POINT SET TOPOLOGY

Introduction

In order to establish a foundation for topological evolution, an introduction to topological ideas and definitions is presented in terms of point set methods for which the topological concepts can be exhibited in terms of simple examples. This expose of topology given in this monograph will not be complete, and will not cover all of topological theory. Only those parts of topology that the author feels are necessary and useful for the development of physical and engineering applications will be presented. A conventional introduction to topology often starts with a metric topology, but herein the concept of a metric is purposely avoided, as the idea of a metric is the essence of those geometrical qualities of size and shape. The conventional procedure is to develop the topological ideas in terms of a space with a euclidean or some Riemannian metric. Then the topological concepts are shown to be independent of the choice of metric. However, the notion of a metric is not needed, and the point set approach takes that point of view that the metric is just extra baggage that can often confuse the issues.

Closed and Open Sets

Consider a set of elements {a,b,c,d,e} and a combinatorial process creating subsets which is symbolized, for example, by the brackets (ab) = (ba) or by (ade). Construct all possible combinations, and include the null set, 0. Define the “whole” set X as the combination X = (abcde), and the empty set as 0. Note that the order of the elements in the combinatorial process is presumed to be unimportant. The possible combinations are

5 singletons:	(a),(b),(c),(d),(e)
10 doubles:	(ab),(ac),(ad),(ae),(bc),(bd),(be),(cd),(ce),(de),
10 triples	(abc),(acd),..
5 quadruples	(abcd),(bcde),(cdea),(deab),(eabc).

From the set of all 32 possible combinations, it is possible to select many subset collections. Certain of these subset collections have the remarkable property of logical “closure” with respect to the process of set intersection. Closure means that the application of the operation on any number of elements of the collection produces some element that is already in the collection. A closed operation produces nothing new. As an example, consider the subset collection, or class of subsets, given by the 7 elements:

$$T1_{closed} := \{X,0,a,b,(ab),(bcd),(abcd)\}$$

Then note that the intersection of (ab) with say (abcd) produces the set (ab) which is an element of the original collection, T1. Similarly the intersection of the set (ab) with the singleton set (a) produces the singleton set (a), which is a member of the original collection. The process can be repeated for *any number* of intersections, and the result is that each resultant intersection is already an element of the collection, T1.

Now also note that the union of *any two* elements of the set is also contained within the set. The idea that a closed algebra can be built upon the notions of union and intersection, and that this algebra be an associative division algebra, is at the heart of the theory of logic. This idea that an arbitrary collection of subsets exhibits logical closure with respect to *arbitrary* intersection and *finite* union of its elements is said to define a topology of closed sets, $T1_{closed}$, on the set X.

Definition:

A topology $T1_{closed}$ on a set X is a collection or class of subsets that obey the following axioms:

Axiom1(closed set topology): X and the null set 0 are elements of the collection.
Axiom2(closed set topology): The arbitrary intersection of any number of elements of the collection belongs to the collection.
Axiom3(closed set topology): The arbitrary union of any pair of elements of the collection belongs to the collection.

The elements of these special collections, $T1_{closed}$, are defined to be "closed" sets. The compliments of the closed sets are defined as "open" sets. The open sets of the topology are the collection of subsets given by

$$T1_{open} = \{0,X,(bcde),(acde),(cde),(ae),e\}.$$

It is important to note that the same set of all combinations of subsets can support many topologies. For example, the subsets of the collection,

$$T2_{closed} = \{X,0,(bcde),(cde),(de)\},$$

are closed with respect to both logical intersection and union, and satisfy the axioms given above. Hence $T2_{closed}$ is a **different** topology built on the **same** set of points, X. The open sets of this topology are

$$T2_{open} = \{0,X,a,(ab),(abc)\}.$$

The compliments of closed sets are defined to be "open" sets, and they too can be used to define a topology axiomatically. In fact there are many different ways to define topologies. A subset can be open, or closed, or both, or neither relative to a specified topology. For example, with respect to the topology given by the closed sets,

$$T3_{closed} = \{X,0,a,(bcde)\},$$

(bcde) is both open and closed, and the set (bc) is neither open nor closed.

The topology of closed sets given by the collection,

$$T4_{closed} = \{X,0,(bcde),(abe),(be),(a)\},$$

has its dual as the topology of open sets

$$T4_{open} = \{0,X,a,(cd),(acd),(bcde)\}.$$

Note that this topology, T4, is a refinement of the topology, T3, in that T4 contains the closed (or open) sets of T3.

