Adventures in Applied Topology Series

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Non-Equilibrium and Irreversible Thermodynamics - from a Topological Perspective.

Sections 3.5 and 3.6 are still very rough

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Chapter 1
A TOPOLOGICAL PERSPECTIVE

1.1 Introduction

A major objective of this monograph is to establish a topological, non-statistical, link between thermodynamics and mechanical, electrical, or hydrodynamic physical systems. A particular goal is to develop a method of describing the differences, and how and when such differences occur, between

Equilibrium and non-equilibrium physical systems, and

Reversible and irreversible evolutionary processes acting on such systems.

The methods which have been developed are based upon Cartan's calculus of exterior differential forms [Flanders 1963], [Cartan 1899]. Exterior differential forms are objects, which, in contrast to tensors, are well behaved with respect to differentiable (continuous) mappings that do not have an inverse (and do not preserve topological properties), as well as with respect to diffeomorphisms, which are differentiable invertible continuous mappings (and which preserve topological properties). Evolutionary processes will be defined in terms of the action of the Lie differential with respect to vector direction fields acting on differential forms [Marsden 1994]. The Lie differential acting on differential forms is not confined by the diffeomorphic constraints of tensor analysis, and can treat problems of topological change. The method goes beyond the more standard "extremal" techniques based upon the calculus of variations. In most of that which follows, the functions used to define the physical systems will be assumed to be C2 differentiable. The functions that describe processes most often will be assumed to be C2 differentiable as well, but certain C1 (tangential discontinuities) and C0 (shocks) processes are of physical interest.

A fundamental result can be expressed by the statement: "Topological change is a necessary condition for a thermodynamic process to be irreversible". Irreversible processes, related to the arrow of time and the biological aging process, require topological evolution and topological change. Current physical theories that describe evolutionary processes (for example, Hamiltonian or Unitary dynamics) usually are formulated in terms of homeomorphisms that do not permit topological change.

The presentation herein is not meant to be a textbook on Cartan's theory of exterior differential forms, nor a textbook on abstract topology. Instead an effort has been made meld Cartan's methods and topological ideas in a manner that would be useful to the applied researcher and engineer. At the time of writing, not too many physicists, and almost no engineers are conversant in the exterior calculus or topology. Of course, some familiarity with the fundamentals of each said discipline is required, and to that end several terse, to the point, presentations (with examples) are given in the Appendix. Most of the useful topological ideas can be rapidly absorbed in terms of point set topology with its metric de-emphasis. In fact, one of the beauties of using the Cartan calculus is that it constructs a differential topology that is free of the metric and connection constraints of differential geometry. It is extraordinary that the Maxwell theory of Electromagnetism (when based on the fields, **E** and **B**, distinct from the fields, **D** and **H**) was one of the first physical theories to be recognized as being a topological theory [Vandantzig 1934], independent from a choice of metric, or connection. Geometric constraints (such as constitutive constraints between the two distinct sets of fields) merely refine the topological features of the fundamental theory. In this monograph, it will be demonstrated how thermodynamics may be considered fundamentally as a topological theory, also independent from metric and connection.

Pick up a modern text in classical thermodynamics and note the appearance of the following words used in describing fundamental thermodynamic concepts:

1. Isolated
2. Closed
3. Open
4. Number of moles (disconnected parts)
5. Closed Cycles
6. Integrability
7. Extensive (homogeneous degree 1) variables
8. Intensive (homogeneous degree 0) variables

Now go to Schaum's Outline, "General Topology" [Lipschutz 1965], or some other textbook on elementary topology, and check the index for these terms. All of these terms have precise definitions in topology, without the imposition of geometric constraints of size, shape or scales. In short, it would appear that Thermodynamics has its foundations in topology, and should be treated as a topological theory from the outset. This is the topological perspective of thermodynamics adopted herein.

In 1974 it was suggested that a certain extension to Hamilton's principle [RMK 1974], [RMK 1975 d] could be made such that the evolutionary processes considered would describe dissipative mechanical systems. Cartan had proved that extremal vector fields, which satisfy the Cartan-Hamilton equation, $i(\mathbf{V})dA = 0$, are generators of Hamiltonian dynamical processes [Cartan 1958 (1922)]. Rather than study such "extremal" vector fields, it was suggested to consider those processes that satisfy the extended equation: $i(\mathbf{V})dA = \Gamma A + d\theta$. Throughout this current presentation, and in the older articles, it is subsumed that a physical system may be described adequately by a 1-form of Action, A, and a physical process may be defined in terms of a dynamical system generated by a vector field, $\mathbf{V}$. It was not appreciated in 1974 that the topological domain of the extremal conservative (Hamiltonian) systems was a contact manifold of odd topological dimension, while the topological domain of the suggested dissipative extension was a symplectic manifold of even topological dimension. (Extremal solutions do not exist on even dimensional manifolds of maximal rank). Currently, the concept of topological dimension seems to be intimately connected to the differences between conservative versus dissipative processes. In fact, as is developed in that which follows, irreversible turbulent thermodynamic processes are artifacts of Pfaff topological dimension 4 (or more). Irreversibility requires that the evolutionary topology of the initial state is not the same as the topology of the final state. It sometimes comes as a surprise to realize that such changes can occur continuously. These features of continuous topological evolution are discussed in detail in Chapter 5, below.

A symplectic manifold is defined by the non-zero domain of an exact 2-form, $F = dA$. The concept of Hamiltonian mechanics can be extended to symplectic (even dimensional) manifolds, where the Bernoulli-Hamiltonian constraint is of the form $i(\mathbf{V})dA = d\Theta$ or the more general Helmholtz constraint, $di(\mathbf{V})dA = 0$. Such processes (although reversible) are of interest for they admit topological defects of the Bohm-Aharanov type. These topological defects are related to the Work 1-form, $W = i(\mathbf{V})A$, produced by the process $\mathbf{V}$ acting on the physical A. system, are not due to the closed but not exact parts of the 1-form of Action.

Almost symplectic manifolds are defined by a 2-form G which is closed but not exact. These even dimensional manifolds G can have compact defect domains without boundary. Such topological structures have been studied by Fomenko [Bolsinov 1990]. From electromagnetic theory, it becomes apparent that the non-exact 2-form (almost symplectic) G is to be associated with the defect structures called charge, and extensive thermodynamic variables, $\mathbf{D}$ and $\mathbf{H}$, while the exact 2-form (symplectic) F is to be associated with extensive thermodynamic variables, $\mathbf{E}$ and $\mathbf{B}$.

The 1-form of Action (Lagrangian) point of view has its advantages, for the fundamental 2-form of a symplectic domain is deduced by construction, $\omega \Rightarrow F = dA$. The disadvantage is that almost all symplectic domains so constructed are not compact without boundary. This apparent flaw becomes an advantage when it is appreciated that such non-compact domains are precisely that which is needed to describe closed (but not isolated) or open thermodynamic systems.

Most classical "laws of physics" are based upon the dogma that a useful physical theory of evolution must give a unique prediction starting from a given set of initial conditions. Combining this constraint with a mathematical description of physical systems in terms of geometrical tensor fields leads to evolutionary processes which preserve topological properties and are (therefore) "reversible". The ubiquitous assumption of uniqueness of predicted solutions, and/or homeomorphic evolution, are topological constraints on "classical mechanics" that eliminates any time asymmetry. The point of departure in this article assumes that reasonable physical laws must be capable of describing topological change, and when this feature is encoded in mathematical form, the laws of physics are no longer necessarily reversible. Hence, in this article, the Boltzmann paradox will be resolved in terms of a theory based upon *Continuous Topological Evolution* [Chapter 5]. It is presumed that the presence of a physical system establishes a *Topological Structure* [Chapter 4] on a base space of independent (but ordered) variables. When a specific evolutionary process is applied to this physical system, the topology becomes refined. This topological perspective is different from, but similar to, the geometric perspective of general relativity, whereby the presence of a physical system is presumed to establish a *metric* on a base space of independent variables. Note that a given base may support many different topological structures; hence a given base may support many different physical systems.

It will become evident that physical systems require two topological structures, one based upon an exact 2-form, $F = dA$, and its associated symplectic manifold, and the other based upon a non-exact 2-form, G, which may or may not be closed, but when closed, $dG = 0$, leads to an *almost* symplectic structure with compact topological defects.

1.2 Topological properties

The idea that the presence of a physical system establishes a *topological structure* on a base space of independent variables is different from, but similar to, the geometric perspective of general relativity, whereby the presence of a physical system is presumed to establish a *metric* on a base space of independent variables. The topological features of the physical system are presumed to be encoded in terms of exterior differential forms, which - unlike tensors - are functionally well behaved with respect to differentiable maps that are not invertible. Note that a given base may support many different topological structures; hence a given base may support many different physical systems. In particular, the topology associated with a 1-form of Action need not be the same as the topology associated with the 1-form of heat or the 1-form of Work, even though the base variables are the same for each 1-form. The Pfaff topological dimension can be different for each of the 1-forms.

For maps, between base sets, that are C1 differentiable (hence smooth), but are not invertible, it is impossible to predict uniquely the functional forms of covariant or contravariant vector fields, constructed over a final base set, in terms of functional forms given on an initial base set [RMK 1976 b]. Point-wise values of the tensor fields in certain cases may be predicted, but the functional forms are never predictable with respect to such non-invertible maps. Hence, classical theories based on tensor fields, which can describe geometrical evolution, will fail to describe topological evolution. It may be surprising to note that (with respect to such non-invertible, non-homeomorphic, maps) it is possible to retrodict the functional forms of covariant vectors and contravariant vector densities on the initial base set in terms of the given functional forms on the final base set. For differentiable evolutionary processes that are diffeomorphisms, topology does not change and both prediction and retrodiction of tensor fields is possible. For differentiable evolutionary processes which are not homeomorphisms, topology changes, and deterministic prediction fails, but deterministic retrodiction remains possible. Hence the feature of topological evolution imposes a sense of asymmetry with respect to an evolutionary parameter - the arrow of time is an artifact of topological change.

Although C1 non-invertible maps are not homeomorphisms, and therefore the topology of the initial state and the topology of the final state are not the same, such maps can be continuous. Continuous topological evolution is not an oxymoron, for topological continuity is defined such that the limit points of every subset in the domain (relative to the topology on the initial state) permute into the closure of the subsets in the range (relative to the topology on the final state). The initial and final state topologies need not be the same!

A physical property which is independent of continuous deformation, and is independent from geometric concepts of size and shape, is a primitive example of a topological property. However, not all topological properties fit this useful, but imprecise, description. As examples, note that the number of holes in a rubber sheet is a topological property, and is independent of the continuous deformation of the rubber sheet into different sizes and shapes. The Planck black body radiation distribution of a hot body is a topological property, for the distribution of radiation frequencies depends only on temperature, but not on the size and shape of the heated sample. Deformation invariants often can be encoded in terms of multi-dimensional integrals. As the elements of the integrand and the integration chain evolve, the value of the integral may be an evolutionary invariant, even though the domain and integration chain are deformed by the evolutionary process. Of special interest are those integral deformation invariants where the integration chain is a closed cycle. Such objects lead to topological "quantum-like" concepts, for the values of the integrals of closed but not exact exterior differential forms over different cycles have (by deRham's theorems) rational ratios. If the evolutionary process causes the topological quantum number to change, then the process describes a topological quantum transition. Surprisingly, processes of topological evolution can change topology in a continuous manner. A soap film connected to a double loop of wire will form the non-orientable surface of a Moebius band. Deformation of the wire into a single loop will cause the soap film to form a disk surface which is orientable. The topological property of orientability has changed continuously.

More precisely, a topological property is defined as an invariant of a homeomorphism. A homeomorphism is a map from initial to final state, which is continuous and has a continuous inverse. If the homeomorphism is C1 differentiable both ways, then the map is called a diffeomorphism. Diffeomorphisms are the transformations used to define tensors and most geometric properties (and invariance with respect to diffeomorphisms is a constraint employed in many physical theories which are based upon the calculus of variations). Recall Klein's concept of a (euclidean) geometric property as being defined in terms of the invariants of rotations and translations (which are diffeomorphisms). Yet diffeomorphisms are specialized homeomorphisms which preserve topology. It follows that tensor analysis, so useful in studying geometric concepts, cannot be used effectively to describe topological change, and therefor tensor analysis is inadequate to describe irreversible evolution, where topological change is a necessary condition. However, continuous C1 processes need not be homeomorphisms, and therefor can be used to describe topological change. Exterior differential forms are mathematical objects that are well behaved in a retrodictive sense with respect to functional substitution of C1 continuous, but not invertible, maps; tensor fields are not. It follows that Cartan's exterior differential forms become the mathematical objects of choice for describing continuous topological evolution, and therefor Cartan's mathematics is the mathematics of choice for a theory of irreversible Thermodynamic processes.

A key topological property is that of dimension. However, the concept of topological dimension is somewhat different from the concept of geometrical dimension. For purposes of the theory developed herein, the topological structure imposed upon a base variety of m independent variables can be used to determine the "Pfaff topological dimension", n, which is to be distinguished from the "geometric dimension" of the base variety, $n \leq m$. The primary feature of a topological structure is that it can be used to determine when an evolutionary process involving topological change (such as the change in topological dimension) is continuous. Topological change can occur both continuously and discontinuously. However, in this article, the focus is on *continuous* topological evolution. Herein it will be demonstrated that thermodynamic "relaxation" from some initial configuration to a state of "equilibrium" can be described by a sequence of continuous processes that cause the topological dimension to change from some initial value n to a final value $n \leq 2$.

1.3 Evolutionary Processes

Evolutionary processes can preserve the topological properties of a physical system, or they can change them. In this monograph, those processes which can continuously change the topology of a physical system are of major interest, for topological change is a necessary requirement of thermodynamic irreversibility. Intuitively, a process applied to a physical system can be arbitrary, which implies that the process V is not necessarily dictated by the topological structure of the physical system, A. This intuitive idea is not precise. As noted above, certain unique extremal processes, V_E, are determined by the unique null eigen vector of the anti-symmetric matrix of maximal rank, dA. Such null eigen vectors exist only on topological domains of odd maximal rank, typically in this monograph, equal to 3. The Extremal vector satisfies the equation:

Extremal Vectors $\mathbf{V}_E$

$$i(\mathbf{V}_E)dA = 0, \tag{1.1}$$
$$\Omega = dx^1\hat{}dx^2\hat{}...dx^n \quad \text{(n odd)} \tag{1.2}$$
$$L(\mathbf{V}_E)A = d(i(\mathbf{V}_E)A). \tag{1.3}$$

On the other hand, on topological domains of even maximal rank, extremal vectors do not exist, and yet there is a unique vector, the Topological Torsion vector, V_T, that is determined by the topological structure of the physical system, A. Such unique vectors exist only on topological domains of even dimensional maximal rank, typically in this monograph, equal to 4. The Topological Torsion vector satisfies the equation:

Topological Torsion Vector $\mathbf{V}_T$

$$i(\mathbf{V}_T)\Omega = A\hat{}dA...dA, \tag{1.4}$$
$$\Omega = dx^1\hat{}dx^2\hat{}...dx^n \quad \text{(n even)} \tag{1.5}$$
$$L(\mathbf{V}_T)A = i(\mathbf{V}_T)dA = \sigma A \tag{1.6}$$
$$i(\mathbf{V}_T)A = 0 \tag{1.7}$$

When $\sigma = 1$, the Topological Torsion vector has been called the "Liouville vector field" (see page 65 [Liebermann 1987]). If for some reason, the topological structure of the physical system, A, evolves (or decays) from a domain where the rank of dA is even to a domain where the maximal rank of dA is odd, then the components of $\mathbf{V}_T$ become proportional to a characteristic direction field, which has extremal properties.

The extremal processes, $\mathbf{V}_E$, can always be put into correspondence with a Hamiltonian process, but those processes represented by a direction field component proportional to V_T do not have a Hamiltonian representation, unless the divergence of V_T is zero. These features, apply not only to topological dimension 3 and 4, but are also valid for topological dimensions which are odd (V_E for $n = 2k+1$) or even ($\mathbf{V}_T$ for $n = 2k+2$). The topological refinement induced by the process forms two categories related to a Contact structure ($n = 2k+1$) or a Symplectic structure

$(n = 2k + 2)$. Note that the concept of "uniqueness" relates to the direction field V that represents a process, but such direction fields as a vector field are unique only to within an arbitrary factor.

Continuous evolution includes two equivalence classes of processes: those processes that preserve topological features (homeomorphisms) and those processes that do not (non-homeomorphisms). The latter class is the class of processes that describe continuous topological evolution, and it is this class which is studied extensively in this article. As will be demonstrated below, the topological structure of a physical system leads to the consideration of odd dimensional integrals of the type $\int_{2k+1} A\hat{}dA...$ and even dimensional integrals of the type $\int_{2k+2} dA\hat{}dA....$ If these integrals are deformation invariants they represent a topological property that is an evolutionary invariant. Of particular interest is the set of even and odd dimensional integrals where the integration chain is a closed cycle. The class of continuous processes that describe topological change can be divided into two distinct classes, A and B.

Class A. This equivalence class of non-homeomorphic continuous processes preserves the even dimensional integrals as deformation invariants, but causes the values of the odd dimensional integrals to change. The Helmholtz conservation of vorticity concept is a classic example of when an even dimensional topological property is preserved. Such processes will be called Helmholtz processes, in general. The Poincare integral invariants of classical mechanics are further examples of even dimensional integral invariants. Extremal and Hamiltonian processes are special cases of Helmholtz processes. However, all such Helmholtz processes, which can produce topological change of the odd dimensional integrals, are thermodynamically reversible. Topological change is a necessary, but not sufficient, condition for thermodynamic irreversibility.

Class B. This equivalence class of non-homeomorphic continuous processes causes the values of both the even and the odd dimensional integrals to change. Both the odd and the even topological features of the physical system are modified. It is this equivalence class that contains those processes which are thermodynamically irreversible. Without being too precise, both energy and angular momentum must change if a process is to be thermodynamically irreversible. Pasting together is a continuous process for which the topology of the final system state is not necessarily the same as the topology of the initial system state. Separation or cutting into parts is a discontinuous process for which the system topology of the final state is not the same as the system topology of the initial state. The obvious topological property that changes is the number of parts. Projections from higher dimensions to lower dimensions are classic examples of many to one differentiable maps that are not invertible. The obvious topological property that changes is the property of dimension. Consider a flat putty disc in the shape of an annulus. Deform the putty continuously such that the points that make up the central hole are pasted together. On the other hand make an interior cut in a disk of putty and discontinuously separate the points to make a hole. The obvious topological property that changes is the number of holes. (Discontinuous processes are more or less ignored in this presentation.)

1.4 Fundamental Axioms and Notable Results

1.4.1 Axioms

The topological view of thermodynamics described herein is based on three axioms.

1. Thermodynamic physical systems can be encoded in terms of a 1-form of covariant Action Potentials, $A_k(x, y, z, t)$, on a 4 dimensional abstract variety of ordered independent variables, $\{x, y, z, t\}$. The variety supports a volume element $\Omega_4 = dx\hat{}dy\hat{}dz\hat{}dt$.

2. Thermodynamic processes are assumed to be encoded, to within a factor, $\rho(x, y, z, t)$, in terms of contravariant vector direction fields, $\mathbf{V}_4(x, y, z, t)$.

3. Continuous topological evolution of the thermodynamic system can be encoded in terms of Cartan's magic formula (see p. 122 in [Marsden 1994]). The Lie differential, when applied to a exterior differential 1-form of Action, $A = A_k dx^k$, is equivalent *abstractly* to the first law of thermodynamics.

In effect, Cartan's methods can be used to formulate precise mathematical definitions for many thermodynamic concepts in terms of topological properties - without the use of statistics or metric constraints. Moreover, the method applies to non-equilibrium thermodynamical systems and irreversible processes, again without the use of statistics or metric constraints.

In order to make the equations more suggestive to the reader, the symbolism for the variety of independent variables has been chosen to be of the format $\{x, y, z, t\}$, but be aware that no constraints of metric or connection are imposed upon this variety. For instance, it is NOT assumed that the base variety is euclidean.

1.4.2 Notable results

The result of employing these axioms will be to demonstrate that:

1. Thermodynamics is a topological theory.

2. Topological change is a necessary condition for thermodynamic irreversibility.

3. Pfaff topological dimension 2, or less, defines an equilibrium system. The topological structure on domains of topological dimension $n \leq 2$ never admit a continuous process which is thermodynamically irreversible.

4. The topological structure on domains of odd topological dimension $n = 3, 5, 7..$ can be used to deduce (to within a factor) a unique continuous extremal process, V_E, which may be chaotic, but yet is thermodynamically reversible, and can have a Hamiltonian representation.

5. Thermodynamic irreversibility is an artifact of Pfaff topological dimension 4 or more. The topological structure on domains of even topological dimension $n = 4, 6, 8...$can be used to deduce (to within a factor) a unique continuous process, V_T, which is thermodynamically irreversible. In this sense, thermodynamic irreversibility is an artifact of topological dimension $n \geq 4$.

6. The change of the Pfaff topological dimension can produce topological defects and thermodynamic phase changes.

7. The assumption of uniqueness of predicted solutions to a Pfaffian equation (which implies a Pfaff Topological dimension 2 or less) and Homeomorphic evolution are different constraints on classical mechanics that eliminates any time asymmetry.

8. All Hamiltonian and Symplectic-Bernoulli processes are thermodynamically reversible.

9. The functional forms of tensor fields with arguments in terms of the base variables of the final state are not well defined in terms of the functional forms of tensor fields with arguments in terms of the base variables of the initial state, unless the map from initial to final state is a diffeomorphism (which preserves topology). On the other hand, the functional forms of those alternating tensor fields which are coefficients of exterior differential forms, and with arguments in terms of the base variables of the initial state, are well defined in terms of the functional forms of tensor fields with arguments in terms of the base variables of the final state, even when the map from initial to final state describes topological evolution. Hence an Arrow of Time asymmetry is a logical result when topological evolution is admitted, but does not appear if the evolution is restricted to be homeomorphic, and therefor topologically invariant.

10. The topological structure of domains of Pfaff dimension 2 or less creates a connected, but not necessarily simply connected topology. Evolution solution uniqueness is possible.

11. The topological structure of domains of Pfaff dimension 3 or more creates a disconnected topology of multiple components. If solutions to a particular evolutionary problem exist, then the solutions are not unique. Envelope solutions, such as Huygen wavelets and propagating discontinuities (called signals) are classic examples of solution non-uniqueness.

12. Cartan's Magic formula, in terms of the Lie differential acting on exterior differential 1-forms establishes the long sought for combination of dynamics and thermodynamics, enabling non-equilibrium systems and irreversible processes to be computed in terms of continuous topological evolution, and without resort to probability theory and statistics.

13. The Lie differential acting on differential forms is not necessarily the same as a linear affine covariant differential acting on differential forms. It is possible to demonstrate that if the process is adiabatic (no heat flow in the direction of the evolutionary process), then the Lie differential and the covariant differential can be made to coincide, as they both satisfy the Koszul axioms for an affine connection. This is a surprising result, for when reversed the theorem implies that the ubiquitous affine covariant differential of tensor analysis acting on a 1-form of Action can always be cast into a form representing an adiabatic process. However, this adiabatic process need not be reversible.

14. If the evolutionary process described by the Lie differential, affine equivalent or not, leaves the 1-form of Action invariant, then the process is thermodynamically reversible. If the affine covariant differential of tensor analysis induces parallel transport (the covariant differential is zero), then the affine process is adiabatic and reversible.

15. The concept of topological entropy is deduced (without statistics) from the fact that Cartan-Hilbert 1-form of Action is of Pfaff Topological dimension 2n+2. The perfect differential of entropy, dS, is composed of the interior product of the non-canonical components of momentum with the components of the differential velocities. An equilibrium system is a Lagrange submanifold of the 2n+2 topological space, upon which the change in entropy is zero. $dS_{equil} = 0$.

The combined thermodynamic-topological perspective presented herein uses the mathematical tools of exterior differential forms to describe the topological features of physical systems, and vector fields to describe the continuous evolutionary processes that may or may not change the topology of the physical system. Examples will demonstrate that topological change is a necessary condition for thermodynamic irreversibility.

1.5 Topological Universality

It is a remarkable fact that the physical theories of Thermodynamics, Electrodynamics and Hydrodynamics all have similar topological foundations. These similarity features become evident, and useful, when the different disciplines are expressed in the universal language of Cartan's theory of exterior differential forms.

1. Each discipline utilizes the concept that a physical system can be encoded in terms of an exterior differential 1-form of Action, A.

2. Each discipline utilizes the concept that a process, or current, acting on the physical system, can be encoded to within a factor, ρ, by a contravariant direction field, V.

3. Each discipline has a dynamics that can be expressed in terms of continuous topological evolution based upon the Lie differential with respect to V. Warning: this topological dynamics is not always fully equivalent to that dynamics generated by the covariant differential of tensor analysis. The geometric dynamics of tensor analysis is a subset of the topological dynamics.

The arguments of the functions that define the physical system, the process, and the induced additional 1-forms, in this article are limited (with some exceptions) to an ordered variety of $n = 4$ independent base variables, abstractly specified as $\{x, y, z, t\}$, and their differentials, $\{dx, dy, dz, dt\}$. It is presumed that other varieties of base variables $\{\xi^1, \xi^2, \xi^3, \xi^4\}$ can be represented in terms of diffeomorphic maps from $\{\xi^1, \xi^2, \xi^3, \xi^4\}$ to $\{x, y, z, t\}$. To a physicist, the base variables play the role of admissible coordinates if they are diffeomorphically related. However no specific geometric metric or connection is (necessarily) imposed on these varieties of pre-geometric dimension $n = 4$ base variables.

Although the main thrust of this article is to study useful applications of non-equilibrium thermodynamic systems, some knowledge of Cartans' topological structure, and the topological properties of the continuum are required. This material has been developed in Cartan's language of exterior differential forms in Chapter 4 and Chapter 5.

1.6 Closure and Continuity

1.6.1 Closure

An important topological idea to be used in this monograph is the idea or concept of closure. The idea of closure is an invariant of a continuous but irreversible process. From set theoretic ideas, the idea of closure means that any pair of elements of a subset can be combined by a rule such that the resultant is still an element of the subset. Closure is perhaps the most fundamental property of a group. Elements of a vector space can be added together such that each sum is an element of the set of all basis elements multiplied by real numbers. The process of addition is closed. However, if two polar elements of a vector space are multiplied together by the method of the Gibbs cross product of engineering science, the resultant axial vector is not an element of the original subset of polar vectors. The Gibbs product is not closed. No engineer would ever add a torque to a force, or a linear momentum vector to an angular momentum vector, because they are not vectors of the same species.

In the Cartan formalism, a new concept of closure is defined in terms of what is called a differential ideal. A differential ideal is the union of a system of differential forms and their exterior differentials. For one forms, Chapter 4 details how to construct a topology in terms of the closure of a 1-form, A, the exterior differential of dA, the closure of A (equal to the union of A and dA), the exterior product (similar to the intersection) of A and dA (equal to $A\hat{}dA$), and the closure of the exterior product $A\hat{}dA$. Relative to this "Cartan topology", the exterior differential becomes an operator that creates the topological limit sets of each subset of the topology (see Chapter 4).

At this beginning level, the thing to remember is that limit sets are topological properties, and they can be determined for a particular exterior differential p-form by constructing the exterior differential of the particular p-form. This differential process is much easier to compute than is the integration process. In this manner, global (integral) topological information appears at a local (differential) level. Chapter 4 describes these and many other details of the Cartan topology.

Key observables in the understanding of the aging process are related to the concepts of closure and connectivity of the non-equilibrium states. Experimental methods to observe "closure" concepts must be devised if the notion of topological evolution is to be made practical. These notions may sound abstract and not useful, but when it is realized that the production of defects in a physical system, and the change of phase from solid to liquid, are exhibitions of topological evolution, then the ideas become more concrete.

The topological methods employed herein can be used to determine when a physical system is in an equilibrium or non-equilibrium state. The topological methods employed herein can be used to distinguish thermodynamically irreversible from reversible evolutionary processes. The topological methods employed herein can be used to describe the irreversible dissipative decay processes from open systems into excited stationary states far from equilibrium, and the further decay from excited non-equilibrium states into equilibrium ground states.

1.6.2 Continuity

An intuitive idea of continuity is built on the notion of a single valued function, or transformation, without breaks. The formal* topological definition [**?**] of a continuous transformation between a set X with topology $T1$ to a set Y with a topology $T2$ states that the transformation is continuous if and only if the inverse image of open sets of $T2$ are open sets of $T1$. The important point is that the topologies $T1$ and $T2$ need not be the same for a continuous transformation. A space is said to have a topological structure if it is possible to determine if a transformation on the space is continuous [Gellert 1977].

There exists another more useful method of defining continuity which does not depend explicitly on being able to define open sets and their inverse images. This second method of defining continuity is based on the concept of closure, discussed in the preceding subsection. The formal topological closure of a set can be defined in

*These formal defintions are most rapidly learned in terms of point set topology, and are presented in the appendix in more detail.

(at least) two ways:

1. The closure of a set is the union of the interior and the boundary of a set.
2. The closure of a set is the union of the set and its limit points.

The first definition of closure is perhaps the most common, and is often exploited in geometric situations, where a metric has been defined and a boundary can be computed easily. The second definition of closure is independent from metric and is the method of choice in this article, both for defining continuity and establishing a topological structure. In terms of the concept of closure, a transformation is continuous if and only if for every subset, the image of the closure of the initial subset is included in the closure of the image of that subset [Lipschutz 1965]. Another way of stating this idea is

3. A map is continuous iff the limit points of every subset in the domain permute into the closure of the subsets in the range.

It will be demonstrated that relative to the Cartan Topology base on a 1-form of Action, the exterior derivative becomes equivalent to a limit point generator. Hence the Cartan Calculus gives an easy method for utilizing the concept of limit points.

1.7 Pfaff Topological Dimension

Perhaps one of the most important topological tools to be used within the theory of continuous topological evolution is the concept of Pfaff topological dimension. The maximum Pfaff dimension is equal to number of independent variables in the base variety, which in this article has been limited to $n = 4$. For a given 1-form of Action, $A = A_k(x, y, z, t)dx^k$ defined on the base variety of $\{x, y, z, t\}$, it is possible to ask what is the irreducible minimum number of independent functions $\theta(x, y, z, t)$ required to describe the topological features that can be generated by the specified 1-form, A. This irreducible number of functions is defined here in as the "Pfaff topological dimension" of the 1-form, A. For example, if

$$A = A_k dx^k \Rightarrow d\theta(x, y, z, t)_{irreducible}, \tag{1.8}$$

$$\text{such that } A_k = \partial\theta(x, y, z, t)/\partial x^k, \tag{1.9}$$

then only one function $\theta(x, y, z, t)$ is required to describe the Action, not four. In this example the irreducible Pfaff topological dimension of the 1-form, A, is 1, although the dimension of base variety is 4. In a sense, the Pfaff topological dimension defines the existence of a domain of "topological" base variables (topological coordinates) as submersions from the original base variety (geometric coordinates) to the irreducible base variety (topological coordinates). Differential forms constructed on the irreducible base variety of functions, are functionally well defined on the original base variety. (See Chapter 4 and)

Relative to the Cartan topology [Baldwin 1991], the "Pfaff topological dimension" can be generated by each of the Pfaffian forms associated with each discipline. The irreducible Pfaff topological dimension for any given 1-form A is readily computed by constructing the Pfaff sequence of forms:

Pfaff sequence

$$\{A, dA, A\hat{}dA, dA\hat{}dA\}. \tag{1.10}$$

The Pfaff topological dimension is equal to the number of non-zero terms in the Pfaff sequence. For example, if the Pfaff sequence for a given 1-form A is $\{A, dA, 0, 0\}$ in a region $U \subset \{x, y, z, t\}$, then the Pfaff topological dimension of A is 2 in the region, U. The 1-form A, in the region U, then admits description in terms of only two, but not less than 2, independent variables, say $\{u^1, u^2\}$. For a differentiable map φ from $\{x, y, z, t\} \Rightarrow \{u^1, u^2\}$, the exterior differential 1-form defined on the target variety U of 2 pre-geometry dimensions as

$$A(u^1, u^2) = A_1(u^1, u^2)du^1 + A_2(u^1, u^2)du^2, \tag{1.11}$$

has a functionally well defined pre-image $A(x, y, z, t)$ on the base variety $\{x, y, z, t\}$ of 4 pre-geometric dimensions. This functionally well defined pre-image is obtained by functional substitution of u^1, u^2, du^1, du^2 in terms of $\{x, y, z, t\}$ as defined by the mapping ϕ. The process of functional substitution is called the pull-back.

$$A(x, y, z, t) = A_k dx^k = \varphi^*(A(u^1, u^2)) = \varphi^*(A_\sigma du^\sigma) \tag{1.12}$$

It may be true that the functional form of A yields a Pfaff topological dimension equal to 2 globally over the domain $\{x, y, z, t\}$, except for sub regions where the Pfaff dimension of A is 3 or 4. These sub regions represent topological defects in the almost global domain of Pfaff dimension 2. Conversely, the Pfaff dimension of A could be 4 globally over the domain, except for sub regions where the Pfaff dimension of A is 3, or less. These sub regions represent topological defects in the almost global domain of Pfaff dimension 4. Applications of both viewpoints will be described below. The important concept of Pfaff topological dimension also can be used to define equivalence classes of physical systems and processes.

The concept of "Pfaff topological dimension" was developed more than 110 years ago (see page 290 of Forsyth [Forsyth (1890) 1959]), and has been called the "class"† of a differential 1-form in the mathematical literature. More recent mathematical developments can be found in [Schouten 1949]. The method and its properties have been little utilized in the applied world of physics and engineering. Of key importance is the fact that the non-zero existence of the 3-form $A\hat{}dA = A\hat{}F$

†The term "Pfaff topological dimension" (instead of class) was introduced by the present author in order to emphasize the topological foundations of the concept.

of *Topological Torsion*, and the 4-form of *Topological Parity*, $dA\hat{}dA = F\hat{}F$ implies that the Pfaff topological dimension of the region is 3 or 4, respectively. Either value is an indicator that the physical system (in the sub region) is NOT in thermodynamic equilibrium. The concept of *topological parity*, $F\hat{}F$, has its foundations in the theory of Pfaff's problem, with a recognizable 4 dimensional formulation appearing in Forsyth [Forsyth (1890) 1959] page 100. The idea of *Topological Torsion*, $A\hat{}F$, is associated with the idea of magnetic helicity density, a concept that apparently had its electromagnetic genesis with the study of plasmas in WWII. However, the concept of helicity density is but one component of the four dimensional *Topological Torsion 4 vector*.

Recall that a space curve with non-zero Frenet - Serret torsion does not reside in a two dimensional plane. Non-zero Frenet - Serret torsion of a space curve is an indicator that the *geometrical* dimension of the space curve is at least 3. The fact that the Pfaff *topological* dimension of the 1-form, A, is at least 3, when $A\hat{}F$ is non-zero, is the basis of why the 3-form, $A\hat{}F$, was called "Topological Torsion". The idea of non-zero $A\hat{}F$ also appears in the theory of the Hopf Invariant [Bott 1994].

The concept of $A\hat{}F$ has also appeared in the differential geometry of connections, where a matrix valued 3-form is known as the Chern-Simons 3-form. However, on varieties without connection or metric, the Chern-Simons concept is not well defined, but the Topological Torsion concept exists and is acceptable, for it does not depend upon the geometric features of metric and/or connection.

1.8 Deformation Invariants as Topological Properties

Topological properties are defined as invariants with respect to homeomorphisms. A more mundane definition is that a topological property is an invariant of a continuous deformation. Certain integral properties of an electromagnetic system are deformation invariants with respect to those continuous evolutionary processes that can be described by a singly parameterized vector field. The absolute deformation invariants lead to the fundamental topological conservation laws described in the physical literature of electromagnetism as the conservation of charge and the conservation of flux. Recall the definitions used to describe processes of continuous topological evolution.

> A continuous process is defined as a map from an initial state of topology $T_{initial}$ into a final state of perhaps different topology T_{final} such that the limit points of the initial state are permuted among the limit points of the final state (see p. 97 et.seq. [Lipschutz 1965]). If the ordering of the limit points is invariant, the process is uniformly continuous. If the ordering (as in a folding of a boundary) or the number of the limit sets is changed the process is non-uniformly continuous.

A simple description of a topological property (invariant of a homeomorphism) is an object that is a deformation invariant. Consider a rubber sheet with three holes.

Stretch the rubber sheet. The holes may be deformed but the fact that there are 3-holes stays the same under small deformations. The concept of three holes is a topological property. It is remarkable that topologically coherent objects can be constructed in terms of open and closed integrals which are deformation invariants.

> A topological deformation invariant is defined as an integral of an exterior differential p-form over a p dimensional manifold, or cycle, *zpd*, such that the Lie differential of the integral of the p-form ω with respect to a singly parameterized vector field, ρV^k, vanishes, for any choice of deformation parameter, ρ.

Integral Deformation Invariant

$$L_{(\rho V^k)} \int_p \omega = 0 \quad any\ \rho \tag{1.13}$$

The requirements that a given p-form becomes a deformation invariant (and therefor a topological property, invariant with respect to homeomorphisms) is expressed in terms of certain topological constraints. Those objects that remain the same under continuous deformation represent topological, not geometric, properties. However, if the topological constraints required for continuous deformation are not satisfied, then topological change takes place. Topological change would require that the number of holes in the thin rubber sheet example were to change. Topological change can occur continuously or discontinuously. The focus in this article is on continuous topological change, and as will be demonstrated below, topological change is a necessary requirement for thermodynamic irreversibility [RMK 1976 b].

1.8.1 Absolute Integral Invariants

There are two types of invariant integrals, Absolute and Relative integral invariants. If the exterior p-form that forms the integrand is exact, the Absolute integral invariant places conditions only on the boundary of the domain of integration. It is these types of objects (Absolute integral invariants) that give a formality to those thermodynamic concepts whereby a physical system reaches equilibrium uniformly within its interior, and yet may couple with its exterior environment via fluxes across its boundary. Only effects related to the boundary are of consequence. For example, consider physical systems that can be defined by a 1-form of Action, A, such that the derived 2-form $F = dA$, is exact. It follows from Stokes theorem that the 2-dimension integral of F is an absolute integral deformation invariant with respect to *all* continuous processes that can be defined by a singly parameterized vector field, subject to a boundary condition that the net flux, $i(\rho V^k)F$, of F, across the 1-dimensional boundary of M is zero:

$$L_{(\rho V^k)} \int\int_M F = \int\int_M i(\rho V^k)dF + \int\int_M d(i(\rho V^k)F) \quad (1.14)$$

$$= 0 + \int_{boundary\ of\ M} i(\rho V^k)F \quad (1.15)$$

This concept is at the basis of the Helmholtz theorems of vorticity conservation (or angular momentum per unit mass) in hydrodynamics, and the conservation of flux in classical electromagnetism. Herein, this concept of deformation invariance of a topologically coherent structure will be written in the form of an exterior differential system [Bryant 1991], $F - dA = 0$. The exterior differential system is to be recognized as topological constraint. From Stokes theorem, the 2 dimensional domain of finite support for F can not, in general, be compact without boundary, unless the Euler characteristic vanishes. There are two exceptional cases for absolute invariance of the integral, and they occur when the integration domain is compact without boundary. Such two dimensional domains which have a zero Euler characteristic are the torus and the Klein-Bottle, but these situations require the additional topological constraint that $F\hat{}F \Rightarrow 0$. The fields in these exceptional cases must reside on these exceptional compact surfaces without boundary, which form topological coherent structures. Note that an evolutionary process could start with $F\hat{}F \neq 0$, and possibly evolve to a state with $F\hat{}F = 0$. If such residue states are compact without boundary, then they must be either tori or Klein bottles.

The same technique can be applied to non-exact but closed p-forms.

1.8.2 Relative Integral Invariants

If the integration of the exact 2-form, F, is over a closed two dimensional chain, designated as a 2 dimensional cycle, z2d (which may or may not be a 2 dimensional boundary), then the Integral is invariant for any deformation factor, ρ :

$$L_{(\rho V^k)} \int\int_{z2d} F = \int\int_{z2d} i(\rho V^k)dF + \int\int_{z2d} d(i(\rho V^k)F) = 0 + 0. \quad (1.16)$$

The two integrals on the right vanish, the first due to the fact that $dF = 0$, and the second due the fact that the closed integral over an exact form vanishes. Close integrals of exact p-forms are always relative deformation integral invariants. However, the same technique can be applied to non-exact but closed p-forms. For electromagnetism, there are several exact p-forms, each producing a relative deformation integral invariant. For example, the 3-form of charge-current density is exact, $J = dG$. The 4-forms that define the Poincare Invariants are exact: $F\hat{}F = d(A\hat{}G)$ and $F\hat{}G - A\hat{}J = d(A\hat{}G)$.

If the conditions of relative integral invariance are applied to an arbitrary 1-form of Action, then the relative integral invariance condition becomes

$$L_{(\rho V^k)} \int_{z1d} A = \int_{z1d} i(\rho V^k) dA + \int_{z1d} d(i(\rho V^k)A) \tag{1.17}$$

$$= \int_{z1d} i(\rho V^k) F + 0 \Rightarrow 0. \tag{1.18}$$

It follows the $i(\rho V^k)dA$ must be zero on the cycle $z1d$ for any deformation parameter ρ. Cartan has shown that this is the condition that implies the process ρV^k has a "Hamiltonian" representation [Cartan 1958 (1922)].

1.8.3 Period Integrals and Topological Quantization

Besides the invariant structures considered above, the Cartan methods may be used to generate other sets of topological invariants. Realize that over a domain of Pfaff dimension n less than or equal to N, the Cartan criteria admits a submersive map to be made from N to a space of minimal dimension n. The map may be viewed as a vector field of functional components,

$$[V^x(x, y, z..), V^y(x, y, z..), V^z(x, y, z..), ...],$$

of dimension n, and will have a representation in the projective geometry of n+1 homogeneous coordinates. The n+1 component will be generated by a function λ, related to the Holder norm,

$$\rho = 1/\lambda = 1/\{a(V^x)^p + b(V^y)^p + c(V^z)^p + ...\}^{n/p}. \tag{1.19}$$

For any vector field, construct the n dimensional volume element,

$$Vol = \rho(V)\ dV^x\hat{}dV^y\hat{}dV^z... \tag{1.20}$$

and the n-1 form density (current) J as:

$$J = i(V^x, V^y, V^z, ...)Vol = \rho\{V^x\ dV^y\hat{}dV^z...\ - V^y\ dV^x\hat{}dV^z... + V^z\ dV^x\hat{}dV^y... - ...\} \quad . \tag{1.21}$$

It is remarkable that the current J so defined has a vanishing exterior differential, independent of the value of p for a given n, and for all values of the constants, plus or minus a,b,c...). All such currents thereby define a "conservation law". As the map defining the components of the vector field in terms of the base {x,y,z..} is presumed to be differentiable, then the n-1 form, *J*, has a well defined pull back on the base space (almost everywhere), and its exterior differential on the base space also vanishes everywhere mod the defects. That is, the form J is locally exact.

In the expression for λ, the factors $\{a, b, c, d...\}$ are arbitrary constants of either sign. The most familiar format is when $p = 2$, and then the function λ has a null set which is a conic. For positive isotropic signature, the only defect is the origin in the space defined by the functions, V. The construction produces the algebraic dual or adjoint vector field from the functional components of the original vector field with integrating factors $\rho = 1/\lambda$ that create conservation laws for physical systems.

The integrals of these closed currents, when integrated over closed N-1 dimensional chains, form deformation invariants, with respect to any evolutionary process that can be described by a vector field, for

$$L_{(\rho\mathbf{V})}\int_{z(n-1)d} J = \int_{z(n-1)d} i(\rho\mathbf{V})dJ + \int_{z(n-1)d} d(i(\rho\mathbf{V}))J) = 0 + 0 = 0 \qquad (1.22)$$

These integral objects appear as "topological coherent" structures (which may have defects or anomalous sources, when the integrating factor $1/\lambda$ is not defined).

The compliment to the zero sets of the function λ determine the domain of support associated with the specified vector field. The closed n-1 form, J, that satisfies the conservation law, $dJ = 0$, has integrals over closed domains that have rational fraction ratios. As this n-1 current is closed globally, it may be deduced on a connected local domain from a n-2 form, G. In every case J has a well defined pull-back to the base variety, x,y,z,t. Note that the n functions $[V^x\,(x, y, z..), V^y(x, y, z..), V^z(x, y, z..), ...]$ represent the minimum number of Clebsch variables that are equivalent to the original action, A, over the domain of support. As each of these integrals is intrinsically closed, the Lie differential with respect to any C1 vector field, $\rho\mathbf{V}$, is a perfect differential, such that (when integrated over closed domains that are p-1 boundaries) the evolutionary variation of these closed integrals vanishes. These n-1 integrals are relative integral invariants for any C1 evolutionary processes, or flows. The values of the integrals are zero if the closed integration domains are boundaries, or completely enclose a simply connected region. If the closed integration domains encircle the zeros of the function λ, then the values of the integrals are proportional to the integers; i.e., their ratios are rational.

In general, by deRham's theorems, these values of these period integrals, for different closed integration chains in domains where $dJ = 0$, have rational ratios [RMK 1977].

Topological Quantization

$$L_{(\rho\mathbf{V})}\int_{z(n-1)d} J = n \text{ constant.} \qquad (1.23)$$

Note that each signature of λ must be investigated. For the elliptic (positive definite) signature, the singular points are the stagnation points, and the domain of support excludes those singularities. For the hyperbolic signatures, the domain of support excludes the hyperbolic singularities of lower dimension, such as the light cone. Further note that a given vector field may not generate real domains of support for all possible signatures of the quadratic form, λ.

Details and applications of homogeneous constructions that give rise to period integrals are presented in Chapter 6.

1.9 The Arrow of Time

On a given domain, Baldwin has shown that the existence of a Cartan 1-form and its Pfaff sequence (1.10) may be used to define a "Cartan" topology over the domain (See Chapter 4 for details). If the Cartan 1-form is integrable in the sense of Frobenius, then the Cartan topology is a connected topology. If the Cartan 1-form is not integrable then the Cartan topology is a disconnected topology. Evolution from a connected topology to a disconnected topology can proceed only by means of discontinuous processes. However, evolution from a disconnected topology to a connected topology can be accomplished continuously. It is this latter class of continuous processes that will be in focus in this article. An important practical result of this fact is that the continuous hydrodynamic evolution from a streamline (integrable and reversible) state to a turbulent (non-integrable and irreversible) state is impossible, while the continuous evolution from a turbulent state to a streamline state is permissible [RMK 2000]. The creation of irreversible turbulence is necessarily discontinuous, but the decay of turbulence can be continuous.

It has been demonstrated ([RMK 1976 b]) that for continuous but non-homeomorphic maps (C1 maps without continuous inverse) it is **impossible to predict** the functional form of either covariant or contravariant vector fields. That is, the functional form of tensor fields on the final state is not well defined in terms of the functional form of the field on the initial state, if the map from initial to final state is continuous but not homeomorphic. On the other hand it can be shown that covariant antisymmetric tensor fields are deterministic in a retrodictive sense, even though the continuous maps from initial to final state are not invertible. That is, the functional form of the components of differential forms defined on the final state are well defined on the initial state even if the map from the initial state to the final state is C1 smooth, but not a homeomorphism. With respect to continuous topological evolution there then exists a natural, logical, arrow of time, which is not observable with respect to diffeomorphic geometric evolution that preserves topology. Therefore, to understand irreversible phenomena, a retrodictive point of view seems to be of some value [Tisza 1966] and it is this non-statistical retrodictive point of view constructed on Cartan's theory of exterior differential systems that is the point of departure in this article.

The methods will be restricted at first to those evolutionary processes and systems which are C2 continuous. It is appreciated that this restriction does not cover all physical situations, where in the opinion of this author "true" discontinuities, not just mathematical artifacts, are possible. The continuous evolutionary processes to be considered will permit topology to change in a continuous but irreversible manner (example: the pasting together of two blobs, or the collapse of a hole). Discontinuous processes are, at present, excluded.

1.10 Applied topology versus applied geometry

It was mentioned above that the presentation herein is not meant to be a textbook on Cartan's theory of exterior differential forms, nor a textbook on abstract topology. Instead, this monograph is an attempt to use the simpler features of topology contained in the exterior calculus and apply the Cartan methods to interesting non-equilibrium problems in the disciplines of mechanics, electrodynamics, or hydrodynamics, without the use of probability theory or statistics. Although the concept of equilibrium has had many useful scientific successes, the truth of the matter is that the real world is rarely in equilibrium, and the methods of evolution that have been developed so far give no details as how the change from non-equilibrium to equilibrium takes place. Even Quantum Mechanics fails to describe the details of the decay of an excited state to the ground state. Paraphrasing Bohr, "a miracle takes place".

The topological methods employed herein can be used to determine when a physical system is in an equilibrium or non-equilibrium state. The topological methods employed herein can be used to distinguish thermodynamically irreversible from reversible evolutionary processes. The topological methods employed herein can be used to describe the irreversible dissipative decay processes from open systems into excited stationary states far from equilibrium, and the further decay from excited states into equilibrium ground states.

Since before the beginning of the 20th century , advances in physical theories have been predicated upon a geometrical approach. (It should be mentioned that another interesting point of view about thermodynamics based upon algebra was presented by [Zeleznik 1976].) Several attempts to better understand thermodynamics in terms of geometrical ideas have been attempted [Blashke], [Weinhold 1975], but without notable success. It was pointed out by Tisza [Tisza 1966] that metrical based properties can not be used to distinguish between the two classes of intensive and extensive thermodynamic variables, and the hint was offered that perhaps topological methods, rather than geometrical methods, might prove to be suitable. Finsler spaces have been examined by Anotelli and Ingevarden [Antonelli (1993)]. At the Aspen conference in 1977, the present author suggested that the methods based upon the first fundamental form (metric) should be replaced by methods based on the second fundamental form (the shape matrix). In Chapter 2, these ideas will be exploited in describing how in 4 dimensional space, the van der Waals gas is a universal topological artifact.

Much of the motivation for development of a topological view of thermodynamics was based upon the concept of topological defects being related to domains or points where topological change took place. The idea that a phase change was a realization of topological evolution and change was very influential in the struggle over the years to develop a dynamics of such a thermodynamics process. The work of van der Kulk and Schouten inspired the use of the concept of Pfaff dimension, and its change, as being one of the key tools for use in topological thermodynamics. The later work Martinet and Zhitomarski [Martinet (1970)] [Zhitomirski (1991)] has yet to be fully exploited. The concept that smoothness could influence thermodynamic evolution has only recently been appreciated. How C1 smoothness interplays with the Nash and Gromov axioms has yet to be exploited, but it was a surprise to find out that C1 sequences of processes in 3D could be reversible, where C2 processes of rotation and expansion could be thermodynamically irreversible.

Herein the emphasis is on topological properties and features of physical systems, and how the topology can change in a continuous evolutionary manner. What is meant by this statement is that the topology of the initial state need not be the same as the topology of the final state. Such topological change can take place either continuously (pasting) or discontinuously (cutting). The result to be demonstrated is the topological change is a necessary artifact of thermodynamic irreversibility.

