + \frac{Z\,\mathrm{d}(Z)}{X^2+Y^2+Z^2+S^2} + \frac{X\,\mathrm{d}(X)}{X^2+Y^2+Z^2+S^2} + \frac{Y\,\mathrm{d}(Y)}{X^2+Y^2+Z^2+S^2}$$

$$\Gamma 23 := -\frac{ch\,X\,\mathrm{d}(S)}{X^2+Y^2+Z^2+S^2} - \frac{Y\,\mathrm{d}(Z)}{X^2+Y^2+Z^2+S^2} + \frac{ch\,S\,\mathrm{d}(X)}{X^2+Y^2+Z^2+S^2} + \frac{Z\,\mathrm{d}(Y)}{X^2+Y^2+Z^2+S^2}$$

$$\Gamma 24 := -\frac{Y\,\mathrm{d}(S)}{X^2+Y^2+Z^2+S^2} + \frac{ch\,X\,\mathrm{d}(Z)}{X^2+Y^2+Z^2+S^2} - \frac{ch\,Z\,\mathrm{d}(X)}{X^2+Y^2+Z^2+S^2} + \frac{S\,\mathrm{d}(Y)}{X^2+Y^2+Z^2+S^2}$$

```
> Gamma31:=wcollect(cartan[3,1]);Gamma32:=wcollect(cartan[3,2]);Gamma33:=wcollect(cartan[3,3]);G
```

amma34:=wcollect(cartan[3,4]);

$$\Gamma31 := -\frac{Y\,ch\,d(S)}{X^2+Y^2+Z^2+S^2}+\frac{X\,d(Z)}{X^2+Y^2+Z^2+S^2}-\frac{Z\,d(X)}{X^2+Y^2+Z^2+S^2}+\frac{ch\,S\,d(Y)}{X^2+Y^2+Z^2+S^2}$$

$$\Gamma32 := \frac{ch\,X\,d(S)}{X^2+Y^2+Z^2+S^2}+\frac{Y\,d(Z)}{X^2+Y^2+Z^2+S^2}-\frac{ch\,S\,d(X)}{X^2+Y^2+Z^2+S^2}-\frac{Z\,d(Y)}{X^2+Y^2+Z^2+S^2}$$

$$\Gamma33 := \frac{S\,d(S)}{X^2+Y^2+Z^2+S^2}+\frac{Z\,d(Z)}{X^2+Y^2+Z^2+S^2}+\frac{X\,d(X)}{X^2+Y^2+Z^2+S^2}+\frac{Y\,d(Y)}{X^2+Y^2+Z^2+S^2}$$

$$\Gamma34 := \frac{Z\,d(S)}{X^2+Y^2+Z^2+S^2}-\frac{S\,d(Z)}{X^2+Y^2+Z^2+S^2}-\frac{Y\,ch\,d(X)}{X^2+Y^2+Z^2+S^2}+\frac{ch\,X\,d(Y)}{X^2+Y^2+Z^2+S^2}$$

```
> Gamma41:=(wcollect(cartan[4,1]));Gamma42:=wcollect(cartan[4,2]);Gamma43:=wcollect(cartan[4,3]);
  Gamma44:=wcollect(cartan[4,4]);
```

$$\Gamma41 := \frac{X\,d(S)}{X^2+Y^2+Z^2+S^2}+\frac{Y\,ch\,d(Z)}{X^2+Y^2+Z^2+S^2}-\frac{S\,d(X)}{X^2+Y^2+Z^2+S^2}-\frac{ch\,Z\,d(Y)}{X^2+Y^2+Z^2+S^2}$$

$$\Gamma42 := \frac{Y\,d(S)}{X^2+Y^2+Z^2+S^2}-\frac{ch\,X\,d(Z)}{X^2+Y^2+Z^2+S^2}+\frac{ch\,Z\,d(X)}{X^2+Y^2+Z^2+S^2}-\frac{S\,d(Y)}{X^2+Y^2+Z^2+S^2}$$

$$\Gamma43 := -\frac{Z\,d(S)}{X^2+Y^2+Z^2+S^2}+\frac{S\,d(Z)}{X^2+Y^2+Z^2+S^2}+\frac{Y\,ch\,d(X)}{X^2+Y^2+Z^2+S^2}-\frac{ch\,X\,d(Y)}{X^2+Y^2+Z^2+S^2}$$

$$\Gamma44 := \frac{S\,d(S)}{X^2+Y^2+Z^2+S^2}+\frac{Z\,d(Z)}{X^2+Y^2+Z^2+S^2}+\frac{X\,d(X)}{X^2+Y^2+Z^2+S^2}+\frac{Y\,d(Y)}{X^2+Y^2+Z^2+S^2}$$

All the elements of the right Cartan matrix have a common factor (1/R4). Note that the diagonal matix elements are perfect differentials, and if the R4 space is constrained to a 3 sphere, the diagonal elements vanish.
(These matrix elements are related to dilatations - "the expanding universe".)

>

Now the components of the right Cartan matrix will be computed by the tensor method, as a check

```
> dim:=4;coord:=[X,Y,Z,S];
```

$$dim := 4$$

$$coord := [X, Y, Z, S]$$

First compute the differentials of the inverse matrix [GG]

```
> for i from 1 to dim do for j from 1 to dim do for k from 1 to dim do d1GG[i,j,k] := (diff(GG[i,j],coord[k]))
  od od od:
```

Compute the elements of the matrix product of - d[G][F]. The notation is such that (a,-b,-c,) implies (upper,lower,lower) index.

```
> for b from 1 to dim do for a from 1 to dim do for k from 1 to dim do  ss:=0;for m from 1 to dim do ss :=
  ss+(d1GG[a,m,k]*FF[m,b]); CC[a,b,k]:=simplify(-ss) od od od od ;
>
> for a from 1 to dim do for b from 1 to dim do for k from 1 to dim do if CC[a,b,k]=0 then else
  print(`Cartan_RIGHT`(a,-b,-k)=factor(subs(ch^2=1,CC[a,b,k]))) fi od od od ;
```

THE non zero CARTAN RIGHT CONNECTION coefficients. (Hopf map)

$$\text{Cartan\_RIGHT}(1, -1, -1) = \frac{X}{X^2+Y^2+Z^2+S^2}$$

$$\text{Cartan\_RIGHT}(1, -1, -2) = \frac{Y}{X^2+Y^2+Z^2+S^2}$$

$$\text{Cartan\_RIGHT}(1, -1, -3) = \frac{Z}{X^2+Y^2+Z^2+S^2}$$

$$\text{Cartan\_RIGHT}(1, -1, -4) = \frac{S}{X^2 + Y^2 + Z^2 + S^2}$$

$$\text{Cartan\_RIGHT}(1, -2, -1) = -\frac{Y}{X^2 + Y^2 + Z^2 + S^2}$$

$$\text{Cartan\_RIGHT}(1, -2, -2) = \frac{X}{X^2 + Y^2 + Z^2 + S^2}$$

$$\text{Cartan\_RIGHT}(1, -2, -3) = -\frac{ch\, S}{X^2 + Y^2 + Z^2 + S^2}$$

$$\text{Cartan\_RIGHT}(1, -2, -4) = \frac{ch\, Z}{X^2 + Y^2 + Z^2 + S^2}$$

$$\text{Cartan\_RIGHT}(1, -3, -1) = \frac{Z}{X^2 + Y^2 + Z^2 + S^2}$$

$$\text{Cartan\_RIGHT}(1, -3, -2) = -\frac{ch\, S}{X^2 + Y^2 + Z^2 + S^2}$$

$$\text{Cartan\_RIGHT}(1, -3, -3) = -\frac{X}{X^2 + Y^2 + Z^2 + S^2}$$

$$\text{Cartan\_RIGHT}(1, -3, -4) = \frac{Y\, ch}{X^2 + Y^2 + Z^2 + S^2}$$

$$\text{Cartan\_RIGHT}(1, -4, -1) = \frac{S}{X^2 + Y^2 + Z^2 + S^2}$$

$$\text{Cartan\_RIGHT}(1, -4, -2) = \frac{ch\, Z}{X^2 + Y^2 + Z^2 + S^2}$$

$$\text{Cartan\_RIGHT}(1, -4, -3) = -\frac{Y\, ch}{X^2 + Y^2 + Z^2 + S^2}$$

$$\text{Cartan\_RIGHT}(1, -4, -4) = -\frac{X}{X^2 + Y^2 + Z^2 + S^2}$$

$$\text{Cartan\_RIGHT}(2, -1, -1) = \frac{Y}{X^2 + Y^2 + Z^2 + S^2}$$

$$\text{Cartan\_RIGHT}(2, -1, -2) = -\frac{X}{X^2 + Y^2 + Z^2 + S^2}$$

$$\text{Cartan\_RIGHT}(2, -1, -3) = \frac{ch\, S}{X^2 + Y^2 + Z^2 + S^2}$$

$$\text{Cartan\_RIGHT}(2, -1, -4) = -\frac{ch\, Z}{X^2 + Y^2 + Z^2 + S^2}$$

$$\text{Cartan\_RIGHT}(2, -2, -1) = \frac{X}{X^2 + Y^2 + Z^2 + S^2}$$

$$\text{Cartan\_RIGHT}(2, -2, -2) = \frac{Y}{X^2 + Y^2 + Z^2 + S^2}$$

$$\text{Cartan\_RIGHT}(2, -2, -3) = \frac{Z}{X^2 + Y^2 + Z^2 + S^2}$$

$$\text{Cartan\_RIGHT}(2, -2, -4) = \frac{S}{X^2 + Y^2 + Z^2 + S^2}$$

$$\text{Cartan\_RIGHT}(2, -3, -1) = \frac{ch\, S}{X^2 + Y^2 + Z^2 + S^2}$$

$$\text{Cartan\_RIGHT}(2, -3, -2) = \frac{Z}{X^2 + Y^2 + Z^2 + S^2}$$

$$\text{Cartan\_RIGHT}(2, -3, -3) = -\frac{Y}{X^2 + Y^2 + Z^2 + S^2}$$

$$\text{Cartan\_RIGHT}(2, -3, -4) = -\frac{ch\, X}{X^2 + Y^2 + Z^2 + S^2}$$

$$\text{Cartan\_RIGHT}(2, -4, -1) = -\frac{ch\, Z}{X^2 + Y^2 + Z^2 + S^2}$$

$$\text{Cartan\_RIGHT}(2, -4, -2) = \frac{S}{X^2 + Y^2 + Z^2 + S^2}$$

$$\text{Cartan\_RIGHT}(2, -4, -3) = \frac{ch\, X}{X^2 + Y^2 + Z^2 + S^2}$$

$$\text{Cartan\_RIGHT}(2, -4, -4) = -\frac{Y}{X^2 + Y^2 + Z^2 + S^2}$$

$$\text{Cartan\_RIGHT}(3, -1, -1) = -\frac{Z}{X^2 + Y^2 + Z^2 + S^2}$$

$$\text{Cartan\_RIGHT}(3, -1, -2) = \frac{ch\, S}{X^2 + Y^2 + Z^2 + S^2}$$

$$\text{Cartan\_RIGHT}(3, -1, -3) = \frac{X}{X^2 + Y^2 + Z^2 + S^2}$$

$$\text{Cartan\_RIGHT}(3, -1, -4) = -\frac{Y\, ch}{X^2 + Y^2 + Z^2 + S^2}$$

$$\text{Cartan\_RIGHT}(3, -2, -1) = -\frac{ch\, S}{X^2 + Y^2 + Z^2 + S^2}$$

$$\text{Cartan\_RIGHT}(3, -2, -2) = -\frac{Z}{X^2 + Y^2 + Z^2 + S^2}$$

$$\text{Cartan\_RIGHT}(3, -2, -3) = \frac{Y}{X^2 + Y^2 + Z^2 + S^2}$$

$$\text{Cartan\_RIGHT}(3, -2, -4) = \frac{ch\, X}{X^2 + Y^2 + Z^2 + S^2}$$

$$\text{Cartan\_RIGHT}(3, -3, -1) = \frac{X}{X^2 + Y^2 + Z^2 + S^2}$$

$$\text{Cartan\_RIGHT}(3, -3, -2) = \frac{Y}{X^2 + Y^2 + Z^2 + S^2}$$

$$\text{Cartan\_RIGHT}(3, -3, -3) = \frac{Z}{X^2 + Y^2 + Z^2 + S^2}$$

$$\text{Cartan\_RIGHT}(3, -3, -4) = \frac{S}{X^2 + Y^2 + Z^2 + S^2}$$

$$\text{Cartan\_RIGHT}(3, -4, -1) = -\frac{Y\, ch}{X^2 + Y^2 + Z^2 + S^2}$$

$$\text{Cartan\_RIGHT}(3, -4, -2) = \frac{ch\, X}{X^2 + Y^2 + Z^2 + S^2}$$

$$\text{Cartan\_RIGHT}(3, -4, -3) = -\frac{S}{X^2 + Y^2 + Z^2 + S^2}$$

$$\text{Cartan\_RIGHT}(3, -4, -4) = \frac{Z}{X^2 + Y^2 + Z^2 + S^2}$$

$$\text{Cartan\_RIGHT}(4, -1, -1) = -\frac{S}{X^2 + Y^2 + Z^2 + S^2}$$

$$\text{Cartan\_RIGHT}(4, -1, -2) = -\frac{ch\, Z}{X^2 + Y^2 + Z^2 + S^2}$$

$$\text{Cartan\_RIGHT}(4, -1, -3) = \frac{Y\, ch}{X^2 + Y^2 + Z^2 + S^2}$$

$$\text{Cartan\_RIGHT}(4, -1, -4) = \frac{X}{X^2 + Y^2 + Z^2 + S^2}$$

$$\text{Cartan\_RIGHT}(4, -2, -1) = \frac{ch\, Z}{X^2 + Y^2 + Z^2 + S^2}$$

$$\text{Cartan\_RIGHT}(4, -2, -2) = -\frac{S}{X^2 + Y^2 + Z^2 + S^2}$$

$$\text{Cartan\_RIGHT}(4, -2, -3) = -\frac{ch\, X}{X^2 + Y^2 + Z^2 + S^2}$$

$$\text{Cartan\_RIGHT}(4, -2, -4) = \frac{Y}{X^2 + Y^2 + Z^2 + S^2}$$

$$\text{Cartan\_RIGHT}(4, -3, -1) = \frac{Y\, ch}{X^2 + Y^2 + Z^2 + S^2}$$

$$\text{Cartan\_RIGHT}(4, -3, -2) = -\frac{ch\, X}{X^2 + Y^2 + Z^2 + S^2}$$

$$\text{Cartan\_RIGHT}(4, -3, -3) = \frac{S}{X^2 + Y^2 + Z^2 + S^2}$$

$$\text{Cartan\_RIGHT}(4, -3, -4) = -\frac{Z}{X^2 + Y^2 + Z^2 + S^2}$$

$$\text{Cartan\_RIGHT}(4, -4, -1) = \frac{X}{X^2 + Y^2 + Z^2 + S^2}$$

$$\text{Cartan\_RIGHT}(4, -4, -2) = \frac{Y}{X^2 + Y^2 + Z^2 + S^2}$$

$$\text{Cartan\_RIGHT}(4, -4, -3) = \frac{Z}{X^2 + Y^2 + Z^2 + S^2}$$

$$\text{Cartan\_RIGHT}(4, -4, -4) = \frac{S}{X^2 + Y^2 + Z^2 + S^2}$$

These results agree with matrix method given above.

