1 Spin as the Source of Torsion is a Myth.

1.1 Introduction

R. M. Kiehn – last update 07/15/99 – from Puget France.

There continue to appear in the literature claims to the effect that torsion does not propagate, and that spin is the source of torsion. Where these statements originated is not clear, but in this article these dogmatic claims are challenged.

Gockeler and Shuecker "Differential Geometry, Gauge theory, and Gravity" Cambridge, (1987) p.68 state that

> " 'Spin is the source of torsion'. Contrary to curvature, torsion does not propagate in any dimension".

However, careful examination of their statement demonstrates a circular argument involving a redefinition of two tensor forms, based upon the assumption of an asymmetric affine connection, and with the additional constraint of SO(1,3) symmetry.

Horic (hep-th/9601066) in his thesis quotes W. Stoeger (Gen Rel Grav **17** 7981 (1985))

> "The only source of torsion is intrinsic particle spin... thus torsion is a fundamentally microscopic quantum mechanically related phenomenon"

Horic also develops the idea of spin and its relationship to torsion in an ad hoc way. After much formalism, which is not needed, his fundamental assumption is similar to the London theory of superconductivity, where the charge current density, J, is conjectured by London to be proportional to the vector potential, A. The London construction is in effect a constitutive constraint of the form $[J_\mu - \lambda A_\mu = 0]$, and implies an additional metric constraint such that the tensor density, J_μ, can be related to a tensor.

The Horic construction (based upon the assumption that the constitutive tensor is complex, is of the form (his equation 2.47)

$$S_\mu - (ie/\hbar c)A_\mu = 0 \tag{1}$$

Let us check the physical units. A_μ has the dimensions of $(\hbar/e)$ per unit length. Hence, the compatible dimension of Horic's S_μ is reciprocal time. This does not lead to a compatible identification of S_μ with spin.

1.2 A topological perspective of Torsion and Spin

The implication (in many of the historical treatments and above) is that torsion depends upon some (microscopic) length scale. In this article, a completely

different point of view is formulated. Torsion is considered to be a topological issue, and therefor is independent from geometrical issues, such as length scales. In fact in this article, a conclusion is reached that the rational, quantized, values of electromagnetic spin, and charge, as well as the rational values of electromagnetic torsion and the flux quantum are topological properties of nature, or better said, topological features of electromagnetism. As deformation invariants, these properties do not depend upon length scales. The notion of a Planck length, or of a Compton length, is a geometrical issue, and is not of importance to this topological perspective. The quantized features relating measurables to rational numbers will follow automatically from the topological view of counting topological defects (you can not have half a hole). These ideas date back to R. M. Kiehn "Periods on Manifolds, Quantization and Gauge" Jour. Math. Phys 18, no.4 1977 p.614. Other quantum features, such as energy, are not included in this topological perspective, and indeed may depend on classes of geometrical constraints placed upon the physical aggregate of matter. These geometrical issues involving size and shape are not of immediate interest to the topological point of view.

Herein, the Maxwell-Faraday law of induction is demonstrated to be a topological statement, again independent from scales or lengths or choice of coordinates, metric, or connection. In fact, the Maxwell-Faraday equations form a nested set of equations for any choice of dimension > 3. (The first four PDE equations of the exterior differential system are the same no matter what the dimension of the embedding). The Maxwell-Ampere equations (the second half of electromagnetic theory) are also considered to be a topological statements, unfettered by metric or connection, but, different from the Maxwell-Faraday equations, the Maxwell-Ampere equations are dependent upon a density weighted volume element, or measure. That is, both the spin 3-form $S = A\hat{}G$, the charge excitation 2-form G, as well as the charge-current density, J, and the Maxwell-Ampere equations, are associated with a density n-form, ρ- $_x$, or volume element. (The volume element is not a metrical idea.) In coordinate notation, the components of aformentioned field quantities are tensor densities. (A great deal of effort in theoretical physics goes into finding the group of diffeomorphisms that leave the volume element invariant. Such restrictive sets are not of immediate interest to this article.)

On the other hand, the Action 1-form, A, the Torsion 3-form, $H = A\hat{}F$, and the Maxwell-Faraday equations do not depend upon such a volume element. In coordinate notation, the components of these field intensities are covariant tensors. No constraints of metric or connection are utilized. From this point of view, it is almost rediculous to study the constrained systems, where F (covariant tensor components) and G (contravariant density components) are presumed to be self dual.

It is assumed that the electromagnetic physical system so described above is generated from two topological constraints defined by two exterior differential systems, $F - dA = 0$ (Maxwell-Faraday-cotensor), and $J - dG = 0$ (Maxwell-Ampere-contravariant tensor density). The first topological constraint presumes that the 2 - manifold of support for electromagnetic intensities $\mathbf{E}$ and $\mathbf{B}$

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(created from F) cannot be compact without boundary, except for the torus and the Klein bottle (the Euler characteristic must vanish). The Maxwell-Faraday constraint is on exterior forms which are well behaved functionally under the pullback of the map (defined in terms of functional substitution and the transpose of the Jacobian of the map from initial to final state). In effect, the volume element is invariant with respect to evolution in the direction of the charge-current 4-vector, J. The Maxwell-Ampere constraint (on an n-1 form density) is also well defined under the pullback of the map (with respect to functional substitution and the adjoint matrix of the Jacobian of the map from the initial to final state, if the components are viewed as tensor densities). These (evolutionary) maps need not be diffeomorphisms. These maps need not be homeomorphisms. In fact, that which is of great interest to this article are those processes or maps which describe topological evolution. In terms of exterior differential forms, topological evolution is well defined without the baggage of metric or connection or of symmetry groups.