Chapter 2
TOPOLOGICAL THERMODYNAMICS

2.1 Continuous Topological Evolution and the First Law

2.1.1 Cartan's Magic Formula

The topological thermodynamic methods used herein are based upon Cartan's theory of exterior differential forms. The thermodynamic view assumes that the physical systems to be studied can be encoded in terms of a 1-form of Action Potentials, A, on a 4 dimensional variety of ordered independent variables, $\{\xi^1, \xi^2, \xi^3, \xi^4\}$. The variety supports a volume element $\Omega_4 = d\xi^1\hat{}d\xi^2\hat{}d\xi^3\hat{}d\xi^4$. No metric, no connection, no constraint of gauge symmetry is imposed upon the 4 dimensional variety. Topological constraints will be imposed in terms of exterior differential systems [Bryant 1991].In order to make the equations more suggestive to the reader, the symbolism for the variety of independent variables will be of the format $\{x, y, z, t\}$, but be aware that no constraints of metric or connection are imposed upon this variety. For instance, it is NOT assumed that the variety is euclidean.

The 1-form of Action, A, will have components that form a covariant direction field (relative to diffeomorphisms), $A_k(x, y, z, t)$, to within a non-zero factor. Evolutionary processes on space-time will be determined in terms of 4 dimensional contravariant direction fields, $\mathbf{V}_4(x, y, z, t)$, to within a non-zero factor, ρ. Continuous topological evolution (see Chapter 5) will be defined in terms of Cartan's magic formula for the Lie differential.

Cartan-Lie differential

$$L_{(\rho\mathbf{V}_4)}A = i(\rho\mathbf{V}_4)dA + d(i(\rho\mathbf{V}_4)A) \tag{2.1}$$

The definition is to be interpreted algebraically, using the properties of the exterior differential and the inner product associated with exterior differential forms. Many derivations of the Cartan-Lie differential formula presume a dynamic constraint, such that the vector field $\mathbf{V}_4(x, y, z, t)$ be the generator of a single parameter group; then the topological constraint of kinematic perfection is satisfied:

Kinematic perfection

$$dx^k - V^k dt = 0. \tag{2.2}$$

The topological constraint in effect defines a limit process, and the Lie *differential* then can be considered to be a Lie *derivative* of the form A representing an infinitesimal propagation of the 1-form, A, down the flow lines generated by 1-parameter group. However, such a kinematic constraint is not necessarily imposed in the presentation herein; the vector field may have multiple parameters, which leads to the important concept of topological fluctuations, discussed below.

When acting on an exterior differential 1-form of Action, $A = A_k dx^k$, Cartan's magic (algebraic) formula* is equivalent *abstractly* to the first law of thermodynamics.

$$\begin{aligned}
&\text{Cartan's Magic Formula} & L_{(\rho\mathbf{V}_4)}A &= i(\rho\mathbf{V}_4)dA + d(i(\rho\mathbf{V}_4)A) && (2.3)\\
&\text{First Law of Thermodynamics} & &: W + dU = Q, && (2.4)\\
&\textbf{Inexact 1-form of Heat} & L_{(\rho\mathbf{V}_4)}A &= Q && (2.5)\\
&\textbf{Inexact 1-form of Work} & W &= i(\rho\mathbf{V}_4)dA, && (2.6)\\
&\textbf{Internal Energy} & U &= i(\rho\mathbf{V}_4)A. && (2.7)
\end{aligned}$$

In effect, Cartan's magic formula leads to a topological basis of thermodynamics. When confined to processes on equilibrium systems (such that $dQ = 0$ and $dW = 0$), the First Law is a statement of deRham cohomology theory.

The topological methods to be described below go beyond the notion of processes which are confined to equilibrium systems. Non-equilibrium systems and processes which are thermodynamically irreversible, as well as many other classical thermodynamic ideas, can be formulated in precise mathematical terms using the topological features of the three 1-forms, A, W, and Q - without the use of statistics or metric constraints.

2.1.2 An electromagnetic example

The thermodynamic identification of the terms in Cartan's magic formula are not whimsical. To establish an initial level of credence in the terminology, consider the 1-form of Action where the component functions are the symbols representing the vector and scalar potentials in electromagnetic theory. The coefficient functions have arguments over the independent variables $\{x, y, z, t\}$:

$$A = A_k(x, y, z, t)dx^k = \mathbf{A} \circ d\mathbf{r} - \phi \; dt. \tag{2.8}$$

Construct the 2-form of field intensities

$$\begin{aligned}
F &= dA = (\partial A_k/\partial x^j - \partial A_k/\partial x^j)dx^j\hat{}dx^k && (2.9)\\
&= F_{jk}dx^j\hat{}dx^k = +\mathbf{B}_z dx\hat{}dy... + \mathbf{E}_x dx\hat{}dt... && (2.10)
\end{aligned}$$

*The name Cartan's Magic Formula is due to Marsden [Marsden 1994].

Contract the 2-form F with a 4 vector direction field, J (scaling parameter, ρ)

$$J = [\mathbf{J}, \rho] = \rho[\mathbf{v}, 1] = \rho\mathbf{V}_4, \tag{2.11}$$

to yield the expression,

$$\begin{aligned} \text{Work 1-form, } W &= i(\rho\mathbf{V}_4)dA = i(J)F & (2.12)\\ &= \{\rho\mathbf{E} + \mathbf{J}\times\mathbf{B}\}\circ d\mathbf{r} - \{\mathbf{J}\circ\mathbf{E}\}dt & (2.13)\\ &= \{\mathbf{f}_{Lorentz}\}\circ d\mathbf{r} - \{\mathbf{J}\circ\mathbf{E}_{Dissipative\ Power}\}dt. & (2.14)\end{aligned}$$

For those with experience in electromagnetism, note that the construction yields automatically the expression for the "Lorentz force" without further assumptions. The dot product of a force and a differential displacement defines the elementary concept of differential work. The 4 component Work 1-form includes the spatial contribution related to force times differential distance, $d\mathbf{r}$, and a differential time component, dt, with a coefficient which is recognized to be the dissipative power, $\{\mathbf{J}\circ\mathbf{E}\}$. The Work 1-form is not necessarily a perfect differential, and therefore can be path dependent. Closed cycles of Work need not be zero.

Next compute the Internal Energy term,

Internal Energy

$$U = i(J)A = \mathbf{J}\circ\mathbf{A} - \rho\phi = \rho[\mathbf{v}\circ\mathbf{A} - \phi]. \tag{2.15}$$

The result is to recognized as the "interaction" energy density in electromagnetic plasma systems. It is apparent that the internal energy, U, corresponds to the interaction energy of the physical system; that is, U is the internal stress energy of system deformation. Therefor, the electromagnetic terminology used to demonstrate that Cartan's magic formula is another way to state that the first law of thermodynamics makes practical sense.

The correspondence so established between the Cartan magic formula acting on a 1-form of Action, and the first law of thermodynamics is taken both literally and seriously in this monograph. The Cartan methods are not limited to applications in electromagnetic theory. The thermodynamic phenomena and the associated topological results describe herein have universal qualities, and applicability to all physical theories. The topological methods permit the long sought for integration of mechanical and thermodynamic concepts, with out the constraints of equilibrium systems, and statistical analysis. The methods yield explicit constructions for testing when a process acting on a physical system is irreversible. The methods permit irreversible adiabatic processes to be distinguished from reversible adiabatic processes analytically. Adiabatic processes need not be "slow" or quasi-static.

Given any 1-form, A, W, and Q, the concept of Pfaff topological dimension permits separation of processes and systems into equivalence classes. For example, dynamical process can be classified in terms of the topological Pfaff dimension of the Work 1-form. All Hamiltonian systems have a Work 1-form of topological Pfaff dimension of 1, $(dW = 0)$ and therefor cannot describe irreversible processes for which the topological Pfaff dimension is 4. A discussion of the concept of Pfaff topological dimension appears below.

2.2 Thermodynamic Physical Systems

2.2.1 Applications of the Pfaff Topological Dimension

As mentioned in Chapter 1, one of the most useful topological tools is that defined as the Pfaff topological dimension†. Recall that it is possible to define many topologies on the same set of elements. For any given exterior differential 1-form of functions, say $A = A_k(x, y, z, t)dx^k$, it is possible to construct the Pfaff sequence of terms, $\{A, dA, A\hat{}dA, dA\hat{}dA\}$. These elements of the Pfaff sequence may be used to construct a Cartan Topology and a Cartan Topological structure (see Chapter 4). In the Cartan topology, the exterior differential operator, d, acts as limit point generator. Hence the union of a form and its exterior differential creates the topological (Kuratowski) closure of the form.

For any given 1-form, the Pfaff sequence will contain M successive non-zero terms equal to or less than N, the number of geometric dimensions of the base independent variables. The number M is defined as the "Pfaff topological dimension" or class of the given 1-form. The concept implies that there is submersive map from the space of geometric dimensions, N, to a space of topological dimensions, M. The topological properties of the given 1-form are expressible in terms of this irreducible number, M, of functions and differentials. The results are invariant with respect to the submersive pullback to the geometric space of dimension, N, from the space of topological dimension, M.

The three important 1-forms of thermodynamics, A, W, and Q, can have different Pfaff dimensions, and can generate different topologies on the elements of the same geometric space. Suppose the 1-form of work is defined in terms of two functions as $W = PdV$. The Pfaff sequence consists of the terms $\{W, dW, 0, 0\}$ as $W\hat{}dW = 0$; hence in this example, the Pfaff dimension of W is 2. From the first law, under the assumption that $W = PdV$,

$$\begin{aligned} Q &= W + dU = PdV + dU, && (2.16)\\ dQ &= dW = dP\hat{}dV, && (2.17)\\ Q\hat{}dQ &= W\hat{}dW + dU\hat{}dW = 0 + dU\hat{}dP\hat{}dV && (2.18)\\ dQ\hat{}dQ &= 0. && (2.19) \end{aligned}$$

†The property of Pfaff Topological dimension has also been called the "class" of a 1-form.

Hence, the Pfaff dimension of 2 for the work 1-form can be related to a Pfaff dimension of 3 for the Heat 1-form, unless the Pressure is a function of the internal energy and the volume. In this latter case, the Pfaff dimension of Q and W are both 2.

In this article, attention will be focused on dissipative Turbulent systems with thermodynamic irreversible processes such that the Pfaff topological dimensions of A, W, and Q will be maximal and equal to 4. (The techniques can be extended to higher dimensional geometric spaces.) These Turbulent systems of Pfaff dimension 4 are not topologically equivalent to Equilibrium systems (for which the topological dimension is 2, at most). Topological defects in the Turbulent state will be associated with sets of space time where the Pfaff topological dimensions of A, W, and Q are not maximal. It is remarkable that such topological defect sets can form attractors causing self organization and long lived states of Pfaff dimension 3, which are far from equilibrium. Examples will be presented below.

2.2.2 Physical Systems: Equilibrium, Isolated, Closed and Open

Physical systems and processes are elements of topological categories determined by the Pfaff topological dimension (or class) of the 1-forms of Action, A, Work, W, and Heat, Q. For example, the Pfaff topological dimension of the exterior differential 1-form of Action, A, determines the various species of thermodynamic systems in terms of distinct topological categories. The are two topological thermodynamic categories that are determined by the closure (or differential ideal) of the 1-form of Action, $A \cup dA$, and the closure of the 3-form of topological torsion, $A\hat{}dA \cup dA\hat{}dA$. The first category is represented by a connected Cartan topology, while the second category is represented by a disconnected Cartan topology. The Cartan topology is discussed in detail in Chapter 4.

Connected Topology $A\hat{}F = 0$

1. Equilibrium physical systems are elements such that the Pfaff topological dimension is 1.

2. Isolated physical systems are elements such that the Pfaff topological dimension is 2, or less. Isolated systems of Pfaff dimension 2 need not be in equilibrium, but do not exchange radiation or mass with the environment.

Disconnected Topology $A\hat{}F \neq 0$

1. Closed physical systems are elements such that the Pfaff topological dimension is 3. Closed systems can exchange radiation, but not mass, with the environment.

2. Open physical systems are such that the Pfaff topological dimension is 4. Open physical systems can exchange both radiation and mass with the environment.

$$\begin{aligned}
Systems &: \text{ defined by the Pfaff dimension of } A && (2.20)\\
dA &= 0 \quad \textbf{Equilibrium} \text{ - Pfaff dimension 1} && (2.21)\\
A\hat{}dA &= 0 \quad \textbf{Isolated} \text{ - Pfaff dimension 2} && (2.22)\\
d(A\hat{}dA) &= 0 \quad \textbf{Closed} \text{ - Pfaff dimension 3} && (2.23)\\
dA\hat{}dA &\neq 0. \quad \textbf{Open} \text{ - Pfaff dimension 4.} && (2.24)
\end{aligned}$$

Note that these topological specifications as given above are determined entirely from the functional properties of the physical system encoded as a 1-form of Action, A. The system topological categories do not involve a process, which is encoded (to within a factor) by some vector direction field, $\mathbf{V}_4$. However, the process $\mathbf{V}_4$ does influence the topological properties of the work 1-form W and the Heat 1-form Q.

2.2.3 Equilibrium versus Non-Equilibrium Systems

The intuitive idea for an equilibrium system comes from the experimental recognition that the intensive variables of pressure and temperature become domain constants in an equilibrium state: $dP \Rightarrow 0$, $dT \Rightarrow 0$. A definition made herein is that the Pfaff topological dimension in the interior of a physical system which is in the equilibrium state is at most 1 [Bamberg 1992]. The Cartan topology generated by the elements of the Pfaff sequence for A is then a connected topology of one component, $\{A \neq 0,\ dA = 0,\ A\hat{}dA = 0,\ dA\hat{}dA = 0\}$. Although the Pfaff topological dimension of A is at most 2 in the isolated state, processes in the equilibrium state are such that the Work 1-form and the Heat 1-form must be of Pfaff dimension 1. For suppose $W = PdV$, then $dW = dP\hat{}dV \Rightarrow 0$ if the pressure is a domain constant. Similarly, suppose $Q = TdS$, then $dQ = dT\hat{}dS \Rightarrow 0$ if the temperature is a domain constant. Hence both W and Q are of Pfaff dimension 1 for this equilibrium example. If the Pfaff dimension of the 1-form of Action is 1, then $dA \Rightarrow 0$. It follows in this more stringent case that $W \Rightarrow 0$, hence the Pressure must vanish, and Heat 1-form is a perfect differential, $Q = d(U)$.

Of particular interest herein are those regions of base variables for open, non-equilibrium, Turbulent physical systems, formed by the closure‡ of the 3-forms $A\hat{}dA$, $W\hat{}dW$, and $Q\hat{}dQ$. For such regions, the Pfaff topological dimension of the 1-forms, A, W, and Q, are all initially of Pfaff topological dimension 4, save for defect regions that are of Pfaff dimension 3. For example, evolutionary dissipative irreversible processes in such open systems can describe evolution to regions of base variables where the Pfaff topological dimension of the 1-form of Action, A, changes from 4 to 3. Such processes describe topological change in the physical system. For a given 1-form of Action, A, those regions of Pfaff topological dimension 3, once created, form topological "defect structures" in the closure of the 3-form, $A\hat{}F$. The defect

‡The closure of the p-form Σ is the union of Σ and $d\Sigma$, which Cartan has called a differential ideal.

structures of the 1-form of Action, A, (of Pfaff dimension 3) can behave as long lived (excited) states of the initial physical system, but they are far from equilibrium and are not isolated, for they are not of Pfaff topological dimension equal to 2 or less. Such excited states (of odd topological dimension) can admit extremal processes of kinematic perfection, and can have a Hamiltonian generator for the kinematics represented as first order ordinary differential equations. The Hamiltonian evolution remains contained in the defect structure, unless topological fluctuations destroy the kinematic perfection.

Such concepts can be applied to a model of cosmology (where the stars are the defect structures), to turbulent plasmas and fluids (where wakes are the defect structures), and to a better understanding of the arrow of time. Although the defects in the Turbulent non-equilibrium regime are not necessarily equilibrium structures, once formed and self organized as coherent topological structures of Pfaff dimension 3, they can evolve along extremal trajectories that are not dissipative, and may even have a Hamiltonian representation. These "stationary", if not long lived (excited) states of Pfaff dimension 3, indeed are states "far" from the equilibrium state, which requires a Pfaff dimension of 1. Note that the word "far" does not imply a "distance". The Pfaff dimension 3 and 4 sets are not even "connected" to the equilibrium states in a topological sense. The non-equilibrium states of a physical system that are "near-by" to the equilibrium state, are "connected" to the equilibrium state, and are of Pfaff dimension 2.

The descriptive words of self-organized states far from equilibrium are abstracted from the intuition and conjectures of I. Prigogine [Kondepudi 1998]. However, the topological theory presented herein presents for the first time a solid, formal, mathematical justification (with examples) for the Prigogine conjectures. Precise definitions of equilibrium and non-equilibrium systems, as well as reversible and irreversible processes can be made in terms of the topological features of Cartan's exterior calculus. Thermodynamic irreversibility and the arrow of time are well defined in a topological sense [RMK 2003], a technique that goes beyond (and without) statistical analysis. Thermodynamic irreversibility and the arrow of time requires that the evolutionary process produce topological change.

2.2.4 Change of Pfaff Topological Dimension

It should be noted that the closed components of the 1-form of Action do not effect the components of the 1-form of intensities, $F = dA = d(A_c + A_0) = 0 + d(A_0) = F_0$. However, these "gauge" additions do influence the topological dimension of the 1-form of Action. For example, let A_0 be of Pfaff Topological dimension 2, representing an isolated system where $A_0\hat{}dA_0 = 0$. Then by addition of a closed component to the original action, $A = A_c + A_0$ could have a topological dimension of 3, as

$$A\hat{}dA = (A_c + A_0)\hat{}dA_0 = A_c\hat{}dA_0 \neq 0. \tag{2.25}$$

So the addition of a closed component to the 1-form of Action could change the system from an isolated system of Pfaff dimension 2 to a closed system of Pfaff dimension 3. The 4-form $dA\hat{}dA$ is not influenced by the (gauge) addition to the original 1-form of Action.

$$dA\hat{}dA = dA_0\hat{}dA_0. \tag{2.26}$$

In section (3.3.3) a 1-form representing a Bohm-Aharanov-Abrikosov singular "vortex" string, $\gamma = b(ydx - xdy)/(x^2 + y^2)$, is added to a 1/r potential for a point source. The bare m/r "Coulomb" potential, $A_0 = m/\sqrt{(x^2 + y^2 + z^2)}dt$ exhibits no Topological Torsion but does exhibit Topological Spin. The 1/r potential term implies that $A_0 \neq 0$. Hence the 1-form of Action representing a bare "coulomb" potential, is not in equilibrium, but does represent a connected "isolated" topology of Pfaff dimension 2. The combined 1-form of Action,

$$A = b(ydx - xdy)/(x^2 + y^2) + m/\sqrt{(x^2 + y^2 + z^2)}dt \tag{2.27}$$

even though $d\gamma = 0$, is of Pfaff dimension 3, not 2. The Topological Torsion 3-form $A\hat{}F$ depends on both b and m, and is zero if $b = 0$, or if $m = 0$, reducing the Pfaff dimension of the modified 1-form back to 2. If $b = 0$ and $m \neq 0$, the 3-form $A\hat{}G$ is not zero.

2.2.5 Systems with Multiple Components

One of the most remarkable properties of the Cartan topology generated by a Pfaff sequence is due to the fact that when $A\hat{}dA = 0$, (Pfaff dimension 2 or less) the physical system is reducible to a single connected topological component. This single connected topological component need not be simply connected. The Topological Torsion field vanishes on equilibrium domains.

On the other hand when $A\hat{}dA \neq 0$, (Pfaff dimension 3 or more) the physical system admits more than one topological component. The bottom line is that when the Pfaff dimension is 3 or greater (such that conditions of the Frobenius unique integrability theorem are not satisfied), solution uniqueness to the Pfaffian differential equation, $A = 0$, is lost. If there exist solutions, there is more than one. Such concepts lead to propagating discontinuities (signals), envelope solutions (Huygen wavelets), an edge of regression (the spinodal line of phase transitions) a lack of time reversal invariance, and the existence of irreducible affine torsion in the theory of connections. It is the opinion of this author that a dogmatic insistence that a viable theory must give a unique prediction from a set of given initial conditions historically has hindered the understanding of irreversibility and non-equilibrium systems. Irreversibility and non-equilibrium are concepts that require non-uniqueness, and demand that the dogma mentioned above has to be rejected.

2.3 Thermodynamic Processes

2.3.1 continuous Processes

All continuous processes (see Chapter 5) may be put into equivalence classes as determined by the vector direction fields, V, that locally generate a flow. For example, for the 1-form, A, those vector fields that satisfy the transversal equation,

Associated Class

$$i(\rho\mathbf{V}_{associated})A = 0 \tag{2.28}$$

are said to be elements of the associated class of vector fields relative to the form A. If the direction field of the 1-form of Action is considered to be a fiber, then the associated vectors are also said to be "horizontal". The associated vectors will form a distribution orthogonal to the 1-form, A, but the distribution need not be a smooth foliation. That is, the fiber direction field is not necessarily the normal field to an implicit hypersurface. The requirement for a smooth foliation is that the associated 1-form be of Pfaff topological dimension 2 or less. For such associated processes acting on a 1-form of Action, A, the "internal interaction energy" vanishes. As shown below, processes generated by associated vectors relative to the 1-form of Action, A, are also included in the set of thermodynamically locally adiabatic processes. Other locally adiabatic processes are generated by those processes which are associated vectors of the exterior derivative of the internal energy, U. In both cases, the adiabatic processes are null vectors of the Heat 1-form, in the sense that $i(\rho\mathbf{V}_{adiabatic})Q = 0$.

Those vectors that satisfy the equations,

Extremal Class

$$i(\rho\mathbf{V}_{extremal})dA = 0 \tag{2.29}$$

are said to be elements of the extremal class of vector fields. It should be noted that the 2-form dA admits a unique extremal vector only on spaces of irreducible odd Pfaff topological dimensions. Such domains form an irreducible 2n+1 dimensional state space, which is defined as a contact manifold. The extremal direction field is completely determined (to within a factor) by the component functions of the 1-form utilized in its definition.

If the Pfaff topological dimension of the 1-form A is even, then a unique extremal vector does not exist. The reduced topological domain (not necessarily the geometric domain) is a symplectic manifold of even dimensions, 2n+2. However, on the symplectic manifold of 4 geometric dimensions and 4 topological dimensions, it follows that there does exist a unique vector direction field, the Torsion vector, completely determined in terms of the functions which define the physical system.

Topological Torsion Class

$$i(\rho \mathbf{V}_{Torsion})dA = \sigma A, \tag{2.30}$$
$$i(\rho \mathbf{V}_{Torsion})A = 0, \tag{2.31}$$

Evolution with a component in the direction of the "Topological Torsion" vector will produce an irreversible process on the physical system described by the Action 1-form, if the divergence of the "Topological Torsion" vector is not zero. This "Topological Torsion" vector is always an associated vector, but is not an extremal vector, relative to the Action 1-form. The Torsion vector is identically zero on domains of Pfaff topological dimension 2, hence non-zero values of the Torsion vector are an indication that the physical system, A, is not an equilibrium system. The Topological Torsion vector exists only on domains of Pfaff topological dimension 3 or greater, in the same sense that Frenet-Serret torsion exists only on domains of geometric dimension 3 or greater. With respect to evolution in the direction of the Torsion Current, the symplectic volume is contracting or expanding exponentially unless $\sigma = 0$, and therefore such vector fields cannot represent a symplectic process (which preserves the volume element). The factor, σ, is a Liapunov function and defines the stability of the process (depending on the sign of σ). When $\sigma = 1$, the Torsion vector has been called the "Liouville vector" [Liebermann 1987].

Vectors which are both extremal and associated are said to be elements of the characteristic class of vector fields [Klein 1962].

Characteristic Class

$$i(\rho \mathbf{V}_{characteristic})A = 0, \quad \text{and} \quad i(\rho \mathbf{V}_{characteristic})dA = 0. \tag{2.32}$$

Note that characteristic flow lines generated by $\mathbf{V}_{characteristic}$ of the Characteristic class preserve the Cartan topology, for each form of the Cartan topological base is invariant with respect to the action of the Lie differential to characteristic flows (See Chapter 4). Characteristics are often associated with wave phenomena, and propagating discontinuities. The Topological Torsion vector mentioned above may have zero divergence on certain geometric subsets of space-time, but these domains are of Pfaff topological dimension 3 (although of geometric dimension 4). In such cases, the Topological Torsion vector will be a characteristic vector for the 1-form of Action, A. These and other properties of the "Topological Torsion" vector will be described n detail below.

2.3.2 Reversible and Irreversible Processes

The Pfaff topological dimension of the exterior differential 1-form of Heat, Q, determines important topological categories of processes. From classical thermodynamics "The quantity of heat in a reversible process always has an integrating factor" [Goldenblatt 1962] [Morse 1964] . Hence, from the Frobenius unique integrability theorem, which requires $Q\hat{}dQ = 0$, all reversible processes are such that the Pfaff dimension of Q is less than or equal to 2. Irreversible processes are such that the Pfaff dimension of Q is greater than 2. A dissipative irreversible topologically *turbulent* process is defined when the Pfaff dimension of Q is 4.

Processes defined by the Pfaff dimension of Q

$$\begin{aligned} Processes &: \text{ as defined by the Pfaff dimension of } Q && (2.33) \\ Q\hat{}dQ &= 0 \quad \textbf{Reversible} \text{ - Pfaff dimension 2} && (2.34) \\ d(Q\hat{}dQ) &\neq 0. \quad \textbf{Turbulent} \text{ - Pfaff dimension 4.} && (2.35) \end{aligned}$$

Note that the Pfaff dimension of Q depends on both the choice of a process, $\mathbf{V}_4$, and the system, A, upon which it acts. As reversible thermodynamic processes are such that $Q\hat{}dQ = 0$, and irreversible thermodynamic processes are such that $Q\hat{}dQ \neq 0$, Cartan's formula of continuous topological evolution can be used to determine if a given process, $\mathbf{V}_4$, acting on a physical system, A, is thermodynamically reversible or not:

Processes defined by the Lie differential of A

$$\left[\begin{array}{l} \textbf{Reversible} \text{ Processes } \rho\mathbf{V}_4 : \; L_{(\rho\mathbf{V}_4)}A\hat{}L_{(\rho\mathbf{V}_4)}dA = 0, \\ \textbf{Irreversible} \text{ Processes } \rho\mathbf{V}_4 : L_{(\rho\mathbf{V}_4)}A\hat{}L_{(\rho\mathbf{V}_4)}dA \neq 0. \end{array} \right] \qquad (2.36)$$

Remarkably, Cartan's magic formula can be used to describe the continuous dynamic possibilities of both reversible and irreversible processes, in equilibrium or non-equilibrium systems, even when the evolution induces topological change, transitions between excited states, and changes of phase, such as condensations.

It is important to note that the direction field, $\mathbf{V}_4$, need not be topologically constrained such that it is singularly parameterized. That is, the evolutionary processes described by Cartan's magic formula are not necessarily restricted to vector fields that satisfy the topological constraints of kinematic perfection, $dx^k - V^k dt = 0$. A discussion of topological fluctuations, where $dx^k - V^k dt = \Delta^k \neq 0$, and an example fluctuation process is described below.

2.3.3 Adiabatic Processes - Reversible and Irreversible

The topological formulation of thermodynamics in terms of exterior differential forms permits a precise definition to be made for both reversible and irreversible adiabatic processes in terms of the topological properties of Q. On a geometrical space of N dimensions, a 1-form, Q, will admit N-1 (null) vector fields, $V_{Associated}$, such that $i(V_{Associated})Q = 0$. Such null vector associated processes $V_{Associated}$ relative to Q are defined as (locally) adiabatic processes, $V_{adiabatic}$ [Bamberg 1992]. The N-1 null vectors will form a distribution of adiabatic processes orthogonal to the 1-form Q. The distribution of adiabatic processes will not form a smooth hypersurface, unless the Pfaff dimension of Q is 2 or less. In other words the null curves (adiabats) form an smooth hypersurface only in the equilibrium state. Note that all adiabatic processes are defined by vector direction fields, to within an arbitrary factor, $\beta(x, y, z, t)$. That is, if $i(V_A)Q = 0$, then it is also true that $i(\beta V_A)Q = 0$. The adiabatic direction fields and the 1-form of Action can be used to construct an interesting basis frame related to projective connections. This possibility will be discussed in a later section.

The differences between the inexact 1-forms of Work and Heat become obvious in terms of the topological format. Both 1-forms, W and Q, depend on the process, $\mathbf{V}_4$, and on the physical system, A. However, Work is always transversal to the process, but Heat is not unless the process is adiabatic:

$$\text{Work is transversal:} \quad i(\mathbf{V}_4)W = i(\mathbf{V}_4)i(\mathbf{V}_4)dA = 0, \tag{2.37}$$

$$\text{Heat is NOT transversal :} \quad i(\mathbf{V}_4)Q = i(\mathbf{V}_4)dU \nRightarrow 0, \tag{2.38}$$

$$\text{unless the process is adiabatic.} \tag{2.39}$$

It is this fundamental difference between Heat, Q, and Work, W, that lead to the Carnot-like statements that it is possible to convert work into heat with 100% efficiency, but it is not possible to convert heat into work with 100% efficiency.

Adiabatic direction fields, so defined as null curves of Q, do not imply that the Pfaff dimension of Q must be 2. That is, it is not obvious that Q can be written in the form, $Q = TdS$, as is possible on the manifold of equilibrium states. From the Cartan formulation it is apparent that if Q is not zero, then

$$L_{(\mathbf{V}_A)}A = Q \neq 0, \tag{2.40}$$

$$\begin{aligned} i(\mathbf{V}_A)L_{(\mathbf{V}_A)}A &= i(\mathbf{V}_A)i(\mathbf{V}_A)dA + i(\mathbf{V}_A)d(i(\mathbf{V}_A)A) \\ &= 0(transversality) + i(\mathbf{V}_A)d(i(\mathbf{V}_A)A) = i(\mathbf{V}_A)Q \end{aligned} \tag{2.41}$$

Hence, an Adiabatic process requires that if Q is not zero,

$$\text{An adiabatic process requires } i(\mathbf{V}_A)Q = i(\mathbf{V}_A)d(i(\mathbf{V}_A)A)) \Rightarrow 0, \quad Q \neq 0 \tag{2.42}$$

$$\text{with a sufficient condition given by} \ : \ U = i(\mathbf{V}_A)A \Rightarrow 0. \tag{2.43}$$

The sufficient condition implies that the process $\mathbf{V}_A$ is an associated vector of the 1-form of Action. All associated vectors to the 1-form, A, are adiabatic. The necessary condition for a process to be adiabatic is the statement that the process is an "associated" vector to the exact exterior differential of the internal energy.

$$\text{An adiabatic process requires } i(\mathbf{V}_A)Q = i(\mathbf{V}_A)d(i(\mathbf{V}_A)A)) \Rightarrow 0, \quad Q \neq 0 \tag{2.44}$$

$$\text{with a necessary condition given by} \ : \ i(\mathbf{V}_A)dU \Rightarrow 0. \tag{2.45}$$

Note that the Topological Torsion vector is an associated vector relative to the Action 1-form, A, and therefore defines an adiabatic (but irreversible) process on domains of Pfaff topological dimension 4.

If the heat 1-form is zero, then the process is a reversible adiabatic process of a special type. A reversible process is defined such that the Pfaff dimension of Q is less than 3; or, $Q\hat{}dQ = 0$. Hence $i(\mathbf{V}_A)(Q\hat{}dQ) = 0$ for reversible processes. But

$$i(\mathbf{V}_A)(Q\hat{}dQ) = (i(\mathbf{V}_A)Q)\hat{}dQ - Q\hat{}i(\mathbf{V}_A)dQ, \tag{2.46}$$

which permits reversible and irreversible adiabatic processes to be well distinguished [§] when $Q \neq 0$:

$$\begin{aligned} \text{Reversible Adiabatic Process} &= -Q\hat{}i(\mathbf{V}_A)dQ \Rightarrow 0,\ i(\mathbf{V}_A)Q \Rightarrow 0, && (2.47)\\ \text{Irreversible Adiabatic Process} &= -Q\hat{}i(\mathbf{V}_A)dQ \neq 0, \quad i(\mathbf{V}_A)Q \Rightarrow 0. && (2.48)\end{aligned}$$

It is certainly true that if $L_{(\mathbf{V})}A = Q = 0$, *identically*, then all such processes are adiabatic, and reversible. (In the next section, it will be demonstrated how these thermodynamic ideas can be associated with the tensor processes of covariant differentiation and parallel transport.) In such special adiabatic cases, the Cartan formalism implies that $W + dU = 0$. Such systems are elements of the Hamiltonian-Bernoulli class of processes, where $W = d\Theta$.

2.3.4 Processes classified by topological constraints on the Work 1-form.

Cartan has shown that all Hamiltonian processes (systems with a generator of ordinary differential equations), $\rho\mathbf{V}_H$, satisfy the following equations of topological constraint on the work 1-form, W :

$$\begin{aligned} & \text{A Hamiltonian } \mathbf{V}_H \text{ is either } \mathbf{V}_E \text{ or } \mathbf{V}_B && (2.49)\\ & \textbf{Extremal Hamiltonian } \mathbf{V}_E && (2.50)\\ W_E &= i(\rho\mathbf{V}_E)dA = 0 \qquad \text{Pfaff dimension} = 0 && (2.51)\\ & \textbf{Bernoulli-Casimir Hamiltonian } \mathbf{V}_B && (2.52)\\ W_B &= i(\rho\mathbf{V}_B)dA = d\Theta \quad \text{Pfaff dimension} = 1 && (2.53)\end{aligned}$$

More details about Cartan's development of Hamiltonian systems appears in Chapter 4.1.

A special case occurs when the Bernoulli function is equal to the negative of the internal energy, for then the heat 1-form produced by this special Hamiltonian process vanishes.

For Helmoltz processes (which are not strictly Hamiltonian) the situation is a bit more intricate, but in all cases the Pfaff dimension of the Work 1-form is at most 1. Hamiltonian processes are subsets of Helmholtz processes.

[§] It is apparent that i(V)Q= 0 defines an adiabatic process, but not necessarily a reversible adiabatic process. This topological point clears up certain misconceptions that appear in the literature.

Helmholtz (Symplectic) Process $\mathbf{V}_S$

$$\begin{aligned} W_H &= i(\rho\mathbf{V}_S)dA = d\Theta + \gamma \text{ Pfaff dimension} = 1 & (2.54)\\ dW_H &= 0 \text{ as } \gamma \text{ is closed but not exact.} & (2.55)\\ & \quad \text{Symplectic when} & (2.56)\\ dA &\neq 0, \quad W_H \neq 0 & (2.57) \end{aligned}$$

Helmholtz-Symplectic processes satisfy the following equations

Helmholtz Conservation of Vorticity

$$L_{(\rho\mathbf{V}_S)}dA = dW_H + ddU = 0 + 0 = dQ, \tag{2.58}$$

However, the closed but not exact component of work can have finite period integrals, so the evolutionary Helmholtz process can involve changing topology. The close integrals of Action are not invariant with respect to $\rho\mathbf{V}_S$ unless $\gamma = 0$.

$$L_{(\rho\mathbf{V}_S)}\int_{z1d} A = \int_{z1d}\gamma = \int_{z1d} Q \neq 0 \tag{2.59}$$

The closed but not exact forms, γ, introduce non-uniqueness into the definition of the work 1-form. As $dQ = d(d\Theta + \gamma + dU) = 0$ for all three processes defined above, all three processes are thermodynamically reversible (see equation (**??**)). So an example is presented which demonstrates that changing topology is a necessary but not sufficient condition for a process to be thermodynamically irreversible. Each of the three reversible processes must satisfy an additional topological constraint if the process is to be locally adiabatic:

$$\begin{aligned} \text{Adiabatic process } i(\mathbf{V}_A)Q &= i(\mathbf{V}_A)d(i(\mathbf{V}_A)A)) \Rightarrow 0, \quad Q \neq 0 & (2.60)\\ \text{with a sufficient condition } &= i(\mathbf{V}_A)A \Rightarrow 0. & (2.61) \end{aligned}$$

If $d\Theta = 0$, then $\rho\mathbf{V}_E$ is a characteristic process relative to the 2-form F. If the work 1-form is of Pfaff topological dimension 0, then the process is an extremal process relative to A (see equation **??**).

Extremal processes cannot exist on a non-singular symplectic domain, because a non-degenerate anti-symmetric matrix (the coefficients of the 2-form dA) does not have null eigenvectors on space of even dimensions . Although unique extremal stationary states do not exist on the domain of Pfaff dimension 4, there can exist evolutionary invariant Bernoulli-Casimir functions, Θ, that generate non-extremal, "stationary"states. Such Bernoulli processes can correspond to energy dissipative Helmholtz processes, but they, as well as all Helmholtz processes, are reversible in the thermodynamic sense described below. The mechanical energy need not be constant, but the Bernoulli-Casimir function(s), Θ,are evolutionary invariant(s), and may be used to describe non-unique stationary state(s).

The equations, above, that define several familiar categories of processes, are in effect constraints on the topological evolution of any physical system represented by an Work 1-form, A. The Pfaff dimension of the 1-form of virtual work, $W = i(\mathbf{V})dA$ is 1 or less for all three categories. The Extremal constraint of equation (2.50) that the Pfaff dimension of W can be used to generate the Euler equations of hydrodynamics for a incompressible fluid. The Bernoulli-Casimir constraint of equation (2.52) can be used to generate the equations for a barotropic compressible fluid. The Helmholtz constraint of equation (**??**) can be used to generate the equations for a Stokes flow. All such processes are thermodynamically reversible as $dQ = 0$. None of these constraints on the Work 1-form, W, above will generate the Navier-Stokes equations, which require that the topological dimension of the 1-form of virtual work must be greater than 2.

Note that for a given 1-form of Action, A, it is possible to construct a matrix of N-1 null vectors, and then to compute the adjoint matrix of cofactors transposed to create the unique direction field (to within a factor), $\mathbf{V}_{NullAdjoint}$. Evolution in the direction of $\mathbf{V}_{NullAdjoint}$ does not represent an adiabatic process path, as $i(\mathbf{V}_{NullAdjoint})A \neq 0$. For a given A, the N-1 null vectors need not span a smooth hypersurface whose surface normal is proportional to a gradient field. The components of the 1-form may be viewed as the normal vector to an implicit hypersurface, but the implicit hypersurface is not necessarily defined as the zero set of some function.

A crucial idea is the recognition that irreversible processes must on domains of Pfaff dimension which support Topological Torsion, $A\hat{}dA \neq 0$, with its attendant properties of non-uniqueness, envelopes, regressions, and projectivized tangent bundles. Such domains are of Pfaff dimension 3 or greater. Moreover, as described below, it would appear that thermodynamic irreversibility must support a non-zero Topological Parity 4-form, $dA\hat{}dA \neq 0$. Such domains are of Pfaff dimension 4 or greater.

Although there does not exist a unique gauge independent stationary state on the symplectic manifold of Pfaff dimension 4, remarkably there does exist a unique vector field on the symplectic domain, with components that are generated by the 3-form $A\hat{}dA$. This unique (to within a factor) vector field is defined as the Torsion Current, $\mathbf{T}_4$, and satisfies (on the 2n+2=4 dimensional manifold) the equation,

$$i(\mathbf{T}_4)dx\hat{}dy\hat{}dz\hat{}dt = A\hat{}dA \tag{2.62}$$

This (four component) vector field, $\mathbf{T}_4$, has a non-zero divergence almost everywhere, for if the divergence is zero, then the 4-form $dA\hat{}dA$ vanishes, and the domain is no longer a symplectic manifold! The Torsion vector, $\mathbf{T}_4$, can be used to generate a dynamical system that will decay to the stationary states ($div_4(\mathbf{T}_4) \Rightarrow 0$) starting from arbitrary initial conditions. These processes are irreversible in the thermodynamic sense. It is remarkable that this unique evolutionary vector field, $\mathbf{T}_4$, is completely determined (to within a factor) by the physical system itself; e.g., the components of the 1-form, A, determine the components of the Torsion vector.

To understand what is meant by thermodynamic irreversibility, realize that Cartan's magic formula of topological evolution is equivalent to the first law of thermodynamics.

$$L_{(\mathbf{V})}A = i(\mathbf{V})dA + d(i(\mathbf{V})A) = W + dU = Q. \tag{2.63}$$

A is the "Action" 1-form that describes the hydrodynamic system. $\mathbf{V}$ is the vector field that defines the evolutionary process. W is the 1-form of (virtual) work. Q is the 1-form of heat. From classical thermodynamics, a process is irreversible when the heat 1-form Q does not admit an integrating factor. From the Frobenius theorem, the lack of an integrating factor implies that $Q\hat{}dQ \neq 0$. Hence a simple test may be made for any process, $\mathbf{V}$, relative to a physical system described by an Action 1-form, A.

Conclusion 1 *If* $L_{(\mathbf{V})}A\hat{}L_{(\mathbf{V})}dA \neq 0$ *then the process* $\mathbf{V}$ *relative to the system* A *is thermodynamically irreversible.*

This topological definition implies that the three categories (above) of Helmholtz, Hamiltonian-Bernoulli, or Hamiltonian-extremal processes, $\subset \mathbf{S}$, are reversible (*as* $L_{(\mathbf{S})}dA{=}dQ = 0$). However, for evolution in the direction of the Torsion vector, $\mathbf{T}_4$, direct computation demonstrates that the fundamental equations lead to a conformal evolutionary process, a process which is thermodynamically irreversible:

$$L_{(\mathbf{T}_4)}A = \sigma A \quad \textit{and} \quad i(\mathbf{T}_4)A = 0, \tag{2.64}$$

such that

$$L_{(\mathbf{T}_4)}A\hat{}L_{(\mathbf{T}_4)}dA = Q\hat{}dQ = \sigma^2 A\hat{}dA \neq 0. \tag{2.65}$$

2.4 The Lie differential $L_{(V)}$ and the Covariant differential $\nabla_{(V)}$

The covariant derivative of tensor analysis, and as used in General Relativity, is often defined in terms of isometric diffeomorphic processes (that preserve the differential line element) and can be used to describe rigid body motions and isometric bendings, but not deformations and shear processes associated with convective fluid flow.

Another definition of the covariant derivative is based on the concept of a connection, such that the differential process acting on a tensor produces a tensor. The definition of the covariant derivative usually depends upon the additional structure (or constraint) of a metric or a connection placed on a given variety, while the Lie differential does not. As the Lie differential is not so constrained, it may be used to describe non-diffeomorphic processes for which the topology changes continuously. The covariant derivative is avoided in this article.

Koszul has give a set of axioms that can be used to define a linear connection and a covariant derivative. The covariant derivative axioms require that

$$\nabla_{(fV)}\omega = f\ \nabla_{(V)}\omega, \tag{2.66}$$
$$\nabla_{(V)}f\omega = (\nabla_{(V)}f\ )\omega + f\ \nabla_{(V)}\omega. \tag{2.67}$$

This axiomatic representation of a covariant derivative and an affine connection should be compared to the Lie differential,

$$L_{(fV)}A = f\ L_{(V)}A + df\ (i(V)A), \tag{2.68}$$
$$L_{(V)}fA = (L_{(V)}f\ )A + f\ L_{(V)}A. \tag{2.69}$$

Only if the last term in the expansion of the Lie differential, $df\ (i(V)A)$, is zero does the formula for the Lie differential have an equivalent representation as a covariant derivative in terms of a connection. Suppose that $i(V)A = 0$,such that the Lie differential and the covariant differential are equivalent.

$$L_{(fV)}A = f\ L_{(V)}A = f\ \nabla_{(V)}A. \tag{2.70}$$

Then it follows that

$$L_{(fV)}A = f\ L_{(V)}A + df\ (i(V)A) \tag{2.71}$$
$$= f\ L_{(V)}A = f\ Q. \tag{2.72}$$
$$\text{But } i(V)Q = f\ i(V)i(V)dA = 0 \Rightarrow i(V)Q, \tag{2.73}$$
$$\text{where } i(V)Q = 0 \text{ defines an adiabatic process.} \tag{2.74}$$

Conclusion 2 *Hence, all covariant derivatives with respect to a connection have an equivalent representation as an adiabatic process!!! Such covariant adiabatic processes need not be thermodynamically reversible.*

Suppose that the adiabatic process is such that

$$L_{(V)}A = Q = 0 \tag{2.75}$$

Then

$$dL_{(V)}A = L_{(V)}dA = \ dQ = 0, \tag{2.76}$$

and it follows that the adiabatic process is reversible. However, the condition that Q be zero is the equivalent to the condition of parallel transport:

$$L_{(V)}\omega = \nabla_{(V)}\omega = 0. \tag{2.77}$$

Conclusion 3 *The remarkable conclusion is that the concept of parallel transport in tensor analysis is in effect an adiabatic, reversible process!!!*

2.5 Topological Torsion

For maximal, non-equilibrium, turbulent systems in space-time, the maximal element in the Pfaff sequence generated by A, W, or Q, is a 4-form. On the geometric space of 4 independent variables, every 4-form is globally closed, in the sense that its exterior differential vanishes everywhere. It follows that every 4-form is exact and can be generated by the exterior differential of a 3-form. The exterior differential of the 3-form is related to the concept of a divergence of a contravariant vector field. Most of the development in this article will be devoted to the study of such 3-forms, and their kernels. It is a remarkable fact that all 3-forms admit integrating denominators, such that their exterior differential of a rescaled 3-form is zero almost everywhere. Space time points upon which the integrating denominator has a zero value form singularities defined as topological defect structures.

When the Action for a physical system is of Pfaff dimension 4, there exists a unique direction field, $\mathbf{T}_4$, defined as the Topological Torsion 4-vector, that can be evaluated *entirely* in terms of those component functions of the 1-form of Action which define the physical system. To within a factor, this direction field¶ has the four components of the 3-form $A\hat{}dA$, with the following properties:

Properties of the Topological Torsion vector T_4

$$\begin{aligned}
i(\mathbf{T}_4)\Omega_4 &= A\hat{}dA &(2.78)\\
W &= i(\mathbf{T}_4)dA = \sigma\ A, &(2.79)\\
U &= i(\mathbf{T}_4)A = 0, &(2.80)\\
L_{(\mathbf{T}_4)}A &= \sigma\ A, &(2.81)\\
Q\hat{}dQ &= L_{(\mathbf{T}_4)}A\hat{}L_{(\mathbf{T}_4)}dA = \sigma^2 A\hat{}dA \neq 0 &(2.82)\\
dA\hat{}dA &= 2\ \sigma\ \Omega_4. &(2.83)
\end{aligned}$$

¶A direction field is defined by the components of a vector field which establish the "line of action" of the vector in a projective sense. An arbitrary factor times the direction field defines the same projective line of action, just reparameterized. In metric based situations, the arbitrary factor can be interpreted as a renormalization factor.

Hence, by equation (2.82) evolution in the direction of $\mathbf{T}_4$ is thermodynamically irreversible, when $\sigma \neq 0$ and A is of Pfaff topological dimension 4. The kernel of this vector field is defined as the zero set under the mapping induced by exterior differentiation. In engineering language, the kernel of this vector field are those point sets upon which the divergence of the vector field vanishes. The Pfaff topological dimension of the Action 1-form is 3 in the defect regions defined by the kernel of $\mathbf{T}_4$.

The Topological Torsion vector vanishes when the Pfaff topological dimension is 2 or less. Note that the Frenet-Serret geometric torsion of a space curve vanishes when the geometric dimension is 2 or less. It is this analog dependence on dimension 3 or more that led to the name "Topological Torsion" for the 3-form $A\hat{}dA$. Solution uniqueness is lost when the Topological Torsion vector is not zero. In 4D, the three form $A\hat{}(dA)$ has been defined as the Topological Torsion 3-form. The Torsion current depends only on the system (the Action) and not upon a process. The divergence of this Torsion current is proportional to the measure of the 4D volume, that defines the symplectic space, and cannot be zero on the symplectic domain. The components of the Topological Torsion vector $\mathbf{T}_4$ generate what is called the "subsidiary Pfaffian system" by Forsyth [Forsyth (1890) 1959].

For purposes of more rapid comprehension, consider a 1-form of Action, A, with an exterior differential, dA, and a notation that admits an electromagnetic interpretation ($\mathbf{E} = -\partial\mathbf{A}/\partial t - \nabla\phi$, and $\mathbf{B} = \nabla \times \mathbf{A}$)[‖]. The explicit format of $\mathbf{T}_4$ becomes:

The Electromagnetic Topological Torsion 4 vector

$$\mathbf{T}_4 = -[\mathbf{E} \times \mathbf{A} + \mathbf{B}\phi,\ \mathbf{A} \circ \mathbf{B}] \tag{2.84}$$

$$A\hat{}dA = i(\mathbf{T}_4)\Omega_4 \tag{2.85}$$

$$= T_4^x dy\hat{}dz\hat{}dt - T_4^y dx\hat{}dz\hat{}dt + T_4^z dx\hat{}dy\hat{}dt - T_4^t dx\hat{}dy\hat{}dz, \tag{2.86}$$

$$dA\hat{}dA = 2(\mathbf{E} \circ \mathbf{B})\ \Omega_4 \tag{2.87}$$

$$= \{\partial T_4^x/\partial x + \partial T_4^y/\partial y + \partial T_4^z/\partial z + \partial T_4^t/\partial t\}\ \Omega_4. \tag{2.88}$$

When the divergence of the topological torsion vector is not zero, $\sigma = (\mathbf{E}\circ\mathbf{B}) \neq 0$, and A is of Pfaff dimension 4, W is of Pfaff dimension 4, and Q is of Pfaff dimension 4. The process generated by $\mathbf{T}_4$ is thermodynamically irreversible, as

$$Q\hat{}dQ = L_{(\mathbf{T}_4)}A\hat{}L_{(\mathbf{T}_4)}dA = \sigma^2 A\hat{}dA \neq 0. \tag{2.89}$$

The evolution of the volume element relative to the irreversible process $\mathbf{T}_4$ is given by the expression,

[‖]The bold letter **A** represents the first 3 components of the 4 vector of potentials, with the order in agreement with the ordering of the independent variables. The letter A represents the 1-form of Action.

$$L(\mathbf{T}_4)\Omega_4 = i(\mathbf{T}_4)d\Omega_4 + d(i(\mathbf{T}_4)\Omega_4) \quad (2.90)$$
$$= 0 + d(A\hat{}dA) = 2(\mathbf{E} \circ \mathbf{B})\ \Omega_4. \quad (2.91)$$

Hence, the differential volume element is expanding or contracting depending on the sign and magnitude of $\mathbf{E}\circ\mathbf{B}$, a useful fact when topological thermodynamics is applied to cosmology.

If A is (or becomes) of Pfaff dimension 3, then $dA\hat{}dA \Rightarrow 0$ which implies that $\sigma^2 \Rightarrow 0$, but $A\hat{}dA \neq 0$. The differential geometric volume element Ω_4 is subsequently an evolutionary invariant, and evolution in the direction of the topological torsion vector is thermodynamically reversible. The physical system is not in equilibrium, but the divergence free $\mathbf{T}_4$ evolutionary process forces the Pfaff dimension of W to be zero, and the Pfaff dimension of Q to be at most 1. Indeed, a divergence free $\mathbf{T}_4$ evolutionary process has a Hamiltonian representation. In the domain of Pfaff dimension 3 for the Action, A, the subsequent continuous evolution of the system, A, relative to the process $\mathbf{T}_4$, proceeds in an energy conserving manner, representing a "stationary" or "excited" state far from equilibrium. These excited states can be interpreted as the evolutionary topological defects in the Turbulent dissipative system of Pfaff dimension 4. The Topological Torsion vector becomes an adiabatic, extremal, characteristic direction field in the space of geometric dimension 4, but where the Pfaff dimension of the physical system, A, is of Pfaff topological dimension 3.