Remarks: In the definition of a topology when the number of elements of the set in not finite, the logical intersection of open sets is restricted to any pair, and the logical union of closed sets to restricted to any pair. There are many other ways to define a topology, but the concepts always come back to the idea of logical closure.

Limit Points

The next idea to be presented is the concept of a limit point. A standard definition states that a point p is a limit point of a subset, A, iff every open set that contains p contains another point of A. Given a subset, A, each point of X must be tested to see if it is a limit point of A relative to the topology specified on the points. If A is a singleton, it can have no limit points, for there are no other points of A. It follows that the limit points of a limit point (a singleton) is the null set. If the limit point of A consists of the singletons or points symbolized by dA, then d(dA) = 0. The set of limit points as a collection of singletons, {a,b,c..} will be denoted by dA, where the union of all limit points will be denoted by A^{limit}.

The symbol d may be viewed as a limit point operator; the symbol d. when applied to a set, A, means that each point p of the domain is tested against the specified topology to see if another point of A is included in each open set of the topology. Consider the subset A = (ab) and the topology given by $T1_{open}$. Now test each point relative to the collection $T1_{open}$:

$$T1_{open} = \{0,X,(bcde),(acde),(cde),(ae),e\}$$

a is not a limit point of (ab) because the open set (acde) does not contain b;
dA = 0 at a.
b is not a limit point of (ab) because the open set (bcde) does not contain a;
dA = 0 at b.
c is not a limit point of (ab) because the open set (cde) does not contain either a or b;
dA = 0 at c
d is not a limit point of (ab) because the open set cde does not contain either a or b;
dA = 0 at d
e is not a limit point of (ab) because the open set cde does not contain either a or b;
dA = 0 at e.

In other words, the subset (ab) has no limit points in the topology given by T1. The limit point set, designated in this monograph as A^{limit}, is given by

$$A^{limit} = \{0\} \text{ for } dA = 0 \text{ at all points of X.}$$

Now make the same tests with regard to the same subset A = (ab), but this time relative to the topology given by $T2_{open}$.

$$T2_{open} = \{0,X,a,(ab),(abc)\}$$

a is not a limit point of (ab) because the open set a is a singleton;
dA = 0 at a.
b is a limit point of (ab) because all the open sets that contain b also contain a;
dA ≠ 0 at b.
c is a limit point of (ab) because the open sets abc and X which contain c also contain a or b, which are points of A;
dA ≠ 0 at c.
d is a limit point of (ab) because the open set X which contains d contains another point of A;
dA ≠ 0 at d.
e is a limit point of (ab) because the open set X which contains e contains another point of A;
dA ≠ 0 at e.

Hence, the points b,c,d and e are limit points of A = (ab) relative to the topology $T2_{open}$

$$A^{limit} = (bcde)$$

With respect to the topology of $T4_{open}$, the set (ab) is such that

$$T4_{open} = \{0,X,a,(cd),(acd),(bcde)\}$$

a is not a limit point of (ab) because the open set a is a singleton;
dA = 0 at a.

b is not a limit point of (ab) because the open set (bcde) does not contain a;
$dA = 0$ at b.
c is not a limit point of (ab) because the open set (cd) which contains c does not contain a or b;
$dA = 0$ at c.
d is not a limit point of (ab) because the open set (cd) which contains d does not contain a or b;
$dA = 0$ at d.
e is a limit point of (ab) because the open set (bcde) which contains e also contains b of A;
$dA \neq 0$ at e.

The limit set of (ab) relative to $T4_{open}$ becomes

$$A^{limt} = \{e\}.$$

Note that the set of limit points as a collection, or a class of sets, may or may not have limit points. If the limit set is a singleton, then the limit points of the set of limit points is the null set. However, consider the limit set $A^{limit} = \{bcde\}$ of the set (ab) relative to the topology T2. Then the limit points of A^{limit} are the points (b,c,d,e). In other words $dA^{limit} \neq 0$ necessarily, but $ddA = 0$.

Closure

The closure of a set is defined to be the union of the set and its limit points. Another way to state this idea is to say that that the closure of a set is the smallest closed set that contains the set of interest. Note that a closed set contains its limit points, if any exist. In the example topologies given above, either definition gives the same result. For example, the closure of the set A = (ab) relative to the topology T1 is equal to the union of A = (ab) and its limit points, which is the null set.

$$A^c = A \cup A^{limit} = (ab).$$

Also (ab) is the smallest closed set that contains (ab). Note that A = (ab) is a closed set, and has no limit points relative to the topology T1. Hence the closure of the closure is the same set (ab).