Next check for Affine Torsion of the Right Cartan matrix using the tensor methods:

Torsion coefficients for the Right Cartan matrix are defined as the difference between Gamma(a,-b,-c)-Gamma(a,-c,-b) times 1/2;

```
> for j from 1 to dim do for i from 1 to dim do for k from 1 to dim do  ss := (CC[i,j,k]-CC[i,k,j])/2;
```

```
CCTTS[i,j,k]:=ss od od od  ;
>
> for i from 1 to dim do for j from 1 to dim do for k from 1 to dim do if CCTTS[i,j,k]=0 then else
print(`RIGHT_AffineTorsion`(i,-k,-j)=simplify(subs(ch^2=1,CCTTS[i,k,j]))) fi od od od ;
```

$$\text{RIGHT\_AffineTorsion}(1,-2,-1) = -\frac{Y}{X^2+Y^2+Z^2+S^2}$$

$$\text{RIGHT\_AffineTorsion}(1,-3,-1) = 0$$

$$\text{RIGHT\_AffineTorsion}(1,-4,-1) = 0$$

$$\text{RIGHT\_AffineTorsion}(1,-1,-2) = \frac{Y}{X^2+Y^2+Z^2+S^2}$$

$$\text{RIGHT\_AffineTorsion}(1,-3,-2) = 0$$

$$\text{RIGHT\_AffineTorsion}(1,-4,-2) = 0$$

$$\text{RIGHT\_AffineTorsion}(1,-1,-3) = 0$$

$$\text{RIGHT\_AffineTorsion}(1,-2,-3) = 0$$

$$\text{RIGHT\_AffineTorsion}(1,-4,-3) = -\frac{Y\,ch}{X^2+Y^2+Z^2+S^2}$$

$$\text{RIGHT\_AffineTorsion}(1,-1,-4) = 0$$

$$\text{RIGHT\_AffineTorsion}(1,-2,-4) = 0$$

$$\text{RIGHT\_AffineTorsion}(1,-3,-4) = \frac{Y\,ch}{X^2+Y^2+Z^2+S^2}$$

$$\text{RIGHT\_AffineTorsion}(2,-2,-1) = \frac{X}{X^2+Y^2+Z^2+S^2}$$

$$\text{RIGHT\_AffineTorsion}(2,-1,-2) = -\frac{X}{X^2+Y^2+Z^2+S^2}$$

$$\text{RIGHT\_AffineTorsion}(2,-4,-3) = \frac{ch\,X}{X^2+Y^2+Z^2+S^2}$$

$$\text{RIGHT\_AffineTorsion}(2,-3,-4) = -\frac{ch\,X}{X^2+Y^2+Z^2+S^2}$$

$$\text{RIGHT\_AffineTorsion}(3,-2,-1) = -\frac{ch\,S}{X^2+Y^2+Z^2+S^2}$$

$$\text{RIGHT\_AffineTorsion}(3,-3,-1) = 0$$

$$\text{RIGHT\_AffineTorsion}(3,-4,-1) = 0$$

$$\text{RIGHT\_AffineTorsion}(3,-1,-2) = \frac{ch\,S}{X^2+Y^2+Z^2+S^2}$$

$$\text{RIGHT\_AffineTorsion}(3,-3,-2) = 0$$

$$\text{RIGHT\_AffineTorsion}(3,-4,-2) = 0$$

$$\text{RIGHT\_AffineTorsion}(3,-1,-3) = 0$$

$$\text{RIGHT\_AffineTorsion}(3,-2,-3) = 0$$

$$\text{RIGHT\_AffineTorsion}(3,-4,-3) = -\frac{S}{X^2+Y^2+Z^2+S^2}$$

$$\text{RIGHT\_AffineTorsion}(3,-1,-4) = 0$$

$$\text{RIGHT\_AffineTorsion}(3,-2,-4) = 0$$

$$\text{RIGHT\_AffineTorsion}(3, -3, -4) = \frac{S}{X^2 + Y^2 + Z^2 + S^2}$$

$$\text{RIGHT\_AffineTorsion}(4, -2, -1) = \frac{ch\, Z}{X^2 + Y^2 + Z^2 + S^2}$$

$$\text{RIGHT\_AffineTorsion}(4, -3, -1) = 0$$

$$\text{RIGHT\_AffineTorsion}(4, -4, -1) = 0$$

$$\text{RIGHT\_AffineTorsion}(4, -1, -2) = -\frac{ch\, Z}{X^2 + Y^2 + Z^2 + S^2}$$

$$\text{RIGHT\_AffineTorsion}(4, -3, -2) = 0$$

$$\text{RIGHT\_AffineTorsion}(4, -4, -2) = 0$$

$$\text{RIGHT\_AffineTorsion}(4, -1, -3) = 0$$

$$\text{RIGHT\_AffineTorsion}(4, -2, -3) = 0$$

$$\text{RIGHT\_AffineTorsion}(4, -4, -3) = \frac{Z}{X^2 + Y^2 + Z^2 + S^2}$$

$$\text{RIGHT\_AffineTorsion}(4, -1, -4) = 0$$

$$\text{RIGHT\_AffineTorsion}(4, -2, -4) = 0$$

$$\text{RIGHT\_AffineTorsion}(4, -3, -4) = -\frac{Z}{X^2 + Y^2 + Z^2 + S^2}$$

IF NO ENTRIES APPEAR ABOVE, THE AFFINE TORSION IS ZERO

For the Hopf map and the frame constructed above, it is remarkable that there is a Torsion component that depends on the chirality factor, ch, and another torsion component which does not.!

The Right Affine torsion does not depend upon the orientation, B, but has components that depend upon chirality, ch

****

Now compute the CARTAN LEFT CONNECTION

```
> for a from 1 to dim do for j from 1 to dim do for k from 1 to dim do d1GG[a,j,k] :=
  simplify(diff(GG[a,j],coord[k])) od od od:
```

Compute the elements of the matrix product of [F]d[G]

```
> for i from 1 to dim do for j from 1 to dim do for k from 1 to dim do  ss:=0;for m to dim do ss :=
  ss+FF[i,m]*(d1GG[m,j,k]); DD[i,j,k]:=simplify(ss) od od od od ;
> for i from 1 to dim do for j from 1 to dim do for k from 1 to dim do if DD[i,j,k]=0 then else
  print(`Cartan_LEFT`(i,-j,-k)=simplify(subs(ch^2=1,DD[i,j,k]))) fi od od od ;
```

$$\text{Cartan\_LEFT}(1, -1, -1) = -\frac{X}{X^2 + Y^2 + Z^2 + S^2}$$

$$\text{Cartan\_LEFT}(1, -1, -2) = -\frac{Y}{X^2 + Y^2 + Z^2 + S^2}$$

$$\text{Cartan\_LEFT}(1, -1, -3) = -\frac{Z}{X^2 + Y^2 + Z^2 + S^2}$$

$$\text{Cartan\_LEFT}(1, -1, -4) = -\frac{S}{X^2 + Y^2 + Z^2 + S^2}$$

$$\text{Cartan\_LEFT}(1, -2, -1) = -\frac{Y\, ch}{X^2 + Y^2 + Z^2 + S^2}$$

$$\text{Cartan\_LEFT}(1, -2, -2) = \frac{ch\, X}{X^2 + Y^2 + Z^2 + S^2}$$

$$\text{Cartan\_LEFT}(1, -2, -3) = \frac{S}{X^2 + Y^2 + Z^2 + S^2}$$

$$\text{Cartan\_LEFT}(1, -2, -4) = -\frac{Z}{X^2 + Y^2 + Z^2 + S^2}$$

$$\text{Cartan\_LEFT}(1, -3, -1) = \frac{ch\, Z}{X^2 + Y^2 + Z^2 + S^2}$$

$$\text{Cartan\_LEFT}(1, -3, -2) = \frac{S}{X^2 + Y^2 + Z^2 + S^2}$$

$$\text{Cartan\_LEFT}(1, -3, -3) = -\frac{ch\, X}{X^2 + Y^2 + Z^2 + S^2}$$

$$\text{Cartan\_LEFT}(1, -3, -4) = -\frac{Y}{X^2 + Y^2 + Z^2 + S^2}$$

$$\text{Cartan\_LEFT}(1, -4, -1) = -\frac{S}{(X^2 + Y^2 + Z^2 + S^2)\, B}$$

$$\text{Cartan\_LEFT}(1, -4, -2) = \frac{ch\, Z}{B\, (X^2 + Y^2 + Z^2 + S^2)}$$

$$\text{Cartan\_LEFT}(1, -4, -3) = -\frac{Y\, ch}{B\, (X^2 + Y^2 + Z^2 + S^2)}$$

$$\text{Cartan\_LEFT}(1, -4, -4) = \frac{X}{B\, (X^2 + Y^2 + Z^2 + S^2)}$$

$$\text{Cartan\_LEFT}(2, -1, -1) = \frac{Y\, ch}{X^2 + Y^2 + Z^2 + S^2}$$

$$\text{Cartan\_LEFT}(2, -1, -2) = -\frac{ch\, X}{X^2 + Y^2 + Z^2 + S^2}$$

$$\text{Cartan\_LEFT}(2, -1, -3) = -\frac{S}{X^2 + Y^2 + Z^2 + S^2}$$

$$\text{Cartan\_LEFT}(2, -1, -4) = \frac{Z}{X^2 + Y^2 + Z^2 + S^2}$$

$$\text{Cartan\_LEFT}(2, -2, -1) = -\frac{X}{X^2 + Y^2 + Z^2 + S^2}$$

$$\text{Cartan\_LEFT}(2, -2, -2) = -\frac{Y}{X^2 + Y^2 + Z^2 + S^2}$$

$$\text{Cartan\_LEFT}(2, -2, -3) = -\frac{Z}{X^2 + Y^2 + Z^2 + S^2}$$

$$\text{Cartan\_LEFT}(2, -2, -4) = -\frac{S}{X^2 + Y^2 + Z^2 + S^2}$$

$$\text{Cartan\_LEFT}(2, -3, -1) = -\frac{ch\, S}{X^2 + Y^2 + Z^2 + S^2}$$

$$\text{Cartan\_LEFT}(2, -3, -2) = \frac{Z}{X^2 + Y^2 + Z^2 + S^2}$$

$$\text{Cartan\_LEFT}(2, -3, -3) = -\frac{Y}{X^2 + Y^2 + Z^2 + S^2}$$

$$\text{Cartan\_LEFT}(2, -3, -4) = \frac{ch\, X}{X^2 + Y^2 + Z^2 + S^2}$$

$$\text{Cartan\_LEFT}(2, -4, -1) = -\frac{Z}{(X^2 + Y^2 + Z^2 + S^2)\, B}$$

$$\text{Cartan\_LEFT}(2, -4, -2) = -\frac{ch\, S}{B\, (X^2 + Y^2 + Z^2 + S^2)}$$

$$\text{Cartan\_LEFT}(2, -4, -3) = \frac{X}{B\, (X^2 + Y^2 + Z^2 + S^2)}$$

$$\text{Cartan\_LEFT}(2, -4, -4) = \frac{Y\, ch}{B\, (X^2 + Y^2 + Z^2 + S^2)}$$

$$\text{Cartan\_LEFT}(3, -1, -1) = -\frac{ch\, Z}{X^2 + Y^2 + Z^2 + S^2}$$

$$\text{Cartan\_LEFT}(3, -1, -2) = -\frac{S}{X^2 + Y^2 + Z^2 + S^2}$$

$$\text{Cartan\_LEFT}(3, -1, -3) = \frac{ch\, X}{X^2 + Y^2 + Z^2 + S^2}$$

$$\text{Cartan\_LEFT}(3, -1, -4) = \frac{Y}{X^2 + Y^2 + Z^2 + S^2}$$

$$\text{Cartan\_LEFT}(3, -2, -1) = \frac{ch\, S}{X^2 + Y^2 + Z^2 + S^2}$$

$$\text{Cartan\_LEFT}(3, -2, -2) = -\frac{Z}{X^2 + Y^2 + Z^2 + S^2}$$

$$\text{Cartan\_LEFT}(3, -2, -3) = \frac{Y}{X^2 + Y^2 + Z^2 + S^2}$$

$$\text{Cartan\_LEFT}(3, -2, -4) = -\frac{ch\, X}{X^2 + Y^2 + Z^2 + S^2}$$

$$\text{Cartan\_LEFT}(3, -3, -1) = -\frac{X}{X^2 + Y^2 + Z^2 + S^2}$$

$$\text{Cartan\_LEFT}(3, -3, -2) = -\frac{Y}{X^2 + Y^2 + Z^2 + S^2}$$

$$\text{Cartan\_LEFT}(3, -3, -3) = -\frac{Z}{X^2 + Y^2 + Z^2 + S^2}$$

$$\text{Cartan\_LEFT}(3, -3, -4) = -\frac{S}{X^2 + Y^2 + Z^2 + S^2}$$

$$\text{Cartan\_LEFT}(3, -4, -1) = -\frac{Y}{B\, (X^2 + Y^2 + Z^2 + S^2)}$$

$$\text{Cartan\_LEFT}(3, -4, -2) = \frac{X}{B\, (X^2 + Y^2 + Z^2 + S^2)}$$

$$\text{Cartan\_LEFT}(3, -4, -3) = \frac{ch\, S}{B\, (X^2 + Y^2 + Z^2 + S^2)}$$

$$\text{Cartan\_LEFT}(3, -4, -4) = -\frac{ch\, Z}{B\, (X^2 + Y^2 + Z^2 + S^2)}$$

$$\text{Cartan\_LEFT}(4, -1, -1) = \frac{B\,S}{X^2 + Y^2 + Z^2 + S^2}$$

$$\text{Cartan\_LEFT}(4, -1, -2) = -\frac{B\,ch\,Z}{X^2 + Y^2 + Z^2 + S^2}$$

$$\text{Cartan\_LEFT}(4, -1, -3) = \frac{Y\,B\,ch}{X^2 + Y^2 + Z^2 + S^2}$$

$$\text{Cartan\_LEFT}(4, -1, -4) = -\frac{B\,X}{X^2 + Y^2 + Z^2 + S^2}$$

$$\text{Cartan\_LEFT}(4, -2, -1) = \frac{B\,Z}{X^2 + Y^2 + Z^2 + S^2}$$

$$\text{Cartan\_LEFT}(4, -2, -2) = \frac{B\,ch\,S}{X^2 + Y^2 + Z^2 + S^2}$$

$$\text{Cartan\_LEFT}(4, -2, -3) = -\frac{B\,X}{X^2 + Y^2 + Z^2 + S^2}$$

$$\text{Cartan\_LEFT}(4, -2, -4) = -\frac{Y\,B\,ch}{X^2 + Y^2 + Z^2 + S^2}$$

$$\text{Cartan\_LEFT}(4, -3, -1) = \frac{B\,Y}{X^2 + Y^2 + Z^2 + S^2}$$

$$\text{Cartan\_LEFT}(4, -3, -2) = -\frac{B\,X}{X^2 + Y^2 + Z^2 + S^2}$$

$$\text{Cartan\_LEFT}(4, -3, -3) = -\frac{B\,ch\,S}{X^2 + Y^2 + Z^2 + S^2}$$

$$\text{Cartan\_LEFT}(4, -3, -4) = \frac{B\,ch\,Z}{X^2 + Y^2 + Z^2 + S^2}$$

$$\text{Cartan\_LEFT}(4, -4, -1) = -\frac{X}{X^2 + Y^2 + Z^2 + S^2}$$

$$\text{Cartan\_LEFT}(4, -4, -2) = -\frac{Y}{X^2 + Y^2 + Z^2 + S^2}$$

$$\text{Cartan\_LEFT}(4, -4, -3) = -\frac{Z}{X^2 + Y^2 + Z^2 + S^2}$$

$$\text{Cartan\_LEFT}(4, -4, -4) = -\frac{S}{X^2 + Y^2 + Z^2 + S^2}$$

The components of f the **LEFT CARTAN Connection** appear above. Note that they are not the same as the components of the Right Cartan matrix, moreover they **depend upon the choice of orientation, B, and chirality ch.**