On a geometric domain of 4 dimensions, assume that the evolutionary process generated by $\mathbf{T}_4$ starts from an initial condition (or state) where the Pfaff topological dimension of A is also 4. Depending on the sign of the divergence of $\mathbf{T}_4$, the process follows an irreversible path for which the divergence represents an expansion or a contraction. If the irreversible evolutionary path is attracted to a region (or state) where the Pfaff topological dimension of the 1-form of Action is 3, then $\mathbf{E} \circ \mathbf{B}$ becomes (or has decayed to) zero. The zero set of the function $\mathbf{E} \circ \mathbf{B}$ defines a hypersurface in the 4 dimensional space. If the process remains trapped on this hypersurface of Pfaff dimension 3, $\mathbf{E} \circ \mathbf{B}$ remains zero, and the $\mathbf{T}_4$ process becomes an extremal, adiabatic, characteristic direction field. Such extremal fields are such that the virtual work 1-form vanishes, $W = i(\mathbf{T}_4)dA = 0$, and the now reversible $\mathbf{T}_4$ process has a Hamiltonian representation. The system is conservative in a Hamiltonian sense, but it is in a "excited" state on the hypersurface that is far from equilibrium, as the Pfaff dimension of the 1-form of Action is 3, and not 2. (If the path is attracted to a region where the function $\mathbf{E} \circ \mathbf{B}$ is oscillatory, the system evolutionary path defines a limit cycle.)

The fundamental claim made in this monograph is that it is these topological defects that self organize from the dissipative irreversible evolution of the Turbulent state into "stationary" states far from equilibrium that form the stars and the galaxies of the cosmos at a cosmological level. They represent the long lived remnants or wakes generated from irreversible processes in the dissipative non-equilibrium macroscopic turbulent fluid. They form the excited quantum states at the microscopic level.

2.6 Topological Fluctuations.

2.6.1 Extensions of the Cartan-Hilbert Action 1-form for first order equations.
This subsection considers those physical systems that can be described by a Lagrange function $L(\mathbf{q}, \mathbf{v}, t)$ and a 1-form of Action given by:

$$A = L(\mathbf{q}^k, \mathbf{v}^k, t)dt + \mathbf{p}_k \cdot (d\mathbf{q}^k - \mathbf{v}^k dt), \tag{2.92}$$

The classic Action, $L(\mathbf{q}^k, \mathbf{v}^k, t)dt$, is extended to included fluctuations in the kinematic variables. It is no longer assumed that the equation of Kinematic perfection is satisfied. Fluctuations of the topological constraint of kinematic perfection are permitted:

Topological Fluctuations in differential position

$$\Delta\mathbf{q} = (d\mathbf{q}^k - \mathbf{v}^k dt) \neq 0. \tag{2.93}$$

When dealing with fluctuations, the geometric dimension will not be constrained to 4 independent variables. At first glance it appears that the domain of definition is a 3n+1 dimensional variety of independent base variables, $\{\mathbf{q}^k, \mathbf{v}^k, t\}$. Do not assume that $\mathbf{p}$ is constrained to be a jet; e.g., $\mathbf{p}_k \neq \partial L/\partial \mathbf{v}^k$. Instead, consider $\mathbf{p}_k$ to be a (set of) Lagrange multiplier(s) to be determined later. Note that the Action 1-form has the format used in the Cartan-Hilbert invariant integral [Chern 1948], except that it is not assumed that $\mathbf{p}_k$ is canonical; $\mathbf{p}_k \neq \partial L/\partial \mathbf{v}^k$ necessarily. Also, do not assume at this stage that $\mathbf{v}$ is a kinematic velocity function, such that $(d\mathbf{q}^k - \mathbf{v}^k dt) \Rightarrow 0$. The classical idea is to assert that topological fluctuations in kinematic velocity are related to pressure.

For the given Action, construct the Pfaff sequence $\{A, dA, A\hat{}dA, dA\hat{}dA...\}$ in order to determine the Pfaff dimension or class of the 1-form [Liebermann 1987]. The top Pfaffian is defined as the non-zero p-form of largest degree p in the sequence). The top Pfaffian for the Cartan-Hilbert Action is given by the formula,

$$(dA)^{n+1} = (n+1)!\{\Sigma_{k=1}^{n}(\partial L/\partial v^k - p_k)dv^k\}\hat{}dp_1\hat{}...dp_n\hat{}dq^1\hat{}..dq^n\hat{}dt, \tag{2.94}$$

The formula is a bit surprising in that it indicates that the Pfaff topological dimension of the Cartan-Hilbert 1-form is 2n+2, and not the geometrical dimension 3n+1. For n = 3 degrees of freedom, the top Pfaffian indicates that the topological of Pfaff topological dimension of the 2-form, dA is $2n+2=8$. The value 3n+1 might be expected as the 1-form was defined initially on a space of 3n+1 "independent" base variables. The implication is that there exists an irreducible number of independent variables equal to 2n+2 which completely characterize the differential topology of the first order system described by the Cartan-Hilbert Action. It follows that the exact two form, dA, satisfies the equations

$$(dA)^{n+1} \neq 0, \; but \quad A\hat{}(dA)^{n+1} = 0. \tag{2.95}$$

The format of the top Pfaffian requires that the bracketed factor $\{\Sigma_{k=1}^{n}(\partial L/\partial v^k - p_k)dv^k\}$ can be represented by a perfect differential:

$$dS = \{\Sigma_{k=1}^{n}(\partial L/\partial v^k - p_k)dv^k\} \tag{2.96}$$

The result is also true for any closed addition γ added to A; e.g., the result is "gauge invariant". Addition of a closed 1-form does not change the Pfaff dimension from even to odd. On the other hand the result is not renormalizeable, for multiplication of the Action 1-form by a function can change the algebraic Pfaff dimension from even to odd.

On the 2n+2 domain, the components of 2n+1 form $T = A\hat{}(dA)^n$ generate what is herein defined as the Torsion Current. The coefficients of the 2n+1 form are components of a contravariant vector density $\mathbf{T}^m$ defined as the Topological Torsion vector, the same concept as defined in the previous chapter, but now extended to 2n+2 dimensions. This vector is orthogonal (transversal) to the 2n+2 components of the covector, $\mathbf{A}_m$. In other words,

$$A\hat{}T = A\hat{}(A\hat{}(dA)^n) = 0 \Rightarrow i(\mathbf{T})(A) = \sum \mathbf{T}^m \mathbf{A}_m = 0. \tag{2.97}$$

This result demonstrates that the extended Topological Torsion vector represents an adiabatic process. This topological result does not depend upon geometric ideas such as metric. The Topological Torsion vector has already appeared above, where it was demonstrated on a space of 4 independent variables, that evolution in the direction of the Topological Torsion vector is irreversible in a thermodynamic sense, subject to the symplectic condition of non-zero divergence, or that $dA\hat{}dA$ is not zero.

The 2n+2 symplectic domain so constructed can not be compact without boundary for it has a volume element which is exact. For the 2n+2 domain to be symplectic, the top Pfaffian can never vanish. The domain is therefore orientable. Examination of the constraint that the symplectic space be of dimension 2n+2 implies that the Lagrange multipliers, $\mathbf{p}_k$, cannot be used to define momenta in the classical "conjugate or canonical" manner. Define the non-canonical components of the momentum as

Non-canonical momentum ϖ_j

$$\varpi_j = (p_j - \partial L/\partial v^j) \tag{2.98}$$

such that the top Pfaffian can be written as

$$(dA)^{n+1} = (n+1)!\{\Sigma_{j=1}^{n} \varpi_j dv^j\}\hat{}dp_1\hat{}...dp_n\hat{}dq^1\hat{}..dq^n\hat{}dt. \qquad (2.99)$$

For the Cartan Hilbert Action to be of Pfaff topological dimension 2n+2, the factor $\{\Sigma_{j=1}^{n} \varpi_j dv^j\} \neq 0$. It is important to note, however, that the 1-form $\Sigma_{j=1}^{n} \varpi_j dv^j$ is exact:

$$\Sigma_{j=1}^{n} \varpi_j dv^j = dS_v \qquad (2.100)$$

Tentatively, this function S_v will be defined as the Topological Entropy relative to fluctuations of differential position.

However, if by some evolutionary process, which induces orthogonality,

Orthogonality

$$\{\Sigma_{j=1}^{n} \varpi_j dv^j\} = dS_v \Rightarrow 0, \qquad (2.101)$$

or the topological constraint that the momenta become canonical

Canonical Momenta

$$p_k = \partial L / \partial v^k, \qquad (2.102)$$

it follows that the Cartan-Hilbert Action decreases its topological dimension to 2n+1. This 2n+1 Contact manifold is the state space of classical mechanics. When that Action 1-form generates a Contact manifold, there is always a unique extremal vector field which generates a system of first order ODE's known as Hamilton's equations describing the extremal process. If at subsequent steps the differentials dp_k become zero the dimensionality of the 2n+2 manifold becomes the configuration space manifold of dimension n+1. If the Pfaff dimension of A is equal to 1 when A is restricted to the submanifold, the equilibrium state has been defined in which the entropy function, S, is a constant; e.g., $dSv = 0$.

The important facts are that there are two classes of processes that can represent the topological change from a Pfaff topological dimension 2n+2 to a Pfaff topological dimension of 2n+1. The 2n+2 system supports thermodynamically irreversible dissipative processes. The 2n+1 system supports stationary reversible Hamiltonian processes. The two classes of processes are distinguished by the property that the velocity field is orthogonal to the non-canonical momenta, or the process causes the non-canonical momenta to vanish.

If the domain of definition is constrained such that the momenta are defined canonically, $\partial L/\partial v^k - p_k = 0$, then the 2-form dA does not define a symplectic manifold of Pfaff topological dimension 2n+2, but the 2-form does define Contact structure on 2n+1 with the formula for the Top Pfaffian given by the expression.

$$A\hat{}(dA)^n = n!\{p_k v^k - L(t, q^k, v^k)\}dp_1\hat{}...dp_n\hat{}dq^1\hat{}..dq^n\hat{}dt. \tag{2.103}$$

The coefficient in brackets is recognized as the Legendre transform of the Lagrangian producing the format of the classic Hamiltonian energy. It is this 2n+1 dimensional contact manifold that serves as the arena for most of classical mechanics prior to 1955, especially for those theories which were built from the calculus of variations. The 2n+1 dimensional contact manifold, or state space, admits a unique "extremal" evolutionary field, $i(\mathbf{V})dA = 0$, that satisfies "Hamilton's equations". The coefficient of the state space volume is to be recognized as the Legendre transform of the physicist's Hamiltonian energy function.

$$L(t, q^k, v^k) = p_k v^k - H(t, q^k, v^k, p_k) \tag{2.104}$$

When the constraints of canonical momenta are valid, it follows that

$$\partial H(t, q^k, v^k, p_k)/\partial v^k = 0. \tag{2.105}$$

This result is interpreted by the statement that the "Hamiltonian" is to be expressed in terms of the variables $\{t, q^k, p_k\}$ only. The Top Pfaffian becomes

$$A\hat{}(dA)^n = n!\{H(t, q^k, p_k)\}dp_1\hat{}...dp_n\hat{}dq^1\hat{}..dq^n\hat{}dt. \tag{2.106}$$

The 2n+1 space maintains its contact structure as long as the "total Hamiltonian energy" is never zero, and the momenta are canonically defined.

If further topological evolution causes the Pfaff topological dimension to change from 2n+2 to 2n, then it follows that the Hamiltonian energy must vanish. That is (using the canonical constraint), reduction of the Pfaff dimension from 2n+1 to 2n (state space to phase space) requires that the LaGrange function be homogeneous of degree 1 in the velocities, v^k:

$$\{v^k \partial L(t, q^k, v^k)/\partial v^k - L(t, q^k, v^k)\} \Rightarrow 0. \tag{2.107}$$

The result is remarkable in that the concept of a Finsler space is precisely that where the coefficient $\{p_k v^k - L(t, q, v\}$ of the 2n+1 manifold vanishes. These constraints, of canonical momentum and that the Lagrangian be homogeneous of degree 1 in the velocities, are precisely Chern's constraints used to define a Finsler space [Chern 1948] which admits non-Riemannian geometries (when the Lagrange function contains more than quadratic powers of $\mathbf{v}$) and spaces with torsion [Brillouin 1964]. Note that the processes of topological reduction described above are not equivalent to forming an arbitrary section(s) in the form of holonomic constraints.

2.6.2 Thermodynamics and Topological Fluctuations of Work

Topological fluctuations are admitted when the evolutionary vector direction fields are not singly parametrized. It is historical to consider the interpretations of equilibrium statistical fluctuations in terms of pressure and temperature. These concepts are assumed to be transported to topological fluctuations:

$$\text{Fluctuations in position (pressure)}: \qquad d\mathbf{q} - \mathbf{v}dt = \Delta\mathbf{q} \neq 0 \qquad (2.108)$$
$$\text{Fluctuations in velocity (temperature)}: \qquad d\mathbf{v} - \mathbf{a}dt = \Delta\mathbf{v} \neq 0 \qquad (2.109)$$

These "failures" of kinematic perfection undo the topological refinements imposed by a "kinematic particle" point of view, and place emphasis on the continuum methods inherent in fluids and plasmas.

For the maximal non-canonical symplectic physical system of Pfaff dimension 2n+2, consider evolutionary processes to be representable by vector fields of the form $\gamma V_{3n+1} = \gamma\{\mathbf{v}, \mathbf{a}, \mathbf{f}, 1\}$, relative to the independent variables $\{\mathbf{q}, \mathbf{v}, \mathbf{p}, t\}$. Use the Cartan magic formula definition of the "virtual work" 1-form, W, as $W = i(\mathbf{W})dA$. The Work 1-form must vanish for the extremal case, and be non-zero, but closed, for the symplectic case.

First compute the 2-form, dA from the Cartan-Hilbert Action:

$$dA = \{\partial L/\partial v^k - p_k\}(\Delta v^k)\hat{}dt + \{dp_k - \partial L/\partial x^k dt\}\hat{}(\Delta q^k) \qquad (2.110)$$

Then compute the Work 1-form

$$W = (\gamma V_{3n+1})dA = \{\mathbf{p} - \partial L/\partial \mathbf{v}\} \bullet \Delta\mathbf{v} + \{\mathbf{f} - \partial L/\partial \mathbf{q}\} \bullet \Delta\mathbf{q} \qquad (2.111)$$

Note that $\{\mathbf{p} - \partial L/\partial \mathbf{v}\}$ is the definition of the non-canonical momentum, $\boldsymbol{\varpi}$, and $\{\mathbf{f} - \partial L/\partial \mathbf{q}\}$ represents those components of the force that are not conservative. When the fluctuations in velocity are zero (temperature) and the fluctuations in position are zero (pressure), then the work 1-form will vanish, and the process and physical system admits an extremal Hamiltonian representation. On the other hand if the fluctuations in velocity are not zero and the fluctuations in position are not zero, then the Work 1-form vanishes only if the momenta (the Lagrange multipliers, $\mathbf{p}$, are canonically defined ($\{\mathbf{p} - \partial L/\partial \mathbf{v}\} \Rightarrow 0$) and the Newtonian force is a gradient, $\{\mathbf{f} - \partial L/\partial \mathbf{x}\} \Rightarrow 0$. These topological constraints are ubiquitously assumed in classical conservative Hamiltonian mechanics.

When all topological fluctuations vanish, then the Pfaff dimension of the work 1-form is also zero. This is a sufficient but not necessary condition for equilibrium.** It is possible that when the momenta are canonical, and the force is conservative,

**Inanimate is perhaps a better description of the state with zero fluctuations.

the equilibrium state can admit fluctuations, and yet the Work 1-form vanishes, and the Heat 1-form is exact. This result forms the basis for a statistical analysis of the equilibrium state (which is more or less ignored in this monograph).

Fluctuations in Pfaff topological dimension 2n+2 and 2n+1

When the 2-form dA is non-zero, all processes acting on the Cartan Hilbert Action, generate a work 1-form of the form given in equation (2.111). The maximum topological dimension for the Cartan-Hilbert Action is 2n+2. Suppose that the 2-form dA is constructed in terms of these "2n+2 topological coordinates". The 2-form dA is said to be non-degenerate, or of maximal rank, on the 2n+2 dimensional space in regions where the antisymmetric matrix representing dA has no zero eigenvalues. (Recall that closed non-degenerate 2-forms define a symplectic structure [Liebermann 1987]). However, there may exist singularities in the space of topological coordinates 2n+2 where the 2-form dA becomes "singular". In such regions the 2-form dA becomes degenerate and admits zero eigenvalues. Such regions, say of Pfaff dimension 2n+1, can be considered to be topological defects in the 2n+2 topological domain. In such subspaces, the 2-form dA expressed in terms of the 2n+2 topological coordinates, admits two null eigenvectors. As the eigen values of an anti-symmetric matrix come in pairs, vectors representing topological defects of the symplectic domain are not unique, a well known result of the calculus of variations having envelope solutions. One of these null eigen vectors (of 2n+2 components) is the unique Hamiltonian-extremal field, and the other is the topological torsion vector (of 2n+2 components), which is reduced to a Characteristic vector relative to the Action in the subspace of topological defects. The Characteristic vector is equivalent to the Topological Torsion vector, if the divergence of the topological torsion vector is zero. Processes defined by the extremal field or the characteristic field (degenerate topological torsion vector) are thermodynamically irreversible. The process generated by the topological torsion vector with non-zero divergence is thermodynamically irreversible.

These facts can now be combined with the expression for the work 1-form given in equation (2.111). In the regions where dA is non degenerate, the Work cannot vanish (as this would imply a null eigenvector). It follows that the following 4 situations are NOT allowed when dA is of maximal rank.

1. **Canonical momentum and gradient forces**

$$\{\mathbf{p} - \partial L/\partial \mathbf{v}\} = 0 \; and \; \{\mathbf{f} - \partial L/\partial \mathbf{q}\} = 0 \tag{2.112}$$

2. **Canonical momentum and zero kinematic fluctuations in position.**

$$\{\mathbf{p} - \partial L/\partial \mathbf{v}\} = 0 \; and \; \Delta\mathbf{q} = 0 \tag{2.113}$$

3. **Zero kinematic fluctuations in velocity and gradient forces**

$$\Delta\mathbf{v} = 0 \; and \; \{\mathbf{f} - \partial L/\partial \mathbf{q}\} = 0 \tag{2.114}$$

4. Zero kinematic fluctuations in velocity and Zero kinematic fluctuations in position

$$\Delta\mathbf{v} = 0 \textit{ and } \Delta\mathbf{q} = 0 \tag{2.115}$$

Conversely, when dA generates a contact manifold of Pfaff topological dimension 2n+1, one of the four cases above must be true. In the contact 2n+1 domain, however, there exists a unique vector field with a null eigen value, such that the virtual work 1-form indeed vanishes: $W = i(\mathbf{X})dA = 0$. This result serves as the basis of the d'Alembert principle. An elementary case is based upon the assumption that 4 is valid. That is, there exists a kinematic description of the process at both the first and the second order (velocities and accelerations are singly parameterized). Another case that is common is based on the assumption that the momentum is canonically defined. Then, for the Contact extremal case to exist, and as $\{\mathbf{p} - \partial L/\partial\mathbf{v}\} = 0$, it is necessary that the work 1-form reduces to vanishing expression

$$W = \{\mathbf{f} - \partial L/\partial\mathbf{q}\} \circ \Delta q \Rightarrow 0 \text{ in the extremal case.} \tag{2.116}$$

The extremal constraint is satisfied when the bracket factor vanishes, which is then the equivalent of the Lagrange-Euler equations of classical mechanics. However, the Contact constraints are also satisfied when the force is a gradient field, or there exist zero fluctuations in position, or the non-zero components of the force (the otherwise dissipative components) are orthogonal to the kinematic fluctuations in position.

Bernoulli-Hamiltonian Processes and fluctuations in Work

A Bernoulli-Hamiltonian process is not uniquely defined by the 1-form of Action representing the physical process. (The extremal direction field in 2n+1 and the topological torsion direction field in 2n+2 are uniquely defined by the 1-form of Action representing the physical system.) Recall that a Bernoulli-Hamiltonian process is defined by the Work 1-form being non-zero and exact, $W = i(\mathbf{X})dA = d\Theta \neq 0$, where Θ is an arbitrary function, often called a "Casimir" -or somewhat inappropriately, a "Hamiltonian". In non-singular regions where the 1-form A is of Pfaff dimension 2n+2, and is non-degenerate, the functions Θ are never constant and never without a gradient. Although not constants over the domain, these "potential" or "energy" functions Θ are evolutionary invariants of the process, $\mathbf{X}$.

Most engineers and applied scientists have a greater appreciation for these functions when it is pointed out that they are equivalent to the Bernoulli invariants in hydrodynamics and the thermodynamic potentials in classical thermodynamics. The engineer would call Θ a Bernoulli "constant", a function invariant along a streamline, but which has different values for different neighboring streamlines: $\Theta = (P + \rho gh + \rho v^2/2)Vol$.

To prove that the Bernoulli-Casimirs are always evolutionary invariants with respect to the vector fields, $\mathbf{X}$, construct the Lie differential of Θ with respect to $\mathbf{W}$.

$$L_{(\mathbf{X})}\Theta = i(\mathbf{X})d\Theta + d(i(\mathbf{X})\Theta) = i(\mathbf{X})i(\mathbf{X})dA + d(i(\mathbf{X})\Theta) = 0 + 0. \tag{2.117}$$

Both the first and second terms vanish algebraically. However, for the classic "Hamiltonian" defined above in terms of the Legendre transformation, $H(t, q, v, p) = \{p_k v^k - L(t, q, v\}$, a direct computation indicates that the Hamiltonian need not be an invariant of a symplectic process - even if the Hamiltonian is explicitly time independent. For consider the evolutionary equation,

$$L_{(\mathbf{X})}H = i(\mathbf{X})dH = \{(\partial H/\partial \mathbf{q}) \bullet \mathbf{v} + (\partial H/\partial \mathbf{p}) \cdot \mathbf{f} + (\partial H/\partial \mathbf{v}) \cdot \mathbf{a} + (\partial H/\partial t)\} \tag{2.118}$$

or equivalently

$$L_{(\mathbf{X})}H = \{(\mathbf{p} - \partial L/\partial \mathbf{v}) \bullet \mathbf{a} + (\mathbf{f} - \partial L/\partial \mathbf{q}) \bullet \mathbf{v} - (\partial L/\partial t)\}. \tag{2.119}$$

For the domain of the Cartan-Hilbert Action which is of Pfaff topological dimension 2n+2, the first factor of the first term cannot vanish. The first factor of the second term, when set to zero, is equivalent to the classical Lagrange-Euler equations, and the forces are conservative gradient fields. Suppose that $(\partial L/\partial t) = -(\partial H/\partial t) = 0$, and the non-conservative forces are orthogonal to the velocities, then, even in this case, if the accelerations $\mathbf{a}$ are such that $(\mathbf{p} - \partial L/\partial \mathbf{v}) \cdot \mathbf{a} \neq 0$, the "Hamiltonian energy" H, is not an evolutionary invariant relative to $\mathbf{X}$. Yet the Bernoulli-Casimir energies are evolutionary invariants relative to $\mathbf{X}$. A simple example of this situation is where the mechanical (Hamiltonian) energy of a system decays to perhaps some non-zero value at a singular point of the 2n+2 domain, but the angular momentum stays constant during the process. Numerical simulations of such evolutionary possibilities in fluids have been studied by Carnevale [Carnevale 1990].

Thermodynamic Potentials as Bernoulli evolutionary invariants.

Interesting correspondences can be made between Bernoulli evolutionary invariants and the thermodynamic potentials of classical thermodynamics. Regard the equation for the work 1-form generated by an arbitrary process acting on a physical system encoded buy the Cartan-Hilbert 1-form of Action, which is constrained by a Bernoulli process to be a perfect differential:

$$W = \{\mathbf{p} - \partial L/\partial \mathbf{v}\} \bullet \Delta\mathbf{v} + \{\mathbf{f} - \partial L/\partial \mathbf{q}\} \bullet \Delta\mathbf{q} \Rightarrow d\Theta \tag{2.120}$$

The first term in the expression for W, which depends on differential fluctuations, $\Delta\mathbf{v}$, suggests a relationship to the thermodynamic Helmholtz free energy (functions of the type TS that involve temperature) and the second term , which depends on differential fluctuations, $\Delta\mathbf{q}$, suggest a relationship to the thermodynamic Enthalpy (functions of the type PV that involve pressure). The combination of the two types suggests a relationship to the Gibbs free energy (functions of the type $TS - PV$) of closed thermodynamic systems and reversible processes.

$$W = \{\mathbf{p} - \partial L/\partial\mathbf{v}\} \bullet \Delta\mathbf{v} + \{\mathbf{f} - \partial L/\partial\mathbf{q}\} \bullet \Delta\mathbf{q} = d\Theta \tag{2.121}$$

$$\Rightarrow \quad -d(TS) \quad + \quad d(PV) \quad = d\Theta. \tag{2.122}$$

Note that for locally adiabatic Bernoulli processes,

$$i(X)L_{(\mathbf{X})}A = i(X)i(X)dA + i(X)d(i(X)A = i(X)Q = 0 \tag{2.123}$$

$$= i(X)d(\Theta + U) = L_{(\mathbf{X})}(\Theta + U) = 0. \tag{2.124}$$

In otherwords, depending on the choice of the Bernoulli function, Θ, representing the Work 1-form in terms of constrained topological fluctuations, the following evolutionary invariants are determined.

$$L_{(\mathbf{X})}(U - TS + PV) = L_{(\mathbf{X})}(G) = 0, \text{ Gibbs Potential} \tag{2.125}$$

$$L_{(\mathbf{X})}(U + PV) = L_{(\mathbf{X})}(H) = 0, \text{ Enthalpy Potential} \tag{2.126}$$

$$L_{(\mathbf{X})}(U - TS) = L_{(\mathbf{X})}(A) = 0, \text{ Helmholtz Potential} \tag{2.127}$$

$$L_{(\mathbf{X})}(U) = L_{(\mathbf{X})}(U) = 0, \text{ Internal Potential.} \tag{2.128}$$

Hence, the empirical thermodynamic potentials, more than 100 years old in concept, are to be recognized as the Bernoulli-Casimir evolutionary invariants of processes that admit topological fluctuations. These processes can exist on symplectic spaces of topological dimension 2n+2, where the Work 1-form doe not vanish. The need for recognizing the differences between mechanical energy and the thermodynamic energies was discussed by Stuke [Stuke 1993], where, without mention of symplectic evolution, he deduces the need for "acceleration " potentials in certain dissipative systems. These acceleration potentials, which can be shown to be the equivalent of Bernoulli-Casimir functions, were used by Stuke to construct the Enthalpy and Gibbs free energy in certain hydrodynamic examples.

The thermodynamic concepts of pressure and temperature are explicitly absent from that classical mechanics which has focused attention on the extremal contact manifolds of dimension 2n+1, and which has ignored the concept of topological differential fluctuations on symplectic spaces of dimension 2n+2. It is suggested that the occurrence of a pressure gradient, or a temperature gradient should be taken as the signature of a symplectic process.

On a symplectic domain of dimension 2n+2, unique ubiquitous extremal fields of classical Hamiltonian mechanics do not exist. There are no solutions to the extremal equation $i(\mathbf{V})dA = 0$, on the symplectic domain, but there do exist *non-unique* vector fields $\mathbf{V}$ that satisfy the Helmholtz constraint equation, $d(i(\mathbf{V})dA) = 0$. In the subset of exact cases, where $i(\mathbf{V})dA = d\Theta$, these vector fields generate "Hamiltonian-like" dynamical systems, or processes, (on the 2n+1 submanifold transversal to $d\Theta$), similar to the dynamical systems that are associated with the 2n+1 contact manifolds of classical State Space. The Action integral is a relative (stationary) integral invariant with respect to such Hamiltonian dynamical processes. The function Θ is a Bernoulli-Casimir evolutionary invariant, but these evolutionary invariants (stationary states) are not unique, not independent of gauge conditions, and strongly dependent upon boundary conditions, and are not constants over the domain. The somewhat larger class of vector fields that satisfy the Helmholtz condition $d(i(\mathbf{V})dA) = 0$ are defined as symplectic vector fields, and as dynamical systems they define symplectic processes. However, all such symplectic processes, exact or not, on symplectic domains of dimension 2n+2, still represent *reversible* thermodynamic processes.

Remarkably, on the 2n+2 symplectic domain there exists a *unique* non Hamiltonian vector field which leaves the Action integral a conformal, not stationary, invariant [RMK 1975 d]. This unique vector field, defined as the Torsion current, $\mathbf{T}$, does not satisfy the symplectic condition, but instead satisfies the equation, $i(\mathbf{T})dA = \Gamma A$ as suggested in the 1974 article [RMK 1974]. Moreover, it now can be demonstrated that this unique vector field generates dynamical systems that represent irreversible processes in a thermodynamic sense. This unique vector field (to within a factor) is generated by the formulas

$$A\hat{}(dA)^n = i(\mathbf{T})\Omega_{2n+2\ vol} \tag{2.129}$$

The symplectic space of dimension 2n+2 on which the Torsion current exists is defined as Thermodynamic Space, in order to distinguish it from the classic State Space of dimension 2n+1. The divergence of this Torsion vector field defines a density function on the 2n+2 space. The zero sets of this density function define smooth attractors (inertial manifolds) of dimension 2n+1 on the 2n+2 dimensional domain. The irreversible dynamical system generated by the Torsion vector irreversibly decays to these sets of measure zero which form the "stationary" states of a 2n+1 contact manifold. Once in the stationary state, the evolution can take place by a reversible Hamiltonian process.

2.7 Topological Entropy and Equilibrium Submanifolds

A remarkable achievement of the topological point view of thermodynamics is the ability to define the concept of entropy in an analytic, non-phenomenological way - and without the use of statistics. The concept of entropy has been extremely hard to define, for, like potential energy, classical mechanics does not yield a clear visual picture of "just what is" entropy. Numerous phenomenological constructions have been suggested (such as entropy is a measure of disorder, entropy is the inverse of information, entropy is proportional to area...) but encoding such concepts is difficult. Associated with the concept of entropy is the idea of a system in equilibrium, which at least in approximation is recognized from experience. Cold water poured into a hot bath comes to equilibrium within a perceptibly short time span. On the otherhand, inter-change-ability of kinetic energy and potential energy, ultimately yields a visual perception of "energy", but there seems to be no visual equivalent for "entropy". Moreover, the currently accepted dogma is that entropy always increases on a global scale. These concepts are hard to formulate analytically using physical techniques that have been based upon geometric concepts. It is the purpose of that which follows to demonstrate how a topological, not geometric, point of view enables an analytic coding of the concept of Entropy - without the use of statistics or a phenomenological assumption.

The topological view also gives a mathematical definition of what is meant by an equilibrium physical system. The topological difference between a connected and a disconnected topology is a sufficient topological property which can be used to distinguish an equilibrium system from a non-equilibrium physical system. This concept is based on the Frobenius unique integrability theorem, which is valid for an equilibrium system, but not for (most) non-equilibrium systems. However, the concept of equilibrium is more subtle. Bamberg and Sternberg [Bamberg 1992] suggest that a thermodynamic equilibrium state corresponds to a solution of a Lagrangian submanifold structure to an exterior differential system (in 4D). In 4D, the Lagrangian submanifold of a symplectic manifold generated by a 2-form, dA, is a 2 dimensional submanifold upon which the 2-form dA vanishes. Of more interest to this article is how such a submanifold structure may be viewed in terms of the limit set of topological fluctuations in arbitrary dimension.

The starting point of a topological analysis begins with those physical systems that can be encoded in terms of the Cartan-Hilbert 1-form of Action, which at first glance involves 3n+1 independent variables,

$$A = L(x^k, v^k, t)dt + p_k(dx^k - vdt). \tag{2.130}$$

The Pfaff topological dimension of the Cartan-Hilbert action is, however, 2n+2 form. The top Pfaffian in the Pfaff sequence formed from A, and dA and their interior products ( ) has the form of a 2n+2 dimensional volume element:

$$(dA)^{n+1} = (n+1)!\{\Sigma_{k=1}^{n}(\partial L/\partial v^k - p_k)dv^k\}\hat{}dp_1\hat{}...dp_n\hat{}dq^1\hat{}..dq^n\hat{}dt. \tag{2.131}$$

It follows that the expression in brackets can be represented by a perfect differential, dS :

$$(n+1)!\{\Sigma_{k=1}^{n}(\partial L/\partial v^k - p_k)dv^k\} = \Sigma_{j=1}^{n}\varpi_j dv^j \Rightarrow dS. \tag{2.132}$$

The exact 1-form, dS, represents the differential of the Entropy function, S, such that the Top Pfaffian becomes a 2n+2 volume element,

$$(dA)^{n+1} = dS\hat{}dp_1\hat{}...dp_n\hat{}dq^1\hat{}..dq^n\hat{}dt. \tag{2.133}$$

Conclusion 4 *As the 2n+1 form represents a volume element, the coefficient of the top Pfaffian has a representation as a perfect differential of a function, S, which is independent from the $\{p_k, q^k, t\}$. The differential of entropy function S is explicitly dependent upon the differentials of velocity dv^k and the non-canonical components of momentum $(\partial L/\partial v^k - p_k)$.*

Definition 5 *The change in entropy is given by the expression*

$$dS = (n+1)!\{\Sigma_{k=1}^{n}(\partial L/\partial v^k - p_k)dv^k\} \tag{2.134}$$

Definition 6 *The function S whose differential is the 1-form $(n+1)!\{\Sigma_{k=1}^{n}(\partial L/\partial v^k - p_k)dv^k\}$ is defined as the topological entropy.*

The even dimensional 2n+2 form represents an orientable volume element, and once an orientation has been fixed (say +1), and evolution is constrained to maintain the volume element and its sign, the change in the entropy, dS, must be of one sign. So entropy, if it changes globally, can be only of one sign (chosen to be positive in the historic literature). Also, as dA is presumed to be non-degenerate, then the differential, dS, can not be zero on the 2n+2 dimensional space.

Conclusion 7 *Hence the fact that global changes in entropy must be of one sign ≥ 0 is an artifact of topological orientability.*

Next consider subspaces of the Symplectic 2n+2 space. In particular consider a Lagrangian submanifold, which must be dimension n+1. By definition, on the Lagrangian submanifold (of dimension n+1) of the Symplectic space (of dimension 2n+2), the 2-form dA must vanish. The 2-form can be written as:

$$dA = dS\hat{}dt + \{dp_k - \partial L/\partial x^k dt\}\hat{}(\Delta q^k) \Rightarrow 0. \tag{2.135}$$

Observe that the immersion ψ of the configuration space with differentials $\{dq^1\hat{}..dq^n\hat{}dt\}$ into the top Pfaffian space $\{dS\hat{}dp_1\hat{}...dp_n\hat{}dq^1\hat{}..dq^n\hat{}dt\}$, defines a Lagrangian submanifold when the pullback of the 2-form dA vanishes. The 2-form dA has expression given by the equation above. Consider the case where the immersion into the 3n+1 space is such that the pullback of $(\Delta q^k) \Rightarrow 0$.

$$\psi : (q^1, .., q^n, t) \Rightarrow (S(q, p, t, v), p_1, ..., p_n, q^1, .., q^n, v^1, .., v^n, t) \tag{2.136}$$

Then the 2-form has a pullback realization such that

$$\psi^*(dA) = dS\hat{}dt \Rightarrow 0 \text{ for a Lagrange submanifold.} \tag{2.137}$$

The Pfaff topological dimension of the constrained 1-form of Action is then 2 on configuration space, and induces a connected Cartan Topology. The 2-form vanishes when the entropy is a constant:

Conclusion 8 $dS(q, p, t, v) \Rightarrow 0$ *implies equilibrium.*

It is also remarkable to note that $dS = 0$, when the momenta are canonically defined, such that

$$\{\partial L/\partial v^k - p_k\} \Rightarrow 0 \supset dS = \{\Sigma_{k=1}^n(\partial L/\partial v^k - p_k)dv^k\} \Rightarrow 0. \tag{2.138}$$

The topological concept of entropy is explicitly dependent upon the existence of *non-canonical* momenta.

It is remarkable that the symplectic systems of irreducible topological dimension 2n+2 seem to solve the Boltzmann - Loschmidt-Zermelo paradox of why *canonical* Hamiltonian mechanics does not seem to be able to describe the decay to an equilibrium state, and why the usual (extremal) methods of Hamiltonian mechanics do not give any insight into the concept of Pressure, Temperature, Entropy or the Gibbs free energy. It is extraordinary that answers to these 150 year old paradoxes of physics seem to follow without recourse to statistics if one utilizes a topological perspective. The interpretation of the fact that the top Pfaffian (for a physical system that can be encoded by a Cartan-Hilbert 1-form of Action) is of dimension 2n+2 and not 3n+1 is, at present, not complete. The implication is that there must exist (3n+1)-(2n+2) = n-1 topological invariants in these systems.

2.8 An Irreversible Example: The Sliding Bowling Ball

2.8.1 The Observation

Consider a bowling ball given an initial amount of translational energy and rotational energy. Assume the angular momentum and the linear momentum are orthogonal to themselves and also orthogonal to the ambient gravitational field. Then place the bowling ball, subject to these initial conditions, in contact with the bowling alley. Initially, it is observed that the ball slips or skids, dissipating its linear and angular momentum, until the No-Slip condition is achieved. Note that it is possible for the angular momentum or the linear momentum to change sign during the irreversible phase of the evolution. The dynamical system representing the evolutionary process is irreversible until the No-Slip condition is reached. Thereafter, the dynamical system is reversible, and momentum is conserved.

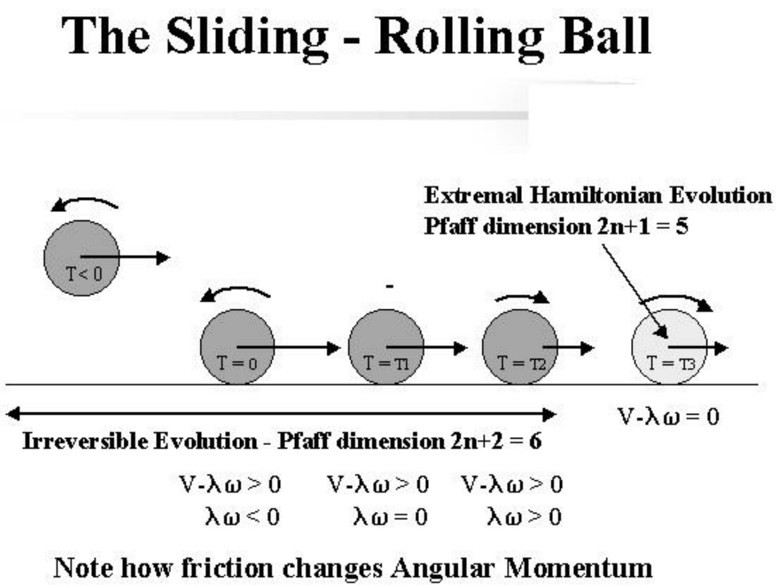


Figure 1

2.8.2 The Analysis

Assume that the physical system may be represented by a 1-form of Action constructed from a Lagrange function:

$$L = L(x, \theta, v, \omega, t) = \{\beta m(\lambda\omega)^2/2 + mv^2/2\} \tag{2.139}$$

The constants are: m=mass, β = moment of inertial factor (2/5 for sphere), λ = effective "radius" of the object, the moment of inertia = $\beta m\lambda^2$.

Let the topological constraints be defined anholonomically by the Pfaffian system:

$$\{dx - vdt\} \Rightarrow 0, \quad \{d\theta - \omega dt\} \Rightarrow 0, \quad \{dx - \lambda d\theta\} \Rightarrow 0 \tag{2.140}$$

Define the constrained 1-form of Action as

$$A = L(x, \theta, v, \omega, t)dt + p\{dx - vdt\} + l\{d\theta - \omega dt\} + ms\{\lambda d\theta - dx\} \tag{2.141}$$

where $\{p, l, s\}$ are Lagrange multipliers. Rearrange the variables to give (in the language of optimal control theory) a pre-Hamiltonian action:

$$A = (-p - ms)dx + (l + \lambda ms)d\theta - \{pv + l\omega - L\}dt. \tag{2.142}$$

It is apparent that the Pfaff dimension of this Action 1-form is 2n+2 = 6. The Action defines a symplectic manifold of dimension 6.

For simplicity, assume initially that the Lagrange multipliers (momenta) are defined canonically; e.g.,

$$p = \partial L/\partial v \Rightarrow mv, \qquad l = \partial L/\partial \omega \Rightarrow \beta m \lambda^2 \omega \tag{2.143}$$

which implies that

$$A = (mv - ms)dx + (\beta m\lambda^2\omega + \lambda s)d\theta - \{-mv^2/2 - \beta m(\lambda\omega)^2/2\}dt. \tag{2.144}$$

The volume element of the symplectic manifold is given by the expression

$$6Vol = 6m^3\beta\lambda^2\{v - \lambda\omega\}dx\hat{}d\theta\hat{}dv\hat{}d\omega\hat{}ds\hat{}dt = dA\hat{}dA\hat{}dA \tag{2.145}$$

The symplectic manifold has a singular subset upon which the Pfaff dimension of the Action 1-form is 2n+1 = 5. The constraint for such a contact manifold is precisely the no-slip condition:

$$\{v - \lambda\omega\} \Rightarrow 0 \tag{2.146}$$

On the 5 dimensional contact manifold there exists a unique extremal (Hamiltonian) field which (to within a projective factor) defines the conservative reversible part of the evolutionary process. As this unique extremal vector satisfies the equation

$$i(\mathbf{V})dA = 0, \tag{2.147}$$

it is easy to show that dynamical systems defined by such vector fields must be reversible in the thermodynamic sense. (As $dQ = d(i(\mathbf{V})dA) = 0$ for all Hamiltonian or symplectic processes, it follows that $Q\hat{}dQ = 0$.)

However, on the 6 dimensional symplectic manifold, there does not exist a unique extremal field, nor a unique stationary field, that can be used to define the dynamical system. The symplectic manifold does support vector fields, **S**, that leave the Action integral invariant, but these vector fields are not unique in the sense that that they depend on an arbitrary gauge addition to the 1-form of Action that may be required to satisfy initial conditions.

There does exist a unique torsion field (or current) defined (to within a projective factor, σ) by the 6 components of the 5 form,

$$Torsion = A\hat{}dA\hat{}dA \tag{2.148}$$

This unique vector, **T**, independent of gauge additions, has the properties that

$$L_{(\mathbf{T})}A = \Gamma \cdot A \quad and \quad i(\mathbf{T})A = 0. \tag{2.149}$$

This "Torsion" vector field satisfies the equation

$$L_{(\mathbf{T})}A\hat{}L_{(\mathbf{T})}dA = Q\hat{}dQ \neq 0. \tag{2.150}$$

Hence a dynamical system having a component constructed from this unique Torsion vector field becomes a candidate to describe the initial irreversible decay of angular momentum and kinetic energy.

Solving for the components of the Torsion vector for the bowling ball problem leads to the (unique) decaying dynamical system:

$$dv/dt = m^3\beta\lambda^2\{-\beta\lambda^2\omega^2 - 2\lambda v\omega + v^2\} \tag{2.151}$$

$$d\omega/dt = m^3\lambda^2\{-\beta\lambda^2\omega^2 + 2\beta\lambda v\omega + v^2\} \tag{2.152}$$

$$ds/dt = m^3\beta\lambda^2\{-\beta\lambda^2\omega^2 - v^2 - 2(\lambda\omega - v)s\} \tag{2.153}$$

This is a Volterra system generated on a Finsler space. (See Antonelli)

It is to be noted that the non-canonical "symplectic momentum" variables, defined by inspection from the constrained 1-form of Action lead to the momentum map:

$$P_x \doteq m(v - s), \qquad P_\theta \doteq m(\beta\lambda^2\omega + s\lambda). \tag{2.154}$$

Substitution in terms of the momentum variables leads to the canonical form (Zhitomirski) for the 1-form of Action:

$$A = P_x dx + P_\theta d\theta - Hdt \tag{2.155}$$

where H is an independent variable on the 6-dimensional manifold. The H map is given by the expression for energy where v and ω are eliminated in terms of the P_x and the P_θ.

$$H = (mv^2/2 + \beta m(\lambda\omega)^2/2) \Rightarrow 1/2m[(P_x/m + s)^2 + \beta\lambda(\frac{P_\theta/m\lambda - s}{\beta\lambda})^2] \tag{2.156}$$

Note that $v = \partial H/\partial P_x$ and $\omega = \partial H/\partial P_\theta$. Each component of "canonical momenta" decays with the same rate in the canonical domain.

2.9 Second order versus first order ODE's

It is also possible to consider those physical systems that can be described by a Lagrange function $L(\mathbf{q}, \mathbf{v}, \mathbf{a}, t)$ and a 1-form of Action given by the extended Cartan-Hilbert Action to include possible fluctuations in differential velocities directly in the 1-form of Action. Consider the Action 1-form of the type:

$$A = L(\mathbf{q}^k, \mathbf{v}^k, t)dt + \mathbf{p}_k \cdot (d\mathbf{q}^k - \mathbf{v}^k dt) + \boldsymbol{\lambda}_k \cdot (d\mathbf{v}^k - \mathbf{a}^k dt), \tag{2.157}$$

Topological Fluctuations in differential velocity

$$\Delta \mathbf{v} = (d\mathbf{v}^k - \mathbf{a}^k dt) \neq 0 \tag{2.158}$$

At first glance it appears that the domain of definition is a 5n+1 dimensional variety of independent variables, $\{\mathbf{q}^k, \mathbf{v}^k, \mathbf{a}^k, \mathbf{p}_k, \boldsymbol{\lambda}_k, t\}$. Do not assume that $\mathbf{p}_k$ or $\boldsymbol{\lambda}_k$ are constrained to be jets; e.g., $\mathbf{p}_k \neq \partial L/\partial \mathbf{v}^k$ and $\boldsymbol{\lambda}_k \neq \partial L/\partial a^k$. Instead, consider $\mathbf{p}_k$ and $\boldsymbol{\lambda}_k$ to be a (set of) Lagrange multiplier(s) to be determined later. Do not assume at this stage that $\mathbf{v}^k$ or $\mathbf{a}^k$ are kinematic velocity functions, such that $(d\mathbf{q}^k - \mathbf{v}^k dt) \Rightarrow 0$ or $(d\mathbf{v}^k - \mathbf{a}^k dt) \Rightarrow 0$. The classical idea is to assert that topological fluctuations in kinematic velocity $(d\mathbf{q} - \mathbf{v}dt) = \Delta_{\mathbf{q}}$ are related to Pressure, and that topological fluctuations in kinematic acceleration $(d\mathbf{v} - \mathbf{a}dt) = \Delta_{\mathbf{v}}$ are related to Temperature.

For the given Action, construct the Pfaff sequence $\{A, dA, A\hat{}dA, dA\hat{}dA...\}$ in order to determine the Pfaff topological dimension or class of the 1-form [Schouten 1949]. The top (non-zero) Pfaffian of this sequence is given by the formula,

$$(dA)^{n+1} = (2n+1)!\{\Sigma_{k=1}^{n}(\partial L/\partial a^k - \lambda_k) \bullet da^k\}\hat{}\Omega_{qpvt} \tag{2.159}$$

$$\Omega_{qpvt} = dv^1\hat{}..dv^n\ \hat{}dp_1\hat{}...dp_n\hat{}\ dq^1\hat{}..dq^n\ \hat{}dt, \tag{2.160}$$

which indicates that the Pfaff topological dimension is $4n+2$ and not the geometrical dimension $5n+1$, which might be expected as the 1-form was defined initially on a space of $5n+1$ "independent" base variables. The implication is that there exists an irreducible number of independent variables equal to $4n+2$ which completely characterize the differential topology of the second order system. It follows that the exact two form dA satisfies the equations

$$(dA)^{2n+1} \neq 0,\ but\quad A\hat{}(dA)^{2n+1} = 0. \tag{2.161}$$

For n = 3 degrees of freedom, the top Pfaffian indicates that the topological of Pfaff dimension of the 2-form, dA is $4n+2 = 14$. The algebra for these systems becomes formidable.

Again, there are two evolutionary routes that lead to a reduction of the even Pfaff dimension to the odd Pfaff dimension, which admits stationary Hamiltonian evolutionary processes. Topological reduction by dissipative irreversible processes on the topological domain of Pfaff dimension 4n+2 reduces to a topological domain of Pfaff dimension 4n+1, if the process

1. induces orthogonality such that

$$\{\Sigma_{j=1}^{n} k_j dv^j\} = 0, \tag{2.162}$$

2. induces the topological constraint such that the Lagrange multipliers are canonical.

$$\lambda_k = \partial L/\partial a^k. \tag{2.163}$$

The format of the top Pfaffian of dimension 4n+2 indicates that the bracketed factor involving the non-canonical components of the Lagrange multipliers, λ_k, must be exact:

$$dS_a = \{\Sigma_{k=1}^{n}(\partial L/\partial a^k - \lambda_k) \bullet da^k\}. \tag{2.164}$$

The procedure permits a definition of entropy, S_a, base upon accelerations. The restriction to a Lagrangian submanifold of the 4n+2 dimensional space implies that $dS_a \Rightarrow 0$. The ubiquitous (in classical mechanics) assumption that the Lagrange multipliers are canonical eliminates thermodynamic irreversible processes in favor of Hamiltonian extremal processes.

2.10 The Ubiquitous van der Walls gas

Up to now the methods have utilized antisymmetric features of the 2-form dA generated by the 1-form of Action, A. Now, the covariant components of the 1-form of Action (or the contravariant components of a Process) will be used to construct a Jacobian matrix on the domain of 4 pre-geometric variables. The characteristic polynomial of the Jacobian matrix will always yield a thermodynamic polynomial phase function of 4th degree in terms of the (possibly complex) eigenvalues of the Jacobian with coefficients in terms of the matrix similarity invariants. The results are therefor universal. It will be demonstrated how this universal result can be put into correspondence with the thermodynamics of a van der Waals gas.

2.10.1 The Jacobian Matrix of the Action 1-form.

The idea is to express the Jacobian matrix of the coefficient functions that define the 1-form if Action, A, in terms of "universal" coordinates. These universal coordinates will be the similarity invariants of the Jacobian matrix. For a 1-form of Action of Pfaff topological dimension 4, the Cayley-Hamilton theorem produces a Universal Phase function as a polynomial of 4th degree. What is remarkable about this Universal Phase function is that it has properties that are homeomorphically deformable into the format of a classic van der Waals gas. It is this universality that gives credence to the idea that the universe could be a non-equilibrium van der Waals gas near its critical point.