It is not assumed herein (as is done in Yang-Mills theories) that F and G are dual, anti-dual or self dual. It fact, as F is a 2-form, and on a 4D base space, and as G is an n-2=2 form density, the two constructions have thermodynamic differences that are suppressed in YM theories. No physicist would claim that the conjugate pairs of thermodynamics variables, such as pressure and volume, are equivalent. Yet the claim of self-dualism, where $\mathbf{D}$ is identified with $\mathbf{E}$ for example, is often made. I plead that such practice be discontinued.

1.3 Three important 3-forms

In the development that follows there appear naturally, from the two topological constraints mentioned above, three different 3-forms. The first three form is a density 3-form (the 3-form of charge-current density, J) and is closed by its very definition, $J - dG = 0 \Rightarrow dJ = 0$.

$$\text{Charge Current density} : J = dG \tag{2}$$

The second three form is defined by the algebraic construction defined as the Topological Torsion 3-form,

$$\text{Topological Torsion 3-form} : H = A\,\hat{}\,F \equiv i(\mathbf{T}_4)\text{-}\,_x \tag{3}$$

The third three form (density) is defined by the algebraic construction

$$\text{Topological Spin 3-form} : S = A\,\hat{}\,G \equiv i(\mathbf{S}_4)\text{-}\,_x \tag{4}$$

These constructions are free from metric, connection, or other imposed geometrical constraints involving size and shape. Note that the physical dimensions of S are that of Planck's constant $\hbar$, and the physical dimensions of H are that of the flux quantum, $\hbar/e$. It is apparent that the two 3 - forms are usually distinct. That is, Torsion is not the same as Spin. In engineering notation, the two 4-vectors are given by the 4 component expressions:

$$\mathbf{T}_4 = [\mathbf{E} \times \mathbf{A} + \mathbf{B}\varphi, \mathbf{A} \cdot \mathbf{B}] \tag{5}$$

and

$$\mathbf{S}_4 = [\mathbf{A} \times \mathbf{H} + \mathbf{D}\varphi, \mathbf{A} \cdot \mathbf{D}]. \tag{6}$$

When either of the latter two 3-forms are closed (have zero 4-divergence), then their closed integrals have rational ratios by deRham's theorems. It is remarkable that the closure condition for the Topological Torsion 3-form is equivalent to the vanishing of the second Poincare invariant. The closure condition for the Spin 3-form is equivalent to the vanishing of the first Poincare invariant, but also is valid for plasma systems where there are interactions between the charge - current density and the 4-vector Action potentials. The formulas for the second and first Poincare "invariants" are:

$$
\begin{aligned}
dH &= d(A\hat{\ }F) = F\hat{\ }F &\tag{7}\\
&= -2\{\mathbf{E} \cdot \mathbf{B}\}dx\hat{\ }dy\hat{\ }dz\hat{\ }dt &\tag{8}\\
&= \{div_4\mathbf{T}_4\}\text{-}_x &\tag{9}
\end{aligned}
$$

and

$$
\begin{aligned}
dS &= d(A\hat{\ }G) = F\hat{\ }G - A\hat{\ }J &\tag{10}\\
&= (\mathbf{B} \cdot \mathbf{H} - \mathbf{D} \cdot \mathbf{E}) - (\mathbf{A} \cdot \mathbf{J} - \rho\varphi)\}dx\hat{\ }dy\hat{\ }dz\hat{\ }dt &\tag{11}\\
&= \{div_4\mathbf{S}_4\}\text{-}_x &\tag{12}
\end{aligned}
$$

Note that the first term in dS is twice the usual Lagrange energy density of the free field (no factor of 1/2) and the second term often appears as the "interaction" term in Lagrangian field theories. Again it is apparent that the two objects of Spin and Torsion are not equivalent in general. Both functions act as different measures on the 4-volume element. As the measures are exact, their integrals over closed integration 4-domains are evolutionary invariants for any evolutionary field.

1.4 Examples of solutions to Maxwell equations

1.4.1 No Spin No Torsion

1.4.2 Spin but not Torsion

1.4.3 Torsion but no Spin

Consider the 4- vector potential, relative to $[x, y, z, t]$:

$$A = [\cos(kz - \omega t), \sin(kz - \omega t), 0, 0]. \tag{13}$$

Compute the magnetic intensity:

$$\mathbf{B} = curl\mathbf{A} = -k[\cos(kz - \omega t), \sin(kz - \omega t), 0] = -k\mathbf{A} \qquad (14)$$

The vector potential is a Beltrami field. Compute the eletric field as

$$\mathbf{E} = -\partial\mathbf{A}/\partial t = -\omega[\sin(kz - \omega t), \cos(kz - \omega t), 0] \qquad (15)$$

Evaluate the non-zero Topological Torsion vector as

$$\mathbf{T}_4 = [0, 0, -\omega, -k] \qquad (16)$$

which has an obvious zero 4 divergence, as ω and k are constants.