Check for asymmetry (LEFT Torsion) defined as {Cartan_LEFT(a,-b,-c) - Cartan_LEFT(a.-c,-b)} times 1/2.

```
> for j from 1 to \dim do for i from 1 to dim do for k from 1 to dim do  ss := (DD[i,j,k]-DD[i,k,j])/2;
  TTS[i,j,k]:=simplify(ss) od od od  ;
>
> for i from 1 to dim do for j from 1 to dim do for k from 1 to dim do if TTS[i,j,k]=0 then else
  print(`LEFT_Torsion`(i,-k,-j)=simplify(subs(ch^2=1,TTS[i,k,j]))) fi od od od ;
```

$$\text{LEFT\_Torsion}(1, -2, -1) = -\frac{1}{2}\frac{Y(ch-1)}{X^2+Y^2+Z^2+S^2}$$

$$\text{LEFT\_Torsion}(1, -3, -1) = \frac{1}{2}\frac{Z(ch+1)}{X^2+Y^2+Z^2+S^2}$$

$$\text{LEFT\_Torsion}(1, -4, -1) = \frac{1}{2}\frac{(-1+B)S}{B(X^2+Y^2+Z^2+S^2)}$$

$$\text{LEFT\_Torsion}(1, -1, -2) = \frac{1}{2}\frac{Y(ch-1)}{X^2+Y^2+Z^2+S^2}$$

$$\text{LEFT\_Torsion}(1, -3, -2) = 0$$

$$\text{LEFT\_Torsion}(1, -4, -2) = \frac{1}{2}\frac{(B+ch)Z}{B(X^2+Y^2+Z^2+S^2)}$$

$$\text{LEFT\_Torsion}(1, -1, -3) = -\frac{1}{2}\frac{Z(ch+1)}{X^2+Y^2+Z^2+S^2}$$

$$\text{LEFT\_Torsion}(1, -2, -3) = 0$$

$$\text{LEFT\_Torsion}(1, -4, -3) = \frac{1}{2}\frac{(-ch+B)Y}{B(X^2+Y^2+Z^2+S^2)}$$

$$\text{LEFT\_Torsion}(1, -1, -4) = -\frac{1}{2}\frac{(-1+B)S}{B(X^2+Y^2+Z^2+S^2)}$$

$$\text{LEFT\_Torsion}(1, -2, -4) = -\frac{1}{2}\frac{(B+ch)Z}{B(X^2+Y^2+Z^2+S^2)}$$

$$\text{LEFT\_Torsion}(1, -3, -4) = -\frac{1}{2}\frac{(-ch+B)Y}{B(X^2+Y^2+Z^2+S^2)}$$

$$\text{LEFT\_Torsion}(2, -2, -1) = \frac{1}{2}\frac{X(ch-1)}{X^2+Y^2+Z^2+S^2}$$

$$\text{LEFT\_Torsion}(2, -3, -1) = -\frac{1}{2}\frac{S(ch-1)}{X^2+Y^2+Z^2+S^2}$$

$$\text{LEFT\_Torsion}(2, -4, -1) = -\frac{1}{2}\frac{(B+1)Z}{B(X^2+Y^2+Z^2+S^2)}$$

$$\text{LEFT\_Torsion}(2, -1, -2) = -\frac{1}{2}\frac{X(ch-1)}{X^2+Y^2+Z^2+S^2}$$

$$\text{LEFT\_Torsion}(2, -3, -2) = \frac{Z}{X^2+Y^2+Z^2+S^2}$$

$$\text{LEFT\_Torsion}(2, -4, -2) = \frac{1}{2}\frac{(-ch+B)S}{B(X^2+Y^2+Z^2+S^2)}$$

$$\text{LEFT\_Torsion}(2, -1, -3) = \frac{1}{2}\frac{S(ch-1)}{X^2+Y^2+Z^2+S^2}$$

$$\text{LEFT\_Torsion}(2, -2, -3) = -\frac{Z}{X^2+Y^2+Z^2+S^2}$$

$$\text{LEFT\_Torsion}(2, -4, -3) = -\frac{1}{2}\frac{(B\,ch-1)X}{B(X^2+Y^2+Z^2+S^2)}$$

$$\text{LEFT\_Torsion}(2, -1, -4) = \frac{1}{2}\frac{(B+1)Z}{B(X^2+Y^2+Z^2+S^2)}$$

$$\text{LEFT\_Torsion}(2,-2,-4) = -\frac{1}{2}\frac{(-ch+B)\,S}{B\,(X^2+Y^2+Z^2+S^2)}$$

$$\text{LEFT\_Torsion}(2,-3,-4) = \frac{1}{2}\frac{(B\,ch-1)\,X}{B\,(X^2+Y^2+Z^2+S^2)}$$

$$\text{LEFT\_Torsion}(3,-2,-1) = \frac{1}{2}\frac{S\,(ch+1)}{X^2+Y^2+Z^2+S^2}$$

$$\text{LEFT\_Torsion}(3,-3,-1) = -\frac{1}{2}\frac{X\,(ch+1)}{X^2+Y^2+Z^2+S^2}$$

$$\text{LEFT\_Torsion}(3,-4,-1) = -\frac{1}{2}\frac{Y\,(B+1)}{B\,(X^2+Y^2+Z^2+S^2)}$$

$$\text{LEFT\_Torsion}(3,-1,-2) = -\frac{1}{2}\frac{S\,(ch+1)}{X^2+Y^2+Z^2+S^2}$$

$$\text{LEFT\_Torsion}(3,-3,-2) = -\frac{Y}{X^2+Y^2+Z^2+S^2}$$

$$\text{LEFT\_Torsion}(3,-4,-2) = \frac{1}{2}\frac{X\,(B\,ch+1)}{B\,(X^2+Y^2+Z^2+S^2)}$$

$$\text{LEFT\_Torsion}(3,-1,-3) = \frac{1}{2}\frac{X\,(ch+1)}{X^2+Y^2+Z^2+S^2}$$

$$\text{LEFT\_Torsion}(3,-2,-3) = \frac{Y}{X^2+Y^2+Z^2+S^2}$$

$$\text{LEFT\_Torsion}(3,-4,-3) = \frac{1}{2}\frac{S\,(B+ch)}{B\,(X^2+Y^2+Z^2+S^2)}$$

$$\text{LEFT\_Torsion}(3,-1,-4) = \frac{1}{2}\frac{Y\,(B+1)}{B\,(X^2+Y^2+Z^2+S^2)}$$

$$\text{LEFT\_Torsion}(3,-2,-4) = -\frac{1}{2}\frac{X\,(B\,ch+1)}{B\,(X^2+Y^2+Z^2+S^2)}$$

$$\text{LEFT\_Torsion}(3,-3,-4) = -\frac{1}{2}\frac{S\,(B+ch)}{B\,(X^2+Y^2+Z^2+S^2)}$$

$$\text{LEFT\_Torsion}(4,-2,-1) = \frac{1}{2}\frac{B\,Z\,(ch+1)}{X^2+Y^2+Z^2+S^2}$$

$$\text{LEFT\_Torsion}(4,-3,-1) = -\frac{1}{2}\frac{B\,Y\,(ch-1)}{X^2+Y^2+Z^2+S^2}$$

$$\text{LEFT\_Torsion}(4,-4,-1) = \frac{1}{2}\frac{X\,(-1+B)}{X^2+Y^2+Z^2+S^2}$$

$$\text{LEFT\_Torsion}(4,-1,-2) = -\frac{1}{2}\frac{B\,Z\,(ch+1)}{X^2+Y^2+Z^2+S^2}$$

$$\text{LEFT\_Torsion}(4,-3,-2) = 0$$

$$\text{LEFT\_Torsion}(4,-4,-2) = \frac{1}{2}\frac{Y\,(B\,ch-1)}{X^2+Y^2+Z^2+S^2}$$

$$\text{LEFT\_Torsion}(4,-1,-3) = \frac{1}{2}\frac{B\,Y\,(ch-1)}{X^2+Y^2+Z^2+S^2}$$

$$\text{LEFT\_Torsion}(4,-2,-3) = 0$$

$$\text{LEFT\_Torsion}(4, -4, -3) = -\frac{1}{2}\frac{Z(B\,ch+1)}{X^2+Y^2+Z^2+S^2}$$

$$\text{LEFT\_Torsion}(4, -1, -4) = -\frac{1}{2}\frac{X(-1+B)}{X^2+Y^2+Z^2+S^2}$$

$$\text{LEFT\_Torsion}(4, -2, -4) = -\frac{1}{2}\frac{Y(B\,ch-1)}{X^2+Y^2+Z^2+S^2}$$

$$\text{LEFT\_Torsion}(4, -3, -4) = \frac{1}{2}\frac{Z(B\,ch+1)}{X^2+Y^2+Z^2+S^2}$$

The Right and the Left Cartan matrices are negative similarity transforms, **but the LEFT AND RIGHT torsion terms appear to be diffferent, and the left "torsion" depends upon the both the chirality factor, ch, and the orientation, B !!!!!.**

****

Next the Christoffel symbols will be computed for the **subsumed pullback metric** on the initial state. The pullback mnetric is conformal to the indentity matrix.

Christoffel Connection coefficients from the induced metric

It is assumed that the "metric" is the pull back metric given below, which is conformal.

```
> metric:=evalm( pullbackmetric);
```

$$metric := \begin{bmatrix} X^2+Y^2+Z^2+S^2 & 0 & 0 & 0 \\ 0 & X^2+Y^2+Z^2+S^2 & 0 & 0 \\ 0 & 0 & X^2+Y^2+Z^2+S^2 & 0 \\ 0 & 0 & 0 & X^2+Y^2+Z^2+S^2 \end{bmatrix}$$

```
> metricinverse:=inverse(metric):
> for i from 1 to dim do for j from 1 to dim do for k from 1 to dim do d1gun[i,j,k] := (diff(metric[i,j],coord[k])) od od od:
> #for i from 1 to dim do for j from 1 to dim do for k from 1 to dim do if d1gun[i,j,k]=0 then  else print(`dgun`(i,j,k)=d1gun[i,j,k]) fi od od od;
> for i from 1 to dim do for j from i to dim do for k from 1 to dim do C1S[i,j,k] := 0  od od od; for i from 1 to dim do for j from 1 to dim do for k from 1 to dim do C1S[i,j,k] := 1/2*d1gun[i,k,j]+1/2*d1gun[j,k,i]-1/2*d1gun[i,j,k]  od od od;
>  for k from 1 to dim do for i from 1 to dim do for j from 1 to dim do ss := 0; for m to dim do ss := ss+metricinverse[k,m]*C1S[i,j,m] od; C2S[k,i,j] := simplify(factor(ss),trig) od od od;
> for i from 1 to dim do for j from 1 to dim do for k from 1 to dim do if C2S[i,j,k]=0 then  else print(`Christoffel_Gamma2`(i,-j,-k)=C2S[i,j,k]) fi od od od;
```

The non zero Christoffel Connection coefficients 2nd kind on the initial space (domain)

Gamma2(i,j,k) index (1,-1,-1)

$$\text{Christoffel\_Gamma2}(1, -1, -1) = \frac{X}{X^2+Y^2+Z^2+S^2}$$

$$\text{Christoffel\_Gamma2}(1, -1, -2) = \frac{Y}{X^2+Y^2+Z^2+S^2}$$

$$\text{Christoffel\_Gamma2}(1, -1, -3) = \frac{Z}{X^2 + Y^2 + Z^2 + S^2}$$

$$\text{Christoffel\_Gamma2}(1, -1, -4) = \frac{S}{X^2 + Y^2 + Z^2 + S^2}$$

$$\text{Christoffel\_Gamma2}(1, -2, -1) = \frac{Y}{X^2 + Y^2 + Z^2 + S^2}$$

$$\text{Christoffel\_Gamma2}(1, -2, -2) = -\frac{X}{X^2 + Y^2 + Z^2 + S^2}$$

$$\text{Christoffel\_Gamma2}(1, -3, -1) = \frac{Z}{X^2 + Y^2 + Z^2 + S^2}$$

$$\text{Christoffel\_Gamma2}(1, -3, -3) = -\frac{X}{X^2 + Y^2 + Z^2 + S^2}$$

$$\text{Christoffel\_Gamma2}(1, -4, -1) = \frac{S}{X^2 + Y^2 + Z^2 + S^2}$$

$$\text{Christoffel\_Gamma2}(1, -4, -4) = -\frac{X}{X^2 + Y^2 + Z^2 + S^2}$$

$$\text{Christoffel\_Gamma2}(2, -1, -1) = -\frac{Y}{X^2 + Y^2 + Z^2 + S^2}$$

$$\text{Christoffel\_Gamma2}(2, -1, -2) = \frac{X}{X^2 + Y^2 + Z^2 + S^2}$$

$$\text{Christoffel\_Gamma2}(2, -2, -1) = \frac{X}{X^2 + Y^2 + Z^2 + S^2}$$

$$\text{Christoffel\_Gamma2}(2, -2, -2) = \frac{Y}{X^2 + Y^2 + Z^2 + S^2}$$

$$\text{Christoffel\_Gamma2}(2, -2, -3) = \frac{Z}{X^2 + Y^2 + Z^2 + S^2}$$

$$\text{Christoffel\_Gamma2}(2, -2, -4) = \frac{S}{X^2 + Y^2 + Z^2 + S^2}$$

$$\text{Christoffel\_Gamma2}(2, -3, -2) = \frac{Z}{X^2 + Y^2 + Z^2 + S^2}$$

$$\text{Christoffel\_Gamma2}(2, -3, -3) = -\frac{Y}{X^2 + Y^2 + Z^2 + S^2}$$

$$\text{Christoffel\_Gamma2}(2, -4, -2) = \frac{S}{X^2 + Y^2 + Z^2 + S^2}$$

$$\text{Christoffel\_Gamma2}(2, -4, -4) = -\frac{Y}{X^2 + Y^2 + Z^2 + S^2}$$

$$\text{Christoffel\_Gamma2}(3, -1, -1) = -\frac{Z}{X^2 + Y^2 + Z^2 + S^2}$$

$$\text{Christoffel\_Gamma2}(3, -1, -3) = \frac{X}{X^2 + Y^2 + Z^2 + S^2}$$

$$\text{Christoffel\_Gamma2}(3, -2, -2) = -\frac{Z}{X^2 + Y^2 + Z^2 + S^2}$$

$$\text{Christoffel\_Gamma2}(3, -2, -3) = \frac{Y}{X^2 + Y^2 + Z^2 + S^2}$$

$$\text{Christoffel\_Gamma2}(3, -3, -1) = \frac{X}{X^2 + Y^2 + Z^2 + S^2}$$

$$\text{Christoffel\_Gamma2}(3, -3, -2) = \frac{Y}{X^2 + Y^2 + Z^2 + S^2}$$

$$\text{Christoffel\_Gamma2}(3, -3, -3) = \frac{Z}{X^2 + Y^2 + Z^2 + S^2}$$

$$\text{Christoffel\_Gamma2}(3, -3, -4) = \frac{S}{X^2 + Y^2 + Z^2 + S^2}$$

$$\text{Christoffel\_Gamma2}(3, -4, -3) = \frac{S}{X^2 + Y^2 + Z^2 + S^2}$$

$$\text{Christoffel\_Gamma2}(3, -4, -4) = -\frac{Z}{X^2 + Y^2 + Z^2 + S^2}$$

$$\text{Christoffel\_Gamma2}(4, -1, -1) = -\frac{S}{X^2 + Y^2 + Z^2 + S^2}$$

$$\text{Christoffel\_Gamma2}(4, -1, -4) = \frac{X}{X^2 + Y^2 + Z^2 + S^2}$$

$$\text{Christoffel\_Gamma2}(4, -2, -2) = -\frac{S}{X^2 + Y^2 + Z^2 + S^2}$$

$$\text{Christoffel\_Gamma2}(4, -2, -4) = \frac{Y}{X^2 + Y^2 + Z^2 + S^2}$$

$$\text{Christoffel\_Gamma2}(4, -3, -3) = -\frac{S}{X^2 + Y^2 + Z^2 + S^2}$$

$$\text{Christoffel\_Gamma2}(4, -3, -4) = \frac{Z}{X^2 + Y^2 + Z^2 + S^2}$$

$$\text{Christoffel\_Gamma2}(4, -4, -1) = \frac{X}{X^2 + Y^2 + Z^2 + S^2}$$

$$\text{Christoffel\_Gamma2}(4, -4, -2) = \frac{Y}{X^2 + Y^2 + Z^2 + S^2}$$

$$\text{Christoffel\_Gamma2}(4, -4, -3) = \frac{Z}{X^2 + Y^2 + Z^2 + S^2}$$

$$\text{Christoffel\_Gamma2}(4, -4, -4) = \frac{S}{X^2 + Y^2 + Z^2 + S^2}$$

If no entries appear above the Christoffel symbols on the domain space vanish

Note that the Christoffel Symbols for the Conformal metric are not zero, and are not the same as the Right or Left Cartan Connection matrices. More over, the **Christoffel symbols built upon the metric defined above are independent from the choice of chirality and orientation.** The metric is symmetric and has a positive definite determinant.

