PHYSICS and TOPOLOGICAL PROPERTIES

Introduction

G. Boole, an English high school teacher, not only brought to the attention of the world the idea of algebraic invariants, but also developed the concept of what is now called Boolean Algebra. Recall that an abstract algebra consists of two combinatorial processes that obey certain axiomatic rules. Grade school children learn these concepts by rote, where the two combinatorial processes are called addition and multiplication of real numbers. What Boole did was to build an algebra on the logical concepts of **OR** and **AND** which is at the basis of modern digital computer system hardware and operating systems. Note that the Boolean process of **AND** can be translated to the idea of set intersection ($\cap$) and the process of **OR** can be translated to the idea of set union ($\cup$). These combinatorial concepts are the basic operations of what is called point set topology. In this sense, topology is strongly connected to logic.

In physical sciences we use a lot of "intuitive" terms that have precise topological meaning. A door to a room is either **open** or **closed**, implying that entrance to the **interior** or egress to the **exterior** can be permitted or denied. An island is said to be **isolated** from other countries if its **boundary** with the ocean **separates** it from being **connected** to another land mass.

A gas is modeled as a set of **disconnected** "points", or atoms or molecules, whose **interiors** are empty, or ignorable, and whose parts are weakly interacting if at all (The ideal gas is a set of non-interacting points). A fluid is (in a sense) a system of **neighborhoods**, which have finite non-ignorable **interiors,** which are only weakly disconnected from their environment. The neighborhoods define domains of "short" range order or coherence. A solid is a set where the **neighborhood** extends over the whole domain of interest. The neighborhoods define domains of "long" range coherence or order. The idea of coherence implies a cooperative interaction, or synergism, between the "points" that make up the neighborhood.

In the study of the solar system, in which it is presumed that energy and angular momentum are conserved, it is rather remarkable that the neighborhoods are almost "flat". The planets are mostly organized into the "equatorial plane". Spiral galaxies (which are in a crude sense thinner than this piece of paper) are also exhibitions of this two dimensionality, or flatness. That is, such systems are completely described by two dimensions, or two functions that play the role of two dimensional coordinates. Yet the solar system is visibly contained in a domain of intrinsically three degrees of freedom, at least. The concept of **dimension**, or degrees of freedom, is an important physical concept. Especially two dimensions is characteristic of many, if not all, *uniquely* soluble dynamical systems in physics. Most of the example problems in text books are reducible to such two dimensional, integrable systems. That does not mean that is all there is! In fact, most real physical systems are NOT uniquely integrable.

In certain biological systems, the neighborhoods form collections such as in sugar, that are predominately right handed. Such forms cannot be continuously deformed into their left hand counterparts. The two species are said to form **isotopes**. The idea of handedness is related to the concept of **orientability.** It is a remarkable fact that the Fermion spin 1/2 systems of the microphysical world can be put into correspondence with non-orientable sets such as the Moebius band. Both the Boson and Fermion spin systems have a strong connection to the two (and only two) species of non-orientable structures that can be constructed in terms of two dimensional surfaces (The Klein bottle is an example of the first species of a non-oriented surface. The Moebius strip is an example of the second species of a non-oriented surface. The Klein bottle is **closed**, the Moebius band is not. The "isotopes" of the Moebius band have left or right handed odd multiples of a π twist, that can be put into correspondence with the experimental evidence of Fermion spin multiplets in quantum mechanics or chemical systems.). Note that the idea of **orientability,** or any of the other boldfaced terms above, do NOT depend on any constraint involving the concepts of size or shape.

All of the above bold faced lower case terms are non-geometrical, topological properties.

A simple interpretation of a geometrical property is that it is a property that is invariant to rotations and translations. A the same simplistic level, a topological property is a property that is invariant with respect to continuous reversible deformations.

Geometrical Invariants and Physics

Euclidean Geometry - Invariant Size and Shape - Rigid Body motion

Historically, fundamental physical theories have been based on geometrical models, geometrical relationships and geometrical properties of physical objects. In fact the very idea of a physical measurement is associated
with the concepts of how big, how far, or how long. These ideas of measurement are intuitively geometrical ideas, and all involve a comparison of something of immediate interest to some legislated standard. However
the idea of a comparison is not restricted to pure geometrical concepts of size and shape. An abstract comparison may be viewed as a mapping from a range to a domain, or a transformation from an initial state
to a final state. The mapping or transformation may be an element of an equivalence class of mappings, and that class is determined by its invariant properties. Recall that, according to Felix Klein, a (euclidean) geometrical property is defined to be an invariant of a translation or rotation. A simple observation demonstrates that size and shape are such geometrical properties, and these properties are the fundamental invariants of that branch of physics that deals with rigid body motion.