The Universal Characteristic Phase Function

The 1-form of Action, used to encode a physical system, contains other useful topological information, as well as geometric information. Consider the Turbulent thermodynamic state generated by a 1-form of Action, A, of Pfaff topological dimension 4. The component functions of the Action 1-form can be used to construct a 4x4 Jacobian matrix of partial derivatives, $[\mathbb{J}_{jk}] = [\partial(A)_j/\partial x^k]$. In general, this Jacobian matrix will be a 4 x 4 matrix that satisfies a 4th order Cayley-Hamilton characteristic polynomial $\Theta(x, y, z, t;\ \Psi)$ with 4 perhaps complex roots representing the perhaps complex eigenvalues, ρ_k, of the Jacobian matrix.

$$\Theta(x, y, z, t;\ \Psi) = \Psi^4 - X_M \Psi^3 + Y_G \Psi^2 - Z_A \Psi^1 + T_K \Rightarrow 0. \tag{2.165}$$

The functions $X_M(x, y, z, t), Y_G(x, y, z, t), Z_A(x, y, z, t), T_K(x, y, z, t)$ are the similarity invariants of the Jacobian matrix. If the eigenvalues are distinct, then the similarity invariants are given by the expressions:

$$X_M = \rho_1 + \rho_2 + \rho_3 + \rho_4, \tag{2.166}$$
$$Y_G = \rho_1\rho_2 + \rho_2\rho_3 + \rho_3\rho_1 + \rho_4\rho_1 + \rho_4\rho_2 + \rho_4\rho_3, \tag{2.167}$$
$$Z_A = \rho_1\rho_2\rho_3 + \rho_4\rho_1\rho_2 + \rho_4\rho_2\rho_3 + \rho_4\rho_3\rho_1, \tag{2.168}$$
$$T_K = \rho_1\rho_2\rho_3\rho_4. \tag{2.169}$$

The similarity invariants may be considered as a coordinate map from the original variety of independent variables, $\{x, y, z, t\} \Rightarrow \{X_M, Y_G, Z_A, T_K\}$. When the similarity invariants are treated as generalized coordinates, then the characteristic polynomial becomes a Universal Phase function, and will be used to encode universal thermodynamic properties.

Minimal surfaces

The Universal Phase function, Θ, may be considered as a family of hypersurfaces in the 4 dimensional space, $\{X_M, Y_G, Z_A, T_K\}$ with a complex family (order) parameter, Ψ. Moreover, it should be realized that the Universal Phase Function is a holomorphic function, $\Theta = \phi + i\chi$ in the complex variable $\Psi = u + iv$. That is

$$\Theta(X_M, Y_G, Z_A, T_K;\ \Psi) \Rightarrow \phi + i\chi, \tag{2.170}$$

where

$$\phi = u^4 - 6u^2v^2 + v^4 - X_M(u^3 - 3uv^2) + Y_G(u^2 - v^2) - Z_A u + T_K \tag{2.171}$$
$$\chi = 4u^3v - 4uv^3 - X_M(3u^2v - v^3) + 2Y_G uv - Z_A v. \tag{2.172}$$

As such, in the 4D space of two complex variables, $\{\phi + i\chi, u + iv\}$, according to the theorem of Sophus Lie, any such holomorphic function produces a pair of conjugate *minimal* surfaces in the 4 dimensional space $\{\phi, \chi, u, v\}$. It follows that there exist a sequence of maps,

$$\{x, y, z, t\} \Rightarrow \{X_M, Y_G, Z_A, T_K\} \Rightarrow \{\phi, \chi, u, v\} \tag{2.173}$$

such that the family of hypersurfaces can be decomposed into a pair of conjugate minimal surface components. The criteria for a minimal surface is equivalent to the idea that $X_M = 0$. By suitable renormalization, the similarity invariant X_M is equivalent to the Mean Curvature of the hypersurface.

Envelopes

The theory of implicit hypersurfaces focuses attention upon the possibility that the Universal Phase function has an envelope. The existence of an envelope depends upon the possibility of finding a simultaneous solution to the two implicit surface equations of the family:

$$\Theta(x,y,z,t;\ \Psi) = \Psi^4 - X_M\Psi^3 + Y_G\Psi^2 - Z_A\Psi + T_K \Rightarrow 0. \tag{2.174}$$

$$\partial\Theta/\partial\Psi = \Theta_\Psi = 4\Psi^3 - 3X_M\Psi^2 + 2Y_G\Psi - Z_A \Rightarrow 0. \tag{2.175}$$

For the envelope to be smooth, it must be true that $\partial^2\Theta/\partial\Psi^2 = \Theta_{\Psi\Psi} \neq 0$, and that the exterior 2-form, $d\Theta\hat{}d\Theta_\Psi \neq 0$ subject to the constraint that the family parameter is a constant: $d\Psi = 0$. The envelope as a smooth hypersurface does not exist unless both conditions are satisfied. Recall that the envelope, if it exists, is a hypersurface in the space of similarity coordinates, $\{X_M, Y_G, Z_A, T_K\}$.

The envelope is determined by the discriminant of the Phase Function polynomial, which as a zero set is equal to a universal hypersurface in the 4 dimensional space of similarity variables $\{X_M, Y_G, Z_A, T_K\}$. This function can be written in terms of the similarity "coordinates" (suppressing the subscripts) :

$$\begin{aligned}
& \text{Discriminant of the Universal Phase Function} && (2.176)\\
= \; & 18X^3ZYT - 27Z^4 + Y^2X^2Z^2 - 4Y^3X^2T + 144YX^2T^2 && (2.177)\\
& +18XZ^3Y - 192XZT^2 - 6X^2Z^2T + 144TZ^2Y - 4X^3Z^3 && (2.178)\\
& -27X^4T^2 - 4Y^3Z^2 + 16Y^4T - 128Y^2T^2 + 256T^3 - 80XZY^2T. && (2.179)
\end{aligned}$$

The discriminant has eliminated the family order parameter. Remarkably, choosing the constraint condition in terms of the dual condition that the Mean Curvature vanishes, $X_M \Rightarrow 0$, leads to a reduced discriminant, which defines a universal swallow tail surface homeomorphic (deformable) to the Gibbs surface of a van der Waals gas (subscripts suppressed):

$$\begin{aligned}
& \text{Universal Gibbs Swallowtail Envelope } (X = 0, Y, Z, T) && (2.180)\\
= \; & -27Z^4 + 144TZ^2Y - 4Y^3Z^2 + 16Y^4T - 128Y^2T^2 + 256T^3 \Rightarrow 0.
\end{aligned}$$

In other words, the Gibbs function for a van der Waals gas is a universal idea associated with minimal hypersurfaces, $X_K = 0$, of thermodynamic systems of Pfaff topological dimension 4. The similarity coordinate T_K plays the role of the Gibbs free energy, in terms of the Pressure ($\tilde{}Z_A$) and the Temperature ($\tilde{}Y_G$). The Spinodal line as a limit of phase stability, and the critical point are ideas that come from the study of a van der Waals gas, but herein it is apparent that these concepts are universal topological concepts that remain invariant with respect to

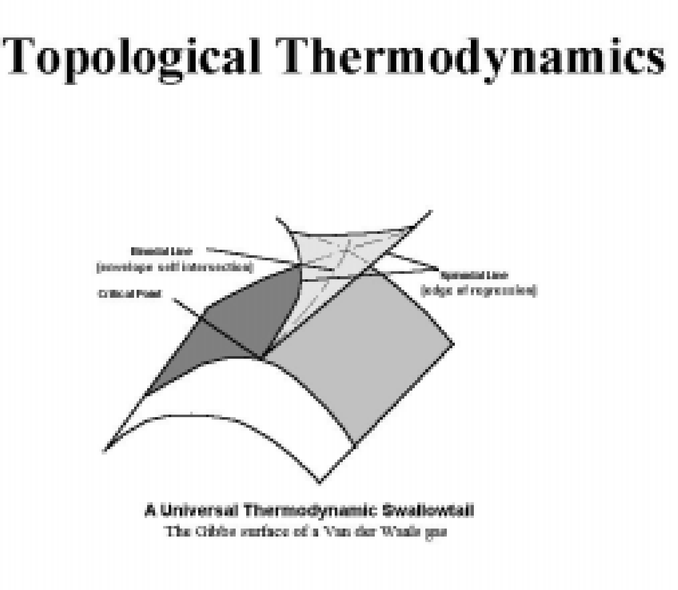


Figure 2

deformations. The universal formulas for such constraints are presented in the next section. The result is that all thermodynamic systems of Pfaff topological dimension 4 are deformably equivalent to a van der Waals gas.

It is important to recognize that the development of a universal non-equilibrium van der Waals gas has not utilized the concepts of metric, connection, statistics, relativity, gauge symmetries, or quantum mechanics.

The Edge of Regression and Self Intersections

The envelope is smooth as long as $\partial^2\Theta/\partial\Psi^2 = \Theta_{\Psi\Psi} \neq 0$, and that the exterior 2-form, $d\Theta\hat{}d\Theta_\Psi \neq 0$ subject to the constraint that the family parameter is a constant: $d\Psi = 0$. If $d\Theta\hat{}d\Theta_\Psi \neq 0$, but $\Theta_{\Psi\Psi} = 0$, then the envelope has a self intersection singularity. If $d\Theta\hat{}d\Theta_\Psi = 0$, but $\Theta_{\Psi\Psi} \neq 0$, there is no self intersection, and no envelope.

If the envelope exists, further singularities are determined by the higher order partial derivatives of the Universal Phase function with respect to Ψ.

$$\partial^2\Theta/\partial\Psi^2 = \Theta_{\Psi\Psi} = 12\Psi^2 - 6X_M\Psi + 2Y_G. \tag{2.181}$$

$$\partial^3\Theta/\partial\Psi^3 = \Theta_{\Psi\Psi\Psi} = 24\Psi - 6X_M \tag{2.182}$$

When $\partial^3\Theta/\partial\Psi^3 = \Theta_{\Psi\Psi\Psi} \neq 0$, and $d\Theta\hat{}d\Theta_\Psi\hat{}d\Theta_{\Psi\Psi} \neq 0$, the envelope terminates in a edge of regression. The edge of regression is determined by the simultaneous solution of $\Theta = 0, \Theta_\Psi = 0$ and $\Theta_{\Psi\Psi} = 0$. For the minimal surface representation of the Gibbs surface for a van der Waals gas, the edge of regression defines the Spinodal line of ultimate phase stability. The edge of regression is evident in the Swallowtail figure (Figure 2.1) describing the Gibbs function for a van der Waals gas.

If $\Theta_{\Psi\Psi\Psi} = 0$, then for $X_M = 0$, it follows that $Y_G = 0$, $Z_A = 0$, $T_K = 0$, which defines the critical point of the Gibbs function for the van der Waals gas. In other words, the critical point is the zero of the 4-dimensional space of similarity coordinates.

If $\Theta_{\Psi\Psi} = 0$, then for $X_M = 0$ the envelope has a self intersection. It follows from $\Theta_{\Psi\Psi} = 0$, that $\Psi^2 = -Y_G/6$, which when substituted into

$$\Theta_{\Psi} = 4\Psi^3 + 2Y_G\Psi - Z_A \Rightarrow 0, \tag{2.183}$$

yields the

$$\text{Universal Gibbs Edge of Regression: } Z_A^2 + Y_G^3(8/27) = 0, \tag{2.184}$$

which defines the Spinodal line, of the minimal surface representation for a universal non-equilibrium van der Waals gas, in terms of "similarity" coordinates.

Within the swallow tail region the "Gibbs" surface has 3 real roots and outside the swallow tail region there is a unique real root. The edge of regression furnished by the Cardano function defines the transition between real and imaginary root structures. The details of the universal non-equilibrium van der Waals gas in terms of envelopes and edges of regression with complex molal densities or order parameters will be presented elsewhere. These systems are not equilibrium systems for the Pfaff dimension is not 2. Of obvious importance is the idea that the a zero value for both Z_G and T_K are required to reduce the Pfaff dimension to 2, the necessary condition for an equilibrium system.

Ginsburg Landau Currents

The Universal Phase function can be solved for the determinant of the Jacobian matrix, which is equal to the similarity invariant T_K,

$$T_K = -\{\Psi^4 - X_M\Psi^3 + Y_G\Psi^2 - Z_A\Psi\}. \tag{2.185}$$

All determinants are in effect N - forms on the domain of independent variables. All N-forms can be related to the exterior derivative of some N-1 form or current, J. Hence

$$dJ = K\Omega_4 = div\mathbf{J} + \partial\rho/\partial t = -(\Psi^4 - X_M\Psi^3 + Y_G\Psi^2 - Z_A\Psi)\Omega_4. \tag{2.186}$$

For currents of the form

$$\mathbf{J} = grad\ \Psi, \tag{2.187}$$
$$\rho = \Psi, \tag{2.188}$$

the Universal Phase function generates the universal Ginsburg Landau equations

$$\nabla^2\Psi + \partial\Psi/\partial t = -(\Psi^4 - X_M\Psi^3 + Y_G\Psi^2 - Z_A\Psi). \tag{2.189}$$

2.10.2 Singularities as defects of Pfaff dimension 3

The family of hypersurfaces can be topologically constrained such that the topological dimension is reduced, and/or constraints can be imposed upon functions of the similarity variables forcing them to vanish. Such regions in the 4 dimensional topological domain indicate topological defects or thermodynamic changes of phase. It is remarkable that for a given 1-form of Action there are an infinite number rescaling functions, λ, such that the Jacobian matrix $\left[\mathbb{J}_{jk}^{scaled}\right] = [\partial(A/\lambda)_j/\partial x^k]$ is singular (has a zero determinant). For if the coefficients of any 1-form of Action are rescaled by a divisor generated by the Holder norm,

$$\text{Holder Norm:}\quad \lambda = \{a(A_1)^p + b(A_2)^p + c(A_3)^p + e(A_4)^p\}^{m/p}, \tag{2.190}$$

then the rescaled Jacobian matrix

$$\left[\mathbb{J}_{jk}^{scaled}\right] = [\partial(A/\lambda)_j/\partial x^k] \tag{2.191}$$

will have a zero determinant, for any index p, any set of isotropy or signature constants, a, b, c, e, if the homogeneity index is equal to unity: $m = 1$. This homogeneous constraint implies that the similarity invariants become projective invariants, not just equi-affine invariants. Such species of topological defects can have the image of a 3-dimensional implicit characteristic hypersurface in space-time:

$$\text{Singular hypersurface in 4D:}\quad \det[\partial(A/\lambda)_j/\partial x^k] \Rightarrow 0 \tag{2.192}$$

The singular fourth order Cayley-Hamilton polynomial of $[\mathbb{J}_{jk}]$ then will have a cubic polynomial factor with one zero eigenvalue.

For example, consider the simple case where the determinant of the Jacobian vanishes: $T_K \Rightarrow 0$. Then the Phase function becomes

$$\text{Universal Equation of State} \ :\ \Theta(\{X_M, Y_G, Z_A, T_K = 0\};\ \Psi) \tag{2.193}$$
$$= \Psi(\Psi^3 - X_M\Psi^2 + Y_G\Psi - Z_A) \Rightarrow 0. \tag{2.194}$$

The space has been topologically reduced to 3 dimensions (one eigen value is zero), and the zero set of the resulting singular Universal Phase function becomes a universal cubic equation that is homeomorphic to the cubic equation of state for a van der Waals gas.

When the rescaling factor λ is chosen such that $p = 2, a = b = c = 1, m = 1$, then the Jacobian matrix, $[\mathbb{J}_{jk}]$, is equivalent to the "Shape" matrix for an implicit hypersurface in the theory of differential geometry. (See appendix 1.) Recall that the homogeneous similarity invariants can be put into correspondence with the linear Mean curvature, $X_M \Rightarrow C_M$, the quadratic Gauss curvature, $Y_G \Rightarrow C_G$, and the cubic Adjoint curvature, $Z_A \Rightarrow C_A$, of the hypersurface. The characteristic cubic polynomial can be put into correspondence with a nonlinear extension of an ideal gas *not necessarily* in an equilibrium state.

The Universal van der Waals gas

More that 100 years ago van der Waals introduced into the science of thermodynamics the equation of state now called the van der Waals gas:

$$P = \rho RT/(1 - b\rho) + a\rho^2 \tag{2.195}$$

The van der Waals equation may be considered as a cubic constraint on the space of variables $\{n; P, V, T\}$ where $\rho = n/V$ is defined as the molar density.

$$\rho^3 - (1/b)\rho^2 + \{-(RT + bP)/ab\}\rho + P/ab = 0. \tag{2.196}$$

This cubic equation is to be compared with the characteristic polynomial written in terms of the similarity invariants, M, G, and A. Note that the roots of the characteristic polynomial are not necessarily real. This observation leads to a well defined procedure for treating non-equilibrium thermodynamics systems as complex deviations from the real, or equilibrium, systems. The reality condition is determined by the Cardano function that describes an edge of regression discontinuity.

For a transformation such that

$$(8T + P)/3 = Y_G/(M/3)^2, \tag{2.197}$$

$$\begin{aligned} P &= Z_K/(M/3)^3, && (2.198)\\ \lambda &= -\rho/(M/3), && (2.199) \end{aligned}$$

the characteristic polynomial becomes an equation in terms of dimensionless parameters,

$$U(\lambda, T, P) = (\lambda)^3 - 3(\lambda)^2 + [(8T + P)/3](\lambda) - P = 0. \tag{2.200}$$

The last format given above is to be recognized as the Equation of State of a van der Waals Gas (compare to 2.193), in terms of dimensionless Pressure, Temperature relative to their values at the critical point.

2.10.3 Non-Equilibrium Examples.

In order to demonstrate content to the thermodynamic topological theory, two algebraically simple examples are presented below. The first corresponds to a Jacobian characteristic equation that has a cubic polynomial factor, and hence can be identified with a van der Waals gas. The second example exhibits the features associated with a Hopf bifurcation, where the characteristic equation has a quadratic factor with two pure imaginary roots, and two null roots. The third example demonstrates how a bowling ball, given initial angular momentum and energy, skids and/or slips changing its angular momentum and kinetic energy irreversibly via friction effects, until the dynamics is such that the ball rolls with out slipping. Once that "excited" state is reached, and topological fluctuations are ignored, the motion continues without dissipation. The system is in an excited state far from equilibrium.

Example 1: van der Waals properties from rotation and contraction

In this example,the Action 1-form is presumed to be of the form

$$A_0 = a(ydx - xdy) + b(tdz + zdt). \tag{2.201}$$

The 1-form of Potentials depends on the coefficients a and b. The results of the topological theory are (for $r^2 = x^2 + y^2 + z^2 + t^2$):

$$\begin{aligned}
\text{Mean curvature} &: C_M = -2btz/(r^2)^{3/2} && (2.202)\\
\text{Gauss curvature} &: C_G = -\{b^2(x^2+y^2) - a^2(z^2+t^2)\}/(r^2)^2 && (2.203)\\
\text{Adjoint curvature} &: C_A = A\hat{}J_s \ = -2a^2btz/(r^2)^{5/2} && (2.204)\\
Top_Torsion &= 2ab\ \cdot [0,0,z,-t]/(r^2) && (2.205)\\
\text{Adjoint Current} &: J_s = (a^2b^2 \cdot [x,y,z,t])\ /(r^2)^2 && (2.206)\\
\text{Pfaff Dimension 4} &: dA\hat{}dA = 2ba(t^2 - z^2)/(r^2)^2\ \Omega_4 && (2.207)
\end{aligned}$$

The Jacobian matrix has 1 zero eigen value and three non-zero eigenvalues. Hence, the cubic polynomial will yield an interpretation as a van der Waals gas. The Adjoint current represents a contraction in space-time, while the flow associated with the 1-form has a rotational component about the z axis.

Example 2: A Hopf 1-form

In this example,the Hopf 1-form is presumed to be of the form

$$A_0 = a(ydx - xdy) + b(tdz - zdt). \tag{2.208}$$

The 1-form of Potentials depends on the coefficients a and b. There are two cases corresponding to left and right handed "polarizations": $a = b$ or $a = -b$. The results of the topological theory are (for $r^2 = x^2 + y^2 + z^2 + t^2$):

$$\begin{aligned}
\text{Mean curvature} &: C_M = 0, & (2.209)\\
\text{Gauss curvature} &: C_G = \{b^2(x^2+y^2) + a^2(z^2+t^2)\}/(r^2)^2 & (2.210)\\
\text{Adjoint Cubic curvature} &: C_A = A\hat{}J_s = 0 & (2.211)\\
Top_Torsion &= 2ab \cdot [x,y,z,t]/(r^2) & (2.212)\\
\text{Adjoint Current} &: J_s = (ab/2) \cdot Top_Torsion & (2.213)\\
\text{Pfaff Dimension 4} &: dA\hat{}dA = 4ab/(r^2)\ \Omega_4 & (2.214)
\end{aligned}$$

What is remarkable for this Action 1-form is that both the mean curvature and the Adjoint curvature of the implicit hypersurface in 4D vanish, for any choice of a or b. The Gauss curvature is non-zero, positive real and is equal to the inverse square of the radius of a 4D euclidean sphere, when $a^2 = b^2$. The Adjoint cubic interaction energy density is zero. The two non-zero curvatures are pure imaginary conjugates equal to

$$\rho = \pm\sqrt{-b^2(x^2+y^2) - a^2(z^2+t^2)}/(r^2). \tag{2.215}$$

The Hopf surface is a 2D imaginary *minimal* two dimensional hyper surface in 4D and has two non-zero imaginary curvatures! Strangely enough the charge-current density is not zero, but it is proportional to the Topological Torsion vector that generates the 3 form $A\hat{}F$. The topological Parity 4 form is not zero, and depends on the sign of the coefficients a and b. In other words the 'handedness' of the different 1-forms determines the orientation of the normal field with respect to the implicit surface. It is known that a process described by a vector proportional to the topological torsion vector in a domain where the topological parity is non-zero $4ba/(x^2+y^2+z^2+t^2) \neq 0$ is thermodynamically irreversible.

2.10.4 The Adjoint Current and Topological Spin

From the singular Jacobian matrix, $\left[\mathbb{J}_{jk}^{scaled}\right] = [\partial(A/\lambda)_j/\partial x^k]$, it is always possible to construct the Adjoint matrix as the matrix of cofactors transposed:

$$\text{Adjoint Matrix}: \left[\widehat{\mathbb{J}}^{kj}\right] = adjoint\left[\mathbb{J}_{jk}^{scaled}\right] \tag{2.216}$$

When this matrix is multiplied times the rescaled covector components, the result is the production of an adjoint current,

$$\text{Adjoint current}: \left|\widehat{\mathbf{J}}^k\right\rangle = \left[\widehat{\mathbb{J}}^{kj}\right] \circ |\mathbf{A}_j/\lambda\rangle \tag{2.217}$$

It is remarkable that the construction is such that the Adjoint current 3-form, if not zero, has zero divergence globally:

$$\begin{aligned}
\widehat{J} &= i(\widehat{\mathbf{J}}^k)\Omega_4 & (2.218)\\
d\widehat{J} &= 0. & (2.219)
\end{aligned}$$

From the realization that the Adjoint matrix may admit a non-zero globally conserved 3-form density, or current, $\widehat{J}$, it follows abstractly that there exists a 2-form density of "excitations", $\widehat{G}$, such that

$$\text{Adjoint current}: \widehat{J} = d\widehat{G}. \tag{2.220}$$

$\widehat{G}$ is not uniquely defined in terms of the adjoint current, for $\widehat{G}$ could have closed components (gauge additions $\widehat{G}_c$, such that $d\widehat{G}_c = 0$), which do not contribute to the current, $\widehat{J}$.

From the topological theory of electromagnetism [RMK 2004] [RMK 1999 b] there exists a fundamental 3-form, $A\hat{}G$, defined as the "topological Spin" 3-form,

$$\text{Topological Spin 3-form} \;:\; A\hat{}G. \tag{2.221}$$

The exterior derivative of this 3-form produces a 4-form, with a coefficient energy density function that is composed of two parts:

$$d(A\hat{}G) = F\hat{}G - A\hat{}\widehat{J}. \tag{2.222}$$

The first term is twice the difference between the "magnetic" and the "electric" energy density, and is a factor of 2 times the Lagrangian usually chosen for the electromagnetic field in classic field theory:

$$\text{Lagrangian Field energy density}: F\hat{}G = \mathbf{B} \circ \mathbf{H} - \mathbf{D} \circ \mathbf{E} \tag{2.223}$$

The second term is defined as the "interaction energy density"

$$\text{Interaction energy density}: A\hat{}\widehat{J} = \mathbf{A} \circ \widehat{\mathbf{J}} - \rho\phi. \tag{2.224}$$

For the special (Gauss) choice of integrating denominator, λ with $(p = 2, a = b = c = 1, m = 1)$ it can be demonstrated that the Jacobian similarity invariants are equal to the classic curvatures:

$$\{X_M, Y_G, Z_A, T_K\} \Rightarrow \{C_{M(mean\_linear)}, C_{G(gauss\_quadratic)}, C_{A(adjoint\_cubic)}, 0\}. \tag{2.225}$$

It can be demonstrated that the interaction density is exactly equal to the Adjoint curvature energy density:

$$\text{Interaction energy } A\hat{}\widehat{J} = C_A\ \Omega_4 \quad \text{(The Adjoint Cubic Curvature)}. \tag{2.226}$$

The conclusion reached is that a non-zero interaction energy density implies the thermodynamic system is not in an equilibrium state.

However, it is always possible to construct the 3-form, $\widehat{S}$:

$$\text{Topological Spin 3-form}: \widehat{S} = A\hat{}\widehat{G} \tag{2.227}$$

The exterior derivative of this 3-form leads to a co-homological structural equation similar the first law of thermodynamics, but useful for non-equilibrium systems. This result, now recognized as a statement applicable to non-equilibrium thermodynamic processes, was defined as the "Intrinsic Transport Theorem" in 1969 [RMK 1969] :

$$\text{Intrinsic Transport Theorem (Spin)} \; : \; d\widehat{S} = F\hat{}\widehat{G} - A\hat{}\widehat{J}, \tag{2.228}$$

$$\text{First Law of Thermodynamics (Energy)} \; : \; dU = Q - W \tag{2.229}$$

If one considers a collapsing system, then the geometric curvatures increase with smaller scales. If Gauss quadratic curvature, C_G, is to be related to gravitational collapse of matter, then at some level of smaller scales a term cubic in curvatures, C_H, would dominate. It is conjectured that the cubic curvature produced by the interaction energy effect described above could prevent the collapse to a black hole. Cosmologists and relativists apparently have ignored such cubic curvature effects.

2.11 The Falaco Soliton - A Topological String in a Swimming Pool

2.11.1 Visual Topological Defects in a Fluid

Although of importance to the cosmological concept of a universe expressible as a low density (non-equilibrium) van der Waals gas near its critical point, the factorization of the Jacobian characteristic polynomial into a cubic is not the only cosmological possibility. Of particular interest is the factorization that leads to a Hopf bifurcation. In this case the characteristic determinant vanishes, the Adjoint cubic curvature vanishes, the mean curvature vanishes (indicating a minimal surface), but the Gauss curvature is positive, and the two remaining eigenvalues of the characteristic polynomial are pure imaginary conjugates. Such results indicate rotations or oscillations (as in the chemical Brusselator reactions) and the possibility of spiral concentration or density waves on such minimal surfaces. Such structures at a cosmological level would appear to explain the origin of spiral arm galaxies. The Hopf type minimal surfaces of positive Gauss curvature do not represent thermodynamic equilibrium systems, for their curvatures, although two in number, are pure imaginary. The molal density distributions (or order parameters) are complex.

The idea that stars could be long lived topological defects at a cosmological level gains credence at the macroscopic level by the creation of Falaco Solitons (topological defects) in a swimming pool.

See the photo taken by D. Radabaugh of the 3 pairs of Falaco Solitons created in a swimming pool in the late afternoon. The lighting and optics enables the dimpled surface structures to be seen. Note the vestiges of mushroom spirals in the surface structures around each pair. The surface spiral arms can be enhanced by spreading chalk dust on the free surface of the pool. The photo demonstrates the existence

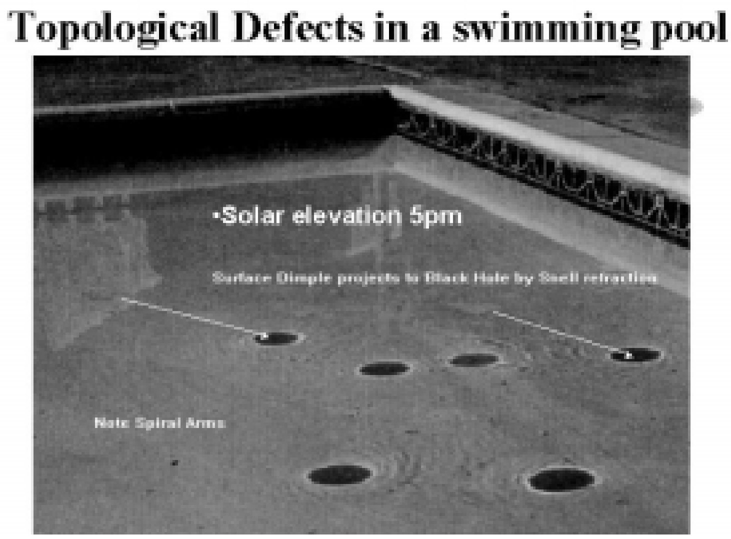
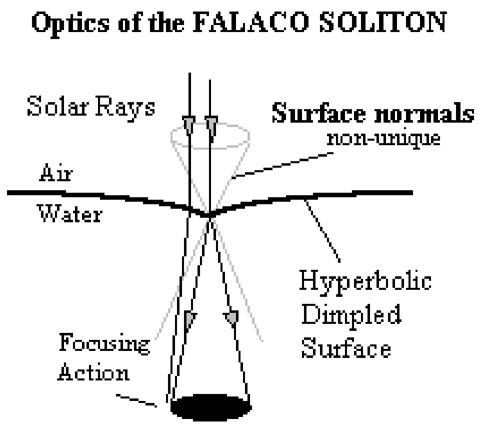
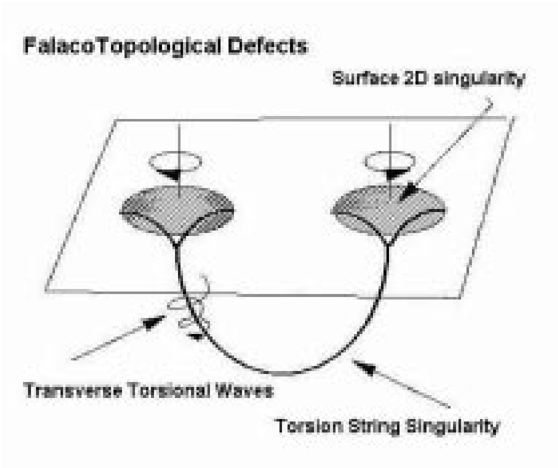


Figure 3

of Falaco Solitons, a few minutes after creation. The kinetic energy and the angular momentum initially given to a pair of Rankine vortices (of positive Gauss curvature) created in the free surface of water quickly decay into dimpled, locally unstable, singular surfaces (of negative Gauss curvature) that have an extraordinary lifetime of more than 15 minutes in a still pool. The Falaco Soliton is observed by means of the unique optics of Snell refraction from a surface of negative Gauss curvature. The dimpled surface is almost a minimal surface, and the projection to the floor of the pool is almost conformal, preserving the circular appearance of the black disc, independent from the angle of solar incidence.





A remarkable feature of the Falaco Soliton [RMK 1986] is that it consists of a pair of two dimensional topological defects, in a surface of discontinuity, which apparently are connected by means of a topological singular thread. It is conjectured that the tension in the singular thread provides the force that maintains the pair of dimpled structures. The equilibrium mode for the free surface is that it be flat, without dimples. The solitons are representative of non-equilibrium long lived structures. If dye drops are injected into the water, the dye particles will execute a *torsional* wave motion that oscillates up and down until the dye maps out the thread singularity (a circular arc) that connects the two vertices of the Falaco Soliton. The singular thread acts as a guiding center for the torsion waves. If the thread is severed, the endcap singularities disappear almost immediately, and not diffusively. The long lifetime of the Falaco Soliton is due to this global stabilization of the connecting string singularity.

2.11.2 Falaco Solitons as Landau Ginsburg structures in micro, macroscopic and cosmological systems

The Falaco experiments demonstrate that such topological defects are available at all scales. The Falaco Solitons consist of spiral "vortex defect" structures (analogous to CGL theory) on a two dimensional minimal surface, one at each end of a 1-dimensional "vortex line" or thread (analogous to GPG theory). Remarkably the topological defect surface structure is locally unstable, as the surface is of negative Gauss curvature. Yet the pair of locally unstable 2-D surfaces is *globally* stabilized by the 1-D line defect attached to the "vertex" points of the minimal surfaces.

For some specific physical systems it can be demonstrated that period (circulation) integrals of the 1-form of Action potentials, A, lead to the concept of "vortex defect lines". The idea is extendable to "twisted vortex defect lines" in three dimensions. The "twisted vortex defects" become the spiral vortices of a Complex Ginsburg Landau (CGL) theory , while the "untwisted vortex lines" become the defects of Ginzburg-Pitaevskii-Gross (GPG) theory [Tornkvist 1997].

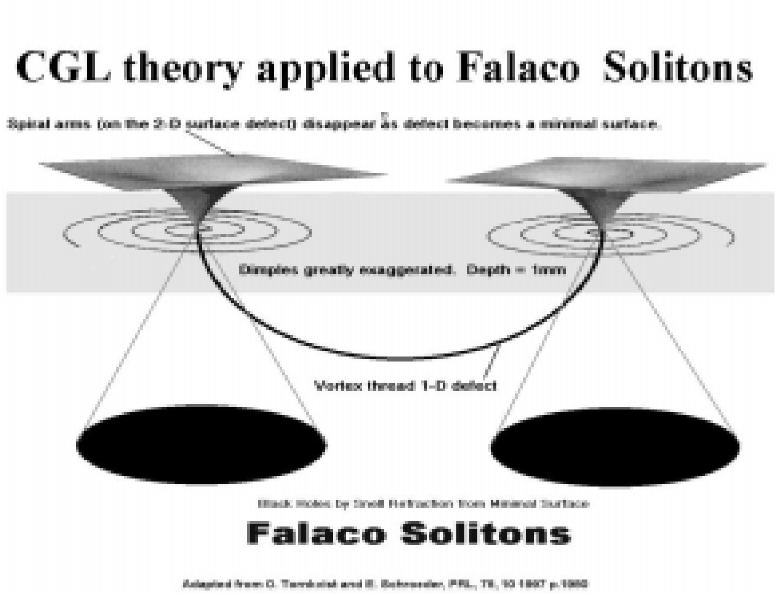


Figure 4

In the macroscopic domain, the experiments visually indicate "almost flat" spiral arm structures during the formative stages of the Falaco solitons. In the cosmological domain, it is suggested that these universal topological defects represent the ubiquitous "almost flat" spiral arm galaxies. Based on the experimental creation of Falaco Solitons in a swimming pool, it has been conjectured that M31 and the Milky Way galaxies could be connected by a topological defect thread [RMK 1986]. Only recently has photographic evidence appeared suggesting that galaxies may be connected by strings.

At the other extreme, the rotational minimal surfaces of negative Gauss curvature which form the two endcaps of the Falaco soliton, like quarks, apparently are confined by the string. If the string (whose "tension" induces global stability of the unstable endcaps) is severed, the endcaps (like unconfined quarks in the elementary particle domain) disappear (in a non-diffusive manner). In the microscopic electromagnetic domain, the Falaco soliton structure offers an alternate, topological, pairing mechanism on a Fermi surface, that could serve as an alternate to the Cooper pairing in superconductors.

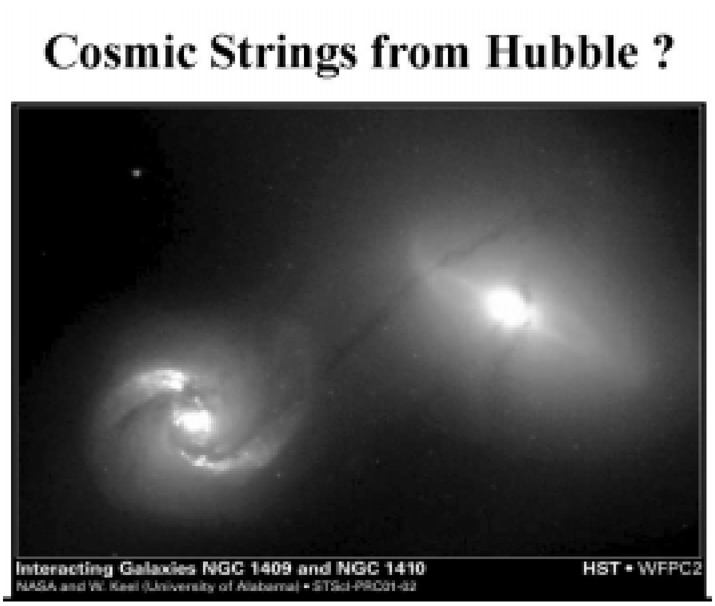

Cosmic Strings from Hubble ?
Interacting Galaxies NGC 1409 and NGC 1410
HST • WFPC2
NASA and W. Keel (University of Alabama)

Figure 5

Chapter 3
APPLICATIONS OF TOPOLOGICAL THERMODYNAMICS.

In this chapter, the topological foundations of thermodynamics developed in the preceding chapter will be applied to several different physical systems. The objective is to demonstrate how the common thread of topology resides in the different disciplines. To repeat the fundamental ideas:

Criterion 9 *Physical Systems can be encoded in terms of exterior differential forms.*

Criterion 10 *Physical processes can be encoded in terms of vector direction fields.*

Criterion 11 *Continuous topological evolution is encoded in Cartan's Magic formula.*

Criterion 12 *Equivalence classes of systems and processes can be defined in terms of the Pfaff topological dimension.*

3.1 Physical Systems of Pfaff dimension 3

Consider those physical systems that are represented by 1-forms, A, of Pfaff topological dimension 3. The concept implies that the topological features can be describe in terms of 3 functions and their differentials. For example, if one presumes the fundamental independent base variables are the set $\{P, q, \tau\}$, with an exterior differential volume element consisting of a product* of exact 1-forms $\Omega_3 = dP\hat{}dq\hat{}dt$, then a Darboux representation for a physical system could have the appearance:

$$A = Pdq + d\tau. \tag{3.1}$$

The objective is to use the features of Cartan's magic formula to compute the possible evolutionary features of such a system. The evolutionary dynamics is essentially the first law of thermodynamics.

$$L_{\rho\mathbf{V}}A = i(\rho\mathbf{V})dA + di(\rho\mathbf{V})A) = W + dU = Q. \tag{3.2}$$

*More abstract systems could be constructed form differential forms which are not exact.

The elements of the Pfaff sequence for this Action become:

$$A = Pdq + d\tau \tag{3.3}$$
$$dA = dP\hat{}dq, \tag{3.4}$$
$$A\hat{}dA = dP\hat{}dq\hat{}d\tau, \tag{3.5}$$
$$dA\hat{}dA = 0. \tag{3.6}$$

Relative to the "ordered position" vector $\mathbf{R} = [P, q, \tau]$, consider the 3 linearly independent orthogonal vector direction fields:

$$\mathbf{V} = [0, 1, 0] \tag{3.7}$$
$$\mathbf{V}_\perp = [1, 0, 0] \tag{3.8}$$
$$\mathbf{E} = [0,\ 0, 1]. \tag{3.9}$$

The *extremal* vector $\mathbf{E}$ is the unique eigen vector with eigenvalue zero relative to the anti-symmetric matrix generated by the 2-form, dA. The *associated* vector $\mathbf{V}_\perp$ is orthogonal to the q, τ plane.

Next deform the vector direction fields by an arbitrary function, ρ. Then construct the contractions (the internal energy).

$$U_\mathbf{V} = i(\rho\mathbf{V})A = \rho P \tag{3.10}$$
$$U_{\mathbf{V}_\perp} = i(\rho\mathbf{V}_\perp)A = 0 \tag{3.11}$$
$$U_\mathbf{E} = i(\rho\mathbf{E})A = \rho. \tag{3.12}$$

The linearly independent Work 1-forms for evolution in the direction of the 3 basis vectors,

$$W_\mathbf{V} = i(\rho\mathbf{V})dA = -\rho dP \tag{3.13}$$
$$W_{\mathbf{V}_\perp} = i(\rho\mathbf{V}_\perp)dA = +\rho dq \tag{3.14}$$
$$W_\mathbf{E} = i(\rho\mathbf{E})dA = 0. \tag{3.15}$$

From Cartan's Magic Formula, $L_{(\mathbf{V})}A = i(\rho\mathbf{V})dA + d(i(\rho\mathbf{V})A) \equiv Q$, it becomes apparent that

$$Q_\mathbf{V} = Pd\rho, \qquad dQ_\mathbf{V} = dP\hat{}d\rho \tag{3.16}$$
$$Q_{\mathbf{V}_\perp} = +\rho dq, \qquad dQ_{\mathbf{V}_\perp} = d\rho\hat{}dq$$
$$Q_\mathbf{E} = d\rho \qquad dQ_\mathbf{E} = 0 \tag{3.17}$$

All processes in the extremal direction satisfy the conditions that $Q_{\mathbf{E}}\hat{}dQ_{\mathbf{E}} = 0$. Hence, all extremal processes are reversible. It is also true that evolutionary processes in the direction of the other basis vectors, separately, are reversible, as the 3-form $Q\hat{}dQ$ vanishes. Hence all such *piecewise* continuous processes are thermodynamically reversible.

However, evolution in the direction of smooth combinations of the base vectors may not satisfy the reversibility conditions, $Q\hat{}dQ = 0$. For example, it is possible to consider expansions or rotations in the P, q plane.

$$V_{\text{expansion}} = \mathbf{V}_{\perp} + \mathbf{V}_{V}, \tag{3.18}$$
$$Q\hat{}dQ = -\rho d\rho\hat{}dP\hat{}dq \tag{3.19}$$

$$V_{\text{rotation}} = \mathbf{V}_{\perp} - \mathbf{V}_{V}, \tag{3.20}$$
$$Q\hat{}dQ = +\rho d\rho\hat{}dP\hat{}dq \tag{3.21}$$

The non-zero value of $Q\hat{}dQ$ for the continuous expansions and rotations are related to the non-zero Godbillon-Vey class [Pittie 1976].

Conclusion 13 *The facts that piecewise (sequential) C1 evolution along a set of direction fields in odd (3) dimensions can be thermodynamically reversible,* $Q\hat{}dQ = 0$*, while C2 evolution as composed of linear combinations of these same direction fields is thermodynamically irreversible,* $Q\hat{}dQ = 0$*, is a remarkable result, that appears to have a relationship to Nash's theorem on C1 embeddings. Physically the results are related to tangential discontinuities such as hydrodynamic wakes.*

It should be remarked that for Action 1-forms of odd Pfaff topological dimension, addition of a closed form whose format contains new independent variables does not change the Pfaff topological dimension of the composite. On the other-hand, if the original 1-from is renormalized by some factor, then the Pfaff topological dimension does change. In the next section it will be demonstrated that this piecewise equivalence class of processes does not produce a thermodynamically irreversible process. In this sense it may be said that thermodynamic irreversibility is an artifact of dimension 4.

3.2 Physical Systems of Pfaff dimension 4

Consider those physical systems that are represented by 1-forms, A, of Pfaff dimension 4. For example, if one presumes the fundamental independent base variables are the space time set $\{x, y, z, t\}$, then a representation for physical system would consist of four functions whose arguments are the base variables, and the Action 1-form would have the appearance:

$$A=\Sigma_{k=1}^{3}A_k(x,y,z,t)dx^k-\phi(x,y,z,t)dt. \tag{3.22}$$

This representation has applicability to the study of electromagnetic systems, where the functions $A_k(x,y,z,t)$ and $\phi(x,y,z,t)$ play the role of the vector and scalar potentials of classic electromagnetic theory [RMK 2004] .

However, the Darboux theorem says that there exists a map from the set $\{x,y,z,t\}$ into four independent functions, $\{q,\tau,P,H\}$ such that the elements of the Pfaff sequence become:

$$A = Pdq - Hd\tau, \tag{3.23}$$
$$dA = dP\hat{}dq - dH\hat{}d\tau, \tag{3.24}$$
$$A\hat{}dA = (PdH - HdP)dq\hat{}d\tau, \tag{3.25}$$
$$dA\hat{}dA = 2dP\hat{}dH\hat{}dq\hat{}d\tau, \tag{3.26}$$

Relative to the "position" vector $\mathbf{R} = [q,\tau,P,H]$, consider the 4 linearly independent orthogonal vector direction fields:

$$\mathbf{V} = [P,H,0,0] \tag{3.27}$$
$$\mathbf{V}_\perp = [-H,P,0,0] \tag{3.28}$$
$$\mathbf{T} = [0,0,P,H] \tag{3.29}$$
$$\mathbf{T}_\perp = [0,0,-H,P] \tag{3.30}$$

The vectors $\mathbf{V}$ and $\mathbf{V}_\perp$ reside in the subspace of $\{x,\tau,0,0\}$ and there for will be called "tangent" vectors. The vectors $\mathbf{T}$ and $\mathbf{T}_\perp$ reside in the subspace $\{0,0,P,H\}$ and will be called "normal" vectors These vectors can be used a basis frame for any vector in the 4 dimensional space.

Next construct the contractions

$$i(\mathbf{V})A = P^2+H^2 \tag{3.31}$$
$$i(\mathbf{V}_\perp)A = 0 \tag{3.32}$$
$$i(\mathbf{T})A = 0 \tag{3.33}$$
$$i(\mathbf{T}_\perp)A = 0 \tag{3.34}$$

and note that the vectors form an orthogonal (but not necessarily normalized) set. Then construct the linearly independent Work 1-forms for evolution in the direction of these 4 basis vectors,

$$W_{\mathbf{V}} = i(\mathbf{V})dA = -d(P^2+H^2)/2 \tag{3.35}$$
$$W_{\mathbf{V}_\perp} = i(\mathbf{V}_\perp)dA = PdH - HdP \tag{3.36}$$
$$W_{\mathbf{T}} = i(\mathbf{T})dA = Pdq - Hd\tau = A \tag{3.37}$$
$$W_{\mathbf{T}_\perp} = i(\mathbf{T}_\perp)dA = -Hdq + Pd\tau. \tag{3.38}$$

From Cartan's Magic Formula, $L_{(\mathbf{V})}A = i(\mathbf{V})dA + d(i(\mathbf{V})A) \equiv Q$, it becomes apparent that

$$\begin{aligned}
Q_{\mathbf{V}} &= +d(P^2+H^2)/2, \qquad dQ_{\mathbf{V}} = 0 &(3.39)\\
Q_{\mathbf{V}_\perp} &= PdH - HdP, \qquad dQ_{\mathbf{V}_\perp} = 2dP\hat{}dH &(3.40)\\
Q_{\mathbf{T}} &= Pdq - Hd\tau = A \qquad dQ_{\mathbf{T}} = dP\hat{}dq - dH\hat{}d\tau = dA &(3.41)\\
Q_{\mathbf{T}_\perp} &= -Hdq + Pd\tau \qquad dQ_{\mathbf{T}_\perp} = -dH\hat{}dq + dP\hat{}d\tau, &(3.42)
\end{aligned}$$

and it follows that the three forms $Q\hat{}dQ$ become:

$$\begin{aligned}
Q_{\mathbf{V}}\hat{}dQ_{\mathbf{V}} &= 0 &(3.43)\\
Q_{\mathbf{V}_\perp}\hat{}dQ_{\mathbf{V}_\perp} &= 0 &(3.44)\\
Q_{\mathbf{T}}\hat{}dQ_{\mathbf{T}} &= (HdP - PdH)\hat{}dq\hat{}d\tau = A\hat{}dA &(3.45)\\
Q_{\mathbf{T}_\perp}\hat{}dQ_{\mathbf{T}_\perp} &= (HdP - PdH)\hat{}dq\hat{}d\tau = A\hat{}dA &(3.46)
\end{aligned}$$

The vector $\mathbf{T}$ is known as the Torsion vector, and motion in the direction of the Torsion vector (or its orthogonal compliment $\mathbf{T}_\perp$) yield a non-zero value for the 3-form Q. It follows that evolution in the direction of the Torsion vector is thermodynamically irreversible. For any vector with components constructed from $\mathbf{V}$ and $\mathbf{V}_\perp$ with arbitrary functional coefficients, it follows that the Heat 1-form Q satisfies the Frobenius integrability theorem. Only evolutionary processes with components constructed from $\mathbf{T}_\perp$ and $\mathbf{T}$ will represent thermodynamically irreversible processes.

The difference between the 3D case and the 4D case is that piecewise evolution in the direction of the basis vectors is always reversible in the 3-D case, but piecewise evolution in the direction of the 4D basis vectors is not always thermodynamically irreversible. So the work of the last two subsections establishes the following theorem:

Conclusion 14 *Thermodynamic irreversibility is an artifact of topological Pfaff dimension* $\geqslant 4$.

It is of some interest to recognize that the 1-form $HdP - PdH$ has a rotational polar coordinate representation

$$HdP - PdH = (P^2 + H^2)\delta(\Theta) \tag{3.47}$$

under the mapping

$$\begin{aligned}
H &= \sqrt{(H^2+P^2)}\cos\Theta &(3.48)\\
P &= \sqrt{(H^2+P^2)}\sin\Theta. &(3.49)
\end{aligned}$$

The 1-form

$$\delta(\Theta) = (HdP - PdH)/(P^2 + H^2) \qquad d(\delta(\Theta) = 0 \tag{3.50}$$

is not an exact 1-form, but it is closed mod the origin of H and P. Such closed but not exact forms are called harmonic forms and have values when integrated over closed cycles with rational ratios.

It should be remarked that for Action 1-forms of even Pfaff topological dimension, addition of a closed form whose format contains new independent variables does change the Pfaff topological dimension of the composite. If the closed 1-form is a function is a function of the original 2n+2 variables, it does not change the Pfaff topological dimension of the even dimensional form. On the otherhand, if the original 1-from is renormalized by some factor, then the Pfaff topological dimension does not change.

3.3 Electromagnetism as a topological theory

Maxwell's PDE's are topological statements that can be deduced from an exterior differential system. The two postulates are:

The Postulate of Potentials

$$F - dA = 0. \tag{3.51}$$

The Postulate of conserved Charge Current densities

$$J - dG = 0. \tag{3.52}$$

No constraints of geometrical connection or metric are required. Such geometric constraints can be used to refine the Maxwell topology for different specific physical systems. For example, constitutive equations of constraint between the two 2-forms F and G can be used to distinguish birefringent media from optically active media. The Maxwell-Faraday PDE's are not restricted to spaces of topological dimension N = 4. For an exterior differential system $F - dA = 0$ on a space of any dimension N>3, the closure conditions, $ddA = dF = 0$, always yield the same identical Maxwell-Faraday PDE's for the first 4 variables. Additional PDE's are also generated for N>4, but the system of PDE's form a nested set, with the Maxwell-Faraday equations as topological kernel, of invariant format for any dimension N. A remarkable result is that Faraday induction is a topological idea, and does not depend upon metric or connection. The concept of Faraday induction applies to any system that satisfies the Postulate of Potentials, including the fundamental axiom of topological thermodynamics which encodes a physical system in terms of a 1-form of Action.