The Right Cartan matrix is often defined as the sum of Christoffel Symbols and Rotation coefficients, T(i,j,k)

CartanRight(ijk) := ChristoffelGamma(ijk) + T(ijk)

Using this *definition for the Rotation coefficients,* Compute the T(i,j,k):

```
> for i from 1 to dim do for j from 1 to dim do for k from 1 to dim do ss:=0; ss := (CC[i,j,k]-C2S[i,j,k]);
  SHIPTR[i,j,k]:=simplify(ss)  od od od   ;
>
>
 > for i from 1 to dim do for j from 1 to dim do for k from 1 to dim do if C2S[i,j,k]=0 and CC[i,j,k]=0 then
          else print(`T`(i,-j,-k)=simplify(subs(ch^2=1,SHIPTR[i,j,k]))) fi od od od ;
```

T(ijk) index (1,-1,-1)

$$T(1,-1,-1)=0$$

$$T(1,-1,-2)=0$$

$$T(1,-1,-3)=0$$

$$T(1,-1,-4)=0$$

$$T(1,-2,-1)=-2\frac{Y}{X^2+Y^2+Z^2+S^2}$$

$$T(1,-2,-2)=2\frac{X}{X^2+Y^2+Z^2+S^2}$$

$$T(1,-2,-3)=-\frac{ch\,S}{X^2+Y^2+Z^2+S^2}$$

$$T(1,-2,-4)=\frac{ch\,Z}{X^2+Y^2+Z^2+S^2}$$

$$T(1,-3,-1)=0$$

$$T(1,-3,-2)=-\frac{ch\,S}{X^2+Y^2+Z^2+S^2}$$

$$T(1,-3,-3)=0$$

$$T(1,-3,-4)=\frac{Y\,ch}{X^2+Y^2+Z^2+S^2}$$

$$T(1,-4,-1)=0$$

$$T(1,-4,-2)=\frac{ch\,Z}{X^2+Y^2+Z^2+S^2}$$

$$T(1,-4,-3)=-\frac{Y\,ch}{X^2+Y^2+Z^2+S^2}$$

$$T(1,-4,-4)=0$$

$$T(2,-1,-1)=2\frac{Y}{X^2+Y^2+Z^2+S^2}$$

$$T(2,-1,-2)=-2\frac{X}{X^2+Y^2+Z^2+S^2}$$

$$T(2,-1,-3)=\frac{ch\,S}{X^2+Y^2+Z^2+S^2}$$

$$T(2,-1,-4)=-\frac{ch\,Z}{X^2+Y^2+Z^2+S^2}$$

$$T(2, -2, -1) = 0$$
$$T(2, -2, -2) = 0$$
$$T(2, -2, -3) = 0$$
$$T(2, -2, -4) = 0$$
$$T(2, -3, -1) = \frac{ch\, S}{X^2 + Y^2 + Z^2 + S^2}$$
$$T(2, -3, -2) = 0$$
$$T(2, -3, -3) = 0$$
$$T(2, -3, -4) = -\frac{ch\, X}{X^2 + Y^2 + Z^2 + S^2}$$
$$T(2, -4, -1) = -\frac{ch\, Z}{X^2 + Y^2 + Z^2 + S^2}$$
$$T(2, -4, -2) = 0$$
$$T(2, -4, -3) = \frac{ch\, X}{X^2 + Y^2 + Z^2 + S^2}$$
$$T(2, -4, -4) = 0$$
$$T(3, -1, -1) = 0$$
$$T(3, -1, -2) = \frac{ch\, S}{X^2 + Y^2 + Z^2 + S^2}$$
$$T(3, -1, -3) = 0$$
$$T(3, -1, -4) = -\frac{Y\, ch}{X^2 + Y^2 + Z^2 + S^2}$$
$$T(3, -2, -1) = -\frac{ch\, S}{X^2 + Y^2 + Z^2 + S^2}$$
$$T(3, -2, -2) = 0$$
$$T(3, -2, -3) = 0$$
$$T(3, -2, -4) = \frac{ch\, X}{X^2 + Y^2 + Z^2 + S^2}$$
$$T(3, -3, -1) = 0$$
$$T(3, -3, -2) = 0$$
$$T(3, -3, -3) = 0$$
$$T(3, -3, -4) = 0$$
$$T(3, -4, -1) = -\frac{Y\, ch}{X^2 + Y^2 + Z^2 + S^2}$$
$$T(3, -4, -2) = \frac{ch\, X}{X^2 + Y^2 + Z^2 + S^2}$$
$$T(3, -4, -3) = -2 \frac{S}{X^2 + Y^2 + Z^2 + S^2}$$
$$T(3, -4, -4) = 2 \frac{Z}{X^2 + Y^2 + Z^2 + S^2}$$
$$T(4, -1, -1) = 0$$
$$T(4, -1, -2) = -\frac{ch\, Z}{X^2 + Y^2 + Z^2 + S^2}$$

$$T(4, -1, -3) = \frac{Y\,ch}{X^2 + Y^2 + Z^2 + S^2}$$

$$T(4, -1, -4) = 0$$

$$T(4, -2, -1) = \frac{ch\,Z}{X^2 + Y^2 + Z^2 + S^2}$$

$$T(4, -2, -2) = 0$$

$$T(4, -2, -3) = -\frac{ch\,X}{X^2 + Y^2 + Z^2 + S^2}$$

$$T(4, -2, -4) = 0$$

$$T(4, -3, -1) = \frac{Y\,ch}{X^2 + Y^2 + Z^2 + S^2}$$

$$T(4, -3, -2) = -\frac{ch\,X}{X^2 + Y^2 + Z^2 + S^2}$$

$$T(4, -3, -3) = 2\frac{S}{X^2 + Y^2 + Z^2 + S^2}$$

$$T(4, -3, -4) = -2\frac{Z}{X^2 + Y^2 + Z^2 + S^2}$$

$$T(4, -4, -1) = 0$$

$$T(4, -4, -2) = 0$$

$$T(4, -4, -3) = 0$$

$$T(4, -4, -4) = 0$$

The Rotation matrices also depend upon the chirality factor.

NOW RESTART FOR THE INSTANTON CASE

The objective will be to build another orthogonal frame where three of the induced 1-forms are not integable, and only the fourth is a perfect differential. The FRAME so constructed will have the SAME (pullback) metric !!!

```
> restart:
>
with(linalg):with(liesymm):with(difforms):with(plots):defform(X=0,Y=0,Z=0,S=0,ch=const,B=const,u=0,v=0,w=0,s=0);
Warning, new definition for norm
Warning, new definition for trace
Warning, new definition for close
Warning, new definition for `&^`
Warning, new definition for d
Warning, new definition for mixpar
Warning, new definition for wdegree
```

The idea is to define the normal field as a gradient field of the function (X^2+Y^2+Z^2+S^2)/2, and then to find three other vector fields that annihilate the gradient field. A particulat choice is the "Instanton Map" for which each of the 3 fields is not integrable.

The choice produces a global frame field matrix with components proportional to :

```
> HE3:=[-Y,+X,S,-Z];HE2:=[-ch*Z,-S,X,ch*Y];HE1:=-[ch*S,-Z,ch*Y,-X];HN1:=evalm(B*[X,Y,ch*Z,ch*S]);Mink:=array([[1,0,0,0],[0,1,0,0],[0,0,1,0],[0,0,0,-1]]);
>
>
```

$$HE3 := [-Y, X, S, -Z]$$

$$HE2 := [-ch\, Z, -S, X, ch\, Y]$$

$$HE1 := [-ch\, S, Z, -ch\, Y, X]$$

$$HN1 := [B\, X, B\, Y, B\, ch\, Z, B\, ch\, S]$$

$$Mink := \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

```
> FF:=array([[HE1[1],HE1[2],HE1[3],HE1[4]],[HE2[1],HE2[2],HE2[3],HE2[4]],[HE3[1],HE3[2],HE3[3],HE3[4]],[HN1[1],HN1[2],HN1[3],HN1[4]]]);DET:=factor(subs(ch^2=1,det(FF)));GG:=simplify(subs(ch^2=1,inverse(FF))):
```

$$FF := \begin{bmatrix} -ch\, S & Z & -ch\, Y & X \\ -ch\, Z & -S & X & ch\, Y \\ -Y & X & S & -Z \\ B\, X & B\, Y & B\, ch\, Z & B\, ch\, S \end{bmatrix}$$

$$DET := B\,(S^2 + Z^2 + Y^2 + X^2)^2$$

The Instanton Frame is orthogonal -- but not orthonormal **The Pullback metric is the same as for the Hopf map above.**

```
> pullbackmetric:=subs(ch^2=1,B^2=1,innerprod(transpose(FF),FF));
```

$$pullbackmetric := \begin{bmatrix} S^2+Z^2+Y^2+X^2 & 0 & 0 & 0 \\ 0 & S^2+Z^2+Y^2+X^2 & 0 & 0 \\ 0 & 0 & S^2+Z^2+Y^2+X^2 & 0 \\ 0 & 0 & 0 & S^2+Z^2+Y^2+X^2 \end{bmatrix}$$

```
> CHPOLYP:=simplify(factor(subs(ch^2=1,B^2=1,charpoly(FF,lambda))));CHPOLYM:=simplify(factor(subs(ch^2=1,B^2=1,charpoly(FF,lambda))));simplify(CHPOLYP-CHPOLYM);
```

FRAME MATRIX FF, INVERSE FRAME GG, ch = chirality factor, B=orientation

\>

\>

$$CHPOLYP := \lambda^4 - \lambda^3 B\, ch\, S + \lambda^2 Z^2 B\, ch - B\, Y^2 ch\, \lambda^2 - S^2 \lambda^2 + 2\, S^2 B\, Y^2 + 2\, Z^2 B\, Y^2 - S^2 \lambda^2 B + 2\, S^2 Z^2 B - S\, ch\, Y^2 \lambda - ch\, Y^2 \lambda^2 + \lambda\, B\, Y^2 ch\, S + \lambda\, S\, Z^2 B\, ch + \lambda\, X^2 B\, ch\, S - ch\, S\, X^2 \lambda + \lambda\, S^3 B\, ch - X^2 \lambda^2 + B\, X^4 - ch\, Z^2 S\, \lambda + S^4 B + B\, Y^4 + Z^4 B + 2\, B\, X^2 S^2 - B\, X^2 \lambda^2 + 2\, B\, X^2 Z^2 + 2\, B\, Y^2 X^2 + ch\, Z^2 \lambda^2 - ch\, S^3 \lambda + ch\, S\, \lambda^3$$

$$CHPOLYM := \lambda^4 - \lambda^3 B\, ch\, S + \lambda^2 Z^2 B\, ch - B\, Y^2 ch\, \lambda^2 - S^2 \lambda^2 + 2\, S^2 B\, Y^2 + 2\, Z^2 B\, Y^2 - S^2 \lambda^2 B + 2\, S^2 Z^2 B - S\, ch\, Y^2 \lambda - ch\, Y^2 \lambda^2 + \lambda\, B\, Y^2 ch\, S + \lambda\, S\, Z^2 B\, ch + \lambda\, X^2 B\, ch\, S - ch\, S\, X^2 \lambda + \lambda\, S^3 B\, ch - X^2 \lambda^2 + B\, X^4 - ch\, Z^2 S\, \lambda + S^4 B + B\, Y^4 + Z^4 B + 2\, B\, X^2 S^2 - B\, X^2 \lambda^2 + 2\, B\, X^2 Z^2 + 2\, B\, Y^2 X^2 + ch\, Z^2 \lambda^2 - ch\, S^3 \lambda + ch\, S\, \lambda^3$$

$$0$$

The characteristic polynomial depends upon both the chirality factor, ch, and the sign of the determinant B (orientation).

For B=plus one the Frame has a negative determinant (the discrete reflective cases and global abnormalities or catastrophies).

For B =minus one, the Frame has a positive determinant which implies perturbations about the identity are possible (local abnormalities) .

The orientation is independent from the chirality

Under the interchange S<=>Z, the B=plus 1 CHPOLYP of the Hopf map and the Instanton map are the same.

Under the interchange S<=>Y and X<=> Z the B=-1 CHPOLYL of the Hopf map and the Instanton map are the same.

The next equation checks to see that the specified frame produces the desired differential structures:
[FF]|dR>=|sigma>

```
> sigma:=evalm(innerprod(FF,[d(X),d(Y),d(Z),d(S)])):
> sigma1:=sigma[1];e1;
> sigma2:=sigma[2];e2;
> sigma3:=sigma[3];e3;omega:=sigma[4];
```

$$\sigma 1 := -ch\, S\, \mathrm{d}(X) + Z\, \mathrm{d}(Y) - ch\, Y\, \mathrm{d}(Z) + X\, \mathrm{d}(S)$$

$$e1$$

$$\sigma 2 := -ch\, Z\, \mathrm{d}(X) - S\, \mathrm{d}(Y) + X\, \mathrm{d}(Z) + ch\, Y\, \mathrm{d}(S)$$

$$e2$$

$$\sigma 3 := -Y\, \mathrm{d}(X) + X\, \mathrm{d}(Y) + S\, \mathrm{d}(Z) - Z\, \mathrm{d}(S)$$

$$e3$$

$$\omega := B\, X\, \mathrm{d}(X) + B\, Y\, \mathrm{d}(Y) + B\, ch\, Z\, \mathrm{d}(Z) + B\, ch\, S\, \mathrm{d}(S)$$

The vector of induced 1-forms [FF]|dR>=|sigma> can written as the column vector of components [sigma1,sigma2,sigma3,omega] .

The induced 1-forms created by the Frame acting on the differentials on the domain are tested for integrablility. In the Instanton Map, the first three induced 1-forms are not integrable. The fourth component of the Frame Field is exact.

```
> `vorticity sigma1`:=d(sigma[1]);`vorticity sigma2`:=d(sigma[2]);`vorticity sigma3`:=d(sigma[3]);`omega`:=omega;`vortic of  omega`:=factor(d(omega));`topological torsion of sigma1`:=factor(subs(B^2=1,sigma[1]&^d(sigma[1])));`topological parity of sigma1`:=subs(B^2=1,d(sigma[1])&^d(sigma[1]));`topological torsion of sigma2`:=factor(subs(B^2=1,sigma[2]&^d(sigma[2])));`topological parity of sigma2`:=subs(B^2=1,d(sigma[2])&^d(sigma[2]));`topological torsion of sigma3`:=factor(subs(B^2=1,sigma[3]&^d(sigma[3])));`topological parity of sigma3`:=subs(B^2=1,d(sigma[3])&^d(sigma[3]));
```

$$\textit{vorticity sigma1} := (-ch - 1)\, (\mathrm{d}(S)\ \&\hat{}\ \mathrm{d}(X)) + (1 + ch)\, (\mathrm{d}(Z)\ \&\hat{}\ \mathrm{d}(Y))$$

$$\textit{vorticity sigma2} := (-ch - 1)\, (\mathrm{d}(Z)\ \&\hat{}\ \mathrm{d}(X)) + (-ch - 1)\, (\mathrm{d}(S)\ \&\hat{}\ \mathrm{d}(Y))$$

$$\textit{vorticity sigma3} := 2\, (\mathrm{d}(S)\ \&\hat{}\ \mathrm{d}(Z)) - 2\, (\mathrm{d}(Y)\ \&\hat{}\ \mathrm{d}(X))$$

$$\omega := B\, X\, \mathrm{d}(X) + B\, Y\, \mathrm{d}(Y) + B\, ch\, Z\, \mathrm{d}(Z) + B\, ch\, S\, \mathrm{d}(S)$$

$$\textit{vortic of omega} := 0$$

$$\textit{topological torsion of sigma1} := (1 + ch)\, (-ch\, S\ \&\hat{}(\mathrm{d}(X), \mathrm{d}(Z), \mathrm{d}(Y)) + ch\, Y\ \&\hat{}(\mathrm{d}(Z), \mathrm{d}(S), \mathrm{d}(X)) - Z\ \&\hat{}(\mathrm{d}(Y), \mathrm{d}(S), \mathrm{d}(X)) + X\ \&\hat{}(\mathrm{d}(S), \mathrm{d}(Z), \mathrm{d}(Y)))$$

$$\textit{topological parity of sigma1} := -2\, (1 + ch)^2\ \&\hat{}(\mathrm{d}(S), \mathrm{d}(X), \mathrm{d}(Z), \mathrm{d}(Y))$$

$$\textit{topological torsion of sigma2} := -(1 + ch)\, (ch\, Y\ \&\hat{}(\mathrm{d}(S), \mathrm{d}(Z), \mathrm{d}(X)) - ch\, Z\ \&\hat{}(\mathrm{d}(X), \mathrm{d}(S), \mathrm{d}(Y)) - S\ \&\hat{}(\mathrm{d}(Y), \mathrm{d}(Z), \mathrm{d}(X)) + X\ \&\hat{}(\mathrm{d}(Z), \mathrm{d}(S), \mathrm{d}(Y)))$$

$$\textit{topological parity of sigma2} := 2\, (1 + ch)^2\ \&\hat{}(\mathrm{d}(Z), \mathrm{d}(X), \mathrm{d}(S), \mathrm{d}(Y))$$

$$\textit{topological torsion of sigma3} := -2\, Y\ \&\hat{}(\mathrm{d}(X), \mathrm{d}(S), \mathrm{d}(Z)) + 2\, X\ \&\hat{}(\mathrm{d}(Y), \mathrm{d}(S), \mathrm{d}(Z)) - 2\, S\ \&\hat{}(\mathrm{d}(Z), \mathrm{d}(Y), \mathrm{d}(X)) + 2\, Z\ \&\hat{}(\mathrm{d}(S), \mathrm{d}(Y), \mathrm{d}(X))$$

$$\textit{topological parity of sigma3} := -8\ \&\hat{}(\mathrm{d}(S), \mathrm{d}(Z), \mathrm{d}(Y), \mathrm{d}(X))$$

Itis remarkable that the Topological Torsion and the Topological parity of the non - integrable induced 1-forms, does not depend upon the orientation, B, but does depend upon the chirality, ch.