However, the closure of A relative to the topology T2 is

$$A^c = A \cup A^{limit} = (ab) \cup (bcde) = (abcde)\ ,$$

which is the whole set. When the closure of a subset is the whole set X, the subset is said to be dense in X relative to the specified topology. Again, (abcde) is the smallest closed set of the topology T2 that includes the specified set of interest, (ab).

The closure of (ab) relative to the topology T4 is

$$A^c = A \cup A^{limit} = (ab) \cup (e) = (abe).$$

Note that the closure of a subset is equal to the smallest closed set that contains the subset. Every closed set is its own closure. A closed set may or may not have limit points, but if it does have limit points they are contained within the (closed) set.

Interior

When emphasis is placed on open sets rather that closed sets, other ideas come to the forefront. In particular, the concept dual to the notion of *closure* is the concept of *interior*. While closure asks for the smallest closed set that covers any specified subset, the idea of interior asks for the largest open set included in the specified subset. The interior of a set can be empty (for there may be no open sets other than the null set contained within the specified set)!

For example, the subset (ab) has no interior relative to the topology $T1_{open}$ for there are no open set of T1 contained in (ab). However, relative to the topology T2, the interior of (ab) is itself, (ab). Relative to the topology T4, the interior of (ab) is the singleton, (a):

Int A = {0} relative to T1
Int A = (ab) relative to T2
Int A = (a) relative to T4.

The set (abe) has an interior (ae) relative to T1and an interior (ab) relative to the topology T2.

Exterior

The exterior of a specified set is the interior of the compliment of the specified set. The compliment of (ab) is the set (cde) which has the interior (cde) relative to T1 and has no interior relative to the topology T2 . Relative to the topology T4, the exterior of (ab) is the set (cd).

Ext A = {cde} relative to T1
Ext A = (0) relative to T2
Ext A = (cd) relative to T4.

The Boundary

The points that make up the boundary of a subset are the union of those points that are not included in the interior or the exterior. However, the union of the points that make up the boundary may have subsets that are not connected. Consider a solid disk. The points that make up the rim of the disk forms its boundary. Now punch a hole in the disk. The collection of points that make up the outer rim and the inner hole now form the boundary of the disk. The two sets of boundary points are not connected.

Similar to the limit point operator, d, a boundary operator, ∂, may be defined in terms of a procedure, such that when ∂ is applied to the set A, it implies that a test is made at each point p to see if p is an element of the interior or of the exterior of the selected subset A. If the test fails, then ∂A # 0 and the point is a boundary point. If the point p is an element of the exterior or interior of A, then ∂A = 0 at the point p. The *boundary*, bA, of the set A is defined as the union of all boundary points.

Example 1

As a first example, consider again the set A = (ab) and the T1 topology. The set A = (ab) has no interior, but the exterior of (ab) is the set (cde). Therefor the points not included in the interior or the exterior are the points (a,b). The boundary set, bA, consists of the union of the boundary points (a,b). In this first example, then

$$A^{limit} = \{0\}, \quad \text{but } bA = (ab) \subset A^c$$

It follows (in a non-intuitive manner) that

$$A \cap A^{limit} = 0, \quad \text{but} \qquad A \cap bA \neq 0$$

The boundary set exists even though the limit set does not!

Example 2

Relative to T2, the set (ab) has an interior set (ab), no exterior set, but a boundary set is bA = (cde). In this case the boundary is included in the closure,

$$A^{limit} = (bcde), \quad bA = (cde) \subset A^c = (abcde),$$

and

$$A \cap A^{limit} = b\ , \quad \text{but} \quad A \cap bA = 0.$$

Although in this topology all boundary points are limit points, there exist limit points (b) that are not elements of the boundary.

Example 3:
Relative to T4, the set (ab) has an interior set (a), and exterior set (cd) and therfore a boundary set (be),

$$A^{limit} = (e), \quad bA = (be) \subset A^c\ ,$$
$$A \cap A^{limit} = 0, \text{ but } \quad A \cap bA = b.$$

It is apparent in this topology that the boundary points contain limit points, but there are boundary points which are not limit points!

In all cases, note that the union of the interior and the boundary is equal to the union of the set and its limit points. The boundary is always included in the closure, but the boundary may contain points which are not limit points.

$$A^c = \text{Int}A \cup bA = A \cup A^{limit}.$$

These examples point out that there exist certain correspondences between limit points and boundaries, but they are not necessarily the same concept.