As demonstrated below, the Postulate of Potentials establishes the field intensities, **E** and **B**, (think forces),and the Postulate of Conserved Charge current densities establishes the field excitations, **D** and **H**, (think sources). The topological perspective subsumes that the two species are independent ideas. The experimental justification of such ideas can be demonstrated with a simple parallel plate capacitor experiment. First connect the plates to a battery of constant potential and let it remain connected. Insert a slab of plastic dielectric halfway between the plates. Release the plastic slab. Does the slab remain motionless, or is the motion such that the slab is expelled or attracted? For a second experiment, attach the plates of the capacitor to a battery and then disconnect the battery after charging the capacitor. Now insert the plastic slab halfway, and release it. Does the slab remain motionless, or is the motion such that the slab is expelled or attracted? In the first case, the **E** field remains constant (the potential does not change), and motion of the dielectric slab causes the **D** field to change (the battery adjusts the charge distribution). In the second experiment, the charge distribution is constant, so that the **D** field remains constant, but the **E** field changes. Consider the simple constitutive constraint, $\mathbf{D} = \varepsilon\mathbf{E}$. In the first experiment, insertion would cause the average ε to increase, hence even though **E** remains constant, the **D** field would increase. However, the total energy density $\mathbf{D} \circ \mathbf{E}$ would decrease if the slab was expelled, and that is what happens. In the second experiment, motion of the slab would cause the **E** field to change, as the **D** field remains constant, and the minimum energy density occurs when the slab is fully inserted.

Current electromagnetic dogma presents the idea that from a given charge current density distribution, $[\mathbf{J}, \rho]$, it is possible to deduce the **E** and **B** fields. However, the Postulate of conserved Charge-Current densities indicates that it is **D** and **H** that are the related quantities, not **E** and **B**. The Postulate of Potentials indicates that the field intensities **E** and **B** are deduced from the potentials $[\mathbf{A}, \phi]$. It takes some constitutive constraint to convert **D** and **H** into **E** and **B**, or $[\mathbf{J}, \rho]$ into $[\mathbf{A}, \phi]$. Both types of constraints appear in the literature in great detail and variety. Such assumptions obscure the topological basis and differences between exterior differential forms and exterior differential form densities.

The postulate of potentials indicates that the domain of support for the 2-form F is not compact without boundary†. The postulate also demonstrates that magnetic monopoles are not compatible with the assumption of C2 differentiability. Such a statement does not apply to the density N-2 form, G , which can have closed and non-closed components. The closed but not exact components of G lead to the quantization of charge as a topological result. As G is a density, it also follows that quantized charge is a pseudo-scalar [Post 1983]. The historical assumptions of charge as a scalar are not compatible with the topological format. Experiments with piezo electric crystals indicate that volume deformations can cause electrical phenomena. If G was not a density, there would be no Piezo electricity.

†There are two exceptions: the Klein bottle and the torus.

3.3.1 The D H field excitations: differential N-2 form densities.

For example consider the exterior differential of the N-1 form density[‡], D, in three dimensions, given by the expression,

$$dD = d(D^x dy\hat{}dz - D^y dz\hat{}dx + D^z dx\hat{}dy) \tag{3.53}$$
$$= div_3(D)dx\hat{}dy\hat{}dz \Rightarrow \rho(x,y,z)dx\hat{}dy\hat{}dz$$

where ρ has been defined as the resultant of the action of the exterior differential, $div_3(\mathbf{D})$. The usual interpretation of Gauss' law is that the field lines of the vector (density) $\mathbf{D}$ terminate (or have a limit or accumulation point) on the charges, Q. Gauss' law generates both the intuitive idea that sources are related to limit points, and demonstrates the novel concept that the exterior differential is a limit point operator. The exterior differential creates limits points when the operation is applied to a differential form. However, as demonstrated above, the concept that the exterior differential is a limit point operator relative to the Cartan topology is a general idea, and is not restricted to Gauss' law.

Extending this idea to four dimensions for the N-2 form density, G, of Maxwell excitations $(\mathbf{D},\mathbf{H})$,

$$G = -D^x dy\hat{}dz + D^y dz\hat{}dx - D^z dx\hat{}dy + H^x dx\hat{}dt + H^y dy\hat{}dt + H^z dz\hat{}dt, \tag{3.54}$$

the exterior differential dG of G yields a three form, J, defined as the electromagnetic current 3-form,

$$J = J^x dy\hat{}dz\hat{}dt - J^y dx\hat{}dz\hat{}dt + J^z dx\hat{}dy\hat{}dt - \rho dx\hat{}dy\hat{}dt \tag{3.55}$$

where in 3-vector language,

$$curl\ \mathbf{H} - \partial\mathbf{D}/\partial t = 0 \qquad div\,\mathbf{D} = \rho. \tag{3.56}$$

The charge current density act as the "limit points" of the Maxwell field excitations. Note that $dJ = 0$ for C2 functions by Poincare's lemma (see the Appendix).

However, consider the N-1 current, C (not necessarily equal to J as defined above) in four dimensions

$$C = \rho\{V^x dy\hat{}dz\hat{}dt - V^y dx\hat{}dz\hat{}dt + V^z dx\hat{}dy\hat{}dt - 1dx\hat{}dy\hat{}dt\} \tag{3.57}$$

[‡]There are two species of differential forms considered in this article. The first specie transforms as a scalar with respect to diffeomorphisms. The second specie transforms as a scalar density, and is proportional to the determinant of the diffeomorphism. The coefficients pull back with respect to the transpose of a differential Jacobian mapping, whether it is a diffeomorphism or not. The second species, the differential form densities, pull back with respect to the adjoint of a differential Jacobian mapping.

and its exterior differential as given by the expression,

$$dC = \{div_3(\rho\mathbf{V}) + \partial\rho/\partial t\}dx\hat{}dy\hat{}dz\hat{}dt. = Rdx\hat{}dy\hat{}dz\hat{}dt = R\,\Omega_{4\_vol} \qquad (3.58)$$

When the 4-form R vanishes, the resultant expression is physically interpreted as the "equation of continuity" or as a "conservation law". Over a closed boundary, that which goes in is equal to that which goes out (when $dC = 0$). Note that the concept of the conservation law is a topological constraint: the "limit points" of the "current 3-form" in four dimensions must vanish if the conservation law is to be true. If the RHS of the above expression (3.58) is not zero, then the current 3-form is said to have an "anomaly", or a source (or sink) . The anomaly acts as the source of the otherwise conserved quantity. The limit points, R, of the 3-form, C, are generated by its exterior differential, $dC = \{div_3(\rho\mathbf{V}) + \partial\rho/\partial t\}\Omega_4$. When the RHS is zero, the current "lines" do not stop or start within the domain. (It is possible for them to be closed on themselves in certain topologies).

3.3.2 The E B Field Intensities: differential 2-forms

On a four dimensional space-time of independent variables, (x, y, z, t) the 1-form of Action (constrained by the postulate of potentials, $F - dA = 0$) can be written in the form

$$A = \Sigma_{k=1}^{3} A_k(x, y, z, t)dx^k - \phi(x, y, z, t)dt = \mathbf{A}\circ d\mathbf{r} - \phi dt. \qquad (3.59)$$

This 1-form of Action defines (part of) the physical system of electromagnetism. Subject to the constraint of the exterior differential system, the 2-form of field intensities, F, becomes:

$$\begin{aligned} F = dA &= \{\partial A_k/\partial x^j - \partial A_j/\partial x^k\}dx^j\hat{}dx^k = F_{jk}dx^j\hat{}dx^k \qquad (3.60)\\ &= \mathbf{B}_z dx\hat{}dy + \mathbf{B}_x dy\hat{}dz + \mathbf{B}_y dz\hat{}dx + \mathbf{E}_x dx\hat{}dt + \mathbf{E}_y dy\hat{}dt + \mathbf{E}_z dz\hat{}dt. \end{aligned}$$

where in usual engineering notation,

$$\mathbf{E} = -\partial\mathbf{A}/\partial t - grad\phi, \qquad \mathbf{B} = curl\ \mathbf{A} \equiv \partial A_k/\partial x^j - \partial A_j/\partial x^k. \qquad (3.61)$$

The closure of the exterior differential system, $dF = 0$, vanishes for C2 differentiable p-forms, to yield

$$dF = ddA = \{curl\ \mathbf{E} + \partial\mathbf{B}/\partial t\}_x dy\hat{}dz\hat{}dt - .. + .. - div\,\mathbf{B}dx\hat{}dy\hat{}dz\} \Rightarrow 0. \qquad (3.62)$$

Equating to zero all four coefficients leads to the Maxwell-Faraday partial derivative equations,

$$\{curl\,\mathbf{E}+\partial\mathbf{B}/\partial t=0,\qquad div\,\mathbf{B}=0\}. \tag{3.63}$$

This topological development of the Maxwell-Faraday equations has made no use of a connection nor of a metric.

The component functions ($\mathbf{E}$ and $\mathbf{B}$) of the 2-form, F, transform as covariant tensor of rank 2. The topological constraint that F is exact, implies that the domain of support for the field intensities cannot be compact without boundary, unless the Euler characteristic vanishes. These facts distinguish classical electromagnetism from Yang-Mills field theories. Moreover, the fact that F is subsumed to be exact and C1 differentiable excludes the concept of magnetic monopoles from classical electromagnetic theory on topological grounds.

This now almost classic generation of the Maxwell field equations has another less familiar interpretation: The $\mathbf{E}$ and $\mathbf{B}$ field intensities are the topological limit "points" of the 1-form of potentials, $\{\mathbf{A},\phi\}$, relative to the Cartan topology! The limit points of the 2-form of field intensities, F, are the null set. For C2 vector fields, the Cartan topology admits flux quanta, charge quanta, and spin quanta, but excludes magnetic monopoles. When the differential system of interest is built upon the forms A, F and G, it is possible to show that superconductivity is to be associated with the constraints on the limit point sets of $A,\ A\hat{}F,\ and\ A\hat{}G$ [RMK 1991 c]. That is, superconductivity has its origins in topological, not geometrical, concepts. This remarkable idea that the exterior differential is a limit point operator is based upon Kuratowski's closure operator is equivalent to the union of the identity and the exterior differential.

3.3.3 Non-Equilibrium Features of Electromagnetism.

The exterior differential forms that make up the electromagnetic system on a geometric domain of 4 dimensions consist of the primitive 1-form, A, and the primitive N-2 form density, G, their exterior derivatives, and their algebraic intersections defined by all possible exterior products. The complete Maxwell system of exterior differential forms (the Pfaff sequence for the Maxwell system on 4 geometric dimensions) is given by the set:

$$\{A;F=dA,G;J=dG,A\hat{}F,A\hat{}G,A\hat{}J;F\hat{}F,G\hat{}G\}. \tag{3.64}$$

These differential forms and their unions may be used to form a topological base on the domain of independent variables. The Cartan topology constructed on this system of forms has the useful feature that the exterior derivative may be interpreted as a limit point, or closure, operator in the sense of Kuratowski (see p. 72 in [Lipschutz 1965]). The exterior differential systems that define the Maxwell-Ampere and the Maxwell-Faraday equations above are essentially topological constraints of closure.

The complete Maxwell system of differential forms (which assumes the existence of A and G and C2 differentiability) also generates two other exterior differential systems,

$$(F\hat{\ }G - A\hat{\ }J) - d(A\hat{\ }G) = 0, \tag{3.65}$$
$$F\hat{\ }F - d(A\hat{\ }F) = 0, \tag{3.66}$$

which prolong the primary (exact) exterior differential systems,

$$F - dA = 0, \tag{3.67}$$
$$J - dG = 0. \tag{3.68}$$

The terms $A\hat{\ }F$ and $A\hat{\ }G$ are zero for equilibrium systems. The existence of these 3-forms are indicators that the electromagnetic system is NOT in equilibrium.

Each of the forms, $A,\ G,\ A\hat{\ }G,\ A\hat{\ }F$, can have closed but not exact components. The two 4-forms $(F\hat{\ }G - A\hat{\ }J)$ and $(F\hat{\ }F)$ are exact and have closed integrals which are evolutionary (relative) invariants of continuous deformations. The closed integrals therefor describe topological properties.

The first 3-form density, $A\hat{\ }G$, with physical units of $\hbar$, is called the "topological spin" (or chirality) [RMK 1977] and the second 3-form, $A\hat{\ }F$, with physical units of $(\hbar/e)^2$, is called the "topological torsion" (or helicity) [RMK 1990]. These two exterior 3-forms, $A\hat{\ }G$ and $A\hat{\ }F$ are not usually found in discussions of classical electromagnetism. The 3-forms are abstractly defined (on a space of 4 geometric dimensions with a volume element, $\Omega_4 = dx\hat{\ }dy\hat{\ }dz\hat{\ }dt$) in terms of exterior multiplication, but can be given realization in terms of 4 component engineering variables, $\mathbf{S}_4$ and $\mathbf{T}_4$.

$$\text{Topological Spin density} \quad : \quad A\hat{\ }G = i(\mathbf{S}_4)\Omega_4 \tag{3.69}$$
$$\mathbf{S}_4 = [\mathbf{S}, \sigma] = [\mathbf{A} \times \mathbf{H} + \mathbf{D}\phi, \mathbf{A} \circ \mathbf{D}], \tag{3.70}$$

$$\text{Topological Torsion vector} \quad : \quad A\hat{\ }F = i(\mathbf{T}_4)\Omega_4 \tag{3.71}$$
$$\mathbf{T}_4 = [\mathbf{T}, h] = -[\mathbf{E} \times \mathbf{A} + \mathbf{B}\phi, \mathbf{A} \circ \mathbf{B}]. \tag{3.72}$$

These constructions should be compared with the exact charge current 4-vector density, J, with a 4 component engineering representation, $\mathbf{J}_4 = [\mathbf{J}, \rho]$. The concepts of the topological Spin density (current) and the topological Torsion vector have had almost no utilization in applications of classical electromagnetic theory. Each construction depends explicitly on the existence of the 1-form of Action-potentials.

Recall that the closed components of the 1-form of Action do not effect the components of the 2-form of intensities, $F = dA = d(A_c + A_0) = 0 + d(A_0)$. However, these "gauge" additions do influence the topological dimension of the 1-form of Action. For example, let A_0 be of Pfaff Topological dimension 2, representing an equilibrium system where $A_0\hat{\ }dA_0 = 0$. Then by addition of a closed component to the original action, $A = A_c + A_0$ could have a topological dimension of 3, as

$$A\hat{\ }dA = (A_c + A_0)\hat{\ }dA_0 = A_c\hat{\ }dA_0 \neq 0. \tag{3.73}$$

So the addition of a closed component to the 1-form of Action could change the system from an equilibrium system to a non-equilibrium system. The 4-form $dA\hat{\ }dA$ is not influenced by the (gauge) addition to the original 1-form of Action.

$$dA\hat{\ }dA = dA_0\hat{\ }dA_0. \tag{3.74}$$

In the example below, a 1-form representing a Bohm-Aharanov-Abrikosov singular "vortex" string, $\gamma = b(ydx - xdy)/(x^2 + y^2)$, is added to a 1/r potential for a point source. The bare m/r "Coulomb" potential, $A_0 = m/\sqrt{(x^2 + y^2 + z^2)}dt$ exhibits no Topological Torsion, but does exhibit Topological Spin. The 1/r potential term implies that $dA_0 \neq 0$. Hence the 1-form of Action representing a bare "coulomb" potential, is not in equilibrium, but does represent a connected "isolated" topology of Pfaff dimension 2. The combined 1-form of Action,

$$A = b(ydx - xdy)/(x^2 + y^2) + m/\sqrt{(x^2 + y^2 + z^2)}dt \tag{3.75}$$

even though $d\gamma = 0$, is of Pfaff dimension 3, not 2. The addition of the BAA term changes the topology of the 1-form from a connected topology (Pfaff dimension 2) to a disconnected topology (Pfaff dimension 3). The Topological Torsion 3-form $A\hat{\ }F$ depends on both b and m, and is zero if $b = 0$, or if $m = 0$, reducing the Pfaff dimension of the modified 1-form back to 2. If $b = 0$ and $m \neq 0$, the 3-form $A\hat{\ }G$ is not zero.

Example: Coulomb Potential plus Vortex singularity.

Consider the Potentials for the combined vortex singularity and Coulomb 1/r potential:

$$A = [by/(x^2 + y^2), -bx/(x^2 + y^2), 0, -(1/4\pi\epsilon)m/r], \tag{3.76}$$

$$r = \sqrt{x^2 + y^2 + z^2}. \tag{3.77}$$

The induced fields (assuming $\mathbf{D} = \varepsilon\mathbf{E}$, $\mathbf{B} = \mu\mathbf{H}$) are:

$$\mathbf{E} = (1/4\pi\epsilon)m[x, y, z]/r^3 \tag{3.78}$$

$$\mathbf{B} = [0, 0, 0] \tag{3.79}$$

$$\text{Topological Torsion } \mathbf{T}_4 = (1/4\pi\epsilon)mb[-zx, -zy, (x^2 + y^2), 0]/(r^3(x^2 + y^2)). \tag{3.80}$$

$$Poincare2 = 0 \tag{3.81}$$

$$\mathbf{J}_4 = [0, 0, 0, 0] \tag{3.82}$$

$$\rho = 0 \tag{3.83}$$

$$\text{Topological spin } \mathbf{S}_4 = (1/4\pi\epsilon)^2 \epsilon m^2 / r^4 [x, y, z, 0]\,. \tag{3.84}$$

$$Poincare1 = -(1/4\pi\epsilon)^2 \epsilon m^2 / r^4 \tag{3.85}$$

$$\mathbf{E} \times \mathbf{H} \ = [0, 0, 0]. \tag{3.86}$$

$$\begin{aligned} (\mathbf{J} \circ \mathbf{E}) &= 0 && (3.87) \\ (\mathbf{E} \circ \mathbf{B}) &= 0, && (3.88) \\ (\mathbf{A} \circ \mathbf{B}) &= 0, && (3.89) \\ (\mathbf{A} \circ \mathbf{D}) &= 0 && (3.90) \end{aligned}$$

3.3.4 The Poincare Topological Invariants

The exterior derivatives of the 3-forms of topological Spin and topological Torsion produce two exact 4-forms, $F\hat{}G - A\hat{}J$ and $F\hat{}F$, whose closed integrals are topological objects which generalize the conformal invariants [Whittaker 1944] of a Lorentz system, as discovered by Poincare and Bateman. Note that these topological properties of invariance with respect to continuous deformations are valid even in the non-equilbrium domain of dissipative charge-currents and radiation.

In the format of independent variables $\{x, y, z, t\}$, with a volume element Ω_4, the exterior derivative, acting on the 3-forms as a topological limit point generator, can be related to the classic 4-divergence of the 4-component Topological Spin and Topological Torsion vectors, $\mathbf{S}_4$ and $\mathbf{T}_4$.

$$\begin{aligned} Poincare\ 1 &= d(A\hat{}G) = F\hat{}G - A\hat{}J \\ &= \{div_3(\mathbf{A} \times \mathbf{H} + \mathbf{D}\phi) + \partial(\mathbf{A} \circ \mathbf{D})/\partial t\}\ \Omega_4 \\ &= \{(\mathbf{B} \circ \mathbf{H} - \mathbf{D} \circ \mathbf{E}) - (\mathbf{A} \circ \mathbf{J} - \rho\phi)\}\ \Omega_4, && (3.91) \\ &= 0 \ \text{ for the example in the previous section.} && (3.92) \end{aligned}$$

$$\begin{aligned} Poincare\ 2 &= d(A\hat{}F) = F\hat{}F \\ &= \{div_3(\mathbf{E} \times \mathbf{A} + \mathbf{B}\phi) + \partial(\mathbf{A} \circ \mathbf{B})/\partial t\}\ \Omega_4 \\ &= \{2\mathbf{E} \circ \mathbf{B}\}\ \Omega_4 && (3.93) \\ &\neq 0 \text{ for the example in the previous section.} && (3.94) \end{aligned}$$

The Poincare invariants are, in effect, the evolutionary source terms for the 3-forms of topological spin, $A\hat{}G$, and topological torsion, $A\hat{}F$. When the Poincare invariants are zero, the closed integrals of the electromagnetic 3-forms of $A\hat{}G$ and $A\hat{}F$

become additional topologically coherent configurations invariant with respect to all evolutionary processes of continuous deformation.

The first term in the first Poincare invariant has a coefficient function which represents twice the difference between the magnetic energy density and the electric energy density of the electromagnetic field in a Lagrangian sense:

$$\text{Topological Field Lagrangian:}\quad F\hat{}G = (\mathbf{B}\circ\mathbf{H} - \mathbf{D}\circ\mathbf{E})\ \Omega_4. \tag{3.95}$$

The second term in the first Poincare invariant has a coefficient function which is defined as the interaction energy density:

$$\text{Topological Interaction:}\quad A\hat{}J = (\mathbf{A}\circ\mathbf{J} - \rho\phi)\ \Omega_4. \tag{3.96}$$

In Lagrangian variational methods, the 4-form $F\hat{}F$, which defines the second Poincare invariant, has been related to the concept of Topological Parity:

$$\text{Topological Parity:}\quad F\hat{}F = +\{2\mathbf{E}\circ\mathbf{B}\}\ \Omega_4. \tag{3.97}$$

Using the example system, (**??**) permits the construction of the Topological Spin current and its divergence relative to the Lorentz-Lorenz constitutive constraint.

3.3.5 Topological Torsion and Spin quanta

When either Poincare deformation invariant vanishes, the corresponding closed 3-dimensional integrals of $A\hat{}G$ and $A\hat{}F$ become deRham period integrals. The closed, but not exact, components of each 3-form can be put into correspondence with "quantized" topological defects.

The topological Spin quantum is defined as the closed integral of those closed but not exact components of the 3-form $A\hat{}G$ (which represent the kernel of the first Poincare 4-form),

$$\text{Spin quantum} = \iiint_{z3d} A\hat{}G \text{ with units n } \hbar. \tag{3.98}$$

The period integrals $\iiint_{z3d} A\hat{}G$ are deformation invariants (hence define a topological property) with rational ratios. The notation $z3d$ designates a closed integration chain defined in regions where $d(A\hat{}G) = 0$.

Similarly, when the second Poincare invariant vanishes, the closed integral of the 3-form of Torsion-Helicity becomes a deformation invariant with quantized values:

$$\text{Torsion quantum} = \iiint_{z3d} A\hat{}F \text{ with units m } (\hbar/e)^2. \tag{3.99}$$

The period integrals $\iiint_{z3d} A\hat{}F$ are deformation invariants (hence define a topological property) with rational ratios. In this case, The notation $z3d$ designates a closed integration chain defined in regions where $d(A\hat{}F) = 0$.

It is important to realize that the topological conservation laws (deformation invariants with respect to homeomorphisms) are valid in a plasma as well as in the vacuum, subject to the conditions of zero values for the Poincare invariants. On the other hand, topological evolution and transitions between "quantized" states of Spin-chirality or Torsion-helicity require that the respective Poincare invariants are not zero.

The 3-forms, $A\hat{}G$, and $A\hat{}F$, are not necessarily closed, nor exact. Their exterior differentials (divergences) are not necessarily zero. The values of the 4-forms created by exterior differentiation of these 3-forms define the integrands of the topological Poincare invariants. As these 4-forms are exact by construction, their closed integrals are always relative integral deformation invariants and thereby define topological properties. The 3-forms are not necessarily, in themselves, deformation invariants.

However, when the Poincare invariants vanish (zero divergence) the closed integral of the corresponding 3-form generates a topological quantity (Topological Spin or Topological Torsion respectively) which is also a deformation invariant. In such situations, the 3-forms are closed, but not necessarily exact. Hence their closed integrals generate deRham period integrals [Flanders 1963] [deRham 1960], and have rational ratios. Such is the stuff of topological quantization, which is independent from scales.

The theory of 3-forms and their period integrals was investigated with respect to electromagnetism and other field theories by the present author, first with respect to the three form defined below as topological spin, $A\hat{}G$, and then later with respect to the 3-form of topological torsion, defined as $A\hat{}F$. The first application of $A\hat{}F$ was in the field of turbulence [RMK1976], where it was conjectured that the transition from streamline flow (uniquely integrable in the sense of Frobenius, such that $A\hat{}F = 0$) to a turbulent flow (not uniquely integrable in the sense of Frobenius, $A\hat{}F \neq 0$) must involve a topological change. Although the interest was focused on hydrodynamics, the electromagnetic format was always used to establish a credence level in the computations that were done by hand. In the modern world of symbolic calculators on your desktop, this algebraic tedium has been alleviated. See http://www22.pair.com/csdc/pdf/maxwell.pdf

On domains where the Pfaff topological dimension is 3 (and not 4) there exists a 3 dimensional period integral of the topological torsion, which is related to the Hopf invariant (see p. 228 in [Bott 1994]), $\iiint_{3\_cycle} A\hat{}F$. It will be demonstrated below that if the domain is of Pfaff dimension 3, then evolutionary processes in the direction of the electromagnetic charge current 4 vector, J, leave the integral of the Topological Torsion current over a 3 dimensional boundary as an evolutionary invariant. Even more remarkable is the fact that such a statement is valid in domains where the Pfaff dimension is 4, not 3, if the current flow is on the surface defined by $(\mathbf{E} \bullet \mathbf{B}) = 0$.

3.4 Hydrodynamics

3.4.1 Euler flows and Hamiltonian fluids

Consider the Action 1-form,

$$A = \mathbf{v} \circ \mathbf{dr} - \{\mathbf{v}.\mathbf{v}/2\}dt. \tag{3.100}$$

Compute the exterior differntial dA and define the following functions:

$$\boldsymbol{\omega} = curl\ \mathbf{v} \quad and \quad \mathbf{a} = -\partial\mathbf{v}/\partial t - grad\{\mathbf{v}.\mathbf{v}/2\}\} \tag{3.101}$$

such that

$$\begin{aligned} F = dA &= \{\partial A_k/\partial x^j - \partial A_j/\partial x^k\}dx^j\hat{}dx^k = F_{jk}dx^j\hat{}dx^k \qquad (3.102)\\ &= \boldsymbol{\omega}_z dx\hat{}dy + \boldsymbol{\omega}_x dy\hat{}dz + \boldsymbol{\omega}_y dz\hat{}dx + \mathbf{a}_x dx\hat{}dt + \mathbf{a}_y dy\hat{}dt + \mathbf{a}_z dz\hat{}dt. \end{aligned}$$

These vector fields always satisfy the Poincare-Faraday induction equations, $dF = ddA = 0, or,$

$$curl\ \mathbf{a} + \partial\boldsymbol{\omega}/\partial t = 0, \qquad div\ \boldsymbol{\omega} = 0. \tag{3.103}$$

Consider a process created by the vector field, $\mathbf{V}_4 = [\mathbf{V}^x, \mathbf{V}^y, \mathbf{V}^z, 1]$ and use Cartan's magic formula to compute the work 1-form.

$$L_{(\rho\mathbf{V}_4)}A = i(\rho\mathbf{V}_4)dA + d(i(\rho\mathbf{V}_4)A) = W + dU = Q. \tag{3.104}$$

The expression for Work becomes

$$W = i(\rho\mathbf{V}_4)dA = \rho\{\partial\mathbf{v}/\partial t + grad(\mathbf{V}\cdot\mathbf{v}/2) - \mathbf{V}\times\mathbf{w}\} \circ d\mathbf{r} \tag{3.105}$$

$$U = i(\mathbf{V}_4)A = \mathbf{V}\cdot\mathbf{v} - \mathbf{v}.\mathbf{v}/2. \tag{3.106}$$

Constrain the Work 1-form to be of the Bernoulli class such that

$$W = -dP, \tag{3.107}$$

and assume that $\mathbf{V} = \mathbf{v}$. The result is the partial differential equations that represent the Lagrange - Euler fluid.

$$\{\partial\mathbf{v}/\partial t + grad(\mathbf{v}\cdot\mathbf{v}/2) - \mathbf{v}\times\mathbf{w}\} = -grad(P)/\rho \tag{3.108}$$

Note that the pressure, P, is an evolutionary invariant in the Bernoulli sense. The flow is Hamiltonian (but not extremal Hamiltonian) and reversible as $Q\hat{}dQ = 0$.

It also follows that the "Master" equation is valid, with the only difference being that $curl\mathbf{v}$ is defined as $\boldsymbol{\omega}$, the vorticity of the hydrodynamic flow. The master equation becomes,

$$curl(\mathbf{v}\times\boldsymbol{\omega}) = \partial\boldsymbol{\omega}/\partial t, \tag{3.109}$$

and this equation is to be recognized as the equivalent of Helmoltz' equation for the conservation of vorticity. In the hydrodynamic sense, conservation of vorticity implies uniform continuity. In other words, the Eulerian flow is not only Hamiltonian, it is also uniformly continuous, and satisfies the master equation and the conservation of vorticity constraints. In addition, it may be demonstrated that such systems are at most of Pfaff dimension 3, and admit a relative integral invariant which generalizes the hydrodynamic concept of invariant helicity. In the electromagnetic topology, the Hamiltonian constraint is equivalent to the statement that the Lorentz force vanishes, a condition that has been used to define the "ideal" plasma or "force-free" plasma state.

3.4.2 The Navier-Stokes fluid

From the theory of topological fluctuations, it must be true that the 1-form of Work must have a format of the type

$$W = i(\rho\mathbf{V}_4)dA = -dP + \varpi_k(d\mathbf{x}^k - \mathbf{v}^k dt) \tag{3.110}$$

If ϖ_k/ρ is defined as $\upsilon\, curl\, curl\, \mathbf{v}$ then the spatial components of the work 1-form are constrained to yield the partial differential equations for a constant density Navier-Stokes fluid.

$$\{\partial\mathbf{v}/\partial t + grad(\mathbf{v}\cdot\mathbf{v}/2) - \mathbf{v}\times\mathbf{w}\} = -grad(P)/\rho + \upsilon\, curl\, curl\, \mathbf{v}. \tag{3.111}$$

Density variations can be included by adding a term $\lambda div(\mathbf{V})$ to the potential $\{\mathbf{v}\cdot\mathbf{v}/2\}$ to yield

$$\partial\mathbf{v}/\partial t + grad\{\mathbf{v}.\mathbf{v}/2\} - \mathbf{v}\times curl\ \mathbf{v} = -gradP/\rho + \lambda grad(div\ \mathbf{v}) + v\{curl\ curl\ \mathbf{v}\}. \tag{3.112}$$

It is thereby demonstrated that the Navier-Stokes equations correspond to a refinement of the Cartan topology [RMK 1975 a]. The Navier-Stokes constraint implies that the Work 1-form need not be closed. There are solutions to the Navier-Stokes equations that are thermodynamically irreversible.

The 1-form of Action will generate a 3-form of Topological Torsion, $A\hat{}dA = i(\mathbf{T}_4)dx\hat{}dy\hat{}dz\hat{}dt$, of the form,

$$\mathbf{T}_4 = [\mathbf{a}\times\mathbf{v} + \{\mathbf{v}.\mathbf{v}/2\}\ curl\ \mathbf{v}, (\mathbf{v}\circ curl\,\mathbf{v}) = [\mathbf{a}\times\mathbf{v} + \{\mathbf{v}.\mathbf{v}/2\}\ \boldsymbol{\omega}, (\mathbf{v}\circ curl\,\mathbf{v})] \tag{3.113}$$

Use the Navier-Stokes equations (3.112) to solve for $\mathbf{a}$,

$$\mathbf{a} = -[grad\{\mathbf{v}.\mathbf{v}/2\} + \partial\mathbf{v}/\partial t] = -\mathbf{v}\times curl\ \mathbf{v} + gradP/\rho - \lambda grad(div\ \mathbf{v}) + \upsilon\{curl\ curl\ \mathbf{v}\}, \quad (3.114)$$

and then substitute in the expression for $\mathbf{T}_4$, to yield

$$\mathbf{T} = [h\mathbf{v} - \{\mathbf{v}\circ\mathbf{v}/2\}curl\ \mathbf{v} - \mathbf{v}\times(gradPl\rho + \lambda grad(div\ \mathbf{v})) - \upsilon\{\mathbf{v}\times(curl\ curl\ \mathbf{v})\}], \quad (3.115)$$

$$h = \mathbf{v}\cdot curl\,\mathbf{v} \quad (3.116)$$

which persists even for Euler flows, where $\upsilon = 0$, if the flow is baroclinic. The measurement of the components of the Torsion vector, $\mathbf{T}_4$, have been completely ignored by experimentalists in hydrodynamics.

By a similar substitution, the topological parity pseudo-scalar becomes expressible in terms of engineering quantities as,

$$K = -\{2(a.\omega)\}\Omega_4 = \{2\{gradP/\rho - \lambda\ div\ \mathbf{v})\circ curl\ \mathbf{v} + \upsilon\{curl\ \mathbf{v}.(curl\ curl\ \mathbf{v})\}\}\Omega_4. \quad (3.117)$$

From this expression it is apparent that even in the limit of zero viscosity (high Reynolds number), it is still possible to produce torsion defects when the pressure gradient and divergence terms are not zero, and are not orthogonal to the the vorticity, ω. Moreover, if the vorticity field is integrable in the sense of Frobenius, then viscosity does NOT contribute to the creation of torsion defects. The integral of K over {x,y,z,t} gives the Euler Index of the flow.

3.5 Mechanics

3.5.1 Physical Systems on 4 dimensions again

The 4 dimensional variety will be expressed in terms of the 4 conjugate pairs of independent variables $\{p, m, q, t\}$. The 1-form of Action representing the physical system will be written in terms of the Cartan-Hilbert format (or Darboux representation) as

$$A = pdq - \Phi(p, m, q, t)dt \quad (3.118)$$

The Pfaff sequence has the terms:

$$\begin{aligned} &\text{Topological Action} & A &= pdq - \Phi(p, m, q, t)dt, & (3.119)\\ &\text{Topological Vorticity} & dA &= dp\hat{}dq - d\Phi\hat{}dt & (3.120)\\ &\text{Topological Torsion} & A\hat{}dA &= (pd\Phi - \Phi dp)\hat{}dq\hat{}dt & (3.121)\\ &\text{Topological Parity} & dA\hat{}dA &= 2dp\hat{}d\Phi\hat{}dq\hat{}dt. & (3.122) \end{aligned}$$

The 3-form of topological torsion has the expression

$$A\hat{}dA = p\partial\Phi/\partial m\ dm\hat{}dq\hat{}dt + (p\partial\Phi/\partial p - \Phi)dp\hat{}dq\hat{}dt \quad (3.123)$$

such that the Topological Torsion vector, relative to the volume element $\Omega_4 = dp\hat{}dm\hat{}dq\hat{}dt$ has the components

$$\mathbf{T}_4 = [p\partial\Phi/\partial m, -(p\partial\Phi/\partial p - \Phi), 0, 0], \tag{3.124}$$

from which it is obvious that $i(\rho\mathbf{T}_4)A = 0$. The torsion vector is important, for evolution in the direction of the torsion vector is thermodynamically irreversible, if $\partial\Phi/\partial m \neq 0$.

Relative to processes defined by the Topological Torsion vector, the work 1-form becomes

$$W = i(\rho\mathbf{T}_4)dA = (\rho\partial\Phi/\partial m)\ (pdq - \Phi dt) = \sigma A \tag{3.125}$$

Direct evaluation of the Topological Parity 4-form yields

$$dA\hat{}dA = 2dp\hat{}d\Phi\hat{}dq\hat{}dt = 2\{\partial\Phi/\partial m\ \}dp\hat{}dm\hat{}dq\hat{}dt, \tag{3.126}$$

as expected.

There are two choices for reducing the Pfaff topological dimension. Assuming that $dp\hat{}dq\hat{}dt \neq 0$, either

1. The "Hamiltonian" function Φ = constant, $d\Phi = 0$, defining a family of hypersurfaces parameterized by m (on this surface, the mass must be constant, $dm = 0$),

or

2. The "Hamiltonian" function Φ is independent of m: $\partial\Phi(p, m, q, t)/\partial m \Rightarrow 0$, and $dm \neq 0$.

In the case $\partial\Phi/\partial m \Rightarrow 0$, the Topological Torsion vector does not vanish, but has components

$$\rho\widehat{\mathbf{T}}_4 = \rho[0, (\Phi - p\partial\Phi/\partial p), 0, 0]. \tag{3.127}$$

This expression vanishes if the "Hamiltonian" function, Φ, is homogeneous of degree 1 in p. Other wise, the constrained process, $\widehat{\mathbf{T}}_4$ in 4D is an eigenvector of dA with a zero eigenvalue; e.g., a null eigenvector. The standard (Hamiltonian) null eigen vector field is given by the expression,

$$\rho\mathbf{V}_{4\_extremal} = \rho[-\partial\Phi/\partial q, 0, \partial\Phi/\partial p, 1]. \tag{3.128}$$

The two null eigen vectors are orthogonal. It is to be expected in 4D that the 2-form, dA, if degenerate will have two null eigen vectors. One null eigen vector ($\mathbf{V}_{4\_extremal}$) will be confined to the 3-dimensional subspace,$\{p, q, t\}$ of constant m, while the other null eigen vector, $\widehat{\mathbf{T}}_4$, will be orthogonal to this subspace. These results are a different way of stating the presentation given in section 3.2.

3.5.2 Helmholtz Processes with invariant connectivity

The Helmholtz class of processes 2.59 can be split into two types:

H_A. Those processes for which the connectivity of the domain of support for the 1-form A is invariant.

$$\text{Helmholtz type A}: L_{(\rho \mathbf{V})} \int_{z1} A \Rightarrow 0, \text{ any } \rho \neq 0, \int_{z1} W = \int_{z1} Q = 0. \tag{3.129}$$

H_B. Those processes for which the connectivity of the domain of support for the 1-form A can change (the number of holes and handles can change),

$$\text{Helmholtz type B}: L_{(\rho \mathbf{V})} \int_{z1} A \neq 0, \text{ any } \rho \neq 0, \int_{z1} W = \int_{z1} Q \neq 0. \tag{3.130}$$

Cartan proved [Cartan 1958 (1922)] that if the 1-form of Action is taken to be of the classic "Hamiltonian" format,

$$A = p_k dq^k - H(p_k, q^k, t)dt \tag{3.131}$$

on a 2n+1 dimensional domain of variables $\{p_k, q^k, t\}$, there exists a *unique* extremal vector field, ρV_E, that satisfies the conditions of Type A:

Cartan's Constraint for Extremal Hamiltonian processes

$$i(\rho V_E)dA = 0. \tag{3.132}$$

The dynamical system so generated by Cartan's constraint is said to be Hamiltonian, for the components of ρV_E are determined by the partial derivatives of the Hamiltonian function, $H(p_k, q^k, t)$:

$$dq^k - \partial H/\partial p_k dt = 0, \tag{3.133}$$
$$dp_k + \partial H/\partial q^k dt = 0. \tag{3.134}$$

The proof is quite straightforward, and follows by evaluating the Cartan constraint for $V_E = [f_k, v^k, 1]$,

$$i(\rho V_E)dA = i(\rho V_E)\{dp_k\hat{}dq^k - dH\hat{}dt = i(\rho V_E)\{dp_k\hat{}dq^k - (\partial H/\partial p_k dp_k + \partial H/\partial q^k dq^k)\hat{}dt\} \tag{3.135}$$
$$= \rho\{f_k dq^k - v^k dp_k - (\partial H/\partial p_k dp_k + \partial H/\partial q^k dq^k) + (\partial H/\partial p_k f_k + \partial H/\partial q^k v^k)dt\} \tag{3.136}$$

The solution to the Cartan constraint equation requires that the components of $\mathbf{V}_E = [f_k, v^k, 1]$ relative to the coordinates $\{p, q, t\}$ are given by the equations, $[-\partial H/\partial q, \partial H/\partial p, 1]$. The dynamical system is defined by the equations

$$\frac{dp}{-\partial H/\partial q}=\frac{dq}{\partial H/\partial p}=\frac{dt}{1}. \quad (3.137)$$

By direct computation the Pfaff dimension of the Cartan Hamiltonian Action 1-form is 2n+1 and the contact manifold volume element is given by the expression

$$A\hat{}(dA)^n = \{p\partial H/\partial p - H(p,q,t)\}dq^1\hat{}...dq^n\hat{}dp_1\hat{}...dp^n\hat{}dt \quad (3.138)$$

$$dA\hat{}(dA)^n = 0. \quad (3.139)$$

The function $\rho_L(q,p,t)=\{p\partial H/\partial p-H(p,q,t)\}$ defines a Lagrange density on the 2n+1 state space. Its zero set reduces the Pfaff dimension to a 2n dimensional manifold of Phase Space.

If the function L is defined by a Legendre transformation as

$$L(v,q,t)=p_kv^k-H=p\partial H/\partial p-H(p,q,t)\equiv\rho_L(q,p(q,v,t),t) \quad (3.140)$$

then substitution into the Action 1-form leads to the format of the Cartan-Hilbert expression,

$$A=Ldt+p_k(dq^k-\{\partial H(p,q,t)/\partial p_k\}dt)=L(v,q,t)dt+p_k(dq^k-v^kdt). \quad (3.141)$$

The substitutions require that the Hamiltonian is NOT homogeneous of degree 1 in the p_k.

At first glance it would appear that the Cartan-Hamiltonian 1-form of Action (3.131) is equivalent to the primitive Lagrange function integrand of the Calculus of variations, $L(v,q,t)dt$, constrained by the anholonomic constraints, (dq^k-v^kdt), with Lagrange multipliers, p_k, in the format of the Cartan-Hilbert Action (2.130). However, the Cartan-Hilbert Action is of Pfaff topological dimension 2n+2, while the Cartan-Hamiltonian Action is of Pfaff topological dimension 2n+1. Hence, there must be an additional constraint on the Cartan-Hilbert Action to make it equivalent to the Cartan-Hamiltonian Action. That constraint is precisely equal to

$$dS=(n+1)!\{\Sigma_{k=1}^n(\partial L/\partial v^k-p_k)dv^k\}=0. \quad (3.142)$$

It follows that the condition that a Cartan-Hamiltonian Action be equivalent to the Cartan-Hilbert Action is the constraint that the entropy change is zero; $dS=0$. Hence the following theorem has be established:

Conclusion 15 *Hamiltonian extremal processes on contact manifolds of dimension 2n+1 are isentropic.*

3.5.3 Simple Examples of Type A Helmoltz processes

The Action will be written in the format

$$A = pdq - \Phi(p, m, q, t)dt. \tag{3.143}$$

The Harmonic Oscillator (constant mass)

Define in a classical manner the Harmonic Oscillator function as

$$\Phi(p, m, q, t) = p^2/2 + \omega^2 q^2/2 - m^2c^2 \tag{3.144}$$

Note that m_0 is a constant. As $\partial\Phi/\partial m = 0$, the Extremal field is

$$\rho\mathbf{V}_{4\_extremal} = \rho[-\partial\Phi/\partial q, 0, \partial\Phi/\partial p, 1] \tag{3.145}$$
$$= \rho[-\omega^2 q, 0, +p, 1]. \tag{3.146}$$

The work 1-form vanishes for the extremal evolution, but the internal energy becomes

$$U = i(\rho\mathbf{V}_{4\_extremal})A = \rho(p^2/2 - \omega^2 q^2/2 + m^2c^2). \tag{3.147}$$

and the Heat 1-form becomes

$$Q = dU = \rho(pdp - \omega^2 qdq + 2mc^2 dm). \tag{3.148}$$

, but m is not unless the system is closed. The Action 1-form is $\omega = pdq - Hdt$, such that $d\omega = dp\hat{}dq - dH\hat{}dt$ and $\omega\hat{}d\omega = (-Hdp + pdH)\hat{}dq\hat{}dt$. Note that the 4-form $d\omega\hat{}d\omega = 2dp\hat{}dq\hat{}dH\hat{}dt$ under the assumption that $H = H(p, q, t, m)$ =constant..

Then a Hamiltonian process is generated by the vector field

$$\lambda W = \lambda(V, F, 1, y) = \lambda(\partial H/\partial p, -\partial H/\partial q, 1, z) = \lambda(p/m_0, -kq, 1, z) \tag{3.149}$$

The internal energy is equal to

$$U = i(\lambda W)\omega = \lambda(pV - H) = \lambda(p\partial H/\partial p - H) = \lambda(p^2/2m_0 - kq^2/2 - cons\tan t) \tag{3.150}$$

The associated ODE.s are $dq - (p/m_0)dt = 0$, and $dp + kqdt = 0$. The internal energy is not zero unless H is homogeneous of degree 1 in p. If the system is closed $(dm \equiv dH = 0)$, and as $\partial H/\partial t = 0$, the process is adiabatic for constant λ for

$$L_{(\lambda W)}\omega = dU = (pdp/m_0 - kqdq)_{\lambda=1} = (\partial H/\partial p)dp + (\partial H/\partial q)dq = dH - \partial H/\partial t = Q = 0. \tag{3.151}$$

As the closed system $(dm = dH = 0)$ is time independent $(\partial H/\partial t = 0)$, the process is adiabatic. Q is integrable and must be of the form, $Q = TdS$ with $dS = 0$ or $T = 0$. The closed (constant mass) harmonic oscillator is isolated, as the internal energy is a global constant.

The constant mass Damped Harmonic Oscillator

Suppose the Hamiltonian is time dependent and of the form

$$H = (p^2/2m)\exp^{-(b/m)t} + (kq^2/2)\exp^{+(b/m)t}. \quad (3.152)$$

Then a Hamiltonian solution for a closed system ($dm = dH = 0$) is given by the equations

$$\begin{aligned} F_u &= -\partial H/\partial q = -kq\exp^{+(b/m)t}. \\ V^u &= +\partial H/\partial p = +(p/m)\exp^{-(b/m)t} \end{aligned} \quad (3.153)$$

subject to the closure condition

$$d[PV/2 + Fq/2] = 0 \quad (3.154)$$

The equations of motion are

$$\begin{aligned} dp/dt &= F_u = -\partial H/\partial q = -kx\exp^{+(b/m)t} \\ dq/dt &= V^u = +\partial H/\partial p = +(p/m)\exp^{-(b/m)t} \end{aligned} \quad (3.155)$$

Use the second equation to solve for $p = mV\exp^{+(b/m)t}$, and then substitute into the first equation,

$$dp/dt = d(mV\exp^{+(b/m)t})/dt = \{md^2q/dt^2 + bdq/dt\}\exp^{+(b/m)t} = -kq\exp^{(b/m)t}$$

which, upon re-arrangement and dropping the exponential factor, becomes the classic kinematic equation for the damped harmonic oscillator:

$$md^2q/dt^2 + bdq/dt + kq = 0 \quad (3.156)$$

It is important to note that the canonical momentum p is not equal to the classic representation mV, but is explicitly dependent upon time

$$p = (mV\exp^{+(b/m)t}) \quad (3.157)$$

The Internal Energy becomes (to within a constant)

$$U = pV - (p^2/2m\exp^{-(b/m)t} + kq^2/2\exp^{+(b/m)t}) \quad (3.158)$$
$$= (p^2/2m)\exp^{(-b/m)t} - (kq^2/2)\exp^{+(b/m)t} \quad (3.159)$$

such that the evolutionary equation becomes

$$L(\lambda\mathbf{W})\omega = Q = dU = [Vdp + Fdq] - dH = (b/m)[pdq - Hdt] = (b/m)\omega. \quad (3.160)$$

In this case, even under the assumption that the system is closed (constant mass), the system is not isolated, as the internal energy is not an evolutionary invariant. Neither is the system adiabatic, for $Q = (b/m)\omega \neq 0$ by construction.

However, the action and the internal energy are conformal invariants of the evolutionary process:

$$L(\lambda\mathbf{W})\omega{=}(b/m)\omega. \quad L(\lambda\mathbf{W})U{=}(b/m)U. \quad (3.161)$$

To check for reversibility, compute

$$Q\hat{}dQ = (b/m)^2\omega\hat{}d\omega \quad (3.162)$$

However, Q for a Hamiltonian system is a perfect differential, hence $Q\hat{}dQ = 0$;it follows that the process is reversible. By construction, the damping factor is not zero; therefore the 3-form of topological torsion must vanish: $\omega\hat{}d\omega = 0$. As the system is reversible, $Q = TdS$. As the process is a Helmholtz process, $dQ = dT\hat{}dS = 0$. It follows that the process is either isothermal ($dT = 0$),or the temperature T is a function of the entropy. The thermodynamic analogue is an ideal gas at constant volume.

It is remarkable that if the original action $\omega \Rightarrow \overline{\omega} = \exp^{-bt/m}\omega$, then for the vector field $\lambda\mathbf{W} = \{V, F, 1\}$ given above it follows that $L(\lambda\mathbf{W})\overline{\omega} = Q = 0$. The result is that the process acting on the rescaled action is adiabatic! The idea that a rescaling or a renormalization of the action changes a conformal process into an adiabatic process is a general result (Kiehn, J.Math Phys 1974). That is, if

$$L(\lambda\mathbf{W})\mathcal{A} = \Gamma\mathcal{A} \quad (3.163)$$

then

$$L(\lambda\mathbf{W})(\beta\mathcal{A}) = 0 \quad (3.164)$$

where β is a solution of

$$L(\lambda\mathbf{W})(\ln\beta) = \Gamma. \quad (3.165)$$

Rayleigh dissipation

Suppose the Hamiltonian is time independent and of the form

$$H = (p^2/2m)\exp^{-\gamma q/m} + (kq^2/2)\exp^{+\gamma q/m}. \quad (3.166)$$

Then a Hamiltonian solution is of the form

$$\begin{aligned} F_u &= -\partial H/\partial q = [+\gamma p^2/2m^2]\exp^{-\gamma q/m} + [-kq - (k\gamma/2m)q^2]\exp^{+\gamma q/m} \quad (3.167) \\ V^u &= +\partial H/\partial p = +(p/m)\exp^{-\gamma q/m} \end{aligned}$$

The equations of motion are

$$dp/dt = F_u = -\partial H/\partial q = [+\gamma p^2/2m^2]\exp^{-\gamma q/m} + [-kq - (k\gamma/2m)q^2]\exp^{+\gamma q/m} \quad (3.168)$$
$$dq/dt = V^u = +\partial H/\partial p = +(p/m)\exp^{-\gamma q/m}$$

Substituting the solution for the momentum, p, in terms of V leads to the equation

$$dp/dt = d(mV\exp^{+\gamma q/m})/dt = \{md^2q/dt^2 + V\gamma dq/dt\}\exp^{+\gamma q/m} = \quad (3.169)$$
$$[+\gamma(dq/dt)^2/2 - kq - (k\gamma/2m)q^2]\exp^{+\gamma q/m}$$

which becomes the kinematic equation for the quadratically damped non-linear oscillator:

$$md^2q/dt^2 + (\gamma/2)(dq/dt)^2 + kq(1 + (\gamma/2m)q) = 0 \quad (3.170)$$

As $W = i(\lambda\mathbf{W})d\omega = 0$, and as $U = (p^2/2m)\exp^{(-\gamma/m)q} - (kq^2/2)\exp^{+(\gamma/m)q}$, this system

$$dU = p/m\exp^{(-\gamma/m)q}dp - (-\gamma/m)(p^2/2m)\exp^{(-\gamma/m)q}dq - (kq + (k\gamma/2m)q^2)\exp^{+(\gamma/m)q}dq$$

The Eulerian Fluid

The application need not be constrained to state space. Consider the action

$$\mathcal{A} = u_k dx^k - (V^k u_k/2 + P/\rho(x^k))dt \quad (3.171)$$

on the space (x^k, u_k, t) and the evolutionary vector field $\lambda\mathbf{W} = \lambda(V^k, F_k, 1)$. The Hamiltonian function is taken to be $H = -(V^k u_k/2 + P/\rho(x^k))$ and the momenta are the u_k. Then forcing the 1-form of work to be zero in order to create an extremal vector field leads to the equations

$$\partial\mathbf{u}/\partial t + grad(\mathbf{V}\cdot\mathbf{u})/2 + \mathbf{V}\ x\ curl(\mathbf{u}) + grad(P/\rho) = 0 \quad (3.172)$$

and the auxiliary condition,

$$(\partial\mathbf{u}/\partial t + grad(\mathbf{V}\cdot\mathbf{u})/2)\cdot\mathbf{V} = 0 \quad (3.173)$$

By subsuming a euclidean identification between contra and co vectors, $\mathbf{V}(\mathbf{x},t) = \mathbf{u}(\mathbf{x},t)$, the Hamiltonian condition, $i(\lambda\mathbf{W})d\omega = 0$, becomes the equation of motion for a barotropic Eulerian fluid:

$$\partial\mathbf{u}/\partial t + grad(\mathbf{u}\cdot\mathbf{u})/2 + \mathbf{u}\ x\ curl(\mathbf{u}) + grad(P/\rho) = 0. \quad (3.174)$$

The existence of a "Hamiltonian" function, $H(q,p,t)$, does not imply that the process $\lambda\mathbf{W}$ has a Hamiltonian representation, $V^u = \partial H/\partial p_u$, *3.5.4 Helmholtz processes with variable connectivity.*

Helmholtz type A processes imply that the 1-dimensional period integrals (first Betti numbers) are NOT necessarily evolutionary invariants of the Helmholtz class of processes. Such Type A processes then must represent topological evolution and are NOT homeomorphisms of the Cartan topology. In the sense that the first Betti number is related to the "hole" count in some surface, and as the process is assumed to be continuous, the interior "holes" can disappear by collapse and pasting, but new holes can be created only by distortion and entrapment at a piece of a boundary.