IT is also remarkable that the Cholce of ch = minus 1 reduces the complexity of the system to where the Topological Torsion and the Topological parity of of the sigma1 and sigma2 are zero!!! One form of polarization behaves differently from the other form of polarization. !!!!

This result implies that these forms are integrable to within an integrating factor (for only 1 choice of chirality - polarization)

IS this some how connected to left handed neutrinos ???

A possible candidate for a metric on X,Y,Z,S would be the symmetric form (which is conformal to the Euclidean metric on R4.)

```
> pullbackmetric:=subs(ch^2=1,B^2=1,innerprod(transpose(FF),FF));
```

$$pullbackmetric := \begin{bmatrix} S^2+Z^2+Y^2+X^2 & 0 & 0 & 0 \\ 0 & S^2+Z^2+Y^2+X^2 & 0 & 0 \\ 0 & 0 & S^2+Z^2+Y^2+X^2 & 0 \\ 0 & 0 & 0 & S^2+Z^2+Y^2+X^2 \end{bmatrix}$$

Note that the Frame matrix FF is orthogonal but not orthonormal. It is also conformal in that the normalization is the same for all basis elements. It is independent of the orientation, B and the chirality ch. !!!! The Pullbackmetric is the same for the Instanton map and the Hopf map !!!

This shows that the frame matrix and the connection carry much more information than does the metric.

Now Compute the Right Cartan Matrix [CR] and the Left Cartan Matrix. These matrices are negative similarity transforms to within a sign.

```
> cartan:=(subs(ch^2=1,innerprod(GG,d(FF)))):cartanL:=evalm((subs(ch^2=1,innerprod(-d(FF),GG)))):
> CLcong:=(subs(ch^2=1,innerprod(-FF,cartan,GG))):zz:=evalm(FF);inverse(zz):
```

$$zz := \begin{bmatrix} -ch\,S & Z & -ch\,Y & X \\ -ch\,Z & -S & X & ch\,Y \\ -Y & X & S & -Z \\ B\,X & B\,Y & B\,ch\,Z & B\,ch\,S \end{bmatrix}$$

```
> `Should be zero if similarity equivalents`:=wcollect(factor(CLcong[1,2]-cartanL[1,2]));
```

$$Should\ be\ zero\ if\ similarity\ equivalents := \frac{ch\,Y^2\,(ch-1)\,(1+ch)\,X\,\mathrm{d}(Y)}{(S^2+Z^2+Y^2+X^2)^2} + \frac{ch\,Y\,(ch-1)\,(1+ch)\,X\,Z\,\mathrm{d}(Z)}{(S^2+Z^2+Y^2+X^2)^2} + \frac{ch\,Y\,(ch-1)\,(1+ch)\,X\,S\,\mathrm{d}(S)}{(S^2+Z^2+Y^2+X^2)^2} + \frac{ch\,Y\,(ch-1)\,(1+ch)\,(-S^2-Z^2-Y^2)\,\mathrm{d}(X)}{(S^2+Z^2+Y^2+X^2)^2}$$

\>

The matrix elements of the Right Cartan connection matrix using the matrix methods:
Note that the Right Cartan connection does not depend upon B, but does depend upon ch.
These connection matrices are **NOT** the same as the connection matrices for the Hopf map.

\>

```
> Gamma11:=wcollect(cartan[1,1]);Gamma12:=wcollect(cartan[1,2]);Gamma13:=wcollect(cartan[1,3]);Gamma14:=wcollect(cartan[1,4]);
```

$$\Gamma 11 := \frac{Y\,\mathrm{d}(Y)}{S^2+Z^2+Y^2+X^2} + \frac{Z\,\mathrm{d}(Z)}{S^2+Z^2+Y^2+X^2} + \frac{S\,\mathrm{d}(S)}{S^2+Z^2+Y^2+X^2} + \frac{X\,\mathrm{d}(X)}{S^2+Z^2+Y^2+X^2}$$

$$\Gamma 12 := \frac{X\,\mathrm{d}(Y)}{S^2+Z^2+Y^2+X^2} - \frac{ch\,S\,\mathrm{d}(Z)}{S^2+Z^2+Y^2+X^2} + \frac{ch\,Z\,\mathrm{d}(S)}{S^2+Z^2+Y^2+X^2} - \frac{\mathrm{d}(X)\,Y}{S^2+Z^2+Y^2+X^2}$$

$$\Gamma 13 := \frac{S\,\mathrm{d}(Y)}{S^2+Z^2+Y^2+X^2} + \frac{X\,ch\,\mathrm{d}(Z)}{S^2+Z^2+Y^2+X^2} - \frac{Y\,\mathrm{d}(S)}{S^2+Z^2+Y^2+X^2} - \frac{ch\,Z\,\mathrm{d}(X)}{S^2+Z^2+Y^2+X^2}$$

$$\Gamma 14 := -\frac{Z\,\mathrm{d}(Y)}{S^2+Z^2+Y^2+X^2} + \frac{Y\,\mathrm{d}(Z)}{S^2+Z^2+Y^2+X^2} + \frac{X\,ch\,\mathrm{d}(S)}{S^2+Z^2+Y^2+X^2} - \frac{ch\,S\,\mathrm{d}(X)}{S^2+Z^2+Y^2+X^2}$$

****

The Left Cartan matix elements can also be computed by the matrix method (for example)

```
> Gamma11L:=wcollect(cartanL[1,1]);Gamma12L:=wcollect(cartanL[1,2]);Gamma13L:=wcollect(cartanL[1,3]);Gamma14L:=wcollect(cartanL[1,4]);
```

$$Gamma11L := -\frac{Y\,d(Y)}{S^2+Z^2+Y^2+X^2} - \frac{X\,d(X)}{S^2+Z^2+Y^2+X^2} - \frac{S\,d(S)}{S^2+Z^2+Y^2+X^2} - \frac{Z\,d(Z)}{S^2+Z^2+Y^2+X^2}$$

$$Gamma12L := \frac{X\,ch\,d(Y)}{S^2+Z^2+Y^2+X^2} + \frac{S\,d(Z)}{S^2+Z^2+Y^2+X^2} - \frac{Z\,d(S)}{S^2+Z^2+Y^2+X^2} - \frac{ch\,Y\,d(X)}{S^2+Z^2+Y^2+X^2}$$

$$Gamma13L := \frac{S\,ch\,d(Y)}{S^2+Z^2+Y^2+X^2} - \frac{X\,d(Z)}{S^2+Z^2+Y^2+X^2} - \frac{Y\,ch\,d(S)}{S^2+Z^2+Y^2+X^2} + \frac{Z\,d(X)}{S^2+Z^2+Y^2+X^2}$$

$$Gamma14L := \frac{Z\,d(Y)}{(S^2+Z^2+Y^2+X^2)\,B} - \frac{Y\,d(Z)}{(S^2+Z^2+Y^2+X^2)\,B} + \frac{X\,ch\,d(S)}{(S^2+Z^2+Y^2+X^2)\,B} - \frac{ch\,S\,d(X)}{(S^2+Z^2+Y^2+X^2)\,B}$$

Note that the Left Cartan connection is sensitive to both B and ch.
****
****

> Gamma21:=wcollect(cartan[2,1]);Gamma22:=wcollect(cartan[2,2]);Gamma23:=wcollect(cartan[2,3]);Gamma24:=wcollect(cartan[2,4]);

$$\Gamma21 := -\frac{X\,d(Y)}{S^2+Z^2+Y^2+X^2} + \frac{ch\,S\,d(Z)}{S^2+Z^2+Y^2+X^2} - \frac{ch\,Z\,d(S)}{S^2+Z^2+Y^2+X^2} + \frac{d(X)\,Y}{S^2+Z^2+Y^2+X^2}$$

$$\Gamma22 := \frac{Y\,d(Y)}{S^2+Z^2+Y^2+X^2} + \frac{Z\,d(Z)}{S^2+Z^2+Y^2+X^2} + \frac{S\,d(S)}{S^2+Z^2+Y^2+X^2} + \frac{X\,d(X)}{S^2+Z^2+Y^2+X^2}$$

$$\Gamma23 := -\frac{Z\,ch\,d(Y)}{S^2+Z^2+Y^2+X^2} + \frac{Y\,ch\,d(Z)}{S^2+Z^2+Y^2+X^2} + \frac{X\,d(S)}{S^2+Z^2+Y^2+X^2} - \frac{S\,d(X)}{S^2+Z^2+Y^2+X^2}$$

$$\Gamma24 := -\frac{S\,ch\,d(Y)}{S^2+Z^2+Y^2+X^2} - \frac{X\,d(Z)}{S^2+Z^2+Y^2+X^2} + \frac{Y\,ch\,d(S)}{S^2+Z^2+Y^2+X^2} + \frac{Z\,d(X)}{S^2+Z^2+Y^2+X^2}$$

> Gamma31:=wcollect(cartan[3,1]);Gamma32:=wcollect(cartan[3,2]);Gamma33:=wcollect(cartan[3,3]);Gamma34:=wcollect(cartan[3,4]);

$$\Gamma31 := -\frac{S\,d(Y)}{S^2+Z^2+Y^2+X^2} - \frac{X\,ch\,d(Z)}{S^2+Z^2+Y^2+X^2} + \frac{Y\,d(S)}{S^2+Z^2+Y^2+X^2} + \frac{ch\,Z\,d(X)}{S^2+Z^2+Y^2+X^2}$$

$$\Gamma32 := \frac{Z\,ch\,d(Y)}{S^2+Z^2+Y^2+X^2} - \frac{Y\,ch\,d(Z)}{S^2+Z^2+Y^2+X^2} - \frac{X\,d(S)}{S^2+Z^2+Y^2+X^2} + \frac{S\,d(X)}{S^2+Z^2+Y^2+X^2}$$

$$\Gamma33 := \frac{Y\,d(Y)}{S^2+Z^2+Y^2+X^2} + \frac{Z\,d(Z)}{S^2+Z^2+Y^2+X^2} + \frac{S\,d(S)}{S^2+Z^2+Y^2+X^2} + \frac{X\,d(X)}{S^2+Z^2+Y^2+X^2}$$

$$\Gamma34 := \frac{X\,ch\,d(Y)}{S^2+Z^2+Y^2+X^2} - \frac{S\,d(Z)}{S^2+Z^2+Y^2+X^2} + \frac{Z\,d(S)}{S^2+Z^2+Y^2+X^2} - \frac{ch\,Y\,d(X)}{S^2+Z^2+Y^2+X^2}$$

> Gamma41:=(wcollect(cartan[4,1]));Gamma42:=wcollect(cartan[4,2]);Gamma43:=wcollect(cartan[4,3]);Gamma44:=wcollect(cartan[4,4]);

$$\Gamma41 := \frac{Z\,d(Y)}{S^2+Z^2+Y^2+X^2} - \frac{Y\,d(Z)}{S^2+Z^2+Y^2+X^2} - \frac{X\,ch\,d(S)}{S^2+Z^2+Y^2+X^2} + \frac{ch\,S\,d(X)}{S^2+Z^2+Y^2+X^2}$$

$$\Gamma42 := \frac{S\,ch\,d(Y)}{S^2+Z^2+Y^2+X^2} + \frac{X\,d(Z)}{S^2+Z^2+Y^2+X^2} - \frac{Y\,ch\,d(S)}{S^2+Z^2+Y^2+X^2} - \frac{Z\,d(X)}{S^2+Z^2+Y^2+X^2}$$

$$\Gamma43 := -\frac{X\,ch\,d(Y)}{S^2+Z^2+Y^2+X^2} + \frac{S\,d(Z)}{S^2+Z^2+Y^2+X^2} - \frac{Z\,d(S)}{S^2+Z^2+Y^2+X^2} + \frac{ch\,Y\,d(X)}{S^2+Z^2+Y^2+X^2}$$

$$\Gamma44 := \frac{Y\,d(Y)}{S^2+Z^2+Y^2+X^2} + \frac{Z\,d(Z)}{S^2+Z^2+Y^2+X^2} + \frac{S\,d(S)}{S^2+Z^2+Y^2+X^2} + \frac{X\,d(X)}{S^2+Z^2+Y^2+X^2}$$

All the elements of the right Cartan matrix have a common factor (1/R4). Note that the diagonal matix elements are perfect differentials, and if the R4 space is constrained to a 3 sphere, the diagonal elements vanish. (These matrix elements are related to dilatations.)

>

Now the components of the right Cartan matrix will be computed by the tensor method, as a check

```
> dim:=4;coord:=[X,Y,Z,S];
```

$$dim := 4$$

$$coord := [X, Y, Z, S]$$

First compute the differentials of the inverse matrix [GG]

```
> for i from 1 to dim do for j from 1 to dim do for k from 1 to dim do d1GG[i,j,k] := (diff(GG[i,j],coord[k])) od od od:
```

Compute the elements of the matrix product of - d[G][F]. The notation is such that (a,-b,-c,) implies (upper,lower,lower) index.

```
> for b from 1 to dim do for a from 1 to dim do for k from 1 to dim do  ss:=0;for m from 1 to dim do ss := ss+(d1GG[a,m,k]*FF[m,b]); CC[a,b,k]:=simplify(-ss) od od od od ;
>
> for a from 1 to dim do for b from 1 to dim do for k from 1 to dim do if CC[a,b,k]=0 then else print(`Cartan_RIGHT`(a,-b,-k)=factor(subs(ch^2=1,CC[a,b,k]))) fi od od od ;
```

THE non zero CARTAN RIGHT CONNECTION coefficients. (Instanton map)