Assume the Lorentz constitutive relation, $\mathbf{D} = \varepsilon\mathbf{E}$, $\mathbf{B} = \mu\mathbf{H}$, $\varepsilon\mu c^2 = 1$, and compute the Charge-Current 4-vector,

$$J = (kc - \omega)(kc + \omega)/(\mu c^2)[\cos(kz - \omega t), \sin(kz - \omega t), 0, 0], \qquad (17)$$

an expression that vanishes when either of the two direction factors goes to zero. These factors are equivalent to the wave criteria, $\lambda f = c$. Hence subject to the constitutive constraint and the wave condition, the charge-current density vanishes.

Direct computation of the Spin 4 vector demonstrates that the Spin 3-form vanishes identically.

$$\mathbf{S}_4 = [0, 0, 0, 0] \qquad (18)$$

Both Poincare invariants are zero. So the example yields a propagating Topological Torsion wave, that has Zero Topological Spin, and is free from charge-current densities. In otherwords it is a counter example to the statements made in the opening paragraph.

1.4.4   Torsion and Spin

1.5   Renormalization for Zero Divergence

Note that evolution of the natural 4-volume element in the direction of a tangent vector, $\mathbf{V}$, associated with $Z = i(\mathbf{V})$-$_x$ is invariant if Z is closed. As Arnold would say $\mathbf{V}$ is a volume preserving map, ϕ.

$$
\begin{aligned}
L_{(\mathbf{V})\text{-}x} &= i(\mathbf{V})d\text{-}_x + d(i(\mathbf{V})\text{-}_x) \qquad &(19)\\
&= 0 + dJ = 0 + 0 \qquad &(20)
\end{aligned}
$$

However, any tangent vector field, $\mathbf{V}$, can be used to generate an infinite number of volume preserving maps by multiplying the direction field by an integrating factor, such that the resulting vector has zero divergence. To demonstrate this result, and to offer a construction, consider the vector defined in terms of components generated by the map

$$\phi : [x, y, z, t] \Rightarrow [V^1, V^2, V^3, V^4] \qquad (21)$$

Construct the volume element

$$- _V = dV^1 {}^{\wedge} dV^2 {}^{\wedge} dV^3 {}^{\wedge} dV^4 \tag{22}$$

and the n-1 form,

$$Z_{on\ V} = \lambda(V^k) i(\mathbf{V}) - _V \tag{23}$$

with

$$\lambda(V^k) = \{a_1(V^1)^p + a_2(V^2)^p + a_3(V^3)^p + a_4(V^4)^p + ...\}^{m/p} \tag{24}$$

The coefficients a_k are arbitrary anisotropic constants, and m and p are integers. For any values of a_k and any integer, p, when m=n, the n-1=3 form so constructed has zero divergence.

$$d(Z_{on\ V}) = d(\lambda(V^k) i(\mathbf{V}) - _V) = (4 - m)\lambda(V^k) - _V = 0. \tag{25}$$

The map ϕ need not be a diffeomorphism, and yet the pull-back of the 3-form Z is well defined on the space of the volume element,

$$- _x = dx {}^{\wedge} dy {}^{\wedge} dz {}^{\wedge} dt. \tag{26}$$

Hence the n-1 form on $- _V$ has a well defined preimage on $- _x$, and the "vector" so induced is divergence free.

$$(Z_{on\ x}) = \phi^*(Z_{on\ V}), \qquad d(Z_{on\ x}) = \phi^*(dZ_{on\ V}) = 0. \tag{27}$$

This result is valid for any anisotropy, and any exponent p. The usual case is the Gauss map where p=2 and the anisotropic constants are all unity. Riemann himself suggested the study of the case where p = 4 (a problem that has received little attention, although recently publicized by Chern).

1.6   Homogeneous Equivalence classes.

At this time, the physical meaning of these renormalizations is not clear, but they may represent, when interpreted as measures or density distributions, as different states far from equilibrium or coherent structures. The integer m is defined as the homogeneity index of the vector field. Other values of the homogeneity index, $m \neq n$, produce interesting subsets of direction fields. These subsets may not be divergence free, but they can lead to involutive, or conformal equivalence classes. If the homogeneity index is 1, any p, any anisotropy, the map $\lambda\phi$ is not a diffeomorphism, as the determinant of the Jacobian matrix generated by $\mathbf{Z}$ is zero. These special situations imply the tangent space generated by the map is not parallelizable.

If the renormalization factor is isotropic, a1=a2=a3=a4, then the induced Jacobian matrix is symmetric, only if p=2, any m. Note that this special case corresponds to the usual euclidean quadratic form, or the Gauss map for m=1, p=2.