However, as $Q\hat{}dQ = 0$, Helmholtz processes of both type A and type B are thermodynamically reversible, even though the type A processes involve topological evolution. In otherwords, topological change is necessary, but not sufficient for thermodynamic irreversibility.

For C2 functions, the vector fields of the Helmholtz class Part A are constrained by the equations of closure,

$$d[i(\mathbf{W})dA] = dW = 0. \tag{3.175}$$

Although, the Helmoltz process indicates that the Work 1-form is closed, it need not be exact. Closed integrals of the Work, where the integration path is a cycle, need not be zero: $\oint W \neq 0$. According to deRham, the closure condition can be satisfied by exact or harmonic contributions. That is

$$W \equiv dP + \gamma = dP + \Gamma\ [(\psi^* d\psi - \psi d\psi^*)/(\psi^*\psi)] \tag{3.176}$$

The first term dP is the analogue of "pressure gradient" in a hydrodynamic system and gives no contribution to the line integral around a cycle, or a boundary. The second term is a harmonic component that yields integer values for integrations around cycles that enclose the zeros (the holes) of the denominator. The creation or the destruction of the harmonic components represents a change of topology. As the evolution is presumed to be continuous, the production of "holes" can occur only on segments of cycles or boundaries and not in the "bulk" interior (Interior production would be a cutting process which is discontinuous). As dA is an evolutionary invariant for Helmoltz processes, the "holes" must be produced in equal and opposite pairs.

The Master Equation

Consider the 1-form of electromagnetic action, A,given above (3.59). In electromagnetic format, for all processes, it follows that

$$curl\ \mathbf{E} + \partial\mathbf{B}/\partial t = 0,\ div\ \mathbf{B} = 0 \tag{3.177}$$

Consider an abstract process $\mathbf{V}_4 \Rightarrow \rho\{V^k, 1\} = \rho\{\mathbf{V}, 1\}$on the space $\{x^k, t\}$. Then for type A Helmholtz processes,

$$W = i(\mathbf{V})dA = \rho(\mathbf{E} + \mathbf{V} \times \mathbf{B})_k dx^k - \rho(\mathbf{V} \circ \mathbf{E})dt \neq 0 \tag{3.178}$$

The covariant spatial components of the Work 1-form are to recognized as the Lorentz force per unit charge to within the parametrization factor, ρ),

$$\mathbf{f}_{(Lorentz)} = \rho(\mathbf{E} + \mathbf{V} \times \mathbf{B}), \tag{3.179}$$

and the time-like component becomes the "local dissipative" power, $P = \rho(\mathbf{V} \circ \mathbf{E})$. The Lorentz force has been derived on topological grounds, and has not been injected into the the theory.

The Helmholtz closure conditions $d[i(\mathbf{W})d\omega] = 0$ require that

$$curl\ \mathbf{f}_{(Lorentz)} = 0 \tag{3.180}$$

and

$$\partial \mathbf{f}_{(Lorentz)}/\partial t + grad\, P = 0. \tag{3.181}$$

For simplicity, let $\rho = 1$,such that the closure conditions of Type A Helmhotz processes becomes the set of constraints which define the Master Equation of a "perfect plasma":

$$\begin{aligned} curl\ \mathbf{f}_{(Lorentz)} &= curl(\mathbf{E}) + curl(\mathbf{V} \times \mathbf{B}) = 0 & (3.182) \\ \text{Master Equation} &\Rightarrow -\partial \mathbf{B}/\partial t + curl(\mathbf{V} \times \mathbf{B}) = 0. & (3.183) \end{aligned}$$

The same results and conclusions apply to Hydrodynamics as well as all physical systems that can be encoded in terms of 1-form of Action with coefficient functions defined over a 4 dimensional variety. The statement demonstrates the concept of topological universality.

Envelopes

It may be that the fundamental constraint $\Theta = 0$, when considered as a family of hyper-surfaces parametrized by the mass, may exhibit non-uniqueness and envelope phenomena. The requirement for the existence of an envelope is that the two surfaces $\Theta = 0$ and $\partial\Theta/\partial m = 0$ have an intersection. This statement can be written in terms of the inexact 1-form $\varepsilon = d\Theta - \partial(d\Theta)/\partial m$. Then envelope exists when the three form generated by this 1-form does not vanish: $\varepsilon\hat{}d\varepsilon \neq 0$. It follows that the 1-form ε is not integrable in the sense of Frobenius, and cannot be defined globally in terms of an integrating factor and the differential of a unique function. If the 1-form defined as $\varepsilon = d\Theta - \partial(d\Theta)/\partial m) - \partial^2(d\Theta)/\partial m^2$ generates a non-zero three form, $\varepsilon\hat{}d\varepsilon \neq 0$, then the envelope condition indicates that the surfaces are in tangential contact but the line of intersection is an "edge of regression".

3.5.5 Irreversible Processes $Q\hat{}dQ \neq 0$

The Isovector (Conformal or Similarity Projective) Class

$$L(\lambda\mathbf{W})\omega = \Gamma\omega = Q \tag{3.184}$$

from which it follows that

$$\begin{aligned} L(\lambda\mathbf{W})d\omega &= d\Gamma\hat{}\omega + \Gamma d\omega = dQ \\ Q\hat{}dQ &= \Gamma^2\omega\hat{}d\omega \neq 0 \end{aligned} \tag{3.185}$$

The Isovector class of processes can be irreversible, but irreversibility requires that both the three form of topological torsion $\omega\hat{}d\omega$ does not vanish, and the Internal Energy function U must depend upon the "time" parameter, τ, explicitly such that $\Gamma = -(\partial U/\partial\tau)/mc^2 \neq 0$. If the internal energy function is independent from τ such that $\Gamma = 0$,and if the three form of topological torsion $\omega\hat{}d\omega$ is non-zero, the solution trajectories can be chaotic, but the process is reversible.

Helmholtz Isovectors

Isovector processes are continuous, but need not be uniformly continuous. If an isovector process is uniformly continuous and Γ is not zero, then $dQ = 0$ implies that the action 3-form of topological torsion must vanish, $\omega\hat{}d\omega = 0$. The implication is that $d\omega = -d(\ln\Gamma)\hat{}\omega$ or that $(\ln\Gamma)$ is the integrating factor for the action, ω.This case is a special case of a Helmholtz isovector.

Bernoulli Isovectors $\Leftrightarrow$ *characteristics*

$$L(\lambda\mathbf{W})\omega = \Gamma\omega \quad \mathit{and} \quad L(\lambda\mathbf{W})U = \Gamma U = 0 \tag{3.186}$$

Then either U is zero or $\Gamma = 0$. If $\Gamma = 0$ the evolutionary field is not an isovector. If the internal energy U vanishes, then the Bernoulli isovector is a characteristic vector.

Suppose that $\delta x^k = dx^k - u^k d\tau$ are independent such that δx^k and ω form a basis on $\{x^k, \tau\}$. Then the current three form $J = \delta x\hat{}\delta y\hat{}\delta z = i(\lambda\mathbf{W})dx\hat{}dy\hat{}dz\hat{}d\tau/(\lambda) - (\Theta/\lambda - 1)dx\hat{}dy\hat{}dz$. Note that the basis is formed when

$$\delta x\hat{}\delta y\hat{}\delta z\hat{}\omega = (\beta mc^2 - u^k\pi_k)dx\hat{}dy\hat{}dz\hat{}d\tau = (-U/\lambda)dx\hat{}dy\hat{}dz\hat{}d\tau \neq 0 \tag{3.187}$$

This four form volume element vanishes only when the internal energy U vanishes. Hence, if the internal energy U is not zero, the coordinates $(dx\hat{}dy\hat{}dz\hat{}d\tau)$ are independent. If the $\pi_k = mu_k$ then the internal energy becomes equal $U = -mc^2[1 - (u^k u_k/c^2)]$.

Using the relativistic concepts it would appear that the internal energy is the negative of the rest mass energy in state of rest, and becomes less negative approaching zero as the speed v approaches the critical speed c.

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3.5.6 Isovectors

Consider the equivalence class of vectors $\lambda\mathbf{W}$ that are isovectors relative to ω. Isovectors satisfy the constraint:

$$L(\lambda\mathbf{W})\omega = \Gamma\omega \equiv Q \tag{3.188}$$

Isovectors are of interest for they form a maximal Lie subalgebra (Edelen)

The isovector condition becomes

$$\begin{aligned} L(\lambda\mathbf{W})\omega &= \lambda(f_k dx^k - u^k d\pi_k) + dU = \Gamma(\pi_k dx^k - mc^2 d\tau) \\ dU &= (\partial U/\partial x^k)dx^k + (\partial U/\partial \pi_k)d\pi_k + (\partial U/\partial\tau)d\tau \end{aligned} \tag{3.189}$$

which requires that the 2n+1 components of the isovector satisfy the first order partial differential equations

$$\lambda f_k = \Gamma\pi_k - \partial U/\partial x^k \tag{3.190}$$

$$\lambda u^k = \partial U/\partial \pi_k \tag{3.191}$$

and

$$\lambda mc^2 = \pi_k \partial U/\partial \pi_k - U \tag{3.192}$$

with

$$mc^2\Gamma = -\partial U/\partial\tau \tag{3.193}$$

The auxiliary ordinary differential system is (compare to Forsyth vol.1 p182)

$$\frac{\lambda dx^k}{(\partial U/\partial \pi_k)} = \frac{\lambda d\pi_k}{\Gamma\pi_k - \partial U/\partial x^k} = \frac{mc^2\lambda d\tau}{\pi_k \partial U/\partial\pi_k - U} \equiv \frac{d\tau}{1}$$

It is apparent that isovector processes are completely determined from a single function

$$U(x^k, \pi_{k,}\, \tau) \tag{3.194}$$

including the "time-like" component $\lambda\beta$, which is proportional to the Legendre transformation of U relative to the π_k.

$$\lambda\beta mc^2 = \pi_k \partial U/\partial\pi_k - U \tag{3.195}$$

Note that $\lambda\beta$ vanishes if U is homogeneous of degree 1 in the π_k; the problem is reduced to 2n dimensions from 2n+1. (Compare to Forsyth Vol 1 p.182)

A special subclass of processes belong to the characteristic class. In addition to satisfying the isovector condition, characteristic vectors satisfy the "associated" equation

$$U(\pi_k, q^k, \tau) = i(\lambda\mathbf{W})\omega = \lambda(u^k\pi_k - mc^2) = 0. \tag{3.196}$$

It would appear that relativistic prejudices would preclude the existence of $U \Rightarrow 0$.

———

An extremal isovector is one that satisfies the equation $W = i(\lambda\mathbf{W})d\omega = 0$,in addition to the isovector constraint: It follows that for extremal isovectors,

$$\lambda(f_k dx^k - u^k d\pi_k) = \Gamma\pi_k dx^k - (\partial U/\partial x^k)dx^k - (\partial U/\partial \pi_k)d\pi_k = 0 \quad (3.197)$$

$$\begin{aligned} \Gamma\pi_k dx^k + (\partial U/\partial\tau)d\tau &= \Gamma(\pi_k dx^k - mc^2 d\tau) \\ &= \Gamma\omega = dU \; an \; extremal \; isovector \end{aligned} \quad (3.198)$$

In otherwords, for an extremal isovector, the dissipation factor (projective factor) Γ is an integrating factor for the original 1-form. An extremal field always satisfies the Helmholtz theorem (but not visa versa). The Action 1-form is therefor of Pfaff dimension 2 if the process is an extremal isovector, hence although such processes have a time dependent Internal Energy ($\Gamma = (-\partial U/\partial\tau)/mc^2 \neq 0$ by assumption) they are never chaotic (as $\omega\hat{}d\omega = 0$) and never irreversible (as $Q\hat{}dQ = 0$).

Further note that if U *is* not an explicit function of τ then the conformal factor must vanish; $\Gamma = 0$, .if $\lambda\mathbf{W}$ is to be an isovector.

Consider the isovector evolution of $d\omega$:

$$L(\lambda\mathbf{W})d\omega = d(\Gamma\omega) = d\Gamma\hat{}\omega + \Gamma\hat{}d\omega \quad (3.199)$$

If the "vorticity" $d\omega$ is an evolutionary invariant such that $L(\lambda\mathbf{W})d\omega = 0$, then either $\Gamma = 0$ or the torsion vanishes, $\omega\hat{}d\omega = 0$. However, $mc^2\Gamma$ depends upon $-\partial U/\partial\tau$. Hence if there is dissipation in the sense that $\Gamma \neq 0$, and the evolutionary process is described by an isovector, then the torsion vanishes, and the system is reversible!!??

$$L(\lambda\mathbf{W})\omega\hat{}d\omega = 2\Gamma\omega\hat{}d\omega \quad (3.200)$$

If the 3-form is closed

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Consider a map from $M : \{\xi^m\} \Rightarrow N : \{x^k, p_{k,} \; t, \; H\} \; where \; 1 \leq k \leq n$

On the space N (2n+2) consider a 1-form in the Darboux representation:

$$\omega = p_k dx^k - Hdt \quad (3.201)$$

with

$$d\omega = dp_k\hat{}dx^k - dH\hat{}dt. \quad (3.202)$$

Consider evolutionary processes that are described by the contravector $\lambda\mathbf{W} = \lambda\{V^k, F_k, \Theta, \beta\}$ relative to the coordinates $\{x^k, p_k, t, H\}$. Then define

$$U = i(\lambda\mathbf{W})\omega = \lambda(V^k p_k - H\Theta) \equiv definition\,of\,U$$

Consider the equivalence class of vectors $\lambda\mathbf{W}$ that are isovectors relative to ω. Isovectors satisfy the constraint:

$$L(\lambda\mathbf{W})\omega = \Gamma\omega \equiv Q \tag{3.203}$$

Isovectors are of interest for they form a maximal Lie subalgebra (Edelen)

The isovector condition becomes

$$\begin{aligned} L(\lambda\mathbf{W})\omega &= \lambda(a_k dx^k - u^k d\pi_k) + \lambda(\beta d\tau - \varepsilon dh) + dU = \Gamma(\pi_k dx^k - hd\tau) \\ dU &= (\partial U/\partial x^k)dx^k + (\partial U/\partial \pi_k)d\pi_k + (\partial U/\partial \tau)d\tau + (\partial U/\partial h)dh \end{aligned} \tag{3.204}$$

which requires that the 2n+1 components of the isovector satisfy the first order partial differential equations

$$\lambda a_k = \Gamma\pi_k - \partial U/\partial x^k \tag{3.205}$$

$$\lambda u^k = \partial U/\partial \pi_k \tag{3.206}$$

and

$$\lambda\varepsilon h = \pi_k \partial U/\partial \pi_k - U \tag{3.207}$$

with

$$h\Gamma = -\partial U/\partial \tau + \lambda\beta \tag{3.208}$$

$$-\lambda\varepsilon = (\partial U/\partial h) \tag{3.209}$$

The auxiliary ordinary differential system is (compare to Forsyth vol.1 p182)

$$\frac{\lambda dx^k}{(\partial U/\partial \pi_k)} = \frac{\lambda d\pi_k}{\Gamma\pi_k - \partial U/\partial x^k} = \frac{\lambda d\tau}{\pi_k \partial U/\partial \pi_k - U} \tag{3.210}$$

It is apparent that isovector processes are completely determined from a single function

$$U(x^k, \pi_{k,}\ \tau) \tag{3.211}$$

including the "time-like" component $\lambda\varepsilon$, which is the Legendre transformation of U relative to the π_k.

$$\lambda\varepsilon = \pi_k \partial L/\partial \pi_k - U \tag{3.212}$$

Note that $\lambda\varepsilon$ vanishes if U is homogeneous of degree 1 in the π_k; the problem is reduced to 2n dimensions from 2n+1. (Compare to Forsyth Vol 1 p.182)

A special subclass of processes belong to the characteristic class. In addition to satisfying the isovector condition, characteristic vectors satisfy the "associated" equation

$$U = i(\lambda\mathbf{W})\omega = \lambda(u^k\pi_k - \varepsilon) = 0. \tag{3.213}$$

An extremal isovector is one that satisfies the equation $W = i(\lambda\mathbf{W})d\omega = 0$,in addition to the isovector constraint: It follows that for extremal isovectors,

$$\lambda(a_k dx^k - u^k d\pi_k) = \Gamma\pi_k dx^k - (\partial U/\partial x^k)dx^k - (\partial U/\partial \pi_k)d\pi_k = 0 \tag{3.214}$$

$$\begin{aligned} \Gamma\pi_k dx^k + (\partial U/\partial\tau)d\tau &= \Gamma(\pi_k dx^k - d\tau) \\ &= \Gamma\omega = dU \text{ an extremal isovector} \end{aligned} \tag{3.215}$$

In otherwords, for an extremal isovector, the dissipation factor (projective factor) Γ is an integrating factor for the original 1-form. An extremal field always satisfies the Helmholtz theorem (but not visa versa). The Action 1-form is of Pfaff dimension 2 if the process is an extremal isovector.

Further note that if U *is* not an explicit function of τ then the conformal factor must vanish; $\Gamma = 0$, .if $\lambda\mathbf{W}$ is to be an isovector.

Consider the isovector evolution of $d\omega$:

$$L(\lambda\mathbf{W})d\omega = d(\Gamma\omega) = d\Gamma\hat{}\omega + \Gamma\hat{}d\omega \tag{3.216}$$

If the "vorticity" $d\omega$ is an evolutionary invariant such that $L(\lambda\mathbf{W})d\omega = 0$, then either $\Gamma = 0$ or the torsion vanishes, $\omega\hat{}d\omega = 0$. However, Γ depends upon $-\partial U/\partial\tau$. Hence if there is dissipation in the sense that $\Gamma \neq 0$, and the evolutionary process is described by an isovector, then the torsion vanishes, and the system is reversible.

$$L(\lambda\mathbf{W})\omega\hat{}d\omega = 2\Gamma\omega\hat{}d\omega \tag{3.217}$$

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3.5.7 Mass in symplectic systems

However, a second possibility exists for the Frobenius integrability constraint to be valid, but now the condition is not valid globally over the 4D domain. However, the 3-form of topological torsion will vanish when

$$[Ldp + p(p - \partial L/\partial v)dv] = [\alpha(t,q,v,p)dq + \beta(t,q,v,p)dt] \;\Rightarrow 0 \quad (\Rightarrow -vLdm.???) \tag{3.218}$$

This constraint can be satisfied on domains even when $(p - \partial L/\partial v) \neq 0$. Hence such systems admit an integrable submanifold in a *symplectic* 4D domain. It is these integrable submanifolds of a symplectic system that are of interest to this article. Consider the case where the RHS of the above equation vanishes. Then the non-zero function $m(t;q,v,p)$ defined in a symplectic domain as

$$m(t;q,v,p) \doteq p(p - \partial L/\partial v)/L = dp/dv \tag{3.219}$$

plays the role of a "mass" (at least for closed systems). (added 02/21/2002:) The mass can be zero, if canonical (which would be the boson Photon or Phonon case? and is contact) or mass non-zero, which is the non-canonical case (which is symplectic).

For constant mass, the resulting singular hypersurface in the 4D domain yields a constrained subspace of $(t; q, v, p)$ upon which the 1-form of Action satisfies the Frobenius complete integrability theorem. Examples will be given below where the symplectic orbits reside on this constant mass surface. The constant mass hypersurface in symplectic systems is the analogue of the constant energy hypersurface in extremal systems.

3.6 Dynamical Systems

3.6.1 ODE's generated by Hamiltonian processes.

A dynamical system may be viewed as a special case of an exterior differential system. That is, let $\mathbf{R}$ be defined as the set of independent variables $\{x, y, z...\}$ and $\mathbf{V}$ as an ordered set of functions. The dynamical system in Pfaffian form is then the system of constraint equations

$$d\mathbf{R} - \mathbf{V}d\tau = 0,$$

where τ is an arbitrary parameter. Geometrically, $\mathbf{R}$ is a position vector to a point in the space of independent variables, and $\mathbf{V}$ is the tangent vector to a curve through the point. This kinematic idea is at the basis of most mechanical theories in physics. The question for physics is how to deduce the functional format of $\mathbf{V}$ to suit a given physical process on a specified physical system, and then to integrate the dynamical system to find the evolutionary trajectories described by $\mathbf{R}(\tau)$

The sophomoric description starts from Newton's law written in the form of another exterior differential system,

$$md\mathbf{V} - \mathbf{F}d\tau = 0.$$

Given the forces, $\mathbf{F}$, acting on a physical system of mass m, solve for $\mathbf{V}$, then with this $\mathbf{V}$ solve for $\mathbf{R}$. Quite often the two exterior differential systems are combined into a second order differential equation. The specification of the physical system by this procedure is reduced to a specification of the mass(es) and the forces. The combined system is said to define the equations of motion.

Another method of deducing dynamical systems is based on the calculus of variations, where Lagrangian techniques are used to find "the equations of motion". These methods have the advantage that they may be written in a coordinate free manner, valid in every reference system, and not constrained, as are Newton's laws, to special "inertial" systems of reference. In addition, it is possible to group variational problems into equivalence classes where the overall behavior of a specific dynamical system can be deformed into other examples with similar properties. Moreover, the variational method gives a philosophical principle for why nature seems to behave the way it does: the principle of least Action. The technique is to find a unique path generated by a dynamical system that minimizes a certain path integral. The classic result is to obtain a dynamical system equivalent to Newton's equations, which is subsumed to be imbedded in the kinematic Pfaffian system.

In this article, two classes of unique (to within a factor) vector fields will be studied. The first unique vector field will be associated with systems of odd Pfaff dimension (2n+1) and is defined as the Extremal class. The second class of unique vector fields will be associated with systems of even Pfaff dimension (2n+2), and is defined as the Torsion class. Extremal vector fields are thermodynamically reversible, Torsion vector fields are thermodynamically irreversible.

3.6.2 Variational Integrands and Pfaffian forms

The classical problem in the calculus of variations as applied in physics is to evaluate the integral $\int L(\mathbf{R}, \mathbf{V}; t)dt$ along certain paths, and then, subject to certain conditions of constraint, find those special paths among all possibilities for which the value of the integral is extremal. Note that the variational integrand is itself a primitive differential 1-form, the 1-form of Action:

$$A_0 = L(x^k, v^k; t)dt.$$

It is subsumed that a specification of the physical system is through the Lagrange function, $L = L(\mathbf{R}, \mathbf{V}; t)$, and certain conditions of constraint. Various recipes (such as Hamilton's principle where the Lagrange function is defined in terms of the difference between the kinetic and potential energy, L=T-V) for writing the functional form of L have been found to be useful for comparison to experiment. After the extremal path(s) is found, it is tacitly subsumed that the extremal path is embedded in the Pfaffian system (of constrained 1-forms),

$$d\mathbf{R} - \mathbf{V}dt = 0.$$

In principle, the kinematic hypothesis stated above should be treated as a system of (possibly anholonomic constraints) on the variational integrand. This process will be carried out by adding the elements of the Pfaffian system to the integrand of the primitive Action, along with appropriate Lagrange multipliers, p_k.

$$A_0 \Rightarrow A = A_0 dt + p_k(dx^k - v^k dt).$$

The variational integral now becomes a line integral over a 1-form of Action, where the Action consists of two parts, one of which specifies conditions of kinematic constraint.. Other non-holonomic constraints, F^σ, may be introduced by adding terms of the form $s_\sigma F^\sigma(x^k, v^k; t, dx^k; dt)$ with their appropriate Lagrange multipliers, s_σ. In certain treatments, the functional constraints that involve functions of both the coordinates and velocities are stated to be anholonomic. However, that is not the definition used herein. A non-holonomic (anholonomic) constraint is differential constraint that is represented by a Pfaffian 1-form, ω, that is not integrable in the sense of Frobenius; e.g., $\omega\hat{}d\omega \neq 0$. Henceforth, in this article it is assumed that the primitive variational method has been extended to finding minimizing paths for integrals whose Action integrands are arbitrary 1-forms.

The variational procedures considered herein will utilize Cartan's magic formula:

$$L_{(\rho\mathbf{V})}\int_a^b A = \int_a^b i(\rho\mathbf{V})dA + \int_a^b d(i(\rho\mathbf{V})A)$$

The integration path is presumed to be defined by the dynamical system generated by $\rho\mathbf{V}$, where ρ is an arbitrary function that scales the envelope of tangent paths generated by the vector field, $\mathbf{V}$. A stationary path is defined as a path generated by $\mathbf{V}$ such that the Lie differential of the Action vanishes.

$$L_{(\rho\mathbf{V})}\int_a^b A \Rightarrow 0 \quad \supset \mathbf{V} \;\; is\; a\, stationary\;\; path$$

As the same path is determined by the dynamical system for any scaling function, ρ, then most stationary paths must force the two integrals in Cartan's formula to vanish separately. (An exceptional case is defined as where the first integral cancels the second integral.) Integration of the second integral (perfect differential which depends only on the end points) yields the endpoint condition,

$$[i(\rho\mathbf{V})A]_a^b = 0,$$

and leads to the definition of an "extremal" field as that subset of stationary vector fields that satisfy the equation (for any ρ).

$$W = i(\rho\mathbf{V})dA = 0.$$

Vectors for which

$$U = i(\rho\mathbf{V})A = 0$$

along the entire path (not just at the endpoints) are defined as "associated" vectors.

Note that endpoint condition may be satisfied in several distinct ways: Either

1. $i(\mathbf{V})A = 0$ everywhere (paths are constrained to the surface $U = 0$.)
or $[i(\mathbf{V})A]_a = 0$ and $[i(\mathbf{V})A]_b = 0$, on the endpoints only; or
2. $U_a = [i(\rho\mathbf{V})A]_a = U_b = [i(\rho\mathbf{V})A]_b$ independent of the choice of ρ.

The second case is satisfied by paths which leave the surface $U = c$, but return to the surface $U = c$. Note that $U = c$ can have multiple components, or be a family with respect to some parameter (wavefronts).

The exceptional case for a stationary integral would be defined when

$$\rho i(\mathbf{V})dA = -\{[d\rho(i(\mathbf{V})A)\}. \tag{SNU}$$

This exceptional criteria would imply that the virtual work 1-form, defined as $W = i(\mathbf{V})dA$, must satisfy the conditions of Frobenius integrability, $W\hat{}dW = 0$. Then the density functions, ρ, are integrating factors for the virtual work in the exceptional case. A subset of these exceptional cases would be given by the equivalence class of fields $\mathbf{V}$ such that $dW = 0$. Such vector fields are defined as symplectomorphisms, of which the extremal class is a special subset. Symplectomorphisms are thermodynamically reversible.

The Pfaff Sequence

Recall that any 1-form of Action, A, constructed on C1 functions, admits of a Pfaff Sequence: $\{A, dA, A\hat{}dA, dA\hat{}dA, ...\}$. The Pfaff dimension or class of the 1-form, $Pfd(A)$, is the integer equal to the number of non-zero terms in the Pfaff sequence. (See [Liebermann 1987] p. 285) The p-form that forms the largest non-zero element of the Pfaff sequence is defined as the Top Pfaffian. In effect, the Pfaff dimension defines the minimum number of functions required to (locally) define the 1-form (Darboux's theorem). As written above, the Pfaff dimension of the primitive Action, $A_0 = Ldt$, is at most two: $Pfd(A_0) = 2$. That is $dA_0 = dL\hat{}dt \neq 0$, but $A_0\hat{}dA_0 = 0$. The utility of the Pfaff dimension is that it may be use to form equivalence classes of problems involving dynamical systems.

In that which follows, given a variational integrand, its Pfaff dimension will be determined. The Top Pfaffian will define a symplectic (even dimensional) or a contact (odd dimensional) manifold almost everywhere. However, there may exist "singular" sets of points upon which the Top Pfaffian vanishes, thereby producing a manifold of lower Pfaff dimension. This point set, which is not only diffeomorphically invariant but is also invariant with respect to non-invertible C1 maps, may also yield (constrained) paths which are extremal. The top Pfaffian of the "reduced" system may exhibit singular points, upon which the Pfaff dimension is further reduced, and further special extremal paths may be generated. The process sometimes can be continued, until there are no more extremal paths.

The primitive variational integrands (of Pfaff dimension 2 or less) form an equivalence class, but they are applicable to only the most simple of physical problems (those which are completely integrable). Modifications of the Action 1-form are required to create equivalence classes of higher Pfaff dimension. Typical modifications include gauge theories (where locally $A \Rightarrow A_0 + d\Theta(x^k, v^k; t)$), renormalization theories (where locally $A \Rightarrow \rho(x^k, v^k; t)A_0$), and the inclusion of anholonomic constraints by means of Lagrange multipliers,

$$A \Rightarrow A_0 + p_k(dx^k - v^k dt).$$

For example, suppose that on {x,y,z), the 1-form of Action is modified by a gauge addition, $A \Rightarrow A_0 + d\Theta$. ($A_0(x, y, z)$ is presumed to be of Pfaff dimension 2 such that $A_0\hat{}dA_0 = 0$). Then, although $dA = dA_0$ the 3-form $A\hat{}dA = d\Theta\hat{}dA_0 \neq$

$A_0\hat{}dA_0$ need not vanish. The Pfaff dimension of the prolonged 1-form A is equal to $\mathrm{Pfd}(A) = 3$. On the otherhand, if the Pfaff dimension of A_0 is 1, then an arbitrary gauge addition will not change the Pfaff dimension of the original form. Arbitrary gauge additions to a form never change the even dimensional members of the Pfaff sequence, but gauge additions can influence the odd dimensional members of the Pfaff sequence.

$$\{A_0 + d\theta, dA_0, (d\theta + A_0)\hat{}dA_0, dA_0\hat{}dA_0, ...\}.$$

These arguments establish:

Conclusion 16 *1-forms of odd Pfaff dimension are "gauge" invariant, but 1-forms of even Pfaff dimension are not.*

On otherhand the renormalization (contact) transformations imply that 2-form can be modified such that

$$dA_0 \Rightarrow dA = d(\rho A_0) = d\rho\hat{}A_0 + \rho dA_0$$

The Pfaff sequence of A becomes

$$\rho\{A_{0,}d\ln\rho\hat{}A_0 + dA_0, \rho A_0\hat{}dA_0, 2d\ln\rho\hat{}A_0\hat{}dA_0 + \rho(dA_0\hat{}dA_0), \rho A_0\hat{}dA_0\hat{}dA_0, ...\}$$

So if the original Pfaff dimension was odd, then a contact transformation will change the Pfaff dimension by unity, but if the original Pfaff dimension was even, the a contact contribution will not change the Pfaff dimension. The latter invariance in class will be defined as renormalizability.

Conclusion 17 *1-forms of even Pfaff dimension are renormalizable, but 1-forms of odd Pfaff dimension are not.*

The Pfaff dimension of the 1-form of Action constrained by the kinematic equations of a dynamical system,

$$A = A_0 + p_k(dx^k - v^k dt)$$

turns out to be 2N+2, even though at first glance there appears to be 3N+1 independent variables. N is the maximum range of the index k.

Unique Vector Fields

(Extremal vectors) A principle reason for reverting to a variational derivation of a dynamical system is that in certain cases the dynamical system is uniquely determined. In that which follows it will be shown that there are two situations for which an Action 1-form will generate a unique dynamical system. The first situation corresponds to the calculus of variations extremal technique, and the other does not.

First, note that any given 1-form of Action, A, admits several classes of vector fields. There are several sets of (contravariant) vector fields of interest to this article. The first set is defined as the associated set of vector fields such that at any point p, a vector of the associated set satisfies the equations

$$Associated Vectors\,\mathbf{T} \;:\quad i(\mathbf{T})A = 0.$$

The second set of vector fields of interest is the extremal set, such that

$$Extremal\ Vectors\ \mathbf{E} : \quad i(\mathbf{E})dA = 0. \quad (Unique\ on\ \mathrm{Pfd}(A) = odd)$$

For a 1-form of Action which defines a domain of odd Pfaff dimension, the extremal vector field is unique. Vectors that satisfy both equations above are defined as "characteristic" vectors. Note that all of these vectors are defined to within an arbitrary factor (a function).

Consider the integer $\mathrm{Pfd}(A) = N$ as the minimal number of independent variables or functions require to represent the Action A. Then on this domain of n dimensions, the $N-1$ associated vectors form geometrically a "distribution" or a tangent subspace relative to the 1-form, A. The distribution generally does not define a smooth surface globally.

If the integer N is odd such that $N = 2n+1$, then, as mentioned above, the extremal field is uniquely determined (to within a factor) as the null eigen vector of the anti-symmetric matrix of functions that form the coefficients of the 2-form dA. Hence, uniqueness focuses attention on the extremal vector fields for Action 1-forms of odd Pfaff dimension, but such attention is not warranted for Action 1-forms of even Pfaff dimension. It is this extremal field that generates the solution to the classical calculus of variation problems with fixed endpoints.

However, if N is even such that $N = 2n+2$, then extremal vectors (as null eigenvectors) do not exist. The extremal uniqueness argument (based on the calculus of variations) for generating a dynamical system fails. However, it is remarkable that for Action 1-forms of Pfaff dimension 2n+2 (which will have the assumed form, $A = p_k dx^k - hdt$) there exists TWO different unique vector fields. The first unique vector field satisfies the equations

$$i(\mathbf{V})dA = -d\,\{i(\mathbf{V})A\} + \partial\{i(\mathbf{V})A\}/\partial t\,dt = -d(V^k p_k - h) + \partial(V^k p_k - h)/\partial t\,dt. \tag{3.220}$$

The two form, dA, generates a symplectic domain without null eigenvectors, but the existence of the inverse leads to a system of partial differential equations of first order defining the components of the to be determined, $\mathbf{V}$. It will be shown that these vector fields are Hamiltonian in the sense that there exists ??????????

(Torsion vectors) The second unique vector field on a symplectic domain is the Torsion vector field, which satisfies the equation:

$$Torsion\,\mathbf{Z}: \quad i(\mathbf{Z})dA = \Gamma(x,v,t)A. \quad (Unique\ on\ \mathrm{Pfd}(A) = even)$$

The fact the the anti-symmetric matrix of coefficients of the 2-form dA of Pfaff dimension 2n+2 has an inverse makes it is possible to solve for the Torsion vector uniquely, if the coefficient $\Gamma(x,v,t)$ is given. The choice of $\Gamma(x,v,t)$ can be determined algebraically without explicit construction of an inverse matrix.

For the symplectic system, the 2n+2 dimensional Top Pfaffian, $dA\hat{}dA\hat{}dA... = \rho\Omega$, is the closure of 2n+1 dimensional current, $A\hat{}dA\hat{}dA...$ Hence, to within a factor, the physical system determines a unique 2n+1 dimensional current, defined herein as the Torsion Current, $\mathbf{Z}$.

$$i(\mathbf{Z})\{\Omega\} = \rho A\hat{}dA\hat{}dA...$$

Hence, to within a factor, the Torsion current is defined algebraically by the 2n+2 coefficients of the 2n+1 form $A\hat{}dA\hat{}dA....$ (In a space of 4 dimension the Torsion Current is the same as the Topological Torsion 3-form.) It is remarkable that the Torsion Current, constructed from the last odd p-form of the Pfaff sequence, satisfies the equations

$$i(\mathbf{Z})dA = \Gamma(x,v,t)A \quad and \quad i(\mathbf{Z})A = 0.$$

Suppose the 1-form Action, A, has the format

$$A = P_k dq^k - \Theta dt$$

where the 2n+2 variables $\{P,\Theta,q,t\}$ are independent. Define the variables $\{q,t\}$ as horizontal and the variables $\{P,\Theta\}$ as vertical. (A thermal dynamic distinction based upon homogeneity will be described later). By direct construction it is apparent that the Torsion vector $\mathbf{Z}$, constructed form the 2n+1-form, has components only in the vertical domain. Relative to the ordered set $\{P,\Theta,q,t\}$, the Torsion current is proportion to the vector $\mathbf{Z}$ =$[P...,\Theta,0...,0]$. The dilatation , or conformal factor, Γ, becomes

$$\Gamma = "div" Z = d(A\hat{}dA...\hat{}dA) = dA\hat{}dA...\hat{}dA = \rho\Omega$$

Note that by construction the Torsion field is also associated (it resides in the vertical space). That is

$$i(\mathbf{Z})\{\Gamma(x,v,t)A\} = \Gamma(x,v,t)i(\mathbf{Z})A \Rightarrow i(\mathbf{Z})i(\mathbf{Z})dA = 0.$$

The torsion vector, $\mathbf{Z}$, usually does not have a zero divergence on the 2n+2 domain of support for dA. Points where the divergence vanishes are singular points of the 2n+2 domain. However, the torsion vector does admit integrating factors such that the current $\rho\mathbf{Z}$ has zero divergence globally. For a given 1-form of Action on the 2n+2 variety, those points where the divergence goes to zero imply a reduction of the Pfaff dimension. These special point sets are subsets for which the Torsion vector becomes a Characteristic vector for the system. For spaces of dimension 4 or more, the dynamical system defined by the torsion vector with non-zero divergence is usually irreversible in a thermodynamic sense. In a variety of Pfaff dimension $n = 2$, with a 1-form represented as

$$A_0 = \rho(x,y)[\{\partial\phi(x,y)/\partial x\}dx + \{\partial\phi(x,y)/\partial y\}dy],$$

the torsion vector reduces to the "stream function" of complex variable theory,

$$\mathbf{Z} = \gamma(x,y)[-\partial\phi(x,y)/\partial y, \partial\phi(x,y)/\partial x].$$

For Pfaff dimension 4, the Torsion vector is the adjoint generator of the Hopf map.

Dynamical systems on domains of Odd Pfaff dimension. A great deal of physical theory is based upon the calculus of variations. The "equations of motion" for many physical systems are subsumed to be generated by the "extremal" vectors, or their prolongations, acting on physical systems represented by an Action 1-form. Consider the 1-form of Action defined as

$$A = A_0 + p_k(dq^k - v^k dt) = L(q^k, v^k, t)dt + p_k(dq^k - v^k dt). \tag{3.221}$$

The primitive Lagrange action has been prolonged to include the constraints of a dynamical system $dq^k - v^k dt \Rightarrow 0$. In this notation, the p_k are to be treated as Lagrange multipliers. Remarkably, The Pfaff dimension of this system is 2n+2 (even) and dA defines a non-compact symplectic manifold.

Re-arrange the variables to read

$$A = p_k dq^k - \{p_k v^k - L\}dt) = p_k dq^k - H(p_k, q^k, v^k, t)dt. \tag{3.222}$$

If the system is further constrained to satisfy the classical equations defining canonical momenta

$$(\partial L/\partial v^k - p_k) = 0,$$

then the Pfaff dimension of the space is reduced to 2n+1. This vector constraint is sufficient, but not necessary to produce a contact manifold. A necessary and sufficient constraint is given by the Pfaffian expression,

$$\sum_{k=1}^{n}(\partial L/\partial v^k - p_k)dv^k = 0$$

The canonical constraints, $(\partial L/\partial v^k - p_k) = 0$, produce the system of odd Pfaff dimension 2n+1 which is precisely the Cartan "Space of States". When the canonical constraint is regular, the v^k can be determined in terms of the $\{p_k, q^k, t\}$. The Action 1-form then reduces to the classic Cartan 1-form

$$A = p_k dq^k - H(p, q, t)dt.$$

Let $\mathbf{V} = \{f_k, v^k, 1\}$ be a contravariant-vector (to within a factor $\rho(p, q, t)$) on the space of states $\{p_k, q^k, t\}$. Cartan has shown that the unique extremal condition, $i(\mathbf{E})dA = 0$, which is defined by the equation,

$$f_k dq^k - v^k dp_k - [i(\mathbf{E})dH(p, q, t)]dt + d(H) = 0$$

corresponds to the (Hamiltonian) vector field

$$Extremal\ \ E = \{f_k, v^k, 1\} \Rightarrow \{-\partial H/\partial q^k, +\partial H/\partial p_k, 1\}$$

The fundamental physical idea on spaces of odd Pfaff dimension is to examine dynamical systems as generated by this unique Extremal Field.

The extremal field is the evolutionary direction field proportional at each point to the null eigen vector of the anti-symmetric matrix of differential coefficients that make up the 2-form dA. The 2n+1 by 2n+1 matrix is antisymmetric, and of rank 2n. Hence there is one unique null eigen vector at each point. This extremal field, for fixed or closed boundary conditions, leaves the integral of the Action 1-form an evolutionary invariant.

The Unique Extremal field yields a stationary Action Note that the extremal condition asserts that the 1-form

$$\omega = f_k dq^k - v^k dp_k - [i(\mathbf{E})dH(p, q, t)]dt \equiv -dH$$

is exact. Hence, the coefficients of ω can be put into correspondence with the partial derivatives of a function, $H(p, q, t)$. A special requirement is that $\partial H/\partial t = [i(\mathbf{E})dH(p, q, t)]$

If this vector field **E** is presumed to generate a dynamical system, then it is subsumed that

$$dp_k - f_k dt = 0, \quad and \quad dq^k - v^k dt = 0,$$

where $f_k = -\partial H/\partial q^k$, *and* $v^k = +\partial H/\partial p_k$. The definition of canonical momenta implies that the dynamical system

$$dp_k - f_k(q^k, p_k, t)dt = 0$$

is equivalent to the exterior differential system on $\{q^k, v^k, t\}$

$$\{\partial(\partial L/\partial v^j)/\partial q^k\}dq^k + \{\partial(\partial L/\partial v^j)/\partial v^k\}dv^k + \{\partial(\partial L/\partial v^j)/\partial t - F_k(q^k, v^k, t)\}dt = 0$$

Note that $F_k = f_k(q^k, p_k(q^k, v^k, t), t)$. Multiplying by v^k yields the power theorem,

$$v^k dp_k - v^k f_k dt = v^k dp_k - f_k dq^k = 0$$

3.6.3 Dynamical systems on domains of Even Pfaff dimension

However suppose that constraints of canonical momenta are not imposed on the Action constrained by the equations of a dynamical system; e.g.,

$$(\partial L/\partial v^k - p_k) \neq 0.$$

Then the Hamiltonian H can be considered as the last component of a map from the space of 3n+1 variables, $\{q^k, v^k, p_k, t\}$, to the space of 2n+2 independent variables, $\{q^k, p_k, t, \Theta\}$. As the Pfaff dimension is even there are no "unique extremal" fields. However, there is an unique field defined below as the Torsion field. It is unique but not stationary! This vector field is the first case of interest for 1-forms of even Pfaff dimension. What is important to realize is that this unique vector is determined to within a factor by the coefficients of the Action 1-form, and their first partial derivatives.

The second case of interest to systems which are of even Pfaff dimension concerns those vector fields for which the 1-form of virtual work is integrable, but not zero. These special vector fields are stationary fields, but they are not unique. They correspond to the exceptional class given by equations above.

The Unique but Not Stationary Torsion Vector field

Consider the 1-form of Action written as

$$A = L(v, q, t)dt + p_k(dq^k - v^k dt) = p_k dq^k - \Theta(p, v, q, t)dt \tag{3.223}$$

on an domain of 4 independent variables $\{p, v, q, t\}$ and such that

$$\Theta(p,v,q,t) = (p_k v^k - L(v,q,t)).$$

The Pfaff sequence in this notation becomes

$$\begin{aligned}
\text{Topological Action} \quad A &= pdq - \Theta(p,v,q,t)dt, && (3.224)\\
\text{Topological Vorticity} \quad dA &= dp\hat{}dq - d\Theta\hat{}dt && (3.225)\\
\text{Topological Torsion} \quad A\hat{}dA &= (pd\Theta - \Theta dp)\hat{}dq\hat{}dt && (3.226)\\
\text{Topological Parity} \quad dA\hat{}dA &= 2dp\hat{}d\Theta\hat{}dq\hat{}dt. && (3.227)
\end{aligned}$$

The 3-form of topological torsion has the expression

$$A\hat{}dA = p\partial\Theta/\partial v\ dv\hat{}dq\hat{}dt + (p\partial\Theta/\partial p - \Phi)dp\hat{}dq\hat{}dt \qquad (3.228)$$

such that the Topological Torsion vector, relative to the volume element $\Omega_4 = dp\hat{}dv\hat{}dq\hat{}dt$ has the components

$$\mathbf{T}_4 = [p\partial\Theta/\partial v, -(p\partial\Theta/\partial p - \Phi), 0, 0], \qquad (3.229)$$

from which it is obvious that $i(\rho\mathbf{T}_4)A = 0$. The torsion vector is important, for evolution in the direction of the torsion vector is thermodynamically irreversible, if $\partial\Theta/\partial m \neq 0$.

Relative to processes defined by the Topological Torsion vector, the work 1-form becomes

$$W = i(\rho\mathbf{T}_4)dA = (\rho\partial\Theta/\partial v)\ (pdq - \Theta dt) = \sigma A \qquad (3.230)$$

Direct evaluation of the Topological Parity 4-form yields

$$\begin{aligned}
dA\hat{}dA &= 2dp\hat{}d\Theta\hat{}dq\hat{}dt = 2\{\partial\Theta/\partial v\ \}dp\hat{}dv\hat{}dq\hat{}dt && (3.231)\\
&= \{\partial L(v,q,t)/\partial v - p\}dv\hat{}dp\hat{}dq\hat{}dt && (3.232)\\
&= dS\hat{}dp\hat{}dq\hat{}dt && (3.233)
\end{aligned}$$

as expected. These results are exactly the same as the equations in section 3.5.1 except for notation.

The change in entropy is

$$dS = \{\partial L(v,q,t)/\partial v - p\}dv \qquad (3.234)$$

which implies that

$$\begin{aligned}
ddS &= \{d(\partial L(v,q,t)/\partial v) - dp\}\hat{}dv \Rightarrow 0 && (3.235)\\
&= \{-dp + (\partial^2 L(v,q,t)/\partial v\partial q)dq + (\partial^2 L(v,q,t)/\partial v\partial t)dt\}\hat{}dv \Rightarrow 0 && (3.236)
\end{aligned}$$

Now consider an arbitrary process $Z_4 = [f_k, r, v, t]$ such that

$$L_{(Z_4)}A = \Gamma A, \tag{3.237}$$
$$i(Z_4)A = 0. \tag{3.238}$$

with coefficients

dimension if this 1-form of Action is

On this even dimensional space (Pfd$(A) = 2n+2$) there does not exist a unique extremal field, but there does exist a unique torsion field, but it is NOT stationary.

$$Torsion\ Vector\,\mathbf{Z}: \quad i(\mathbf{Z})dA = \Gamma(x, v, t)A.$$

The first fundamental physical idea on systems of even Pfaff dimension is to examine dynamical systems as generated by the unique but not stationary Torsion Field.

(Note: When $\Gamma(x, v, t) = 1$, the torsion vector has been called the "Liouville vector" by P. Libermann p302.)

Let $\mathbf{Z} = \{f_k, v^k, r, 1\}$ be the symbols for the component functions of a contravariant-vector (to within a factor) on the space of "thermodynamic variables" $\{P_k, \Theta, q_k, t\}$. Consider the1-form $A = \{P_k dq^k - \Theta dt\}$, which is presumed to be of Pfd$(A) = 2n + 2$. Then the equations that define the torsion vector are given by the Pfaffian expression:

$$f_k dq^k - v^k dP_k - r dt + d\Theta = \Gamma\{P_k dq^k - \Theta dt\},$$

or,

$$(f_k - \Gamma P_k)dq^k - v^k dP_k - (r - \Gamma\Theta)dt + d\Theta = 0. \tag{FTF}$$

The functions, v^k, must also satisfy the associated (non-linear) equation,

$$\Theta(P, v, q, t) = P_k v^k$$

which identifies the torsion vector map from 3n+1 space to 2n+2 space. Working backwards from the original definition of

$$H(P_k, q^k, v^k, t) \equiv \Theta = \{P_k v^k - L(q^k, v^k, t)\}$$

it is to be observed that the torsion vector map corresponds to the case where the original Lagrange function is null, $\gamma L(q^k, v^k, t) \Rightarrow 0$. Differentiation yields

$$d(\gamma\Theta) - v^k dP_k = P_k dv^k.$$

Replacing Θ and $d\Theta$ in the original Pfaffian equation, FTF, yields the anholonomic Pfaffian equation of constraint (on the 3n+1 space) as a necessary condition that must be satisfied by the components of the Torsion Vector :

$$(f_k - \Gamma P_k)dq^k + p_k dv^k - (r - \Gamma P_k v^k)dt = 0.$$

Note that this expression can be interpreted as a second order equation if it is assumed that $dq^k = \pm v^k dt$. Assume the + sign. Then

$$p_k dv^k - (r - f_k v^k)dt = 0.$$

Next assume the - sign. Then,

$$p_k dv^k - (r + f_k v^k - 2\Gamma p_k v^k)dt = 0.$$

It appears that motion of the dynamical system (defined by the Torsion Current) in the forward direction is not the same as motion in the backwards direction, a result which demonstrates the ***irreversibility*** *of motion along the torsion vector.*

The two Pfaffian equations are the same only when

$$(r - 2\Gamma P_k v^k + f_k v^k) = (r - f_k v^k)$$

which requires that

$$(f_k - \Gamma P_k)v^k = 0.$$

For reversibility, either the force is viscously dissipative, $(f_k - \Gamma P_k) = 0,$ or the velocities v^k are orthogonal to $(f_k - \Gamma P_k)$ defining a special type of "no-slip" condition associated with "rolling". The evolution proceeds from the initial state irreversibly to the "steady state" case of rolling without slipping in the presence of friction. (Somehow $(f_k - \Gamma p_k) \equiv \mu(Normal)$)

This idea should be put into correspondence with a bowling ball which initially is given rotational energy - as overspin or underspin - as well as translation energy before making contact with the bowling alley surface. There are 4 evolutionary possibilities, that come in 2 pairs. In one circumstance the linear momentum reverses sign; in the second circumstance the angular momentum reverses sign. The choice depends upon the relative amounts of rotational and translational energy given in the initial state. In the "steady state" case of no slipping, the motion is such that $v = \lambda\omega$. Then the rotational energy $= \beta m\lambda^2\omega^2/2$ and the translation energy is $m\lambda^2\omega^2/2 = mv^2/2$. At first glance it would appear that the "frictional" forces do not equilibrate to where the two degrees of freedom have "equal" energy.