$$\text{Cartan\_RIGHT}(1, -1, -1) = \frac{X}{S^2 + Z^2 + Y^2 + X^2}$$

$$\text{Cartan\_RIGHT}(1, -1, -2) = \frac{Y}{S^2 + Z^2 + Y^2 + X^2}$$

$$\text{Cartan\_RIGHT}(1, -1, -3) = \frac{Z}{S^2 + Z^2 + Y^2 + X^2}$$

$$\text{Cartan\_RIGHT}(1, -1, -4) = \frac{S}{S^2 + Z^2 + Y^2 + X^2}$$

$$\text{Cartan\_RIGHT}(1, -2, -1) = -\frac{Y}{S^2 + Z^2 + Y^2 + X^2}$$

$$\text{Cartan\_RIGHT}(1, -2, -2) = \frac{X}{S^2 + Z^2 + Y^2 + X^2}$$

$$\text{Cartan\_RIGHT}(1, -2, -3) = -\frac{ch\, S}{S^2 + Z^2 + Y^2 + X^2}$$

$$\text{Cartan\_RIGHT}(1, -2, -4) = \frac{ch\, Z}{S^2 + Z^2 + Y^2 + X^2}$$

$$\text{Cartan\_RIGHT}(1, -3, -1) = -\frac{ch\, Z}{S^2 + Z^2 + Y^2 + X^2}$$

$$\text{Cartan\_RIGHT}(1, -3, -2) = \frac{S}{S^2 + Z^2 + Y^2 + X^2}$$

$$\text{Cartan\_RIGHT}(1, -3, -3) = \frac{ch\, X}{S^2 + Z^2 + Y^2 + X^2}$$

$$\text{Cartan\_RIGHT}(1, -3, -4) = -\frac{Y}{S^2 + Z^2 + Y^2 + X^2}$$

$$\text{Cartan\_RIGHT}(1, -4, -1) = -\frac{ch\, S}{S^2 + Z^2 + Y^2 + X^2}$$

$$\text{Cartan\_RIGHT}(1, -4, -2) = -\frac{Z}{S^2 + Z^2 + Y^2 + X^2}$$

$$\text{Cartan\_RIGHT}(1, -4, -3) = \frac{Y}{S^2 + Z^2 + Y^2 + X^2}$$

$$\text{Cartan\_RIGHT}(1, -4, -4) = \frac{ch\, X}{S^2 + Z^2 + Y^2 + X^2}$$

$$\text{Cartan\_RIGHT}(2, -1, -1) = \frac{Y}{S^2 + Z^2 + Y^2 + X^2}$$

$$\text{Cartan\_RIGHT}(2, -1, -2) = -\frac{X}{S^2 + Z^2 + Y^2 + X^2}$$

$$\text{Cartan\_RIGHT}(2, -1, -3) = \frac{ch\, S}{S^2 + Z^2 + Y^2 + X^2}$$

$$\text{Cartan\_RIGHT}(2, -1, -4) = -\frac{ch\, Z}{S^2 + Z^2 + Y^2 + X^2}$$

$$\text{Cartan\_RIGHT}(2, -2, -1) = \frac{X}{S^2 + Z^2 + Y^2 + X^2}$$

$$\text{Cartan\_RIGHT}(2, -2, -2) = \frac{Y}{S^2 + Z^2 + Y^2 + X^2}$$

$$\text{Cartan\_RIGHT}(2, -2, -3) = \frac{Z}{S^2 + Z^2 + Y^2 + X^2}$$

$$\text{Cartan\_RIGHT}(2, -2, -4) = \frac{S}{S^2 + Z^2 + Y^2 + X^2}$$

$$\text{Cartan\_RIGHT}(2, -3, -1) = -\frac{S}{S^2 + Z^2 + Y^2 + X^2}$$

$$\text{Cartan\_RIGHT}(2, -3, -2) = -\frac{ch\, Z}{S^2 + Z^2 + Y^2 + X^2}$$

$$\text{Cartan\_RIGHT}(2, -3, -3) = \frac{ch\, Y}{S^2 + Z^2 + Y^2 + X^2}$$

$$\text{Cartan\_RIGHT}(2, -3, -4) = \frac{X}{S^2 + Z^2 + Y^2 + X^2}$$

$$\text{Cartan\_RIGHT}(2, -4, -1) = \frac{Z}{S^2 + Z^2 + Y^2 + X^2}$$

$$\text{Cartan\_RIGHT}(2, -4, -2) = -\frac{ch\, S}{S^2 + Z^2 + Y^2 + X^2}$$

$$\text{Cartan\_RIGHT}(2, -4, -3) = -\frac{X}{S^2 + Z^2 + Y^2 + X^2}$$

$$\text{Cartan\_RIGHT}(2, -4, -4) = \frac{ch\, Y}{S^2 + Z^2 + Y^2 + X^2}$$

$$\text{Cartan\_RIGHT}(3, -1, -1) = \frac{ch\, Z}{S^2 + Z^2 + Y^2 + X^2}$$

$$\text{Cartan\_RIGHT}(3, -1, -2) = -\frac{S}{S^2 + Z^2 + Y^2 + X^2}$$

$$\text{Cartan\_RIGHT}(3, -1, -3) = -\frac{ch\, X}{S^2 + Z^2 + Y^2 + X^2}$$

$$\text{Cartan\_RIGHT}(3, -1, -4) = \frac{Y}{S^2 + Z^2 + Y^2 + X^2}$$

$$\text{Cartan\_RIGHT}(3, -2, -1) = \frac{S}{S^2 + Z^2 + Y^2 + X^2}$$

$$\text{Cartan\_RIGHT}(3, -2, -2) = \frac{ch\, Z}{S^2 + Z^2 + Y^2 + X^2}$$

$$\text{Cartan\_RIGHT}(3, -2, -3) = -\frac{ch\, Y}{S^2 + Z^2 + Y^2 + X^2}$$

$$\text{Cartan\_RIGHT}(3, -2, -4) = -\frac{X}{S^2 + Z^2 + Y^2 + X^2}$$

$$\text{Cartan\_RIGHT}(3, -3, -1) = \frac{X}{S^2 + Z^2 + Y^2 + X^2}$$

$$\text{Cartan\_RIGHT}(3, -3, -2) = \frac{Y}{S^2 + Z^2 + Y^2 + X^2}$$

$$\text{Cartan\_RIGHT}(3, -3, -3) = \frac{Z}{S^2 + Z^2 + Y^2 + X^2}$$

$$\text{Cartan\_RIGHT}(3, -3, -4) = \frac{S}{S^2 + Z^2 + Y^2 + X^2}$$

$$\text{Cartan\_RIGHT}(3, -4, -1) = -\frac{ch\, Y}{S^2 + Z^2 + Y^2 + X^2}$$

$$\text{Cartan\_RIGHT}(3, -4, -2) = \frac{ch\, X}{S^2 + Z^2 + Y^2 + X^2}$$

$$\text{Cartan\_RIGHT}(3, -4, -3) = -\frac{S}{S^2 + Z^2 + Y^2 + X^2}$$

$$\text{Cartan\_RIGHT}(3, -4, -4) = \frac{Z}{S^2 + Z^2 + Y^2 + X^2}$$

$$\text{Cartan\_RIGHT}(4, -1, -1) = \frac{ch\, S}{S^2 + Z^2 + Y^2 + X^2}$$

$$\text{Cartan\_RIGHT}(4, -1, -2) = \frac{Z}{S^2 + Z^2 + Y^2 + X^2}$$

$$\text{Cartan\_RIGHT}(4, -1, -3) = -\frac{Y}{S^2 + Z^2 + Y^2 + X^2}$$

$$\text{Cartan\_RIGHT}(4, -1, -4) = -\frac{ch\, X}{S^2 + Z^2 + Y^2 + X^2}$$

$$\text{Cartan\_RIGHT}(4, -2, -1) = -\frac{Z}{S^2 + Z^2 + Y^2 + X^2}$$

$$\text{Cartan\_RIGHT}(4, -2, -2) = \frac{ch\, S}{S^2 + Z^2 + Y^2 + X^2}$$

$$\text{Cartan\_RIGHT}(4, -2, -3) = \frac{X}{S^2 + Z^2 + Y^2 + X^2}$$

$$\text{Cartan\_RIGHT}(4, -2, -4) = -\frac{ch\, Y}{S^2 + Z^2 + Y^2 + X^2}$$

$$\text{Cartan\_RIGHT}(4,-3,-1)=\frac{ch\,Y}{S^2+Z^2+Y^2+X^2}$$

$$\text{Cartan\_RIGHT}(4,-3,-2)=-\frac{ch\,X}{S^2+Z^2+Y^2+X^2}$$

$$\text{Cartan\_RIGHT}(4,-3,-3)=\frac{S}{S^2+Z^2+Y^2+X^2}$$

$$\text{Cartan\_RIGHT}(4,-3,-4)=-\frac{Z}{S^2+Z^2+Y^2+X^2}$$

$$\text{Cartan\_RIGHT}(4,-4,-1)=\frac{X}{S^2+Z^2+Y^2+X^2}$$

$$\text{Cartan\_RIGHT}(4,-4,-2)=\frac{Y}{S^2+Z^2+Y^2+X^2}$$

$$\text{Cartan\_RIGHT}(4,-4,-3)=\frac{Z}{S^2+Z^2+Y^2+X^2}$$

$$\text{Cartan\_RIGHT}(4,-4,-4)=\frac{S}{S^2+Z^2+Y^2+X^2}$$

These results agree with matrix method given above.

Next check for Affine Torsion of the Right Cartan matrix using the tensor methods: Torsion coefficients for the Right Cartan matrix are defined as the difference between Gamma(a,-b,-c)-Gamma(a,-c,-b) times 1/2;

```
> for j from 1 to dim do for i from 1 to dim do for k from 1 to dim do  ss := (CC[i,j,k]-CC[i,k,j])/2;
  CCTTS[i,j,k]:=ss od od od  ;
>
> for i from 1 to dim do for j from 1 to dim do for k from 1 to dim do if CCTTS[i,j,k]=0 then else
  print(`RIGHT_AffineTorsion`(i,-k,-j)=simplify(subs(ch^2=1,CCTTS[i,k,j]))) fi od od od ;
```

$$\text{RIGHT\_AffineTorsion}(1,-2,-1)=-\frac{Y}{S^2+Z^2+Y^2+X^2}$$

$$\text{RIGHT\_AffineTorsion}(1,-3,-1)=-\frac{1}{2}\frac{Z\,(1+ch)}{S^2+Z^2+Y^2+X^2}$$

$$\text{RIGHT\_AffineTorsion}(1,-4,-1)=-\frac{1}{2}\frac{S\,(1+ch)}{S^2+Z^2+Y^2+X^2}$$

$$\text{RIGHT\_AffineTorsion}(1,-1,-2)=\frac{Y}{S^2+Z^2+Y^2+X^2}$$

$$\text{RIGHT\_AffineTorsion}(1,-3,-2)=\frac{1}{2}\frac{S\,(1+ch)}{S^2+Z^2+Y^2+X^2}$$

$$\text{RIGHT\_AffineTorsion}(1,-4,-2)=-\frac{1}{2}\frac{Z\,(1+ch)}{S^2+Z^2+Y^2+X^2}$$

$$\text{RIGHT\_AffineTorsion}(1,-1,-3)=\frac{1}{2}\frac{Z\,(1+ch)}{S^2+Z^2+Y^2+X^2}$$

$$\text{RIGHT\_AffineTorsion}(1,-2,-3)=-\frac{1}{2}\frac{S\,(1+ch)}{S^2+Z^2+Y^2+X^2}$$

$$\text{RIGHT\_AffineTorsion}(1,-4,-3)=\frac{Y}{S^2+Z^2+Y^2+X^2}$$

$$\text{RIGHT\_AffineTorsion}(1,-1,-4) = \frac{1}{2}\frac{S(1+ch)}{S^2+Z^2+Y^2+X^2}$$

$$\text{RIGHT\_AffineTorsion}(1,-2,-4) = \frac{1}{2}\frac{Z(1+ch)}{S^2+Z^2+Y^2+X^2}$$

$$\text{RIGHT\_AffineTorsion}(1,-3,-4) = -\frac{Y}{S^2+Z^2+Y^2+X^2}$$

$$\text{RIGHT\_AffineTorsion}(2,-2,-1) = \frac{X}{S^2+Z^2+Y^2+X^2}$$

$$\text{RIGHT\_AffineTorsion}(2,-3,-1) = -\frac{1}{2}\frac{S(1+ch)}{S^2+Z^2+Y^2+X^2}$$

$$\text{RIGHT\_AffineTorsion}(2,-4,-1) = \frac{1}{2}\frac{Z(1+ch)}{S^2+Z^2+Y^2+X^2}$$

$$\text{RIGHT\_AffineTorsion}(2,-1,-2) = -\frac{X}{S^2+Z^2+Y^2+X^2}$$

$$\text{RIGHT\_AffineTorsion}(2,-3,-2) = -\frac{1}{2}\frac{Z(1+ch)}{S^2+Z^2+Y^2+X^2}$$

$$\text{RIGHT\_AffineTorsion}(2,-4,-2) = -\frac{1}{2}\frac{S(1+ch)}{S^2+Z^2+Y^2+X^2}$$

$$\text{RIGHT\_AffineTorsion}(2,-1,-3) = \frac{1}{2}\frac{S(1+ch)}{S^2+Z^2+Y^2+X^2}$$

$$\text{RIGHT\_AffineTorsion}(2,-2,-3) = \frac{1}{2}\frac{Z(1+ch)}{S^2+Z^2+Y^2+X^2}$$

$$\text{RIGHT\_AffineTorsion}(2,-4,-3) = -\frac{X}{S^2+Z^2+Y^2+X^2}$$

$$\text{RIGHT\_AffineTorsion}(2,-1,-4) = -\frac{1}{2}\frac{Z(1+ch)}{S^2+Z^2+Y^2+X^2}$$

$$\text{RIGHT\_AffineTorsion}(2,-2,-4) = \frac{1}{2}\frac{S(1+ch)}{S^2+Z^2+Y^2+X^2}$$

$$\text{RIGHT\_AffineTorsion}(2,-3,-4) = \frac{X}{S^2+Z^2+Y^2+X^2}$$

$$\text{RIGHT\_AffineTorsion}(3,-2,-1) = \frac{S}{S^2+Z^2+Y^2+X^2}$$

$$\text{RIGHT\_AffineTorsion}(3,-3,-1) = \frac{1}{2}\frac{X(1+ch)}{S^2+Z^2+Y^2+X^2}$$

$$\text{RIGHT\_AffineTorsion}(3,-4,-1) = -\frac{1}{2}\frac{Y(1+ch)}{S^2+Z^2+Y^2+X^2}$$

$$\text{RIGHT\_AffineTorsion}(3,-1,-2) = -\frac{S}{S^2+Z^2+Y^2+X^2}$$

$$\text{RIGHT\_AffineTorsion}(3,-3,-2) = \frac{1}{2}\frac{Y(1+ch)}{S^2+Z^2+Y^2+X^2}$$

$$\text{RIGHT\_AffineTorsion}(3,-4,-2) = \frac{1}{2}\frac{X(1+ch)}{S^2+Z^2+Y^2+X^2}$$

$$\text{RIGHT\_AffineTorsion}(3, -1, -3) = -\frac{1}{2}\frac{X(1+ch)}{S^2+Z^2+Y^2+X^2}$$

$$\text{RIGHT\_AffineTorsion}(3, -2, -3) = -\frac{1}{2}\frac{Y(1+ch)}{S^2+Z^2+Y^2+X^2}$$

$$\text{RIGHT\_AffineTorsion}(3, -4, -3) = -\frac{S}{S^2+Z^2+Y^2+X^2}$$

$$\text{RIGHT\_AffineTorsion}(3, -1, -4) = \frac{1}{2}\frac{Y(1+ch)}{S^2+Z^2+Y^2+X^2}$$

$$\text{RIGHT\_AffineTorsion}(3, -2, -4) = -\frac{1}{2}\frac{X(1+ch)}{S^2+Z^2+Y^2+X^2}$$

$$\text{RIGHT\_AffineTorsion}(3, -3, -4) = \frac{S}{S^2+Z^2+Y^2+X^2}$$

$$\text{RIGHT\_AffineTorsion}(4, -2, -1) = -\frac{Z}{S^2+Z^2+Y^2+X^2}$$

$$\text{RIGHT\_AffineTorsion}(4, -3, -1) = \frac{1}{2}\frac{Y(1+ch)}{S^2+Z^2+Y^2+X^2}$$

$$\text{RIGHT\_AffineTorsion}(4, -4, -1) = \frac{1}{2}\frac{X(1+ch)}{S^2+Z^2+Y^2+X^2}$$

$$\text{RIGHT\_AffineTorsion}(4, -1, -2) = \frac{Z}{S^2+Z^2+Y^2+X^2}$$

$$\text{RIGHT\_AffineTorsion}(4, -3, -2) = -\frac{1}{2}\frac{X(1+ch)}{S^2+Z^2+Y^2+X^2}$$

$$\text{RIGHT\_AffineTorsion}(4, -4, -2) = \frac{1}{2}\frac{Y(1+ch)}{S^2+Z^2+Y^2+X^2}$$

$$\text{RIGHT\_AffineTorsion}(4, -1, -3) = -\frac{1}{2}\frac{Y(1+ch)}{S^2+Z^2+Y^2+X^2}$$

$$\text{RIGHT\_AffineTorsion}(4, -2, -3) = \frac{1}{2}\frac{X(1+ch)}{S^2+Z^2+Y^2+X^2}$$

$$\text{RIGHT\_AffineTorsion}(4, -4, -3) = \frac{Z}{S^2+Z^2+Y^2+X^2}$$

$$\text{RIGHT\_AffineTorsion}(4, -1, -4) = -\frac{1}{2}\frac{X(1+ch)}{S^2+Z^2+Y^2+X^2}$$

$$\text{RIGHT\_AffineTorsion}(4, -2, -4) = -\frac{1}{2}\frac{Y(1+ch)}{S^2+Z^2+Y^2+X^2}$$

$$\text{RIGHT\_AffineTorsion}(4, -3, -4) = -\frac{Z}{S^2+Z^2+Y^2+X^2}$$

IF NO ENTRIES APPEAR ABOVE, THE AFFINE TORSION IS ZERO

For the Hopf map and the frame constructed above, it is remarkable that there is a Torsion component that depends on the chirality factor, ch, and another torsion component which does not.!