Isometric Geometry - Invariant distance - Bending

With the advent of tensor calculus, mathematicians relaxed the constraints of pure (euclidean) geometry a bit in order to include the concept of bending without compression or shear. The fundamental physical invariant of interest became the *distance between a pair of points*: size is considered is an invariant in such geometries, but shape is not (necessarily) an invariant of the equivalence class of transformations that describe bending processes. The equivalence class of transformations based on the property of invariant distance are called isometries. The pure geometrical constraint of invariant shape is relaxed to include the possibility of bending. The idea of invariant distance has dominated physical theories since the turn of the century. In fact a new derivative concept, the covariant derivative, was defined in a such a manner that it preserves the concept of distance as an invariant. Displacements via a “covariant derivative” are constrained such the distance between a pair of points is preserved as an invariant! However, be warned that the prescription of a covariant derivative has in its foundations the *impossibility* of intrinsically describing compressions or shears!

Conformal Geometry - Invariant Phase - Twisting

Affine Geometry - Invariant parallel planes - Shear

Homothetic Geometry - Pressure

If the deformation is confined to compressions or expansions, then the distance between a pair of points is no longer invariant, and as both size and shape are not necessarily invariants, the covariant derivative concept becomes obsolete, and the question arises as to what invariants might be used to classify such transformations. Such transformations are defined as conformal transformations that leave invariant the angle between a pair of lines.

The next class of transformations are those that are continuous reversible deformations that admit translations, rotations, bending, expansion, and finally shears. The invariants of such transformations are defined as topological properties.

Topological Invariants and Physics

Independence from size and shape

Indeed the geometric method has served physics well, but there are many things in nature that obey rules that are independent from size and shape. For example, the Planck blackbody radiation distribution in frequency is independent from the size, shape, and even chemical makeup of the hot body that is radiating. The simply connected space of a hollow wave guide of any finite size or shape will always have a low frequency cut-off, but the co-axial cable, which is topologically not simply connected, can support DC currents. A physical system is conservative in a thermodynamic sense if the cyclic work vanishes, independent from the length (size and shape) of the process path. The closed surface integral of Gauss' law does not depend upon the shape or size of the surface but only on the number of charges contained in its interior. A flowing fluid can be in a laminar streamline state which can evolve into a chaotic turbulent state. A measurement with a finite compact apparatus of a infinite or non-compact property will always have an uncertainty associated with the measurement. These qualities of nature that do not seem to depend upon size and shape, and seem to be independent of continuous reversible deformations, can be defined and studied as topological properties.

The idea is that are basic physical properties that do not depend upon legislated standards, but are absolute in the sense that they are answers to questions such as:

How many?, or
Is it possible?, or more technically,
If a solution exists, is the solution unique?
If the solution is not unique, how many solutions are there?

A great deal of engineering and physical theories are built around the deterministic geometrical dogma which supposes that, given initial data, the name of the game is to be able to predict the outcome, and predict it uniquely. From a more topological perspective, it will become apparent that unique prediction may become impossible, but deterministic retrodiction can be achieved.

Observables as invariants of transformations

Concepts that do not depend upon size and shape can still be invariants of an equivalence class of transformations. Again, these invariants, which are not pure geometric invariants, may be used to define an equivalence class of transformations. The issue is how to define and observe these qualities of nature that do not depend upon size and shape. Consider a piece of notebook paper made out of flexible rubber material. The sheet has 3 holes along one side, and can be marked as 1,2,3,4 at its corners, in a prescribed sequence or orientation. Translate the sheet, and ask what are the invariants of the transformation. The answer is: the size, the shape, the number of holes and the orientation sequence, 1,2,3,4. are all invariant properties of the translation.

Now rotate the sheet; what stays the same? Again, size, shape, hole count, and orientation stay the same. However, in the case of rotation there exists one other invariant that is not in the class of translations. This additional rotational invariant is the fixed point that defines where the axis of rotation intersects the sheet. Translations are said to be transitive because there is no fixed point, while rotations are intransitive because there must be one fixed point. Recall that by Klein's definition, the four properties of size, shape, hole count and orientation are geometric properties.

Now take the this rubber sheet and deform it by pulling and stretching the sheet. What stays the same? The answer is not the size and not the shape, but the hole count (distorted holes, of course, in the deformed case) and the orientation sequence 1,2,3,4 do stay the same under the deformation. Those properties that stay the same under continuous and reversible deformation are defined to be topological properties. Note that topological properties are included in the class of geometrical properties, but the class of geometrical transformations are included in the class of topological transformations. Pure geometric properties will be defined as those properties which are invariant under translations and rotations only.

A topological property is defined as an invariant of a homeomorphism, or in more simple terms, a topological property is an invariant of a continuous and reversible deformation, while pure geometrical properties are not. Pure geometrical properties such as size and shape can evolve with respect to homeomorphisms. In this monograph, the process of studying invariants of transformations will be taken one step further,

for of physical interest to dissipative systems are those processes that are continuous but not reversible. Pure topological properties are not invariants of continuous but irreversible transformations. As pure geometrical properties evolve with respect to continuous and reversible transformations, pure topological properties evolve with respect to continuous but irreversible processes. For example, if the rim of one of the holes in the rubber sheet was grasped and pulled out of the sheet into the shape of a long trumpet with the rim becoming smaller and smaller until it collapsed to a point that could be glued together, then the topological property of hole count in the rubber sheet would have been changed from three to two during the deformation and gluing process. Note that it was the absolute number of holes that changed during this process of topological evolution which effectively collapsed one of the holes. It is important to note the topological change is quantized, for you can never have half a hole. The question of how many holes is absolute, for it is in relation to the integers.