Isolated Sets

A set is said to be isolated if its intersection with its limit set is empty. For the examples given above, the set (ab) is isolated with respect to the topologies T1 and T4, but is not an isolated set with respect to the topology T2.

Neighborhoods

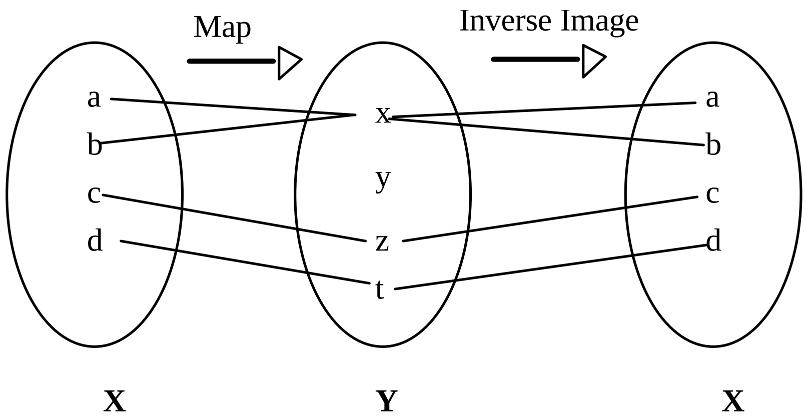
The idea of a neighborhood is the inverse of the idea of an interior. A neighborhood of a point is a superset for which the point in question is an interior point. A point can have many neighborhoods.

Continuity

Now comes a major issue of this monograph. Continuity of a transformation is defined relative to the topologies that may exist on the initial and final states. Let the set of points X with topology $T1_{open}$ be mapped into the set of points Y with the topology $T2_{open}$. Then the **map is defined to be continuous** iff the closure of every subset of the initial state relative to T1 is included in the closure of the image of the final state relative to the topology T2.

Another (more common) definition of continuity is given in most textbooks by the statement that the inverse image of every open set of Y relative to T2 is an open set of X relative to the topology T1. Continuity is a property of a map between the points that make up perhaps two different topologies.

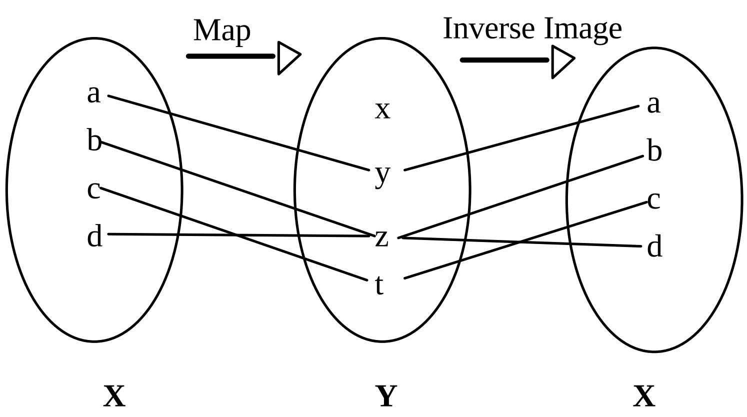
As a first example of the more common definition of continuity, consider the transformations given on X to Y by the following diagram:



Let the open sets of the two topologies be TX = {X,0,(a),(ab),(abc)}   and TY = {Y,0,(x),(y),(xy),(yzt)}. The open set (x) of Y has a preimage (ab) which is open in X. The open set (yzt) of Y does not have a preimage as an open set in X. Hence the Map is not continuous. Note that the open set (y) on Y does not enter into the test for it is not connected to points of X.

Also note that the inverse image map is not continuous either, for every open set TX of the inverse image does not have a preimage which is an open set in TY. The open set (abc) of X does not have a preimage as an open set of Y. The open sets (a) and the open sets (ab) of X do have a preimage as the open set (x) of Y. Hence the inverse image is NOT continuous.

As a second example, consider the mapping:



Let the open sets of the two topologies be TX = {X,0,(a),(ab),(abc)}   and TY = {Y,0,(x),(y),(xy),(yzt)}. The open set (y) has a preimage (a) which is open. The open set (yzt) has a preimage (abc) which is open. Hence the Map is continuous. However, the inverse image mapping is not continuous for the open set (ab) has a preimage as (yz) but (yz) is not an open set of Y.

Now comes the key result:

All topological properties are invariant only with respect to evolutionary processes that are described by continuous maps with continuous inverse images.