The Right Affine torsion does not depend upon the orientation, B, but has components that depend upon chirality, ch

****

Now compute the CARTAN LEFT CONNECTION

```
> for a from 1 to dim do for j from 1 to dim do for k from 1 to dim do d1GG[a,j,k] :=
  simplify(diff(GG[a,j],coord[k])) od od od:
```

Compute the elements of the matrix product of [F]d[G]

```
> for i from 1 to dim do for j from 1 to dim do for k from 1 to dim do  ss:=0;for m to dim do ss :=
  ss+FF[i,m]*(d1GG[m,j,k]); DD[i,j,k]:=simplify(ss) od od od od ;
> for i from 1 to dim do for j from 1 to dim do for k from 1 to dim do if DD[i,j,k]=0 then else
  print(`Cartan_LEFT`(i,-j,-k)=simplify(subs(ch^2=1,DD[i,j,k]))) fi od od od ;
```

$$\mathrm{Cartan\_LEFT}(1,-1,-1) = -\frac{X}{S^2+Z^2+Y^2+X^2}$$

$$\mathrm{Cartan\_LEFT}(1,-1,-2) = -\frac{Y}{S^2+Z^2+Y^2+X^2}$$

$$\mathrm{Cartan\_LEFT}(1,-1,-3) = -\frac{Z}{S^2+Z^2+Y^2+X^2}$$

$$\mathrm{Cartan\_LEFT}(1,-1,-4) = -\frac{S}{S^2+Z^2+Y^2+X^2}$$

$$\mathrm{Cartan\_LEFT}(1,-2,-1) = -\frac{ch\,Y}{S^2+Z^2+Y^2+X^2}$$

$$\mathrm{Cartan\_LEFT}(1,-2,-2) = \frac{X\,ch}{S^2+Z^2+Y^2+X^2}$$

$$\mathrm{Cartan\_LEFT}(1,-2,-3) = \frac{S}{S^2+Z^2+Y^2+X^2}$$

$$\mathrm{Cartan\_LEFT}(1,-2,-4) = -\frac{Z}{S^2+Z^2+Y^2+X^2}$$

$$\mathrm{Cartan\_LEFT}(1,-3,-1) = \frac{Z}{S^2+Z^2+Y^2+X^2}$$

$$\mathrm{Cartan\_LEFT}(1,-3,-2) = \frac{ch\,S}{S^2+Z^2+Y^2+X^2}$$

$$\mathrm{Cartan\_LEFT}(1,-3,-3) = -\frac{X}{S^2+Z^2+Y^2+X^2}$$

$$\mathrm{Cartan\_LEFT}(1,-3,-4) = -\frac{ch\,Y}{S^2+Z^2+Y^2+X^2}$$

$$\mathrm{Cartan\_LEFT}(1,-4,-1) = -\frac{ch\,S}{(S^2+Z^2+Y^2+X^2)\,B}$$

$$\mathrm{Cartan\_LEFT}(1,-4,-2) = \frac{Z}{(S^2+Z^2+Y^2+X^2)\,B}$$

$$\mathrm{Cartan\_LEFT}(1,-4,-3) = -\frac{Y}{(S^2+Z^2+Y^2+X^2)\,B}$$

$$\mathrm{Cartan\_LEFT}(1,-4,-4) = \frac{X\,ch}{(S^2+Z^2+Y^2+X^2)\,B}$$

$$\mathrm{Cartan\_LEFT}(2,-1,-1) = \frac{ch\,Y}{S^2+Z^2+Y^2+X^2}$$

$$\mathrm{Cartan\_LEFT}(2,-1,-2) = -\frac{X\,ch}{S^2+Z^2+Y^2+X^2}$$

$$\text{Cartan\_LEFT}(2, -1, -3) = -\frac{S}{S^2 + Z^2 + Y^2 + X^2}$$

$$\text{Cartan\_LEFT}(2, -1, -4) = \frac{Z}{S^2 + Z^2 + Y^2 + X^2}$$

$$\text{Cartan\_LEFT}(2, -2, -1) = -\frac{X}{S^2 + Z^2 + Y^2 + X^2}$$

$$\text{Cartan\_LEFT}(2, -2, -2) = -\frac{Y}{S^2 + Z^2 + Y^2 + X^2}$$

$$\text{Cartan\_LEFT}(2, -2, -3) = -\frac{Z}{S^2 + Z^2 + Y^2 + X^2}$$

$$\text{Cartan\_LEFT}(2, -2, -4) = -\frac{S}{S^2 + Z^2 + Y^2 + X^2}$$

$$\text{Cartan\_LEFT}(2, -3, -1) = -\frac{S}{S^2 + Z^2 + Y^2 + X^2}$$

$$\text{Cartan\_LEFT}(2, -3, -2) = \frac{ch\, Z}{S^2 + Z^2 + Y^2 + X^2}$$

$$\text{Cartan\_LEFT}(2, -3, -3) = -\frac{ch\, Y}{S^2 + Z^2 + Y^2 + X^2}$$

$$\text{Cartan\_LEFT}(2, -3, -4) = \frac{X}{S^2 + Z^2 + Y^2 + X^2}$$

$$\text{Cartan\_LEFT}(2, -4, -1) = -\frac{ch\, Z}{(S^2 + Z^2 + Y^2 + X^2)\, B}$$

$$\text{Cartan\_LEFT}(2, -4, -2) = -\frac{S}{(S^2 + Z^2 + Y^2 + X^2)\, B}$$

$$\text{Cartan\_LEFT}(2, -4, -3) = \frac{X\, ch}{(S^2 + Z^2 + Y^2 + X^2)\, B}$$

$$\text{Cartan\_LEFT}(2, -4, -4) = \frac{Y}{(S^2 + Z^2 + Y^2 + X^2)\, B}$$

$$\text{Cartan\_LEFT}(3, -1, -1) = -\frac{Z}{S^2 + Z^2 + Y^2 + X^2}$$

$$\text{Cartan\_LEFT}(3, -1, -2) = -\frac{ch\, S}{S^2 + Z^2 + Y^2 + X^2}$$

$$\text{Cartan\_LEFT}(3, -1, -3) = \frac{X}{S^2 + Z^2 + Y^2 + X^2}$$

$$\text{Cartan\_LEFT}(3, -1, -4) = \frac{ch\, Y}{S^2 + Z^2 + Y^2 + X^2}$$

$$\text{Cartan\_LEFT}(3, -2, -1) = \frac{S}{S^2 + Z^2 + Y^2 + X^2}$$

$$\text{Cartan\_LEFT}(3, -2, -2) = -\frac{ch\, Z}{S^2 + Z^2 + Y^2 + X^2}$$

$$\text{Cartan\_LEFT}(3, -2, -3) = \frac{ch\, Y}{S^2 + Z^2 + Y^2 + X^2}$$

$$\text{Cartan\_LEFT}(3,-2,-4)=-\frac{X}{S^2+Z^2+Y^2+X^2}$$

$$\text{Cartan\_LEFT}(3,-3,-1)=-\frac{X}{S^2+Z^2+Y^2+X^2}$$

$$\text{Cartan\_LEFT}(3,-3,-2)=-\frac{Y}{S^2+Z^2+Y^2+X^2}$$

$$\text{Cartan\_LEFT}(3,-3,-3)=-\frac{Z}{S^2+Z^2+Y^2+X^2}$$

$$\text{Cartan\_LEFT}(3,-3,-4)=-\frac{S}{S^2+Z^2+Y^2+X^2}$$

$$\text{Cartan\_LEFT}(3,-4,-1)=-\frac{Y}{(S^2+Z^2+Y^2+X^2)\,B}$$

$$\text{Cartan\_LEFT}(3,-4,-2)=\frac{X}{(S^2+Z^2+Y^2+X^2)\,B}$$

$$\text{Cartan\_LEFT}(3,-4,-3)=\frac{ch\,S}{(S^2+Z^2+Y^2+X^2)\,B}$$

$$\text{Cartan\_LEFT}(3,-4,-4)=-\frac{ch\,Z}{(S^2+Z^2+Y^2+X^2)\,B}$$

$$\text{Cartan\_LEFT}(4,-1,-1)=\frac{B\,ch\,S}{S^2+Z^2+Y^2+X^2}$$

$$\text{Cartan\_LEFT}(4,-1,-2)=-\frac{B\,Z}{S^2+Z^2+Y^2+X^2}$$

$$\text{Cartan\_LEFT}(4,-1,-3)=\frac{B\,Y}{S^2+Z^2+Y^2+X^2}$$

$$\text{Cartan\_LEFT}(4,-1,-4)=-\frac{B\,X\,ch}{S^2+Z^2+Y^2+X^2}$$

$$\text{Cartan\_LEFT}(4,-2,-1)=\frac{B\,ch\,Z}{S^2+Z^2+Y^2+X^2}$$

$$\text{Cartan\_LEFT}(4,-2,-2)=\frac{B\,S}{S^2+Z^2+Y^2+X^2}$$

$$\text{Cartan\_LEFT}(4,-2,-3)=-\frac{B\,X\,ch}{S^2+Z^2+Y^2+X^2}$$

$$\text{Cartan\_LEFT}(4,-2,-4)=-\frac{B\,Y}{S^2+Z^2+Y^2+X^2}$$

$$\text{Cartan\_LEFT}(4,-3,-1)=\frac{B\,Y}{S^2+Z^2+Y^2+X^2}$$

$$\text{Cartan\_LEFT}(4,-3,-2)=-\frac{B\,X}{S^2+Z^2+Y^2+X^2}$$

$$\text{Cartan\_LEFT}(4,-3,-3)=-\frac{B\,ch\,S}{S^2+Z^2+Y^2+X^2}$$

$$\text{Cartan\_LEFT}(4,-3,-4)=\frac{B\,ch\,Z}{S^2+Z^2+Y^2+X^2}$$

$$\text{Cartan\_LEFT}(4, -4, -1) = -\frac{X}{S^2 + Z^2 + Y^2 + X^2}$$

$$\text{Cartan\_LEFT}(4, -4, -2) = -\frac{Y}{S^2 + Z^2 + Y^2 + X^2}$$

$$\text{Cartan\_LEFT}(4, -4, -3) = -\frac{Z}{S^2 + Z^2 + Y^2 + X^2}$$

$$\text{Cartan\_LEFT}(4, -4, -4) = -\frac{S}{S^2 + Z^2 + Y^2 + X^2}$$

The components of f the LEFT CARTAN Connection appear above. Note that they are not the same as the components of the Right Cartan matrix, moreover they depend upon the choice of orientation, B, and chirality ch.

Check for asymmetry (LEFT Torsion) defined as {Cartan_LEFT(a,-b,-c) - Cartan_LEFT(a.-c,-b)} times 1/2.

```
> for j from 1 to \dim do for i from 1 to dim do for k from 1 to dim do  ss := (DD[i,j,k]-DD[i,k,j])/2;
  TTS[i,j,k]:=simplify(ss) od od od  ;
>
> for i from 1 to dim do for j from 1 to dim do for k from 1 to dim do if TTS[i,j,k]=0 then else
  print(`LEFT_Torsion`(i,-k,-j)=simplify(subs(ch^2=1,TTS[i,k,j]))) fi od od od ;
```

$$\text{LEFT\_Torsion}(1, -2, -1) = -\frac{1}{2}\frac{Y(ch-1)}{S^2 + Z^2 + Y^2 + X^2}$$

$$\text{LEFT\_Torsion}(1, -3, -1) = \frac{Z}{S^2 + Z^2 + Y^2 + X^2}$$

$$\text{LEFT\_Torsion}(1, -4, -1) = -\frac{1}{2}\frac{(-B+ch)\,S}{(S^2 + Z^2 + Y^2 + X^2)\,B}$$

$$\text{LEFT\_Torsion}(1, -1, -2) = \frac{1}{2}\frac{Y(ch-1)}{S^2 + Z^2 + Y^2 + X^2}$$

$$\text{LEFT\_Torsion}(1, -3, -2) = \frac{1}{2}\frac{S(ch-1)}{S^2 + Z^2 + Y^2 + X^2}$$

$$\text{LEFT\_Torsion}(1, -4, -2) = \frac{1}{2}\frac{(B+1)\,Z}{(S^2 + Z^2 + Y^2 + X^2)\,B}$$

$$\text{LEFT\_Torsion}(1, -1, -3) = -\frac{Z}{S^2 + Z^2 + Y^2 + X^2}$$

$$\text{LEFT\_Torsion}(1, -2, -3) = -\frac{1}{2}\frac{S(ch-1)}{S^2 + Z^2 + Y^2 + X^2}$$

$$\text{LEFT\_Torsion}(1, -4, -3) = \frac{1}{2}\frac{(B\,ch-1)\,Y}{(S^2 + Z^2 + Y^2 + X^2)\,B}$$

$$\text{LEFT\_Torsion}(1, -1, -4) = \frac{1}{2}\frac{(-B+ch)\,S}{(S^2 + Z^2 + Y^2 + X^2)\,B}$$

$$\text{LEFT\_Torsion}(1, -2, -4) = -\frac{1}{2}\frac{(B+1)\,Z}{(S^2 + Z^2 + Y^2 + X^2)\,B}$$

$$\text{LEFT\_Torsion}(1, -3, -4) = -\frac{1}{2}\frac{(B\,ch-1)\,Y}{(S^2 + Z^2 + Y^2 + X^2)\,B}$$

$$\text{LEFT\_Torsion}(2, -2, -1) = \frac{1}{2} \frac{X(ch - 1)}{S^2 + Z^2 + Y^2 + X^2}$$

$$\text{LEFT\_Torsion}(2, -3, -1) = 0$$

$$\text{LEFT\_Torsion}(2, -4, -1) = -\frac{1}{2} \frac{(B + ch) Z}{(S^2 + Z^2 + Y^2 + X^2) B}$$

$$\text{LEFT\_Torsion}(2, -1, -2) = -\frac{1}{2} \frac{X(ch - 1)}{S^2 + Z^2 + Y^2 + X^2}$$

$$\text{LEFT\_Torsion}(2, -3, -2) = \frac{1}{2} \frac{Z(1 + ch)}{S^2 + Z^2 + Y^2 + X^2}$$

$$\text{LEFT\_Torsion}(2, -4, -2) = \frac{1}{2} \frac{(B - 1) S}{(S^2 + Z^2 + Y^2 + X^2) B}$$

$$\text{LEFT\_Torsion}(2, -1, -3) = 0$$

$$\text{LEFT\_Torsion}(2, -2, -3) = -\frac{1}{2} \frac{Z(1 + ch)}{S^2 + Z^2 + Y^2 + X^2}$$

$$\text{LEFT\_Torsion}(2, -4, -3) = \frac{1}{2} \frac{(-B + ch) X}{(S^2 + Z^2 + Y^2 + X^2) B}$$

$$\text{LEFT\_Torsion}(2, -1, -4) = \frac{1}{2} \frac{(B + ch) Z}{(S^2 + Z^2 + Y^2 + X^2) B}$$

$$\text{LEFT\_Torsion}(2, -2, -4) = -\frac{1}{2} \frac{(B - 1) S}{(S^2 + Z^2 + Y^2 + X^2) B}$$

$$\text{LEFT\_Torsion}(2, -3, -4) = -\frac{1}{2} \frac{(-B + ch) X}{(S^2 + Z^2 + Y^2 + X^2) B}$$

$$\text{LEFT\_Torsion}(3, -2, -1) = \frac{1}{2} \frac{S(1 + ch)}{S^2 + Z^2 + Y^2 + X^2}$$

$$\text{LEFT\_Torsion}(3, -3, -1) = -\frac{X}{S^2 + Z^2 + Y^2 + X^2}$$

$$\text{LEFT\_Torsion}(3, -4, -1) = -\frac{1}{2} \frac{Y(B\, ch + 1)}{(S^2 + Z^2 + Y^2 + X^2) B}$$

$$\text{LEFT\_Torsion}(3, -1, -2) = -\frac{1}{2} \frac{S(1 + ch)}{S^2 + Z^2 + Y^2 + X^2}$$

$$\text{LEFT\_Torsion}(3, -3, -2) = -\frac{1}{2} \frac{Y(1 + ch)}{S^2 + Z^2 + Y^2 + X^2}$$

$$\text{LEFT\_Torsion}(3, -4, -2) = \frac{1}{2} \frac{X(B + 1)}{(S^2 + Z^2 + Y^2 + X^2) B}$$

$$\text{LEFT\_Torsion}(3, -1, -3) = \frac{X}{S^2 + Z^2 + Y^2 + X^2}$$

$$\text{LEFT\_Torsion}(3, -2, -3) = \frac{1}{2} \frac{Y(1 + ch)}{S^2 + Z^2 + Y^2 + X^2}$$

$$\text{LEFT\_Torsion}(3, -4, -3) = \frac{1}{2} \frac{S(B + ch)}{(S^2 + Z^2 + Y^2 + X^2) B}$$

$$\text{LEFT\_Torsion}(3, -1, -4) = \frac{1}{2} \frac{Y(B\, ch + 1)}{(S^2 + Z^2 + Y^2 + X^2) B}$$

$$\text{LEFT\_Torsion}(3,-2,-4) = -\frac{1}{2}\frac{X(B+1)}{(S^2+Z^2+Y^2+X^2)\,B}$$

$$\text{LEFT\_Torsion}(3,-3,-4) = -\frac{1}{2}\frac{S(B+ch)}{(S^2+Z^2+Y^2+X^2)\,B}$$

$$\text{LEFT\_Torsion}(4,-2,-1) = \frac{1}{2}\frac{B\,Z(1+ch)}{S^2+Z^2+Y^2+X^2}$$

$$\text{LEFT\_Torsion}(4,-3,-1) = 0$$

$$\text{LEFT\_Torsion}(4,-4,-1) = \frac{1}{2}\frac{X(B\,ch-1)}{S^2+Z^2+Y^2+X^2}$$

$$\text{LEFT\_Torsion}(4,-1,-2) = -\frac{1}{2}\frac{B\,Z(1+ch)}{S^2+Z^2+Y^2+X^2}$$

$$\text{LEFT\_Torsion}(4,-3,-2) = \frac{1}{2}\frac{B\,X(ch-1)}{S^2+Z^2+Y^2+X^2}$$

$$\text{LEFT\_Torsion}(4,-4,-2) = \frac{1}{2}\frac{Y(B-1)}{S^2+Z^2+Y^2+X^2}$$

$$\text{LEFT\_Torsion}(4,-1,-3) = 0$$

$$\text{LEFT\_Torsion}(4,-2,-3) = -\frac{1}{2}\frac{B\,X(ch-1)}{S^2+Z^2+Y^2+X^2}$$

$$\text{LEFT\_Torsion}(4,-4,-3) = -\frac{1}{2}\frac{Z(B\,ch+1)}{S^2+Z^2+Y^2+X^2}$$

$$\text{LEFT\_Torsion}(4,-1,-4) = -\frac{1}{2}\frac{X(B\,ch-1)}{S^2+Z^2+Y^2+X^2}$$

$$\text{LEFT\_Torsion}(4,-2,-4) = -\frac{1}{2}\frac{Y(B-1)}{S^2+Z^2+Y^2+X^2}$$

$$\text{LEFT\_Torsion}(4,-3,-4) = \frac{1}{2}\frac{Z(B\,ch+1)}{S^2+Z^2+Y^2+X^2}$$

The Right and the Left Cartan matrices are negative similarity transforms, **but the LEFT AND RIGHT torsion terms appear to be diffferent, and the left "torsion" depends upon the both the chirality factor, ch, and the oreintation, B !!!!!.**

****

Next the Christoffel symbols will be computed for the **subsumed pullback metric** on the initial state. The pullback mnetric is conformal to the indentity matrix.