In the underspin case the motion must correspond to the $dq + vdt = 0$ constraint, and the overspin case corresponds to the $dq - vdt = 0$ assumption. The extremal path must ultimately be where $(f_k - \Gamma p_k)v^k = 0$, and it must be approached from two different initial conditions. (If the coordinates were closed this is analogous to a limit cycle. The limit cycle constraint must be the analogue of $(f_k - \Gamma p_k)v^k = 0$.)

The Stationary but Not Unique Vector Fields (Mechanical Equilibrium)

For a given Action 1-form, there may exist vector fields for which the 1-form of virtual work is integrable. In such cases the virtual work 1-form is of Pfaff dimension 2, and can be given the Darboux representation as $W = -dP/\rho$. The density function ρ is to be viewed as an integrating factor, and many such integrating factors can exist. It is tantalizing to conjecture that there will exist a hierarchy of non-zero values for the integral of the virtual work, corresponding to a ground state and a set of "excited" states, each with a different density function or integrating factor. The manifolds of mechanical equilibrium, defined as those sets upon which the virtual work 1-form is integrable, form a set of states that are not reachable by stationary paths. (There is, however, a unique path that can connect the manifolds of stationary states).

The second fundamental physical idea on systems of even Pfaff dimension is to examine dynamical systems as generated by the not unique but stationary fields that generate mechanical equilibrium.

Cartan's magic formula indicates that it is of the form of the first law of Thermodynamics.

$$\mathcal{L}_{(\rho\mathbf{V})}A = i(\rho\mathbf{V})dA + d(i(\rho\mathbf{V})A) = W + dU = Q.$$

For stationary Action, $Q = 0$ (the analogue of an adiabatic process). If the Work 1-form is given the special format of mechanical equilibrium, then $-dP + \rho dU = 0$, an expression that effectively defines an equation of state for mechanical equilibrium. .

The work 1-form is given by the expression:

$$W = f_k dq^k - v^k dp_k - rdt + d\Theta.$$

3.6.4 Van der Pohl Limit cycles

This example is related to the Weinstein theorem that an "overtwisted but not tight" contact submanifold of a symplectic manifold has a periodic orbit. A crucial feature of limit cycles of the Van der Pol type is that the mechanical energy of the system is time dependent, decaying from any set of initial conditions to a "breathing" system, with the power being positive on certain parts of the cycle and negative on others. As such, the limit cycle orbits are not tractable in terms of an extremal Hamiltonian analysis. The objective of this section is to demonstrate that the Van der Pol oscillator can be considered as an example of a symplectic system on a 4-dimensional variety.

First the concept of an "overtwisted but not tight" contact submanifold should be made clear. Consider the example of a hyperboloid of revolution. By looking at the surface, no concept of whether the hyperboloid is "twisted" is evident. However, the hyperboloid my be constructed by the envelope of an array of straight lines from one lower base circle to another upper base circle. If the threads connecting the points on one boundary circle to the other are orthogonal to the two circles, the resulting envelope of threads is a cylinder. If, with the threads attached, the upper circle is rotated by an angle of less than pi with respect to the bottom circle, the envelope of the threads forms a hyperboloid of revolution as a ruled surface with an discernible twist. If the relative rotation of the upper circle is pi, the envelope is a twisted cone. If the relative rotation is more that pi degrees, it is apparent that the twist is "tight". The fact that the hyperboloid is twisted is an artifact that the twisted surface is constrained 2-surface in a non-integrable 3-dimensional "contact submanifold". It is also apparent that on the twisted hyperboloid with a relative rotation of less that pi, there is a closed path around the enveloping hyperboloid which is a minimum in length. It is the circle of constriction. The circle of constriction is the epitome of the "limit cycle". A orbit with initial conditions on the surface will follow some "minimal" trajectory winding around the hyperboloid and being gradually attracted to the limit cycle.

Consider the Lagrangian

$$L(t,x,v) = -1/2kx^2 + 1/2mv^2 - \beta^2(v^3/3c^2 - v)x + m_0c^2$$

on a space of variables {t; x, v, p} The Action 1-form (presuming that $\{\kappa, \sigma = \beta^2, c, m_0\}$ are constants) is:

$$A = L(x,v,t)dt + p(dx - vdt) \tag{3.239}$$

Explicit evaluation of the functions and 1-forms defined above yields:

$$H = 1/2kx^2 + pv - 1/2mv^2 - m_0c^2 + \beta^2 x(v^3/3 - v)$$

$$(h/2\pi)k \doteq p - \partial L/\partial v = p - mv + \beta^2(v^2 - 1)x \neq 0$$

$$\Delta p = dp - \partial L/\partial x dt = dp + [\kappa x + \beta^2(v^3/3c^2 - v)]dt \Rightarrow 0$$

$$d\Theta = [p - mv + \beta^2(v^2/c^2 - 1)x]\{dv - adt\} \neq 0$$

(Note there is a critical velocity: $v^2/c^2 - 1 = 0$) The criteria that $(h/2\pi)k \neq 0$ insures that the 4D space is symplectic in terms of the two form, dA; the two form is of maximal (symplectic) rank 4. The non-zero function k has a covector gradient that is never zero:

$$dk = 1dp - mdv - vdm + \beta^2(v^2/c^2 - 1)dx + 2x\beta^2 v/c^2 dv$$

.This Fomenko symplectic system requires that the Lagrange-Euler constraint be satisfied, hence the equations of motion are given by a Pfaffian equation which acts as a differential constraint on the 4D space:

$$\Delta p = dp + [\kappa x + \beta^2(v^3/3c^2 - v)]dt = 0.$$

The criteria for a subspace upon which the Action 1-form is completely integrable is given by the mass equation,

$$dp = [\{p(p - \partial L/\partial v)\}/L\}]dv \Rightarrow mdv.$$

On the surfaces of constant "mass" in the 4D domain,

$$[\{p(p - \beta^2(v^2/c^2 - 1)x)\}/\{-1/2kx^2 + 1/2mv^2 - \beta^2(v^3/3c^2 - v)x + m_0c^2\}] = m(x, v, p) = cons\tan t,$$

the Pfaffian equation of motion becomes:

$$mdv + [\kappa x + \beta^2(v^3/3c^2 - v)]dt = 0.$$

Adjoining to this form the hypothesis of null differential fluctuations of position ($dx - vdt = 0$) leads to the second order differential system of the Van der Pol equation on the 2D integrable subspace of the 4D symplectic system.

$$m(d^2x/dt^2) + \beta^2[(dx/dt)^2/3c^2 - 1](dx/dt) + \kappa x = 0.$$

Note that the mass constraint implies that the momenta, p, is not canonically defined, and depends on both velocity and coordinate for constant mass. Starting from any initial condition $\{0, q_0, v_0, p_0 \equiv \phi(m, q_0, v_0\}$, the power either decays or increases always being attracted to the limit cycle or its whiskers. When the limit cycle is reached the power $Power = v \bullet dp/dt$ becomes cyclic.

The constant mass function, m, defines a surface in the 3 dimensional subspace of (x, v, p). Orbits of the evolutionary constant mass process reside on this surface. The usual velocity-position display of the Van der Pol Oscillator is not in phase space, but is a projection from an orbit on the constant mass surface to the v-x plane.

3.6.5 Breathers

To be written

Chapter 4
CARTAN'S TOPOLOGICAL STRUCTURE

4.1 Introduction

In this presentation*, a topological perspective will be used to extract those properties of physical systems and their evolution that are independent from the geometrical constraints of connections and/or metrics. It is subsumed that the presence of a physical system establishes a *topological structure* on a (possibly geometric) base space of independent variables. This concept is different from, but similar to, the geometric perspective of general relativity, whereby the presence of a physical system is presumed to establish a *metric* on a base space of independent variables. Note that a given base of independent variables may support many different topological structures; hence a given base may support many different physical systems. A major success of theory is that continuous non-homeomorphic processes of topological evolution establish a logical basis for thermodynamic irreversibility and the arrow of time [RMK 2003] without the use of statistics.

The fundamental axioms utilized in this chapter are:

Axiom 18 *The topological structure of Physical Systems on a domain of independent base variables can be encoded in terms of exterior differential forms (symbolically represented by A).*

Axiom 19 *Physical Processes can be defined in terms of contravariant vector direction fields, which may or may not be generators of 1-parameter groups, and in particular need not be homeomorphisms (symbolically represented by V).*

Axiom 20 *Equations of Continuous Evolution describing both reversible and irreversible Processes acting on Physical Systems are encoded by Cartan's magic formula :*

$$L_{(\mathbf{V})}A = i(\mathbf{V})dA + d(i(\mathbf{V})A) \tag{4.1}$$

*The basis for this article was presented as a talk given in August, 1991, at the Pedagogical Workshop on Topological Fluid Mechanics held at the Institute for Theoretical Physics, Santa Barbara UCSB. Part of the T4 truth table was due to Phil Baldwin. The recognition that the Cartan topology was a disconnected topology is due to the author.

In the period from 1899 to 1926, Eli Cartan developed his theory of exterior differential systems [Cartan 1899], [Cartan 1922], which included the ideas of spinor systems [Cartan 1966] and the differential geometry of projective spaces and spaces with torsion [Cartan 1937]. The theory was appreciated by only a few contemporary researchers, and made little impact on the main stream of the physical sciences until about the 1960's. Even specialists in differential geometry (with a few notable exceptions [Chern 1944]) made little use of Cartan's methods until the 1950's. Even today, many physical scientists and engineers have the impression that Cartan's theory of exterior differential forms is just another formalism of fancy.

However, Cartan's theory of exterior differential systems has several advantages over the methods of tensor analysis that were developed during the same period of time. The principle fact is that differential forms are well behaved with respect to functional substitution of C1 differentiable maps. Such maps need not be invertible even locally, yet differential forms are always deterministic in a retrodictive sense [RMK 1976 b], by means of functional substitution. Such determinism is not to be associated with contravariant tensor fields, if the map is not a diffeomorphism. Cartan's theory of exterior differential systems contains topological information, and admits non-diffeomorphic maps which can describe topological evolution.

Although the word "topology" had not become popular when Cartan developed his ideas (topological ideas were described as part of the theory of analysis situs), there is no doubt that Cartan's intuition was directed towards a topological development. For example, Cartan did not define what were the open sets of his topology, nor did he use, in his early works, the words "limit points or accumulation points" explicitly, but he did describe the union of a differential form and its exterior differential as the "closure" of the form. With this concept, Cartan effectively used the idea that the closure of a subset is the union of the subset with its topological limit points. What was never stated (until 1990) is the idea that the exterior differential is indeed a limit point generator relative to a Cartan topology. The union of the identity operator and the exterior differential satisfy the axioms of a Kuratowski closure operator [Lipschutz 1965], which can be used to define a topology. The other operator of the Cartan calculus, the exterior product, also has topological connotations when it is interpreted as an intersection operator.

In a perhaps over simplistic comparison, it might be said that ubiquitous tensor methods are restricted to geometric applications, while Cartan's methods can be applied directly to topological concepts as well as geometrical concepts. Cartan's theory of exterior differential systems is a topological theory not necessarily limited by geometrical constraints and the class of diffeomorphic transformations that serve as the foundations of tensor calculus. A major objective of this article is to show how limit points, intersections, closed sets, continuity, connectedness and other elementary concepts of modern topology are inherent in Cartan's theory of exterior differential systems. These ideas do not depend upon the geometrical ideas of size and shape. Hence Cartan's theory, as are all topological theories, is renormalizeable (perhaps a better choice of words is that the topological components of the theory are independent from scale). In fact the most useful of Cartan's ideas do not depend explicitly upon the geometric ideas of a metric, nor upon the choice of a differential connection between basis frames, as in fiber bundle theories. The theme of this chapter is to explore the physical usefulness of those topological features of Cartan's methods which are independent from the constraints and refinements imposed by a connection and/or a metric.

In this article the Cartan topology will be constructed explicitly for an arbitrary exterior differential system, Σ. For a particular simple, but useful, system consisting of a single 1-form of Action, A, all elements of the Cartan topology will be evaluated, and the limit points, the boundary sets and the closure of every subset will be computed abstractly. Earlier intuitive results, which utilized the notion that Cartan's concept of the exterior product may be used as an intersection operator, and his concept of the exterior differential may used as a limit point operator acting on differential forms, will be given formal substance in this article. A major result of this article, with important physical consequences in describing topological evolutionary processes, is the demonstration that the Cartan topology is not necessarily a connected topology, unless the property of topological torsion vanishes, and that thermodynamic irreversibility is a consequence of 4 dimensions or more.

4.1.1 A Point Set Topology Example

As an example of a topological ideas, consider the set of 4 elements or points,

$$X : \{a, b, c, d\}. \tag{4.2}$$

and all possible subsets:

$$\emptyset, \tag{4.3}$$
$$\{a\}, \{b\}, \{c\}, \{d\}, \tag{4.4}$$
$$\{a, b\}, \{a, c\}, \{a, d\}, \{b, c\}, \{b, d\}, \{c, d\}, \tag{4.5}$$
$$\{a, b, c\}, \{a, c, d\}, \{b, c, d\}, \{a, b, d\}, \tag{4.6}$$
$$\{a, b, c, d\} = X \tag{4.7}$$

Select the following subset elements as a topological basis,

$$\text{basis selection} \quad \{a\}, \{a, b\}, \{c\}, \{c, d\}, \tag{4.8}$$

and then compose a topology $T4$ of open sets from all possible unions of the selected basis elements:

$$T4\{open\} : \emptyset, \{a\}, \{c\}, \{a, b\}, \{c, d\}, \{a, c\}, \{a, b, c\}, \{a, c, d\}, \{a, b, c, d\} \tag{4.9}$$

The closed sets are the compliments of the open sets:

$$T4\{closed\} : \{a, b, c, d\}, \{b, c, d\}, \{a, b, d\}, \{c, d\}, \{a, b\}, \{b, d\}, \{d\}, \{b\}, \emptyset \qquad (4.10)$$

It is an easy exercise to demonstrate that the collections above indeed satisfy the axioms of a topology. (This is not the only topology that can be constructed over 4 elements).

This simple example of a point set topology permits explicit construction of all the topological concepts, which include limit sets, interiors, boundaries, and closures, for the all of subsets of X, relative to the topology, $T4$. The standard definitions are:

1. A limit point of a subset A is a point p such that all open sets that contain p also contain a point of A not equal to p.

2. The closure of a subset A is the union of the subset and its limit points, and is the smallest closed set that contains A.

3. The interior of a subset is the largest open set contained by the subset.

4. The exterior of a subset is the interior of its compliment.

5. A boundary of a subset is the set of points not contained in the interior or exterior.

6. The closure of a subset is also equal to the union of its interior and its boundary.

The results of applying these definitions to the $T4$ topology of 4 points are:

Table 1. **A T4 Topology of 4 points**

$X = \{a, b, c, d\}$

Basis subsets $\{a\}, \{a, b\}, \{c\}, \{c, d\}$

$T4\{open\} : \emptyset, \{a\}, \{c\}, \{a, b\}, \{c, d\}, \{a, c\}, \{a, b, c\}, \{a, c, d\}, X$

$T4\{closed\} : X, \{b, c, d\}, \{a, b, d\}, \{c, d\}, \{a, b\}, \{b, d\}, \{d\}, \{b\}, \emptyset$

Subset	**Limit Pts**	**Interior**	**Boundary**	**Closure**
$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
$\{a\}$	$\{b\}$	$\{a\}$	$\{b\}$	$\{a, b\}$
$\{b\}$	$\emptyset$	$\emptyset$	$\{b\}$	$\{b\}$
$\{c\}$	$\{d\}$	$\{c\}$	$\{d\}$	$\{c, d\}$
$\{d\}$	$\emptyset$	$\emptyset$	$\{d\}$	$\{d\}$
$\{a, b\}$	$\{b\}$	$\{a, b\}$	$\emptyset$	$\{a, b\}$
$\{a, c\}$	$\{b\}, \{d\}$	$\{a, c\}$	$\{b, d\}$	X
$\{a, d\}$	$\{b\}$	$\{a\}$	$\{b, d\}$	$\{a, b, d\}$
$\{b, c\}$	$\{d\}$	$\{c\}$	$\{b, d\}$	$\{b, c, d\}$
$\{b, d\}$	$\emptyset$	$\emptyset$	$\{b, d\}$	$\{b, d\}$
$\{c, d\}$	$\{d\}$	$\{c, d\}$	$\emptyset$	$\{c, d\}$
$\{a, b, c\}$	$\{b\}, \{d\}$	$\{a, b, d\}$	$\{d\}$	X
$\{b, c, d\}$	$\{d\}$	$\{c, d\}$	$\{b\}$	$\{b, c, d\}$
$\{a, c, d\}$	$\{b\}, \{d\}$	$\{a, c, d\}$	$\{b\}$	X
$\{a, b, d\}$	$\{b\}$	$\{a, b\}$	$\{d\}$	$\{b, c, d\}$
$\{a, b, c, d\}$	$\{b\}, \{d\}$	X	$\emptyset$	X

(4.11)

This $T4$ topology is quite interesting for many demonstrable reasons. First note that the all of the singletons of the topology are not closed. This implies that the topology is NOT a metric topology, NOT a Hausdorf topology, and even does NOT satisfy the separation axioms to be a T_1 topology. Note that all closed sets contain all of their limit points. Some open sets can contain limit points, but some open sets do not contain their limit points. Some subsets have boundaries that are composed of their limit points. Some subsets have limit points which are not boundary points. Certain subsets have a boundary, but do not have limit points, and in other cases there are subsets that have limit points, but do not have a boundary. There are certain subsets with a boundary, but without an interior. There are certain subsets with an interior, but without a boundary. These situations, though topologically correct, are not always intuitive to those accustomed to metric based topological concepts, which impose a number of additional constraints on the sets of interest. Yet all of these topological ideas, including the non-intuitive ones, are easy to grasp from the simple example of the $T4$ point set topology.

One other very important observation is that there are subsets of the $T4$ topology, $\{a, b\}$ and $\{c, d\}$, (other than $\emptyset$ and X) which are both open and closed. The union of these two subsets $\{a, b\}$ and $\{c, d\}$ is X. Topologies with this property are said to be disconnected topologies. What is important is that it is possible to construct a continuous map from a disconnected topology to a connected topology, but it is impossible to construct a continuous map from a connected topology to a disconnected topology. If the mapping process is interpreted as an evolutionary process, these facts establish a logical or topological basis for the arrow of time [RMK 2003]. This idea will be exploited to explain the concept of thermodynamic irreversibility without the use of statistics.

What is even more remarkable is that properties of the $T4$ topology can be replicated in terms of the Pfaff sequence of exterior differential sets,

$$\text{Pfaff Sequence} : \{A, dA, A\hat{}dA, dA\hat{}dA...\}, \tag{4.12}$$

generated from any given 1-form of Action, A, on a N dimensional variety. The Pfaff sequence is readily computed, and will contain $M \leq N$ elements, where M is defined as the Pfaff topological dimension (or class) of the given 1-form, A. The realization of a $T4$ topology in terms of exterior differential forms is herein defined as the "Cartan topology", and is detailed in the next section. The Cartan topology has far reaching consequences in applications to physical problems.

4.1.2 Algebraic and Differential Closure

The concept of closure is one of the most important ideas in Cartan's theory. His methods center on two procedures of closure, one algebraic, and one differential. Both processes are closed in the sense that when they operate on a subset of a set of exterior differential forms, the objects created are also subsets of the set of exterior differential forms. There are no surprises. Cartan utilized the exterior algebra over a variety of dimension N to construct a vector space of exterior differential forms of dimension 2^N. The N subspaces of this (Grassmann) space are vector spaces of dimension equal to N things taken p at a time. The elements of the subspaces are called p-forms. In 4 dimensions, the subspace sets are 1 dimensional, N=4 dimensional, N(N+1)/2=6 dimensional, N=4 dimensional, and 1 dimensional. The elements of the subspaces are often called scalars (0-forms), vectors (1-forms), tensors (2-forms), pseudovectors (3-forms), pseudo-scalars (4-forms) in relativistic physical theories. The Exterior (Grassmann) algebra has a finite 2^N basis (equal to 16 elements in a space of 4 independent variables). The concept of closure means that the operations on elements of the 2^N dimensional space yield results that are contained within the 2^N dimensional space. When the operations are applied to elements of a subspace, the results usually are not contained in the same subspace, but they are contained within the 2^N dimensional vector space of p forms.

The exterior product (with symbol ^) takes elements of the 2^N base space and multiplies them together in a manner such that the result is contained as an element of the 2^N base space. This process of exterior multiplication is closed, for the action of the process on any subset of the 2^N base space produces another subset of the 2^N base space. However, the exterior product takes a p-form times a q-form into a p+q form. The elements of the product can be from different or from the same vector subspaces, but the resultant is always a linear combination of the subspaces of the Exterior algebra.

Similarly the concept of exterior differentiation (with symbol d) is defined such that the operation produces a p+1 form from a p-form. This process of exterior differentiation is "closed", for the action of the process on any subset of the 2^N base space produces another subset of the 2^N base space. A differential ideal is defined as the union of a collection of given p-forms and their exterior derivatives.

An "interior" product with respect to a direction field $\mathbf{V}$ (with symbol $i(\mathbf{V})$ and of dimension N) can be defined on the Grassmann algebra of exterior differential forms. The interior product takes a p-form to a p-1 form, and in this sense is another operation which is closed within the Grassmann algebra. The resultant product is still an element of the 2^N base space. Where the exterior differential raises the rank of a p-form to a p+1 form, the inner product lowers the rank of a p-form to a p-1form. (There are other useful operators that lower the rank of the exterior differential p-form, and involve integration.)

By composition of the exterior derivative and the inner product operators, the Lie differential operator (with symbol $L_{(\mathbf{V})} = i(\mathbf{V})d + di(\mathbf{V})$) can be constructed, such that when the Lie differential operates on an exterior p-form, the resultant object is another p-form. For a 1-form of Action, A, the process reads:

$$L_{(\mathbf{V})}A = i(\mathbf{V})dA + d(i(\mathbf{V})A) = Q. \tag{4.13}$$

The resultant is not only closed relative to the Grassmann algebra, it also remains within the same Grassmann vector subspace. The Lie differential does not depend upon a metric nor upon a connection. When the Lie differential acting on a p-form vanishes, the p-form is said to be an invariant of the process, $\mathbf{V}$. When the Lie differential of a p-form does not vanish, the topological features of the resultant p-form permit the processes, $\mathbf{V}$, that produce such a result, to be put into equivalence classes, depending on the Pfaff dimension of the resultant form. For example, if in the formula given above for a 1-form, A, yields a result Q such that $dQ = 0$, then the process $\mathbf{V}$ belongs to the class of process known as Hamiltonian processes in mechanics, and to the Helmholtz class of processes that conserve vorticity in Hydrodynamics. Of particular interest to this article are processes where Q is of Pfaff dimension greater than 2. The Pfaff sequence constructed from Q contains three or more elements. Such processes are thermodynamically irreversible.

The Lie differential will be used extensively in physical applications of Cartan's theory, especially to the study of processes that involve topological evolution. The perhaps more familiar covariant derivative, highly constrained by connection or metric assumptions, is a special case of the Lie differential. The use of the covariant derivative leads to useful, but limited, physical theories for which the description of topological evolution is awkward, if not impossible.

4.1.3 The Exterior Product and Set Intersection

Cartan's theory of exterior differential systems has its foundations in the Grassmann algebra, where the two combinatorial processes are defined to produce algebraic and differential closure. The algebra is based upon the concepts of vector space addition, and an algebraic closure multiplication process now called the exterior product [Flanders 1963]. The Cartan calculus is defined in terms of the another closure operator now called the exterior differential †. In that which follows the operators of the exterior product and exterior differential will be applied to objects defined as exterior differential p-forms.

An exterior differential p-form is a function of independent variables, x^v, and their differentials, dx^μ. An exterior differential 1-form, A, is given by the expression,

$$A = A_\mu(x^v)dx^\mu. \tag{4.14}$$

The Cartan operations of exterior product (symbol ˆ) and exterior differential (symbol d), when operating on 1-forms, A and B, obey the rules

$$A\hat{}A = 0, \tag{4.15}$$
$$A\hat{}B = -B\hat{}A. \tag{4.16}$$

and

$$dA = d(A_\mu\, dy^\mu) = (dA_\mu)\hat{}dy^\mu + A_\mu\, d(dy^\mu) \tag{4.17}$$
$$= (dA_\mu)\hat{}dy^\mu + 0 \tag{4.18}$$
$$d(A\hat{}B) = dA\hat{}B - A\hat{}dB. \tag{4.19}$$

The non-zero product, $A\hat{}B$, defines an exterior differential 2-form; the product of three 1-forms defines a 3-form; etc.. For more detail consult [Flanders 1963] [**?**] [Bamberg 1992] [Liebermann 1987], [Liebermann 1987].

In simple cases, a 1-form can be constructed from the differential of an ordinary function. In such cases, the coefficients of the 1-form are proportional to the gradient of the function.

$$A = A_\mu dx^\mu = \nabla\phi \cdot d\mathbf{r} = (\partial\,\boldsymbol{\phi}/\partial x^\mu)dx^\mu \quad . \tag{4.20}$$

In surface theory, the gradient is classically interpreted as vector direction field orthogonal to the implicit surface, $\phi(x^\mu) = 0$. Consider the simple case where the 1-forms A and B each have coefficients which form the components of (different) gradient fields,

$$A = A_\mu dx^\mu = \nabla\phi \cdot d\mathbf{r} \qquad B = B_\mu dx^\mu = \nabla\psi \cdot d\mathbf{r}. \tag{4.21}$$

†Cartan originally defined the calculus operation as the exterior derivative. Then in the later years defined calculus operation as the exterior differential.

Do the two implicit (curved) surfaces intersect? The answer is yes if the two surfaces have points in common. The classic analysis in 3D says there is a curve of points in common defined by a non-zero value of the Gibbs cross product of the two gradient fields:

$$\text{Intersection of two implicit surfaces: } \mathbf{J} = \nabla\phi \times \nabla\psi \neq 0. \tag{4.22}$$

Note that (in 3 dimensions) the exterior product of the two 1-forms has coefficients exactly equal to the Gibbs cross product:

$$A\hat{}B = \mathbf{J}_z dx\hat{}dy \ + \mathbf{J}_x dy\hat{}dz + \mathbf{J}_y dz\hat{}dx. \tag{4.23}$$

This result pictorially cements the notion that the exterior product (acting on 1-forms) is an operator related to the concept of intersection. If the two surfaces do not intersect, the exterior product vanishes, and then the direction fields of the gradients of ϕ and χ are proportional to one another. The two functions, ϕ and χ, are not functionally independent if the exterior product vanishes.

These concepts extend to 1-forms which are not representable by gradient fields, and to p-forms of higher rank. If the exterior product of two p-forms is not zero, then the p-forms have non-zero intersections. The coefficient functions are functionally independent.

An exterior differential 1-form A is deterministic, as a predictive (or retrodictive) invariant, with respect to all tensor diffeomorphisms. The coefficient functions, $A_\mu(x^\upsilon)$, are presumed to behave as a covariant vector, and the differentials, dx^μ, behave as a contravariant vector, with respect to tensor diffeomorphisms. (Exterior differential form densities will be discussed later.) However, the exterior differential 1- form, and hence all p-forms, are also well behaved with respect to a larger class of transformations, which contain the tensor diffeomorphisms as special cases. The exterior differential 1- form is deterministic in a retrodictive sense (but not in a predictive sense) with respect to C1 mappings that do not have a local or a global inverse. These C1 mappings do not preserve all topological features, where diffeomorphisms of tensor theory, are special cases of homeomorphisms, which do preserve all topological properties. These extraordinary features demonstrate that Cartan's theory is not just another formalism of fancy, and goes well beyond the theory of tensor analysis. In fact, these features of exterior differential forms can be exploited to develop something that has slipped through the net of tensor analysis: a non-statistical theory of thermodynamic irreversibility.

4.1.4 The Exterior Differential and Limit Points

The second closure operator found in Cartan's theory of exterior differential systems is the exterior differential. The exterior differential, like the exterior product, also has topological connotations when applied to differential forms, but the results are sometimes surprising and unfamiliar. Where the exterior product is related to the topological concept of set intersection, the exterior differential is related to the topological idea of limit points. It will be demonstrated that:

Conjecture 21 *With respect to the Cartan topology, the exterior differential is a limit point generator.*

The exterior differential is a differential operator which takes the p-forms into p+1 forms. Hence, like the exterior product, the exterior differential generates a vector in a different vector subspace of the exterior algebra.

$$d(\omega^p) \Rightarrow \omega^{p+1}. \tag{4.24}$$

The exterior differential of a function (0-form) is equivalent to the total differential of a scalar function, and yields a 1-form with coefficients proportional to the gradient field. The exterior differential of a 1-form is defined as

$$\begin{aligned} d\omega^1 &= d(A_b\, dy^b) = (dA_b)\hat{}dy^b + A_b\, d(dy^b) \qquad (4.25)\\ &= (\partial A_b/\partial y^e\, dy^e)\hat{}dy^b + 0 \\ &= (\partial A_b/\partial y^e - \partial A_e/\partial y^b)\, dy^e\ \hat{}\ dy^b \\ &= F_{[eb]}\, dy^{[eb]} = F_{[H]} dy^{[H]}. \end{aligned}$$

It has been assumed that the coefficients of the forms are C2, such that $dd(\omega^p) = 0$. The collective index notation $[H] = [eb]$ permits the formula defining exterior differentiation to be generalized:

$$\begin{aligned} d\omega^p &= d(A_H\, dy^H) = (dA_H)\hat{}dy^H \qquad (4.26)\\ &= (\{\partial A_H/\partial y^e\}\, dy^e)\hat{}dy^H \qquad (4.27) \end{aligned}$$

Other properties of the exterior differential will be exemplified by the rules for distributing the operator over a product of 1-forms, A and B,

$$d(A\hat{}B) = dA\hat{}B - A\hat{}dB. \tag{4.28}$$

It can be shown that the operator $KCl = I \cup d$, where I is the identity and d is the exterior differential, acting on a system of differential forms satisfies the "Kuratowski closure" axioms [Lipschutz 1965], and therefor can be used to define a topology. Starting from a single 1-form, A, on a 4 dimensional space, it is possible to generate the Pfaff Sequence

$$\begin{aligned} Pfaff\ Sequence &: \{A, dA, A\hat{}dA, dA\hat{}dA\} \qquad (4.29)\\ &= \{A, F, H, K\}. \qquad (4.30) \end{aligned}$$

The subsets of the Cartan topological space consist of all possible unions of the subsets that make up the Pfaff sequence. The Cartan topology will be constructed from a topological basis which consists of the odd elements of the Pfaff sequence, and their closures:

$$\textit{the Cartan topological base}: \{A, K_{Cl}(A), A\hat{}dA, K_{Cl}(A\hat{}dA)\}. \tag{4.31}$$

When applied to the Pfaff sequence generated by a single 1-form of Action, A, on a space of 4 dimensions, the base elements correspond to the set

$$\textit{the Cartan topological base} \quad : \quad \{A, A \cup F, H, H \cup K\} \tag{4.32}$$

$$\text{compare to the point set example above} \quad \approx \quad \{a, b, c, d\}\ . \tag{4.33}$$

When it is realized that the exterior differential acts a limit point generator, it becomes apparent why Cartan referred to the union of Σ and $d\Sigma$ as the closure of Σ,

$$\text{Closure} = (K_{Cl}) \circ \Sigma = (I \cup d) \circ \Sigma = \Sigma + d\Sigma = \text{subset} + \text{limit points}. \tag{4.34}$$

In the next section, the topological features of the Cartan topology, based on the Cartan topological base, will be worked out in detail. It will turn out that the Cartan topology can be put into correspondence with the T4 topology of 4 points displayed in a previous section. It will be evident, indeed, that the exterior differential is a limit point generator for any subset relative to the Cartan topology. This is a remarkable result, for as will be demonstrated below, all C2 vector fields acting through the concept of the Lie differential on a set of differential forms, with C2 coefficients‡, generate continuous transformations with respect to the Cartan topology. Moreover, the Cartan topology is disconnected if $A\hat{}dA \neq 0$. As the conditions for unique integrability of the 1-form A are given by the Frobenius theorem, which requires $A\hat{}dA = 0$, it should be expected that one of the features of the disconnected Cartan topology is that if solutions exist, they are not unique.

4.2 The Cartan "Point Set" Topology.

Cartan built his theory around an exterior differential system, Σ, which consists of a collection of 0- forms, 1-forms, 2-forms, etc. [Cartan 1958 (1922)]. He defined the closure of this collection as the union of the original collection with those forms which are obtained by forming the exterior differentials of every p-form in the initial collection. In general, the collection of exterior differentials will be denoted by $d\Sigma$, and the closure of Σ by the symbol, $K_{Cl}(\Sigma)$, where

$$\text{Kuratowski Closure operator:} \quad K_{Cl}(\Sigma) = \Sigma \cup d\Sigma \tag{4.35}$$

For notational simplicity in this article the systems of p-forms will be assumed to consist of the single 1-form, A. Then the exterior differential of A is the 2-form

‡Adiabatic processes may be C1

$F = dA$, and the closure of A is the union of A and $F : K_{Cl}(A) = A \cup F$. The other logical operation is the concept of intersection, so that from the exterior differential it is possible to construct the set $A\hat{}F$ defined collectively as $H : H = A\hat{}F$. The exterior differential of H produces the set defined as $K = dH$, and the closure of H is the union of H and $K : K_{Cl}(H) = H \cup K$.

This ladder process of constructing exterior differentials, and exterior products, may be continued until a null set is produced, or the largest p-form so constructed is equal to the dimension of the space under consideration. The set so generated is defined as a Pfaff sequence. The largest rank of the sequence determines the Pfaff dimension of the domain (or class of the form, [Schouten 1949]), which is a topological invariant. The idea is that the 1-form A (in general the exterior differential system, Σ) generates on space-time four equivalence classes of points that act as domains of support for the elements of the Pfaff sequence, A, F, H, K . The union of all such points will be denoted by $X = A \cup F \cup H \cup K$. The fundamental equivalence classes are given specific names [RMK 1990]:

$$\text{Topological ACTION:} \quad A = A_\mu dx^\mu \tag{4.36}$$

$$\text{Topological VORTICITY:} \quad F = dA = F_{\mu\nu} dx^\mu\,\hat{}\,dx^\nu \tag{4.37}$$

$$\text{Topological TORSION:} \quad H = A\hat{}dA = H_{\mu\nu\sigma} dx^\mu\,\hat{}\,dx^\nu\,\hat{}\,dx^\sigma \tag{4.38}$$

$$\text{Topological PARITY:} \quad K = dA\hat{}dA = K_{\mu\nu\sigma\tau} dx^\mu\,\hat{}\,dx^\nu\,\hat{}\,dx^\sigma\,\hat{}\,dx^\tau. \tag{4.39}$$

The Cartan topology is constructed from a basis of open sets, which are defined as follows: first consider the domain of support of A. Define this "point" by the symbol A. A is the first open set of the Cartan topology. Next construct the exterior differential, $F = dA$, and determine its domain of support. Next, form the closure of A by constructing the union of these two domains of support, $K_{Cl}(A) = A \cup F$. $A \cup F$ forms the second open set of the Cartan topology.

Next construct the intersection $H = A\hat{}F$, and determine its domain of support. Define this "point" by the symbol H, which forms the third open set of the Cartan topology. Now follow the procedure established in the preceding paragraph. Construct the closure of H as the union of the domains of support of H and $K = dH$. The construction forms the fourth open set of the Cartan topology. In four dimensions, the process stops, but for N > 4, the process may be continued.

Now consider the basis collection of open sets that consists of the subsets,

$$B = \{A,\ K_{Cl}(A),\ H,\ K_{Cl}(H)\} = \{A,\ A \cup F,\ H,\ H \cup K\} \tag{4.40}$$

The collection of all possible unions of these base elements, and the null set, $\emptyset$, generate the Cartan topology of open sets:

$$T(open) = \{X,\ \emptyset,\ A,\ H, A \cup F,\ H \cup K,\ A \cup H,\ A \cup H \cup K,\ A \cup F \cup H\}. \quad (4.41)$$

These nine subsets form the open sets of the Cartan topology constructed from the domains of support of the Pfaff sequence constructed from a single 1-form, A, in 4 dimensions. The compliments of the open sets are the closed sets of the Cartan topology.

$$T(closed) = \{\emptyset,\ X,\ F \cup H \cup K,\ A \cup F \cup K, A \cup F,\ H \cup K, F \cup K, F,\ K\}. \quad (4.42)$$

From the set of 4 "points" $\{A, F, H, K\}$ that make up the Pfaff sequence it is possible to construct 16 subset collections by the process of union. It is possible to compute the limit points for every subset relative to the Cartan topology. The classical definition of a limit point is that a point p is a limit point of the subset Y relative to the topology T if and only if for every open set which contains p there exists another point of Y other than p [Lipschutz 1965]. The results of this and other standard definitions are presented in Table 2, and are to be compared to Table 1.

Table 2. The Cartan T4 Topology

A 1-form in 4D: $A = A_k(x)dx^k$

$X = \{A, \quad F = dA, \quad H = A\hat{}F, \quad K = F\hat{}F\}$

Basis subsets $\{A,\ K_{Cl}(A),\ H,\ K_{Cl}(H)\} = \{A, A \cup F, H, H \cup K\}$

$T(open) = \{X,\ \emptyset,\ A,\ H, A \cup F,\ H \cup K,\ A \cup H,\ A \cup H \cup K,\ A \cup F \cup H\}$

$T(closed) = \{\ \emptyset, X, F \cup H \cup K, A \cup F \cup K,\ H \cup K,\ A \cup F,\ F \cup K,\ F, K\}$

Subset	**Limit Pts**	**Interior**	**Boundary**	**Closure**
σ	$d\sigma$	.	$\partial\sigma$	$\sigma \cup d\sigma$
$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
A	F	A	F	$A \cup F$
F	$\emptyset$	$\emptyset$	F	F
H	K	H	K	$H \cup K$
K	$\emptyset$	$\emptyset$	K	K
$A \cup F$	F	$A \cup F$	$\emptyset$	$A \cup F$
$A \cup H$	F, K	$A \cup H$	$F \cup K$	X
$A \cup K$	F	A	$F \cup K$	$A \cup F \cup K$
$F \cup H$	K	H	$F \cup K$	$F \cup H \cup K$
$F \cup K$	$\emptyset$	$\emptyset$	$F \cup K$	$F \cup K$
$H \cup K$	K	$H \cup K$	$\emptyset$	$H \cup K$
$A \cup F \cup H$	F, K	$A \cup F \cup K$	K	X
$F \cup H \cup K$	K	$H \cup K$	F	$F \cup H \cup K$
$A \cup H \cup K$	F, K	$A \cup H \cup K$	F	X
$A \cup F \cup K$	F	$A \cup F$	K	$A \cup F \cup K$
X	F, K	X	$\emptyset$	X

(4.43)

By examining the set of limit points so constructed for every subset of the Cartan system, and presuming that the functions that make up the forms are C2 differentiable (such that the Poincare lemma is true, $dd\omega = 0, any\ \omega$), it is easy to show that for all subsets of the Cartan topology every limit set is composed of the exterior differential of the subset thereby proving the conjecture that the exterior differential is a limit point operator relative to the Cartan topology.

Conclusion 22 *With respect to the Cartan topology, the exterior differential is a limit point generator.*

For example, the open subset, $A \cup H$, has the limit points that consist of F and K. The limit set consists of $F \cup K$ which can be derived directly by taking the exterior differentials of the elements that make up $A \cup H$; that is, $(F \cup A = d(A \cup H) = (dA \cup dH)$. Note that this open set, $A \cup H$, does not contain its limit points. Similarly for the closed set, $A \cup F$, the limit points are given by F which may be deduced by direct application of the exterior differential to $(A \cup F) : (F) = d(A \cup F) = (dA \cup dF) = (F \cup \emptyset) = (F)$.

4.3 Topological Torsion and Connected versus Non-connected Cartan topologies.

Topological torsion of a 1-form, A, is defined as the exterior product of the 1-form and its exterior derivative, $A\hat{}dA$. Topological torsion is different from, but can be related to, the Frenet torsion of a space curve and the affine torsion of a connection. If non-zero, Topological torsion has important topological properties. The Cartan topology as given in Table 2 is composed of the union of two sub-sets which are both open and closed,

$$(X = K_{Cl}(A) \cup K_{Cl}(H) = \{A \cup F\} \cup \{H \cup K\}), \tag{4.44}$$

a result that implies that the Cartan topology is not necessarily a connected topology. An exception exists if the topological torsion, H, and hence its closure, vanishes, for then the Cartan topology is connected. This extraordinary result has broad physical consequences. The connected Cartan topology based on a vanishing topological torsion is at the basis of most physical theories of equilibrium. In mathematics, the connected Cartan topology corresponds to the Frobenius integrability condition for Pfaffian forms. In thermodynamics, the connected Cartan topology is associated with the Caratheodory concept of inaccessible thermodynamic states [Hermann 1968], and the existence of an equilibrium thermodynamic surface. If the non-exact 1-form, Q, of heat generates a Cartan topology of null topological torsion, $H = Q\hat{}dQ = \emptyset$, then the Cartan topology built on Q is connected. Such systems are "isolated" in a topological sense, and the heat 1-form has a representation in terms of two and only two functions, conventionally written as: $Q = TdS$. Note again that a fundamental physical concept, in this case the idea of equilibrium, is a topological concept independent from geometrical properties of size and shape. Processes that generate the 1-form Q such that $Q\hat{}dQ = \emptyset$ are thermodynamically reversible. If $Q\hat{}dQ \neq \emptyset$, the process that generates Q is thermodynamically irreversible.

When the Cartan topology is connected, it might be said that all forces are extendible over the whole of the set, and that these forces are of "long range". Conversely when the Cartan topology is disconnected, the "forces" cannot be extended indefinitely over the whole domain of independent variables, but perhaps only over a single component. The components are not arc connected. In this sense, such forces are said to be of short range, as they are confined to a specific component. Note that this notion of short or long range forces does not depend upon geometrical size or scale. The physical idea of short or long range forces is a topological idea of connectivity, and not a geometrical concept of how far.

In an earlier article, these ideas were formulated intuitively in order to give an explanation of the "four forces" of physics. The earlier work was based upon experience with differential geometry [RMK 1975 c]. The features of the Pfaff sequence were used to establish equivalence classes for 1-forms constructed from known example metric field solutions, $g_{\mu v}$, to the Einstein field equations. The original ideas, based upon experience with systems in differential geometry, can now be given credence based upon differential topology. The construction of a 1-form, $A = g_{\mu 4}dx^{\mu}$, whose coefficients are the space time components of a metric tensor, will divide the topology into equivalence classes depending upon the number of non-zero elements of its Pfaff sequence. This number has been defined above as the Pfaff topological dimension. Long range parity preserving forces due to gravity (Pfaff dimension 1) and electromagnetism (Pfaff dimension 2) are to be associated with a Cartan Topology that is connected ($H = A\hat{}F = A\hat{}dA = 0$). Both the strong force (Pfaff dimension 3) and the weak force (Pfaff dimension 4) are "short" range ($H \neq 0$) and are to be associated with a disconnected Cartan topology. The strong force is parity preserving ($K = 0$) and the weak force is not ($K \neq 0$). The fact that the Cartan topology is not necessarily connected is the topological (not metrical) basis that may be used to distinguish between short and long range forces.

In much of our physical experience with nature, it appears that the disconnected domains of Pfaff dimension 3 or more are often isolated as nuclei, while the surrounding connected domains of Pfaff dimension 2 or less appears as fields of charged or non-charged molecules and atoms. However, part of the thrust of this article is to demonstrate that such disconnected topological phenomena are not confined to microscopic systems, but also appear in a such mundane phenomena as the flow of a turbulent fluid. Physical examples of the existence of topological torsion (and hence a non-connected Cartan topology) are given by the experimental appearance of what appear to be coherent structures in a turbulent fluid flow.

To prove that a turbulent flow must be a consequence of a Cartan topology that is not connected, consider the following argument: First consider a fluid at rest and from a global set of unique, synchronous, initial conditions generate a vector field of flow. Such flows must satisfy the Frobenius complete integrability theorem, which requires that $A\hat{}dA = 0$. The Cartan topology for such systems is connected, and the Pfaff dimension of the domain is 2 or less. Such domains do not support topological torsion (the Helicity vanishes). Such globally laminar flows are to be distinguished from flows that reside on surfaces, but do not admit a unique set of connected sychronizeable initial conditions. Next consider turbulent flows which, as the anti-thesis of laminar flows, can not be integrable in the sense of Frobenius; such turbulent domains support topological torsion ($A\hat{}dA \neq 0$), and therefore a disconnected Cartan topology [RMK1976]. The connected components of the disconnected Cartan topology can be defined as the (topologically) coherent structures of the turbulent flow.

Note that a domain can support a homogeneous topology of one component and then undergo continuous topological evolution to a domain with some interior holes. The domain is simply connected in the initial state, and multiply connected in the final state, but still connected. However, consider the dual point of view where the originally connected domain consists of a homogeneous space that becomes separated into multiple components. The evolution to a topological space of multiple components is not continuous. It follows that the case of a transition from an initial laminar state ($H = 0$) to the turbulent state ($H \neq 0$) is a transition from a connected topology to a disconnected topology. Therefore the transition to turbulence is NOT continuous. However, note that the decay of turbulence can be described by a continuous transformation from a disconnected topology to a connected topology. Condensation is continuous, gasification is not. It is demonstrated below that relative to the Cartan topology all C2 differentiable, $\mathbf{V}$, acting on C2 p-forms by means of the Lie differential are continuous. The conclusion is reached that the transition to turbulence must involve transformations that are not C2, hence can occur only in the presence of shocks or tangential discontinuities.

4.4 Applications of Cartan's Topological Structure

4.4.1 Continuous processes

A topological structure is defined to be enough information to decide whether a transformation is continuous or not [Gellert 1977]. The classical definition of continuity depends upon the idea that every open set in the range must have an inverse image in the domain. This means that topologies must be defined on both the initial and final state, and that somehow an inverse image must be defined. Note that the open sets of the final state may be different from the open sets of the initial state, because the topologies of the two states can be different.

There is another definition of continuity that is more useful for it depends only on the transformation, and not its inverse, explicitly. A transformation is continuous if and only if the image of the closure of every subset is included in the closure of the image. This means that the concept of closure and the concept of transformation must commute for continuous processes. Suppose the forward image of a 1-form A is Q, and the forward image of the set $F = dA$ is Z. Then if the closure, $K_{Cl}(A) = A \cup F$ is included in the closure of $K_{Cl}(Q) = Q \cup dQ$, for all sub-sets, the transformation is defined to be continuous. The idea of continuity becomes equivalent to the concept that the forward image Z of the limit points, dA, is an element of the closure of Q [Hocking 1961]:

> A function that produces an image $f[A] = Q$ is continuous iff for every subset A of the Cartan topology, $Z = f[dA] \subset K_{Cl}(Q) = (Q \cup dQ)$.

The Cartan theory of exterior differential systems can now be interpreted as a topological structure, for every subset of the topology can be tested to see if the process of closure commutes with the process of transformation. For the Cartan topology, this emphasis on limit points rather than on open sets is a more convenient method for determining continuity. A simple evolutionary process, $X \Rightarrow Y$, is defined by a map Φ. The map, Φ, may be viewed as a propagator that takes the initial state, X, into the final state, Y. For more general physical situations the evolutionary processes are generated by vector fields of flow, $\mathbf{V}$. The trajectories defined by the vector fields may be viewed as propagators that carry domains into ranges in the manner of a convective fluid flow. The evolutionary propagator of interest to this article is the Lie differential with respect to a vector field, $\mathbf{V}$, acting on differential forms, Σ [Bishop 1968].

The Lie differential has a number of interesting and useful properties.

1. The Lie differential does not depend upon a metric or a connection.

2. The Lie differential has a simple action on differential forms producing a resultant form that is decomposed into a transversal and an exact part:

$$L_{(\mathbf{V})}\omega = i(V)d\omega + di(V)\omega. \tag{4.45}$$

   This formula is known as "Cartan's magic formula". For those vector fields V which are "associated" with the form ω, such that $i(V)\omega = 0$, the Lie differential becomes equivalent to the covariant differential of tensor analysis. Otherwise the two differential concepts are distinct.

3. The Lie differential may be used to describe deformations and topological evolution. Note that the action of the Lie differential on a 0-form (scalar function) is the same as the directional or convective differential of ordinary calculus, $L_{(\mathbf{V})}\Phi = i(V)d\Phi + di(V)\Phi = i(V)d\Phi + 0 = \mathbf{V} \cdot grad\Phi$. It may be demonstrated that the action of the Lie differential on a 1-form will generate equations of motion of the hydrodynamic type. In fact Arnold calls the Lie differential the "convective" or "Fisherman's" differential.

4. 4. With respect to vector fields and forms constructed over C2 functions, the Lie differential commutes with the closure operator. Hence, the Lie differential (restricted to C2 functions) generates transformations on differential forms which are continuous with respect to the Cartan topology.

The last statement requires a formal proof:

Proof. First construct the closure, $\{\Sigma \cup d\Sigma\}$. Next propagate Σ and $d\Sigma$ by means of the Lie differential to produce the decremental forms, say Q and Z,

$$L_{(\mathbf{V})}\Sigma = Q \quad and \quad L_{(\mathbf{V})}d\Sigma = Z. \tag{4.46}$$

Now compute the contributions to the closure of the final state as given by $\{Q \cup dQ\}$. If $Z = dQ$, then the closure of the initial state is propagated into the closure of the final state, and the evolutionary process defined by $\mathbf{V}$ is continuous. However,

$$dQ = dL_{(\mathbf{V})}\Sigma = di(V)d\Sigma + dd(i(V)\Sigma) = di(V)d\Sigma \tag{4.47}$$

as $dd(i(V)\Sigma) = 0$ for C2 functions. But,

$$Z = L_{(\mathbf{V})}d\Sigma = d(i(V)d\Sigma) + i(V)dd\Sigma = di(V)d\Sigma \tag{4.48}$$

as $i(V)dd\Sigma = 0$ for C2 p-forms. It follows that $Z = dQ$, and therefore $\mathbf{V}$ generates a continuous evolutionary process relative to the Cartan topology. QED ■

It is to be noticed that this concept of a topological structure is developed in terms of the action of the Lie differential acting on a 1-form. The method does not depend upon metric or connection.