Topological Evolution and Irreversibility

What are the invariants of the equivalence class of continuous, but irreversible transformations? Examples of such invariant properties are connectivity, compactness, and most important to this monograph, the concept of closure. Rather than carrying the words "continuous but irreversible" throughout the monograph, a biological concept will be used to define such processes: A continuous but irreversible process will be defined as an element of an equivalence class of transformations, and will be defined as an aging process. Like all transformations, the equivalence class of aging processes will be defined in terms of its invariants. The ability to develop a physical understanding of the aging process must be built upon the observable invariants
of such processes, and the dynamical theory of those topological invariants that can change during such processes. This dynamical theory will be called the theory of topological evolution.

Physical laws as topological statements

The ultimate goal of this monograph is to establish methods of distinguishing topological effects in physics from geometrical ones, to establish laws describing topological properties of matter, and in particular to establish the laws of physical topological evolution. Note that the first step is to go beyond the constraints of geometry and study strictly topological properties and the evolution of geometrical properties. The second step is to go beyond the constraints of topology to study the evolution of topological properties. The reader may not realize that he or she has often worked with topological concepts without knowing anything about topology, per se. For example, it will be demonstrated herein that the Maxwell theory of electromagnetism, without the geometrical constraint of a Lorentz symmetry group, is a statement about topological properties of space-time. It also will be demonstrated that the first law of thermodynamics is a topological statement of
cohomology. The flow of a Navier-Stokes fluid can admit solutions which are examples of an irreversible but continuous topological evolution.

Cartan's exterior calculus

After the basic concepts of topology are presented, the next step is to develop a thorough understanding of Cartan's theory of Exterior Calculus. Cartan developed his exterior calculus long before the word Topology became fashionable, but the key feature of Cartan's theory is that it transcends the geometrical constraints of tensor calculus and is truly a theory of topology and topological evolution. It was mentioned above that a topological property was an invariant of a homeomorphism. Technically, a homeomorphism is a map from an initial to a final state that has two qualities: 1) it must be continuous, and 2) it must be reversible in the sense that the inverse exists and is continuous. If topological evolution is to take place, then one or both of these qualities must not be true. Of particular interest to the developments in this monograph are those evolutionary processes which are continuous but not reversible. However continuity is not a geometrical idea; it is a concept that does not depend upon size and shape. A major goal will be the development of a useful topological structure, such that it can be decided whether or not a particular process is continuous, or not. Fortunately, the concept of a topological structure can be developed in terms of the Cartan calculus, such that
a decision can be made if a process is continuous or not. If the process is determined to be continuous, and if it can be shown that the topological properties change during the process, then the process is an aging process. That is the process is continuous but irreversible.

As the ultimate interest is with evolutionary processes that do not have continuous inverses, the emphasis on group theoretic methods that have so dominated the development of current physical theory will be given low priority. As the group concept requires the property of an inverse, it seems apparent to this author that such concepts, although very useful to geometrical problems, can not be at the heart of a theory that does not support a continuous inverse.

Perhaps the most important property or idea to this monograph is the concept of closure. The idea of closure is an invariant of a continuous but irreversible process. From set theoretic ideas, the idea of closure means that any pair of elements of a subset can be combined by a rule such that the resultant is still an element of the subset. Closure is perhaps the most fundamental property of a group. Elements of a vector space can be added together such that each sum is an element of the set of all basis elements multiplied by real numbers. The process of addition is closed. However, if two polar elements of a vector space are multiplied together by the method of the Gibbs cross product of engineering science, the resultant axial vector is not an element of the original subset of polar vectors. The Gibbs product is not closed. No engineer would ever add a torque to a force, or a linear momentum vector to an angular momentum vector, because they are not vectors of the same species.

Key observables in the understanding of the aging process are the concepts of closure and connectivity. Experimental methods to observe "closure" concepts must be devised if the notion of topological evolution is to be made practical. These notions may sound abstract and not useful, but when it is realized that the production of defects in a physical system, and the change of phase from solid to liquid, are exhibitions of topological evolution, then the ideas become more concrete.

Point set topology and metric de-emphasis

In order to establish a foundation for topological evolution, an introduction to topological ideas and definitions is presented in terms of point set methods for which the topological concepts can be exhibited in terms of simple examples. This expose of topology given in this monograph will not be complete, and will not cover all of topological theory. Only those parts of topology that the author feels are necessary and useful for the development of physical and engineering applications will be presented. A conventional introduction to topology often starts with a metric topology, but herein the concept of a metric is purposely avoided, as the idea of a metric is the essence of those geometrical qualities of size and shape. The conventional procedure is to develop the topological ideas in terms of a space with a euclidean or some Riemannian metric. Then the topological concepts are shown to be independent of the choice of metric. However, the notion of a metric is not needed, and the point set approach takes that point of view that the metric is just extra baggage that can often confuse the issues.