Christoffel Connection coefficients from the induced metric

It is assumed that the "metric" is the pull back metric given below, which is conformal.

```
> metric:=evalm( pullbackmetric);
```

$$metric := \begin{bmatrix} S^2+Z^2+Y^2+X^2 & 0 & 0 & 0 \\ 0 & S^2+Z^2+Y^2+X^2 & 0 & 0 \\ 0 & 0 & S^2+Z^2+Y^2+X^2 & 0 \\ 0 & 0 & 0 & S^2+Z^2+Y^2+X^2 \end{bmatrix}$$

```
> metricinverse:=inverse(metric):
> for i from 1 to dim do for j from 1 to dim do for k from 1 to dim do d1gun[i,j,k] :=
  (diff(metric[i,j],coord[k])) od od od:
> #for i from 1 to dim do for j from 1 to dim do for k from 1 to dim do if d1gun[i,j,k]=0 then  else
  print(`dgun`(i,j,k)=d1gun[i,j,k]) fi od od od;
> for i from 1 to dim do for j from i to dim do for k from 1 to dim do C1S[i,j,k] := 0  od od od; for i from 1
  to dim do for j from 1 to dim do for k from 1 to dim do C1S[i,j,k] :=
  1/2*d1gun[i,k,j]+1/2*d1gun[j,k,i]-1/2*d1gun[i,j,k]  od od od;
>  for k from 1 to dim do for i from 1 to dim do for j from 1 to dim do ss := 0; for m to dim do ss :=
  ss+metricinverse[k,m]*C1S[i,j,m] od; C2S[k,i,j] := simplify(factor(ss),trig) od od od;
> for i from 1 to dim do for j from 1 to dim do for k from 1 to dim do if C2S[i,j,k]=0 then  else
  print(`Christoffel_Gamma2`(i,-j,-k)=C2S[i,j,k]) fi od od od;
```

The non zero Christoffel Connection coefficients 2nd kind   on the initial space (domain)

Gamma2(i,j,k) index (1,-1,-1)

$$\text{Christoffel\_Gamma2}(1,-1,-1)=\frac{X}{S^2+Z^2+Y^2+X^2}$$

$$\text{Christoffel\_Gamma2}(1,-1,-2)=\frac{Y}{S^2+Z^2+Y^2+X^2}$$

$$\text{Christoffel\_Gamma2}(1,-1,-3)=\frac{Z}{S^2+Z^2+Y^2+X^2}$$

$$\text{Christoffel\_Gamma2}(1,-1,-4)=\frac{S}{S^2+Z^2+Y^2+X^2}$$

$$\text{Christoffel\_Gamma2}(1,-2,-1)=\frac{Y}{S^2+Z^2+Y^2+X^2}$$

$$\text{Christoffel\_Gamma2}(1,-2,-2)=-\frac{X}{S^2+Z^2+Y^2+X^2}$$

$$\text{Christoffel\_Gamma2}(1,-3,-1)=\frac{Z}{S^2+Z^2+Y^2+X^2}$$

$$\text{Christoffel\_Gamma2}(1,-3,-3)=-\frac{X}{S^2+Z^2+Y^2+X^2}$$

$$\text{Christoffel\_Gamma2}(1,-4,-1)=\frac{S}{S^2+Z^2+Y^2+X^2}$$

$$\text{Christoffel\_Gamma2}(1,-4,-4)=-\frac{X}{S^2+Z^2+Y^2+X^2}$$

$$\text{Christoffel\_Gamma2}(2,-1,-1)=-\frac{Y}{S^2+Z^2+Y^2+X^2}$$

$$\text{Christoffel\_Gamma2}(2,-1,-2)=\frac{X}{S^2+Z^2+Y^2+X^2}$$

$$\text{Christoffel\_Gamma2}(2,-2,-1)=\frac{X}{S^2+Z^2+Y^2+X^2}$$

$$\text{Christoffel\_Gamma2}(2, -2, -2) = \frac{Y}{S^2 + Z^2 + Y^2 + X^2}$$

$$\text{Christoffel\_Gamma2}(2, -2, -3) = \frac{Z}{S^2 + Z^2 + Y^2 + X^2}$$

$$\text{Christoffel\_Gamma2}(2, -2, -4) = \frac{S}{S^2 + Z^2 + Y^2 + X^2}$$

$$\text{Christoffel\_Gamma2}(2, -3, -2) = \frac{Z}{S^2 + Z^2 + Y^2 + X^2}$$

$$\text{Christoffel\_Gamma2}(2, -3, -3) = -\frac{Y}{S^2 + Z^2 + Y^2 + X^2}$$

$$\text{Christoffel\_Gamma2}(2, -4, -2) = \frac{S}{S^2 + Z^2 + Y^2 + X^2}$$

$$\text{Christoffel\_Gamma2}(2, -4, -4) = -\frac{Y}{S^2 + Z^2 + Y^2 + X^2}$$

$$\text{Christoffel\_Gamma2}(3, -1, -1) = -\frac{Z}{S^2 + Z^2 + Y^2 + X^2}$$

$$\text{Christoffel\_Gamma2}(3, -1, -3) = \frac{X}{S^2 + Z^2 + Y^2 + X^2}$$

$$\text{Christoffel\_Gamma2}(3, -2, -2) = -\frac{Z}{S^2 + Z^2 + Y^2 + X^2}$$

$$\text{Christoffel\_Gamma2}(3, -2, -3) = \frac{Y}{S^2 + Z^2 + Y^2 + X^2}$$

$$\text{Christoffel\_Gamma2}(3, -3, -1) = \frac{X}{S^2 + Z^2 + Y^2 + X^2}$$

$$\text{Christoffel\_Gamma2}(3, -3, -2) = \frac{Y}{S^2 + Z^2 + Y^2 + X^2}$$

$$\text{Christoffel\_Gamma2}(3, -3, -3) = \frac{Z}{S^2 + Z^2 + Y^2 + X^2}$$

$$\text{Christoffel\_Gamma2}(3, -3, -4) = \frac{S}{S^2 + Z^2 + Y^2 + X^2}$$

$$\text{Christoffel\_Gamma2}(3, -4, -3) = \frac{S}{S^2 + Z^2 + Y^2 + X^2}$$

$$\text{Christoffel\_Gamma2}(3, -4, -4) = -\frac{Z}{S^2 + Z^2 + Y^2 + X^2}$$

$$\text{Christoffel\_Gamma2}(4, -1, -1) = -\frac{S}{S^2 + Z^2 + Y^2 + X^2}$$

$$\text{Christoffel\_Gamma2}(4, -1, -4) = \frac{X}{S^2 + Z^2 + Y^2 + X^2}$$

$$\text{Christoffel\_Gamma2}(4, -2, -2) = -\frac{S}{S^2 + Z^2 + Y^2 + X^2}$$

$$\text{Christoffel\_Gamma2}(4, -2, -4) = \frac{Y}{S^2 + Z^2 + Y^2 + X^2}$$

$$\text{Christoffel\_Gamma2}(4, -3, -3) = -\frac{S}{S^2 + Z^2 + Y^2 + X^2}$$

$$\text{Christoffel\_Gamma2}(4, -3, -4) = \frac{Z}{S^2 + Z^2 + Y^2 + X^2}$$

$$\text{Christoffel\_Gamma2}(4, -4, -1) = \frac{X}{S^2 + Z^2 + Y^2 + X^2}$$

$$\text{Christoffel\_Gamma2}(4, -4, -2) = \frac{Y}{S^2 + Z^2 + Y^2 + X^2}$$

$$\text{Christoffel\_Gamma2}(4, -4, -3) = \frac{Z}{S^2 + Z^2 + Y^2 + X^2}$$

$$\text{Christoffel\_Gamma2}(4, -4, -4) = \frac{S}{S^2 + Z^2 + Y^2 + X^2}$$

If no entries appear above the Christoffel symbols on the domain space vanish

Note that the Christoffel Symbols for the Conformal metric are not zero, but are not the same as the Right or Left Cartan Connection matrices. More over, the Christoffel symbols built upon the metric defined above are independent from the choice of chirality and orientation. The metric is symmetric and has a positive definite determinant.

The Right Cartan matrix is often defined as the sum of Christoffel Symbols and Rotation coefficients, T(i,j,k)

CartanRight(ijk) := ChristoffelGamma(ijk) + T(ijk)

Using this **definition for the Rotation coefficients,** Compute the T(i,j,k):

```
> for i from 1 to dim do for j from 1 to dim do for k from 1 to dim do ss:=0; ss := (CC[i,j,k]-C2S[i,j,k]);
  SHIPTR[i,j,k]:=simplify(ss)  od od od   ;
>
>
 > for i from 1 to dim do for j from 1 to dim do for k from 1 to dim do if C2S[i,j,k]=0 and CC[i,j,k]=0 then
   else print(`T`(i,-j,-k)=simplify(subs(ch^2=1,SHIPTR[i,j,k]))) fi od od od ;
```

T(ijk) index (1,-1,-1)

$$T(1, -1, -1) = 0$$

$$T(1, -1, -2) = 0$$

$$T(1, -1, -3) = 0$$

$$T(1, -1, -4) = 0$$

$$T(1, -2, -1) = -2\frac{Y}{S^2 + Z^2 + Y^2 + X^2}$$

$$T(1, -2, -2) = 2\frac{X}{S^2 + Z^2 + Y^2 + X^2}$$

$$T(1, -2, -3) = -\frac{ch\, S}{S^2 + Z^2 + Y^2 + X^2}$$

$$T(1,-2,-4) = \frac{ch\,Z}{S^2 + Z^2 + Y^2 + X^2}$$

$$T(1,-3,-1) = -\frac{Z\,(1 + ch)}{S^2 + Z^2 + Y^2 + X^2}$$

$$T(1,-3,-2) = \frac{S}{S^2 + Z^2 + Y^2 + X^2}$$

$$T(1,-3,-3) = \frac{X\,(1 + ch)}{S^2 + Z^2 + Y^2 + X^2}$$

$$T(1,-3,-4) = -\frac{Y}{S^2 + Z^2 + Y^2 + X^2}$$

$$T(1,-4,-1) = -\frac{S\,(1 + ch)}{S^2 + Z^2 + Y^2 + X^2}$$

$$T(1,-4,-2) = -\frac{Z}{S^2 + Z^2 + Y^2 + X^2}$$

$$T(1,-4,-3) = \frac{Y}{S^2 + Z^2 + Y^2 + X^2}$$

$$T(1,-4,-4) = \frac{X\,(1 + ch)}{S^2 + Z^2 + Y^2 + X^2}$$

$$T(2,-1,-1) = 2\,\frac{Y}{S^2 + Z^2 + Y^2 + X^2}$$

$$T(2,-1,-2) = -2\,\frac{X}{S^2 + Z^2 + Y^2 + X^2}$$

$$T(2,-1,-3) = \frac{ch\,S}{S^2 + Z^2 + Y^2 + X^2}$$

$$T(2,-1,-4) = -\frac{ch\,Z}{S^2 + Z^2 + Y^2 + X^2}$$

$$T(2,-2,-1) = 0$$

$$T(2,-2,-2) = 0$$

$$T(2,-2,-3) = 0$$

$$T(2,-2,-4) = 0$$

$$T(2,-3,-1) = -\frac{S}{S^2 + Z^2 + Y^2 + X^2}$$

$$T(2,-3,-2) = -\frac{Z\,(1 + ch)}{S^2 + Z^2 + Y^2 + X^2}$$

$$T(2,-3,-3) = \frac{Y\,(1 + ch)}{S^2 + Z^2 + Y^2 + X^2}$$

$$T(2,-3,-4) = \frac{X}{S^2 + Z^2 + Y^2 + X^2}$$

$$T(2,-4,-1) = \frac{Z}{S^2 + Z^2 + Y^2 + X^2}$$

$$T(2,-4,-2) = -\frac{S\,(1 + ch)}{S^2 + Z^2 + Y^2 + X^2}$$

$$T(2, -4, -3) = -\frac{X}{S^2 + Z^2 + Y^2 + X^2}$$

$$T(2, -4, -4) = \frac{Y(1 + ch)}{S^2 + Z^2 + Y^2 + X^2}$$

$$T(3, -1, -1) = \frac{Z(1 + ch)}{S^2 + Z^2 + Y^2 + X^2}$$

$$T(3, -1, -2) = -\frac{S}{S^2 + Z^2 + Y^2 + X^2}$$

$$T(3, -1, -3) = -\frac{X(1 + ch)}{S^2 + Z^2 + Y^2 + X^2}$$

$$T(3, -1, -4) = \frac{Y}{S^2 + Z^2 + Y^2 + X^2}$$

$$T(3, -2, -1) = \frac{S}{S^2 + Z^2 + Y^2 + X^2}$$

$$T(3, -2, -2) = \frac{Z(1 + ch)}{S^2 + Z^2 + Y^2 + X^2}$$

$$T(3, -2, -3) = -\frac{Y(1 + ch)}{S^2 + Z^2 + Y^2 + X^2}$$

$$T(3, -2, -4) = -\frac{X}{S^2 + Z^2 + Y^2 + X^2}$$

$$T(3, -3, -1) = 0$$

$$T(3, -3, -2) = 0$$

$$T(3, -3, -3) = 0$$

$$T(3, -3, -4) = 0$$

$$T(3, -4, -1) = -\frac{ch\, Y}{S^2 + Z^2 + Y^2 + X^2}$$

$$T(3, -4, -2) = \frac{ch\, X}{S^2 + Z^2 + Y^2 + X^2}$$

$$T(3, -4, -3) = -2\frac{S}{S^2 + Z^2 + Y^2 + X^2}$$

$$T(3, -4, -4) = 2\frac{Z}{S^2 + Z^2 + Y^2 + X^2}$$

$$T(4, -1, -1) = \frac{S(1 + ch)}{S^2 + Z^2 + Y^2 + X^2}$$

$$T(4, -1, -2) = \frac{Z}{S^2 + Z^2 + Y^2 + X^2}$$

$$T(4, -1, -3) = -\frac{Y}{S^2 + Z^2 + Y^2 + X^2}$$

$$T(4, -1, -4) = -\frac{X(1 + ch)}{S^2 + Z^2 + Y^2 + X^2}$$

$$T(4, -2, -1) = -\frac{Z}{S^2 + Z^2 + Y^2 + X^2}$$

$$T(4,-2,-2)=\frac{S(1+ch)}{S^2+Z^2+Y^2+X^2}$$

$$T(4,-2,-3)=\frac{X}{S^2+Z^2+Y^2+X^2}$$

$$T(4,-2,-4)=-\frac{Y(1+ch)}{S^2+Z^2+Y^2+X^2}$$

$$T(4,-3,-1)=\frac{ch\,Y}{S^2+Z^2+Y^2+X^2}$$

$$T(4,-3,-2)=-\frac{ch\,X}{S^2+Z^2+Y^2+X^2}$$

$$T(4,-3,-3)=2\,\frac{S}{S^2+Z^2+Y^2+X^2}$$

$$T(4,-3,-4)=-2\,\frac{Z}{S^2+Z^2+Y^2+X^2}$$

$$T(4,-4,-1)=0$$

$$T(4,-4,-2)=0$$

$$T(4,-4,-3)=0$$

$$T(4,-4,-4)=0$$

The Rotation matrices also depend upon the chirality factor.

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