The concept of continuity based on open sets exemplified above is much too hard to apply in physical systems, because it presupposes knowledge of the inverse image. The inverse image often is either hard to define, or it does not exist. The equivalent definition of continuity which is much more useful is based upon the concept of closure. To repeat, a process is said to be continuous if and only if for every subset A of the initial state, X, the closure of the subset, A^c , relative to the topology of X is included in the closure of the image of the subset, with respect to the topology of the final state, Y. In physical evolution, the process mapping can be determined by experiment (or theory), but whether or not it is continuous is a problem to be solved. With the definition based on closure, all that has to be determined is what the closures are on the initial state for every subset relative to the topology of the initial state (which is usually given information). Next construct the forward image of the closures. Then test the forward (not inverse) images of the subsets for their closures with respect to the final state. If the closures are included in the closures of the forward images, the map is continuous. Surprisingly this method turns out to be not too difficult when the evolutionary mapping is at least 2 times differentiable.

Now if it is known that topology of the final state is not the same as the topology of the initial state, and if can be shown that the forward mapping is continuous, then the conclusion is reached that either the inverse image mapping is NOT continuous, or it does not exist. In either case it could be said that the continuous evolutionary process is **irreversible**. Such is the basis of a theory of the aging process.

REMARKS

All of the bold faced concepts defined above , such as the limit set, a boundary set, closure, etc., are what are called topological properties. The list of topological properties is not exhausted by the above collection of ideas. Topological properties are preserved with respect to continuous and reversibly continuous mappings between an initial state and a final state. (Some of the properties are preserved if the mapping is only continuous; that is, the inverse mapping does not have to be continuous and the property is preserved.) The concept of invariance with respect to mappings is at the heart of almost every physical theory. The conservation laws (of energy, momentum, angular momentum, etc.) are concepts of evolutionary invariance.

Much of current physical theory has historically emphasized the **boundary** and **open set** point of view. It turns out that these ideas are natural with respect to what in tensor analysis is called a contravariant point of view. Contravariant things are like velocity and acceleration of a point. In physics, these contravariant notions correspond to the concept of particles. In topology, these ideas lead to a specialization called homology theory.

Another point of view places emphasis on the concepts of **limit points** and **closure.** It may be shown that these ideas are natural with respect to what in tensor analysis is called the covariant point of view. In physics, these notions correspond to the concept of waves and fields. In topology these ideas lead to a specialization called co-homology. Later it will be shown that the first law of thermodynamics is essentially a statement about cohomology theory. All of these ideas above are independent from the concept of distance, which is often an elementary building block of physical measurement.

When such geometrical constraints (such as a metric, or things involving size and shape) are placed on a physical system, then the particle and the wave point of view, or more precisely, the contravariant or covariant point of view become an alias and an alibi. That is, the ideas of one point of view can be converted one into the other; Moreover, it is then impossible to *measure* intrinsic differences between the two points of view. The evolution of the contravariant set can be compared to the evolution of the covariant set, with identical comparisons found in both the initial and the final states, if the topological properties are preserved during the evolution.

However, if the evolutionary process does not preserve the topological properties, then differences between the two points of view lead to measurable differences. The concept of aging is one of these differences.

It should be noted for later reference that the concepts of isolated, closed and open sets form the basis of the theory of thermodynamics. It is therefor apparent that thermodynamics is a study of topological properties of matter. It is unfortunate that almost no engineering literature, and very little physical literature, emphasizes this fact.

Isolated, Closed, and Open physical systems form the basis of Thermodynamics, which is, therefore, a study of the Topological Properties of Matter.

Problems:

1. Deternine which of the following classes of subsets form a topology on X = (abcde).
   Are the sets selected open or closed?

   a. T1 = {X, 0, (a), (cd), (acd), (bcde)}
   b. T2 = {X, 0, (a), (cd), (acd), (bcd)}
   c. T3 = {X, 0, (c), (be), (abe), (bcde)}
   d. T4 = {X, 0, (a), (be), (abe), (bcde)}

2. For those collections above find the limit sets if the subsets form a topology

3. Let X = {A,F,H,K} be four points.

   Show that CT:= $\{ X, 0, A, H, A\cup F, H\cup K, A\cup H, A\cup H\cup K, A\cup F \cup H\}$

   defines a topology of open sets.

   Find all limit points, closures, interiors, and boundaries of all subsets

   This topology will be defined as the Cartan Topology, and is important for the study of relativistic dynamical systems. The "point" A will be put into correspondence

with the Action for the physical system. In electromagnetism it will be defined in terms of the potentials. The set F turns out to be the “limit points” for A.
In electromagnetism it turns out to be the collection of electromagnetic **E** and **B** Fields.