Certain special cases arise for those subsets of vector fields that satisfy the equations, $d(i(\mathbf{V})\Sigma) = 0$. In these cases, only the functions that make up the p-form, Σ, need be C2 differentiable, and the vector field need only be C1. Such processes will be of interest to symplectic processes, with Bernoulli-Casimir invariants, and to the analysis of tangential discontinuities.

By suitable choice of the 1-form of action it is possible to show that the action of the Lie differential on the 1-form of action can generate the Navier Stokes partial differential equations. See section 3.4.2 or [RMK 1975 a] [RMK 1992 a]. The analysis above indicates that C2 differentiable solutions to the Navier-Stokes equations can not be used to describe the transition to turbulence. The C2 solutions can, however, describe the irreversible decay of turbulence to the globally laminar state.

4.4.2 Uniform Continuity, Frozen - in Fields, the Master Equation

A starting point for many discussions of the magnetic dynamo and allied problems in hydrodynamics starts with what has been called the "master equation" [Kochine 1964],

$$\text{Master equation} \;\; Curl(\mathbf{V} \times \mathbf{B}) = \partial\mathbf{B}/\partial t. \tag{4.49}$$

Using the Cartan methods it may be shown that this equation is equivalent to the constraint of "uniform" continuity relative to the Cartan topology. Moreover, it is easy to show these constraints generate symplectic processes which include Hamiltonian evolutionary systems, such as Euler flows, as well as a number of other evolutionary processes which are continuous, but not homeomorphic. In addition a criteria can be formulated to develop an extension of the "helicity" conservation law to a more general setting.

The proof of these results produces a nice exercise in use of the Cartan theory. Consider a 1-form A that satisfies the exterior differential system

$$F - dA = 0, \tag{4.50}$$

where A is a 1-form of Action, with twice differentiable coefficients (potentials proportional to momenta) which induce a 2-form, F, of electromagnetic intensities ($\mathbf{E}$ and $\mathbf{B}$, related to forces). The exterior differential system is a topological constraint that in effect defines field intensities in terms of the potentials.

Now search for all vector fields that leave the 2-form F an absolute invariant of the flow; that is, search for all vectors that satisfy Cartan's magic formula

$$L_{(\mathbf{V})}F = i(V)dF + di(V)F = 0 + di(V)F = 0. \tag{4.51}$$

For C2 functions, the term involving dF vanishes, leaving the expression,

$$\begin{aligned} L_{(\mathbf{V})}F &= di(V)F && (4.52)\\ &= d\{(\mathbf{E} + \mathbf{V} \times \mathbf{B}) \cdot d\mathbf{r} - (\mathbf{E} \cdot \mathbf{V})dt\} && (4.53)\\ &= \{curl(\mathbf{E} + \mathbf{V} \times \mathbf{B})\}_z dy\hat{\ } dz... && (4.54)\\ &+ \{\partial(\mathbf{E} + \mathbf{V} \times \mathbf{B})/\partial t + grad(\mathbf{E} \cdot \mathbf{V})\} \cdot d\mathbf{r}\hat{\ } dt && (4.55)\\ &= 0. && (4.56) \end{aligned}$$

Setting the first three factors to zero yields

$$curl(\mathbf{E} + \mathbf{V} \times \mathbf{B}) = 0 \tag{4.57}$$

From the Maxwell Faraday equations for C2 functions, $curl\mathbf{E} = -\partial\mathbf{B}/\partial t$, and when this expression is substituted into the above equation, the "master" equation given above is the result. Now recall that dF generates the limit points of A, and if $F = dA$ is a flow invariant, then all limit points are flow invariants relative to the Cartan topology. This result implies that the vector fields, $\mathbf{V}$, that satisfy the constraints of the "master equation" are uniformly continuous evolutionary processes, as the limit points, $F = dA$, of the 1-form A are flow invariants, and the lines of vorticity are "frozen-in" the flow. Non-uniform continuity would imply that the limit points are not invariants of the process, but that the closure of the limit points of the target range includes the limit points of the initial domain. Such processes would correspond to a folding of the "lines" of vorticity, which preserve the limit points, but not their sequential order. A second criteria for limit point invariance is given by the equation,

$$\{\partial(\mathbf{E}+\mathbf{V}\times\mathbf{B})/\partial t + grad(\mathbf{E}\cdot\mathbf{V})\} = 0. \tag{4.58}$$

The formula indicates that limit point invariance can occur in the presence of input-output power, $\mathbf{E}\cdot\mathbf{V} \neq 0$.

The criteria for frozen-in fields is established as a constraint of uniform continuity on the admissible vector fields,

$$\text{Uniform Continuity:} \quad di(V)dA = di(V)F = 0. \tag{4.59}$$

The solution vector fields, V, subject to this constraint can be put into three global categories:

1. Extremal (Hamiltonian) $i(V)F = 0.$
2. Bernoulli-Casimir (Hamiltonian) $i(V)F = d\Theta.$
3. Helmholtz-Symplectic $i(V)F = d\Phi + \gamma_{harmonic}$

The first category can exist only on domains of support of F which are of odd Pfaff dimension, but then the solution vector is unique to within a factor. In the other categories, the solution vector need not be unique. Vector fields that satisfy the equation for uniform continuity are said to be symplectic relative to the 1-form, A. Vector fields that belong to categories 1 and 2 have a Hamiltonian representation. Vector fields that belong to category 1, are said to be "extremal" relative to the 1-form, A.

When the concepts are applied to the integrals of the 2-form F, then the criteria for invariance of the flux integral depends on the topology of the integration domain. If the integration area of the 2-form is a boundary or a cycle of a 3 dimensional domain, the flux integral over the closed boundary or cycle is always a flow invariant. If the integration area is bounded, then by Stokes theorem the flux integral depends only on the boundary conditions: F or $i(V)F$ must vanish on the boundary, or when integrated over the boundary.

Chapter 5
CONTINUOUS TOPOLOGICAL EVOLUTION

5.1 Introduction

In this chapter, the theory of Continuous Topological Evolution* will be developed in terms of Cartan's theory of exterior differential forms. The motivation is based upon the concept that thermodynamic irreversibility implies topological change. The basis for such a postulate follows from the fact that if an evolutionary process is described by a map, Φ, between initial and final states, and if the map is not continuously reversible, then the observable topology of the final state is different from the observable topology of the initial state. Cartan's methods can be used to extend these concepts to the dynamics of physical systems that admit description in terms of exterior differential forms. It is remarkable that the mathematical development leads to recognizable thermodynamic features which permit the determination of classes of processes which are reversible or irreversible. For example, all Hamiltonian processes are thermodynamically reversible. In fact, all Helmholtz processes are thermodynamically reversible. An essential feature of irreversible processes is that they involve the evolution of what has been defined as Topological Torsion. The appearance of Topological Torsion is a signal that the thermodynamic system is not in equilibrium.

The observation of topological change, with the production and destruction of defects and holes, lines of self-intersection and other obstructions, will be the signature of topological irreversible evolution. Topological change can occur discontinuously as in a cutting process, or continuously, as in a pasting process. Such continuous but irreversible processes can be used to study the decay of turbulence, but not its creation. The production of disconnected components will be the signature of those discontinuous processes which are necessary to describe the creation and evolution of chaotic but perhaps reversible evolution, or turbulent, irreversible evolution. In this article, emphasis will be place upon those processes which are C2 continuous, but not reversible. The more difficult problem of C1 continuity in producing tangential discontinuities can be applied to the generation of wakes [RMK 1995] [RMK 1993 a]. Remarkably such C1 processes are locally adiabatic.

Processes or maps that preserve topology are technically described as homeomorphisms [Hocking 1961]. Homeomorphisms are both continuous and reversible.

*The fundamental ideas were initially formulated about 1981.

Homeomorphic reversibility means that the inverse function, Φ^{-1}, must exist and must be continuous. Topological properties, such as orientability, compactness, connectivity, hole count, lines of self-intersection, pinch points, and Pfaff dimension are invariants of homeomorphisms, but geometrical properties such as size and shape are not necessarily invariants of homeomorphic deformations. In fact an elementary method of recognizing topological properties is to observe those properties that stay the same under continuous deformations that do not preserve size and shape.

The theory of Continuous Topological Evolution is developed herein in terms of physical systems that undergo certain thermodynamic processes. The physical system is assumed to be modeled in terms of the topological features inherent in Cartan's theory of exterior differential systems. The thermodynamic process will be defined in terms of a vector field, V, and its effect on the differential forms that make up the exterior differential system. The action of the process will be defined in terms of the Lie differential with respect to V acting on the differential forms that make up the exterior differential system, and which in turn approximate the physical system. The methods lead to concepts that are coordinate free and are well behaved in any reference system. A precise non-statistical definition of thermodynamic irreversibility will be stated, and a cohomological equivalent of the first law of thermodynamics will be derived and studied relative to the single constraint of continuous but irreversible topological evolution. Remarkably, many intuitive thermodynamic concepts can be stated precisely, without the use of statistics, in terms of the theory of continuous topological evolution based on the Cartan topology.

Given a topology on the final state and a map from an initial state to the final state it is always possible to define a topology on the initial state such that the given transformation, or even a given set of transformations, is continuous. However, the topologies of the initial and final states need not be the same; hence the map need not be reversible. Recall that with respect to a discrete topology all maps from the initial to final state are continuous, while relative to the concrete topology, only the constant functions are continuous [Bishop 1968]. A first problem of a theory of topological evolution is to devise a rule for constructing a topology that is physically useful and yet is neither too coarse nor too fine. Such a rule is necessary for the concept of continuity of an evolutionary transformation is defined relative to the topologies of the initial and final states. In this article the topological rules will be made by the specification of an exterior differential system that will model the physical system of interest. Many physical systems appear to be adequately modeled by 1-form of Action.

Physical exhibitions of continuous and discontinuous transformations can be achieved through the deformations of a soap film attached to a wire frame. For example, a soap film attached to a single closed, but double, loop of wire can be deformed from a non-orientable surface into an orientable surface continuously (the topological property of orientability is changed). That is, the soap film can be transformed continuously from a Moebius band into a cylindrical strip. As another example, consider an initial state where a soap film is attached to two slightly separated but concentric circular wire loops. The resulting surface is a minimal surface of a single component. As the separation distance of the concentric rings forming the boundary of the soap film is slowly increased, the minimal surface is stretched until a critical separation is reached. Then, without further displacement, the surface spontaneously continues to deform to form a "two sheeted" cone connected at a singular vertex point. The surface separates at the conical singularity, and the two separate sheets of the cone continue to collapse to form a minimal surface of two components. The final state consists of two flat films attached, one each, to each ring. The originally connected minimal surface undergoes a topological (phase) change to where it becomes two disconnected (still minimal) surfaces. An example of this topological transition in the surface of null helicity density has been described in conjunction with the parametric saddle node Hopf bifurcation of a Navier-Stokes flow [RMK 1991 a].

In this article the fundamental set, X, will be the points $\{x, y, z, t, ...\}$ that make up an N-dimensional space. Upon this fundamental set will be constructed arbitrary subsets, such as functions, tensor fields and differential forms. Many different topologies may be constructed on the fundamental set in terms of special classes of subsets that obey certain rules of logical closure. In fact the very existence of subsets can be used to define a course topology on X in terms of a topological base. The topological base consists of those subsets whose unions form a special collection of all possible subsets that is closed under logical union and intersection. This special collection of subsets will be defined as the open sets of a topology. The topological base can be used to define a topological structure. A space is said to have a topological structure if it is possible to determine if a transformation on the space is continuous [Gellert 1977].

5.2 Continuity

The classic definition [Lipschutz 1965] of a continuous transformation between a set X with topology $T1$ to a set Y with a topology $T2$ states that the transformation is continuous if and only if the inverse image of open sets of $T2$ are open sets of $T1$. This definition can be made transparent by use of a simple point set example.

Consider two sets of 4 points, an initial state, $\{a, b, c, d\}$ and a final state $\{x, y, z, t\}$. Define an open set topology on the initial state $T1 = [X, \emptyset, a, ab, abc]$ and a open set topology on the final state $T2 = [X, \emptyset, x, y, xy, yzt]$. The transformation considered is exemplified by the Figure 1.

The open set (y) has a pre-image (a) which is open. The open set (yzt) has a pre-image (abc) which is open. Hence the Map is continuous. (The open sets that involve x are not included as the map does involve x.) However, the Inverse Image mapping is not continuous for the open set (ab) has a preimage as (yz) but (yz) is not an open set of Y. The point set example demonstrates the idea of a continuous but not homeomorphic mapping. The objective herein is to examine such maps in

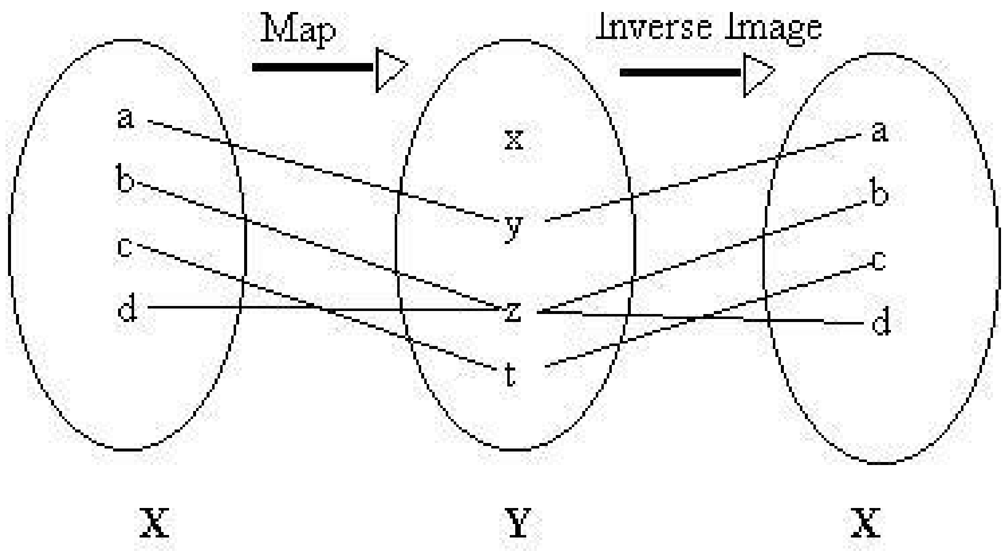

Figure 1 Figure 1

terms of exterior differential systems.

There exists another more useful method of defining continuity which does not depend explicitly on being able to define open sets and their inverse images. This second method of defining continuity is based on the concept of closure. The closure of a set can be defined in (at least) two ways:

1. The closure of a set is the union of the interior and the boundary of a set.
2. The closure of a set is the union of the set and its limit points.

The first definition of closure is perhaps the most common, and is often exploited in geometric situations, where a metric has been defined and a boundary can be computed easily. The second definition of closure is independent from metric and is the method of choice in this article, both for defining continuity and establishing a topological structure. In terms of the concept of closure, a transformation is continuous if and only if for every subset, the image of the closure of the initial subset is included in the closure of the image of that subset [Lipschutz 1965]. Another way of stating this idea is

3. A map is continuous iff the limit points of every subset in the domain permute into the closure of the subsets in the range.

If a method for constructing a closure operator (a Kuratowski closure operator K of a subset relative to a topology) can be defined, then a strong version of continuity would imply that the Kuratowski closure operator commutes with those transformations which are continuous. The test for continuity would be to construct the closure of an arbitrary subset on the initial state, and then to propagate the elements of the closure to the final state by means of a transformation. If this result is the same as the result obtained by first propagating the subset to the final state by means of the transformation, and then constructing its closure on the final state, then the map is continuous. Note that such a procedure has defined a topological structure which will be exploited in this article, for the subsets of interest will be defined as a Cartan system of exterior differential forms, Σ, on X. The topological base defined by this class of sets is too course to be of interest. Hence the Cartan exterior derivative will be used to generate additional sets of forms, $d\Sigma$, which when adjoined to the initial system of forms defines the Kuratowski closure of the Cartan system as the system of forms, $K(\Sigma) = \{\Sigma \cup d\Sigma\}$.

The Cartan exterior product may be used as a convenient intersection operator between sets of differential forms. Starting from the system, $\{\Sigma\}$, the Cartan topology is then determined by the construction of the Cartan-Pfaff sequence, which consists of all possible intersections that may be constructed from the subsets of the closure of the differential system:

$$Pfaff\ \ Sequence: \ \{\Sigma, d\Sigma, \Sigma\hat{}d\Sigma, d\Sigma\hat{}d\Sigma, ...\}. \tag{5.1}$$

The subsets of the Cartan topological space consist of all possible unions of the subsets that make up the Pfaff sequence. The Cartan topology will be constructed from a topological basis which consists of the odd elements of the Pfaff sequence, and their closures:

$$the\ \ Cartan\ \ topological\ base: \{\Sigma, K(\Sigma), \Sigma\hat{}d\Sigma, K(\Sigma\hat{}d\Sigma), ...\}. \tag{5.2}$$

With respect to a topological base constructed from a single 1-form of Action it has been shown [34] that the Cartan exterior derivative may be viewed as a closure or limit point operator. Given any subset of the Cartan topological space, the exterior derivative of that subset generates its limit points, if any. This is a remarkable result, for as will be demonstrated below, all C2 vector fields acting through the concept of the Lie differential on a set of differential forms, with C2 coefficients, generate continuous transformations with respect to the Cartan topology. Moreover, the Cartan topology is disconnected if $\Sigma\hat{}d\Sigma \neq 0$ is not zero.

5.3 The evolutionary process

5.3.1 Cartan's Magic formula

An arbitrary evolutionary process, $X \Rightarrow Y$, is defined by a map Φ. The map, Φ, may be viewed as a propagator that takes the initial state, X, into the final state, Y. In this article the evolutionary processes to be studied are asserted to be generated by vector fields, $\mathbf{V}$. However, evolutionary vector fields need not be topologically constrained such that they are generators of a single parameter group. In other words, kinematics without fluctuations is not imposed a priori. The local trajectories defined by the vector fields may be viewed as propagators that carry domains into ranges in the manner of a convective fluid flow. The evolutionary propagator of interest to this article is the Lie differential with respect to a vector field ,$\mathbf{V}$, acting on differential forms, Σ. The Lie differential has a number of interesting and useful properties.

1. The Lie differential does not depend upon a metric or a connection.

2. The Lie differential has a simple action on differential forms producing a resultant form that is decomposed into a transversal and an exact part:

$$L_{(\mathbf{V})}\Sigma = i(\mathbf{V})d\Sigma + di(\mathbf{V})\Sigma. \tag{5.3}$$

   Marsden [Marsden 1994] calls this Cartan's Magic Formula (see below).

3. The Lie differential may be used to describe both deformations and topological evolution. For vector fields $\mathbf{V}$ that are singly parameterized, Cartan's magic formula has a dynamic interpretation as a propagator down a set of flow trajectories. However, the formula can be used in an algebraic manner such that the vector field need not be singly parameterized, and can admit (topological) fluctuations about the kinematic perfection required by a singly parameterized definition of a limit set.

4. If the Lie differential of Σ is zero, then Σ is a (Bernoulli type) invariant along the flow trajectories generated by $\mathbf{V}$.

5. With respect to vector fields and forms constructed over C2 functions, the Lie differential commutes with the Kuratowski closure operator. In fact $dL_{(\mathbf{V})}\Sigma = L_{(\mathbf{V})}d\Sigma$. Hence, the Lie differential generates transformations on differential forms which are continuous with respect to the Cartan topology.

Note that the action of the Lie differential on a 0-form (scalar function) is the same as the directional derivative of ordinary calculus,

$$L_{(\mathbf{V})}\varphi = i(\mathbf{V})d\varphi + 0 \Rightarrow \mathbf{V}\cdot grad\varphi. \tag{5.4}$$

5.3.2 The Lie differential and C2 continuity

The first four properties of the Lie differential listed above appear in the literature, but the extraordinary property that all C2 vector fields that propagate C2 differential forms in either the dynamic or algebraic manner are continuous relative to the Cartan topology requires proof:

Proof. Given Σ, first construct the closure, $\Sigma \cup d\Sigma$. Next propagate Σ and $d\Sigma$ by means of the Lie differential to produce the decremental or residue forms, say Q and Z,

$$L_{(\mathbf{V})}\Sigma = Q \quad and \quad L_{(\mathbf{V})}d\Sigma = Z. \tag{5.5}$$

Now compute the contributions to the closure of the final state as given by $Q \cup dQ$. If $Z = dQ$, then the closure of the initial state is propagated into the closure of the final state, and the evolutionary process defined by $\mathbf{V}$ is continuous. However,

$$dQ = dL_{(\mathbf{V})}\Sigma = di(\mathbf{V})d\Sigma + dd(i(\mathbf{V})\Sigma) \tag{5.6}$$

and

$$Z = L_{(\mathbf{V})}d\Sigma = (i(\mathbf{V})dd\Sigma) + di(\mathbf{V})d\Sigma. \tag{5.7}$$

The difference becomes

$$Z - dQ = (i(\mathbf{V})dd\Sigma) - dd(i(\mathbf{V})\Sigma). \tag{5.8}$$

The concept of continuity requires that $Z - dQ \Rightarrow 0,$ forming an exterior differential system. For vector fields and differential forms with coefficient functions that are twice differentiable, the continuity condition is always satisfied relative to the Cartan topology (the Poincare lemma states that $dd\omega = 0$ where ω is any differential p-form with C2 coefficients). Therefore subject to the constraint of C2 differentiability, every vector field, $\mathbf{V}$, generates a continuous evolutionary process relative to the Cartan topology. The set $\{\Sigma, d\Sigma\}$ forms a differential ideal (closure) which is permuted into the differential ideal $\{Q, dQ\}$ by the action of the Lie differential with respect to $\mathbf{V}$. *QED.* ■

The Lie differential also can be used to make some sense out of certain classes of discontinuous evolutionary processes (which are not C2). For example, consider a vector field $\mathbf{V} = \rho\mathbf{v}$ where the support function, ρ, is not C2. Then, the action of the Lie differential produces the discontinuity or excess function,

$$Z - dQ = -d(d(i(\rho\mathbf{v})\Sigma)) = d\{d\rho\hat{}(i(\mathbf{v})\Sigma) + d\rho\hat{}d(i(\mathbf{v})\Sigma\} = d(\Theta). \tag{5.9}$$

If $\Theta = i(\rho\mathbf{v})\Sigma \Rightarrow 0$, then the second differential is zero even though the vector field $\rho\mathbf{v}$ is not C2 continuous. This equation is of use in the study of tangential discontinuities, such as wakes in hydrodynamic systems, and shocks, in physical systems, which unlike the processes, are C2 smooth.

5.3.3 C1 Continuity

Note that the special situation arise when $(i(\mathbf{v})\Sigma) = 0$, for then $\{d\rho\hat{}(i(\mathbf{v})\Sigma) + d\rho\hat{}d(i(\mathbf{v})\Sigma\} \Rightarrow 0$ without the second differentiation. The process is continuous even when $\rho\mathbf{v}$ is C1. Such special vector fields were defined above to be *associated* vector fields, and have the properties that the Lie differential has the same abstract form as the covariant derivative. In a thermodynamic system, with $\Sigma \Rightarrow A$, the 1-form of Action that encodes the properties of the physical system, the associated direction field describes a locally adiabatic process. Such adiabatic processes are continuous even though the density function ρ has discontinuities. Also, it can be shown that for even dimensional symplectic manifolds, there is a unique vector direction field that satisfies $i(\mathbf{T}_4)\Sigma = 0$ and $L_{(\mathbf{T}_4)}\Sigma = \Gamma\ \Sigma$. This direction field, $\mathbf{T}_4$, will generate thermodynamically irreversible evolution, and is continuous if C0. The continuity of $\mathbf{T}_4$ is not uniformly continuous.

5.4 Topological Evolution

5.4.1 Evolutionary Invariants.

If the direction field generated by **V** acting on a Cartan system of forms satisfies the equations

$$L_{(\mathbf{V})}\Sigma = 0 \quad and \tag{5.10}$$
$$L_{(\mathbf{V})}d\Sigma = 0. \tag{5.11}$$

then, with respect to such evolutionary processes, the forms of the closure are said to be absolute invariants. It follows that each element that makes up the Cartan topological base [See Chapter 4] is also invariant, such that the whole Cartan topology is invariant. As **V** is continuous, and the topology is preserved, those vector fields, **V**, that satisfy the equations above must be homeomorphisms, and are reversible. In other words, $Q = 0$ and $dQ = 0$ are sufficient conditions that **V** be reversible.

However, for continuous transformations on the elements of the C2 Cartan topology the general equations of topological evolution become,

$$L_{(\mathbf{V})}\Sigma = Q \tag{5.12}$$

and

$$L_{(\mathbf{V})}d\Sigma = dQ, \tag{5.13}$$

from which it follows that

$$L_{(\mathbf{V})}\Sigma\hat{\ }d\Sigma = Q\hat{\ }d\Sigma + \Sigma\hat{\ }dQ \tag{5.14}$$

and

$$L_{(\mathbf{V})}d\Sigma\hat{\ }d\Sigma = 2dQ\hat{\ }d\Sigma. \tag{5.15}$$

As these equations of continuous topological evolution imply that the elements of the topological base may not be constant, then specific tests must be made to determine what features of the topology are changing, if any. For if it can be determined that the topology is indeed modified by the evolutionary process, then the process generated by this class of vector fields, **V**, is continuous, but need not be reversible.

When $dQ \neq 0$, the limit points of the Cartan topological structure are not invariants, and it would be natural to expect that the topology is not constant. However, even if Q is closed, such that $dQ = 0$, it may be true that Q contains harmonic components, such that deRham cohomological classes of Σ are not evolutionary invariants. Even though the topology of the initial state is not the same as the topology of the final state (for the "hole" count of the initial state is not the same as the hole count of the final state) it is not necessarily true that such continuous processes are thermodynamically irreversible.

5.4.2 Deformation Invariants.

Recall the topological requirements that define a continuous process. The interest is to find a method of construction of topological properties and then see if they change under the application of a continuous process.

Definition 23 *A continuous process is defined as a map from an initial state of topology $T_{initial}$ into a final state of perhaps different topology T_{final} such that the limit points of the initial state are permuted among the limit points of the final state [Lipschutz 1965]. If the ordering of the limit points is invariant, the process is uniformly continuous. If the ordering (as in a folding of a boundary) or the number of the limit sets is changed the process is non-uniformly continuos.*

A simple description of a topological property (invariant of a homeomorphism) is an object that is a deformation invariant. Consider a rubber sheet with three holes. Stretch the rubber sheet. The holes may be deformed but the fact that there are 3-holes stays the same under small deformations. The concept of three holes is a topological property. If for some reason a fourth holed should appear under continuous deformation, then the topological property has changed. It is remarkable that topologically coherent objects can be constructed in terms of open and closed integrals which are deformation invariants.

> A topological deformation invariant is defined as an integral of a p-form over a p dimensional manifold, which may or may not be a cycle, *zpd*, such that the Lie differential of the integral of the p-form ω with respect to a singly parameterized vector field, ρV^k, vanishes, for any choice of deformation parameter, ρ.

$$\text{Integral Deformation Invariant}: \quad L_{(\rho V^k)} \int_p \omega = 0 \quad \textit{any } \rho \tag{5.16}$$

Those objects that are integral deformation invariants represent topological, not geometric properties.

Absolute Integral Invariants

There are two types of invariant integrals, Absolute and Relative integral invariants. If the p-form is exact, the Absolute integral invariant places conditions only on the boundary of the domain of integration. For example, consider physical systems that can be defined by a 1-form of Action, A, the derived 2-form $F = dA$, is exact. It follows from Stokes theorem that the 2-dimension integral of F is an absolute integral deformation invariant with respect to *all* continuous processes that can be defined by a singly parameterized vector field, subject to a boundary condition that the net flux of F across the 1-dimensional boundary of M is zero:

$$L_{(\rho V^k)} \int\int_M F = \int\int_M i(\rho V^k) dF + \int\int_M d(i(\rho V^k)F) \tag{5.17}$$

$$= 0 + \int_{boundary\ M} i(\rho V^k)F \tag{5.18}$$

This concept is at the basis of the Helmholtz theorems of vorticity (or angular momentum per unit mass) conservation in hydrodynamics, and the conservation of flux in classical electromagnetism. Herein, this concept of deformation invariance of a topologically coherent structure will be written in the form of an exterior differential system, $F - dA = 0$, which is to be recognized as topological constraint. From Stokes theorem, the (2 dimensional) domain of finite support for F can not, in general, be compact without boundary, unless the Euler characteristic vanishes. There are two exceptional cases for two dimensional domains, the torus and the Klein-Bottle, but these situations require the additional topological constraint that $F\hat{}F \Rightarrow 0$. The fields in these exceptional cases must reside on these exceptional compact surfaces, which form topological coherent structures. Note that an evolutionary process could start with $F\hat{}F \neq 0$, and possibly evolve to a state with $F\hat{}F = 0$. If such residue states are compact without boundary, then they must be either tori or Klein bottles.

The same technique can be applied to non-exact but closed p-forms.

Relative Integral Invariants

If the integration of the exact 2-form, F, is over a closed two dimensional closed chain, designated as a 2 dimensional cycle, z2d (which may or may not be a 2 dimensional boundary), then the Integral is invariant for any deformation factor, ρ :

$$L_{(\rho V^k)} \int\int_{z2d} F = \int\int_{z2d} i(\rho V^k) dF + \int\int_{z2d} d(i(\rho V^k)F) = 0 + 0. \tag{5.19}$$

Close integrals of exact forms are always relative deformation integral invariants. However, the same technique can be applied to non-exact but closed p-forms.

If the conditions of relative integral invariance are applied to an arbitrary 1-form of Action, then the relative integral invariance condition becomes

$$L_{(\rho V^k)} \int_{z1d} A = \int_{z1d} i(\rho V^k) dA + \int_{z1d} d(i(\rho V^k)A) \tag{5.20}$$

$$= \int_{z1d} i(\rho V^k)F + 0. \tag{5.21}$$

It follows the $i(\rho V^k)dA$ must be zero on the cycle $z1d$ for any deformation parameter ρ. Cartan has shown that this is the condition that implies the process ρV^k satisfies the constraint i$(\rho V^k)F = 0$, and has a "Hamiltonian" representation.

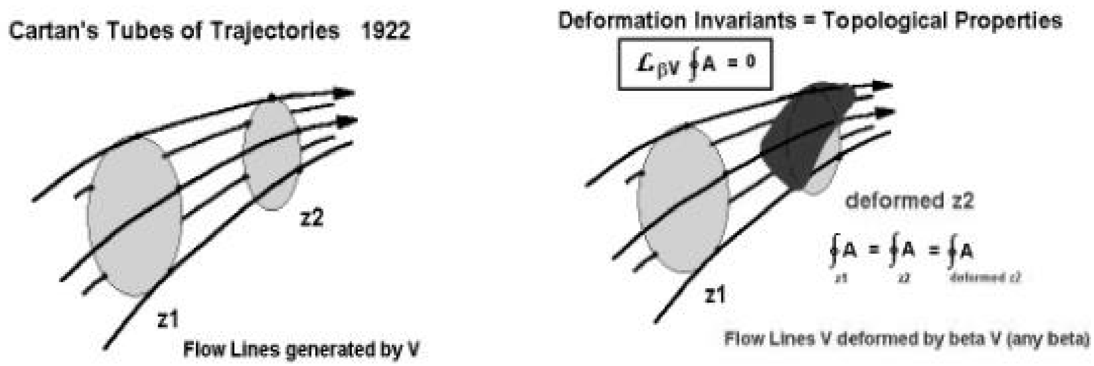

Figure 2

Cartan's Hamiltonian example

Consider the flow lines tangent to a given vector direction field, $\mathbf{V}(x, y, z, t...)$ that generates a dynamical system, $d\mathbf{r} - \mathbf{V}d\tau = 0$. By reparameterization, $\mathbf{V} \Rightarrow \beta(x, y, z, t...)\mathbf{V}$, the "speed" at which points move down the lines of flow can be changed, but the points that start on a particular flow line, remain upon the same flow line. Next consider a closed curve, Z1, intersecting the flow lines transversely for say $\tau = 0$. The flow lines that intersect Z1 form a "tube of trajectories" As τ increases to some value, say $\tau = 1$, the points of the closed curve appear to flow down the "tube of trajectories". The result of this convective evolution is to produce a new closed curve, Z2 Now choose another parameterization function β‘, which is equal to the original β at $\tau = 0$. The points that make up the closed curve Z1 now flow down the same tube of trajectories, but at $\tau = 1$ form a new closed curve $deformed\, Z2$ that may be considered as a deformation of the closed curve Z2.

5.5 Simple Systems

5.5.1 The Action 1-form and its Pfaff Sequence

Consider an arbitrary 1-form, A, on an n dimensional variety of independent functions. The exterior derivative of A produces a 2-form of closure points, $F = dA$, whose components are given by the expression, $F_{\mu\nu} dx^{\mu} \hat{}\, dx^{\nu}$. The combined set $\{A, F\}$ forms the closure of the set $\{A\}$. All possible intersections of the closure, $\{A, F, A\hat{}F, F\hat{}F...\}$, form what is defined herein as the Pfaff sequence for the domain $\{x, y, z, t\}$. In this article (for a 4 dimensional variety) these elements are defined as

$$\text{Topological ACTION} : A = A_{\mu} dx^{\mu} \tag{5.22}$$

$$\text{Topological VORTICITY}: F = dA = F_{\mu\nu} dx^{\mu}\hat{}dx^{\nu} \tag{5.23}$$

$$\text{Topological TORSION}: H = A\hat{}dA = H_{\mu\nu\sigma} dx^{\mu}\hat{}dx^{\nu}\hat{}dx^{\sigma} \tag{5.24}$$

$$\text{Topological PARITY}: K = dA\hat{}dA = K_{\mu\nu\sigma\tau} dx^{\mu}\hat{}dx^{\nu}\hat{}dx^{\sigma}\hat{}dx^{\tau}. \tag{5.25}$$

The union of all elements of the Pfaff sequence and their closures forms the elements of the Cartan topological base:

$$\{A, A \cup F, H, H \cup K...\}. \tag{5.26}$$

In order to take into account projective (and certain discontinuous) features, the vector fields of interest often will be scaled by a support function, ρ, such that $\mathbf{J} = \rho\mathbf{V}$. The fundamental equations of continuous evolution become

$$L_{(\rho\mathbf{V})} A = Q \tag{5.27}$$

$$L_{(\rho\mathbf{V})} F = dQ \tag{5.28}$$

$$L_{(\rho\mathbf{V})} H = Q\hat{}F + A\hat{}Q$$

$$L_{(\rho\mathbf{V})} K = 2(dQ\hat{}F) = 2d(Q\hat{}F) \tag{5.29}$$

Note that for the even dimensional elements of the Pfaff sequence, (F and K), the action of the Lie differential is to produce an exact form: dQ, for the Lie differential of F, and $2d(Q\hat{}F)$ for the Lie differential of K. As integrals of exact forms over closed cycles or boundaries of support vanish, then it is possible to formulate the first theorem.

Theorem 24 ***1.*** *All even dimensional Pfaff classes of p-forms, $dA = F, dA\hat{}dA = K$ ... are relative integral deformation invariants of continuous evolutionary processes relative to the Cartan topology.*

The closed integrals of F, K, ... are invariants of a continuous process as each integrand is exact, and the integral of an exact form over a closed domain vanishes. Hence if the functions are twice differentiable,

$$L_{(\rho\mathbf{V})} \int_{z2} F = \int_{z2} \{i(\rho\mathbf{V})dF + di(\rho\mathbf{V})F\} = \int_{z2} dQ \Rightarrow 0. \tag{5.30}$$

The closed integrals of $F, K, ...$ are invariants of any process generated by $\rho\mathbf{V}$ for integration domains, $z2$, that are boundaries or cycles.

This theorem is an extension of Poincare's theorem for even dimensional p-forms which are absolute integral invariants (the integration domain is not necessarily closed) with respect to the restricted set of Hamiltonian processes. It is important to realize that the theorem expresses the existence of (relative) integral deformation invariants (topological properties) with respect to processes that may be thermodynamically reversible or irreversible. It should be noted that the domains of support of the even dimensional Pfaff classes can not be compact without boundary.

5.5.2 The Action 1-form and Topological Fluctuations

For purposes of expose, the Cartan system, Σ, will be limited to a single 1-form of action, A, and perhaps a single pseudoscalar field, or N form density, ρ. The 1-form of Action, A, can be written in several equivalent formats known as the Cartan-Hilbert action:

$$A = A_\mu dx^\mu = \mathbf{p} \cdot d\mathbf{x} - \mathcal{H}(\mathbf{x}, \mathbf{v}, \mathbf{p}, t)dt = \mathcal{L}(\mathbf{x}, \mathbf{v}, t)dt + \mathbf{p} \cdot (d\mathbf{x} - \mathbf{v}dt) \tag{5.31}$$

The last representation indicates that the Action may be viewed abstractly in terms a Lagrangian function, $\mathcal{L}(\mathbf{x}, \mathbf{v}, t)$, and the kinematic fluctuations in position,

$$\Delta\mathbf{x} = (d\mathbf{x} - \mathbf{v}dt). \tag{5.32}$$

It is to be noted that the usual assumption for physical systems is to assume that there are zero kinematic fluctuations. In this sense, kinematic perfection prevails:

$$\Delta\mathbf{x} = (d\mathbf{x} - \mathbf{v}dt) \Rightarrow 0. \tag{5.33}$$

It is rarely appreciated that kinematic perfection is equivalent to an exterior differential system which imposes topological restrictions on the variety. For this example, the fluctuations, $\Delta\mathbf{x}$,are not presumed to be zero.

A simple count of the independent variables that are used to define the Cartan-Hilbert 1-form of action indicates that the "fluctuation" space is a variety of 3n+1⇒10 dimensions $(t, \mathbf{x}, \mathbf{v}, \mathbf{p})$. (For simplicity, the "particle" index n has been chosen to be unity). The coefficients, $\mathbf{p}$, act as Lagrange multipliers for the fluctuations, $\Delta\mathbf{x}$. However, it can be determined that the maximum Pfaff dimension of the sequence $\{A, dA, A\hat{}dA, dA\hat{}dA...)$ is of dimension 2n+2⇒8 and not dimension 10. Hence the 10 dimensional space is redundant, and an 8 dimensional space is adequate to describe the physical system in terms of a 1-form of Action. The given 1-form of Action therefore generates a non-compact symplectic manifold of dimension 8.

If the Lagrange multipliers $\mathbf{p}$ of the kinematic fluctuations $(d\mathbf{x} - \mathbf{v}dt)$ are restricted to be the canonical momenta, as defined by the ubiquitous formula, $\mathbf{p} = \partial\mathcal{L}/\partial\mathbf{v}$, the maximum Pfaff dimension is 7, forming a contact manifold historically defined as state space. If the Lagrange function $\mathcal{L}(\mathbf{x}, \mathbf{v}, t)$ is homogeneous of degree 1 in $\mathbf{v}$, then the maximal Pfaff dimension is 6, forming a symplectic Finsler manifold of dimension

6, the phase space of classical mechanics. This manifold cannot be compact without boundary.

If the contact manifold of dimension 7 is constrained by the equations of kinematic closure,

$$d(\Delta\mathbf{x}) = d(d\mathbf{x} - \mathbf{v}dt) \Rightarrow 0, \tag{5.34}$$

then the space of interest becomes the configuration space of 4 dimensions, a submanifold of the original symplectic structure of 8 dimensions. The constraints of kinematic closure imply that the velocity field is expressible as functions of a single variable, t; $\mathbf{v} \Rightarrow \mathbf{v}(t)$. Note that the more severe constraint of kinematic perfection, $\Delta\mathbf{x} = (d\mathbf{x} - \mathbf{v}dt) \Rightarrow 0$, implies that the maximal Pfaff dimension is 2, as in this case $A\hat{}dA = \mathcal{L}(\mathbf{x}, \mathbf{v}, t)dt\hat{}d\mathcal{L}(\mathbf{x}, \mathbf{v}, t)\hat{}dt = 0$. The Action defines a completely integrable 2 dimensional submanifold that, in this circumstance, is not compact without boundary. These concepts will be exploited in other examples given below.

5.6 Continuous Processes

5.6.1 Uniform and Non-uniform Continuity

The continuous processes are naturally divided into two main categories: those for which $dQ = 0$,defined as uniformly continuous flows (in the sense that the limit points of A are invariant) and those for which $dQ \neq 0$ (not uniformly continuous). Therefore, relative to the Cartan Topology,

$$\text{Uniform Continuity}: \;\; L_{(\mathbf{V})}dA = dQ = 0 \tag{5.35}$$

defines a uniformly continuous closed process, while

$$\text{Non-Uniform Continuity}: \;\; L_{(\mathbf{V})}dA = dQ \neq 0 \tag{5.36}$$

defines a non-uniformly continuous process. The vector field on a 4 dimensional domain that is in the direction of the current $A\hat{}F$ is a continuous but a not uniformly continuous process.

Uniform continuity implies that the limit sets are invariant. Continuity only requires that the limit points permute amongst themselves. For example a fold into pleats which are then pasted together is a processes that rearranges the limit points and is not therefor uniformly continuous. Hence uniform continuity is a more constrained situation. When $dQ = 0$, it is possible to formulate immediately the following theorem (Poincare) for closed flows:

Theorem 25 ***II.*** *All even dimensional Pfaff classes of p-forms, $dA = F, dA\hat{}dA = K, \ldots$ are invariants of evolutionary processes that satisfy $L_{(\mathbf{V})}(dA) = dQ = 0$ relative to the Cartan topology. The forms $F, K, \ldots$ form a set of absolute integral invariants with respect to uniformly continuous processes.*

The difference between Theorem 1 and Theorem 2 is that in Theorem 2, the integration chains need not be closed. Moreover, the process is uniformly continuous in Theorem 2, while the continuity is non-uniformly continuous in theorem 1. The proof of the theorem follows immediately by application of the Leibniz rule, using the constraint, $dQ = 0$:

$$L_{(\mathbf{V})}(dA\hat{}dA\hat{}dA..\hat{}dA) = \text{integer} \times \{L_{(\mathbf{V})}(dA)\}\hat{}dA\hat{}dA..\hat{}dA) = 0. \tag{5.37}$$

The integrands of the selected integrals are local invariants and so are their convected integrals.

The first application of theorem II gives,

$$L_{(\mathbf{V})}(dA) = L_{(\mathbf{V})}F = 0 \tag{5.38}$$

which is the equivalent of Helmholtz' theorem [Lamb 1945]. The theorem often is interpreted as the local conservation of angular momentum per unit moment of inertia, or the conservation of Topological Vorticity.

The second application of theorem II gives:

$$L_{(\mathbf{V})}(dA\hat{}dA) = L_{(\mathbf{V})}F\hat{}F = L_{(\mathbf{V})}K = 0 \tag{5.39}$$

which leads to the local conservation of Topological Parity, with respect to uniformly continuous flows.

In general,

$$L_{(\mathbf{V})}(dA\hat{}dA\hat{}...dA) = 0 \tag{5.40}$$

which expresses the invariance of a 2N dimensional area with respect to uniformly continuous flows.

5.6.2 Continuous Hydrodynamic Processes

Consider the domain of four independent variables of space time, $\{x, y, z, t\}$, and the three form of topological torsion

$$H = A\hat{}dA = A\hat{}F = i(\mathbf{T}_4)dx\hat{}dy\hat{}dz\hat{}dt. \tag{5.41}$$

The continuous evolution of this 3-form is determined relative to an arbitrary process, $\mathbf{V}_4 = [\mathbf{V}, 1]$, by the equation:

$$L_{(\beta\mathbf{V}_4)}H = L_{(\beta\mathbf{V}_4)}(A\hat{}dA) = i(\beta\mathbf{V}_4)dH + di(\beta\mathbf{V}_4)H = Q\hat{}F + A\hat{}dQ \tag{5.42}$$

For local invariance of the 3-form with respect to arbitrary parameterizations, the evolutionary vector $\beta\mathbf{V}_4$ must be collinear with the topological torsion vector $(\mathbf{T}_4)$

such that the term $i(\beta\mathbf{V}_4)H \Rightarrow 0$. This constraint implies that the three form H then must be of the format:

$$H = A\hat{}F \approx \rho(x,y,z,t)(dx - V^x dt)\hat{}(dy - V^y dt)\hat{}(dz - V^z dt) = \rho \ i(\mathbf{V}_4)dx\hat{}dy\hat{}dz\hat{}dt \tag{5.43}$$

The invariance of the 3-form H then requires that a function $\rho(x,y,z,t)$ exist such that $dH \Rightarrow 0$. But this constraint becomes the equivalent of the famous hydrodynamic equation of continuity:

$$dH = \{div_3\,\rho\mathbf{V} + \partial\rho/\partial t\}dx\hat{}dy\hat{}dz\hat{}dt \Rightarrow 0 \tag{5.44}$$

which is interpreted physically as the conservation of mass. The implication is that those vector fields, $\beta\mathbf{V}_4$, that define a continuous hydrodynamic current, need not satisfy necessarily the formulas of topological kinematic constraint, $d\mathbf{x} - \mathbf{V}dt = 0$, but instead must be collinear with the topological torsion vector, $\mathbf{J}_4 = \lambda(x,y,z,t)\mathbf{T}_4$, if it exists. The important idea is that local deformable conservation of mass is to be associated with the conservation of the 3-form of Topological torsion as an absolute evolutionary invariant.

These results are to be compared with the even dimensional Poincare absolute integral invariants [Whittaker 1944] for the more restrictive case of Hamiltonian (extremal) evolution of a Hamiltonian action,

$$A = A_\mu dx^\mu = \mathbf{p}\cdot d\mathbf{x} - H(\mathbf{x},\mathbf{p},t)dt \tag{5.45}$$

on a 2N+1 dimensional state space. It is the result 5.38 which is interpreted in statistical mechanics as the invariant "area" of phase space with respect to extremal, or Hamiltonian, evolution. The fact of the matter is that uniform continuity alone produces a set of absolute integral invariants for any action, in Hamiltonian format or not. Hamiltonian extremal flows satisfy the equation $dQ = 0$, and are therefore uniformly continuous, but they are not the only flows that satisfy this constraint. The invariance of ”phase space area” is a consequence of uniform continuity alone, and does not require the additional constraints of constant homogeneity that limit the set of continuous flows to that subset of continuous vector fields which are extremal, and Hamiltonian.

5.6.3 DeRham categories of Continuous Vector Fields

DeRham's cohomology theory [deRham 1960] may be used to classify p-forms, and such ideas may be applied to the 1-form W defined by $W = i(\rho\mathbf{V})F$. Correspondingly, the vector fields that are used to construct the 1-forms W of virtual work permit processes to be put into the following categories, depending on whether the virtual work, W, is null, exact, closed, or not closed with respect to exterior differentiation. These categories are defined as:

Uniformly continuous *Categories for Q − W = dU*	$W = i(\rho\mathbf{V})F$	Q	dW	dQ
Hamiltonian − extremal	0	dU	0	0
Bernoulli − Eulerian	$d\Theta$	$d(U+\Theta)$	0	0
Helmholtz − Symplectic	$d\Theta+\gamma$	$d(U+\Theta)+\gamma$	0	0
Non-Uniformly continuous *Navier − Stokes − Torsion*	*arbitrary*	*arbitrary*	$dW \neq 0$	$dQ \neq 0$

(5.46)

The Bernoulli-Casimir functions, Θ, must be first integrals as in general,

$$i(\mathbf{V})W = i(\mathbf{V})d\Theta = 0. \tag{5.47}$$

For uniformly continuous processes the first law insures that the 1-form W is closed, $dW = dQ = 0$, but W need not be exact and may contain harmonic components. That is, the 1-form W is not necessarily representable over the variety x, y, z, t in terms of the gradient of a single scalar function. The classic example of a non-exact 1-form is given by the expression,

$$\Gamma = \sigma_z(ydx - xdy)/(x^2 + y^2) \tag{5.48}$$

for which $d\Gamma = 0, but \int_{z1}\Gamma = 2\pi\sigma_z$. The coefficient σ_z is assumed to be a constant. Such forms, Γ, generate period integrals and the deRham cohomology classes. The number of independent forms of the type given by 5.48 determine the Betti numbers of a variety for which the singular point (at the origin in the example) has been excised. The Betti numbers can be interpreted as a method for counting the number of holes or handles in the variety. It is these contributions to the general differential form that carry topological information about the domain of support. The duals to these forms are also closed, leading to the definition, harmonic forms.

From the first law the harmonic contributions to W are equal to the harmonic contributions to Q. If the harmonic contributions to Q are not zero, then the number of "holes and handles" in the Cartan topology of the final state is different from the number of holes and handles in the Cartan topology of the initial state, and the evolutionary process is continuous but not reversible.

In order to make 5.48 transversal, use the Cartan trick of substituting $dx^i - V^i dt$ for each dx^i. The transversal harmonic form becomes

$$\Gamma = \sigma_z\{ydx - xdy + (\mathbf{r}\times\mathbf{V})_z dt\}/(x^2 + y^2) \tag{5.49}$$

which demonstrates the close relationship to transversal harmonic forms and angular momentum. The format may be extended to a spin vector of components

$$\boldsymbol{\sigma} = [\boldsymbol{\sigma}_1, \boldsymbol{\sigma}_2, \boldsymbol{\sigma}_3] = [\sigma_x/(y^2+z^2), \sigma_y/(z^2+x^2), \sigma_z/(x^2+y^2)] \tag{5.50}$$

such that the harmonic form becomes

$$\Gamma = \boldsymbol{\sigma}_1(zdy - ydz) + \boldsymbol{\sigma}_2(xdz - zdx) + \boldsymbol{\sigma}_3(ydx - xdy) + (\boldsymbol{\sigma} \circ \mathbf{r} \times \mathbf{V})dt. \quad (5.51)$$

The last term is recognized as a "spin orbit" coupling term. The idea of harmonic contributions to a 1-form is closely related to the concept of a complex number or ordered pair representation; i.e., the form cannot be represented by a map to a space of 1 dimension. Other formats for harmonic 1-forms are given by the expressions:

$$\Gamma = \{\phi d\chi - \chi d\phi\}/(a\phi^p + b\chi^p)^{2/p}, \quad (5.52)$$

where ϕ and χ are arbitrary functions on the base space, and for the complex function, ψ,

$$\Gamma = \{\psi d\psi^* - \psi^* d\psi\}/(\psi^*\psi). \quad (5.53)$$

The last representation of a harmonic form is in the format of the "probability current" of quantum mechanics, and gives a clue as how to adapt the formalism of this article to quantum systems. Such a development is deferred to a later article.

For uniformly continuous processes on space time, the fundamental equations of evolution are given by the expressions for the 2-form $dA = F$ and the 4-form $d(A\hat{}F) = K = dF\hat{}dF$. The even forms are invariant. The two fundamental equations of uniformly continuous evolution are:

$$L_{(\rho\mathbf{V})}d(A) = L_{(\rho\mathbf{V})}F = dQ = 0 \;\; and \quad (5.54)$$

$$L_{(\rho\mathbf{V})}d(A\hat{}F) = L_{(\rho\mathbf{V})}K = L_{(\rho\mathbf{V})}F\hat{}F = dQ\hat{}dA = 0 \quad (5.55)$$

It should be noted that the first equation defines the uniformly continuous evolution of the limit points of the Action 1-form, while the second equation defines the uniformly continuous evolution of the limit points of the topological torsion 3-form. If $dQ \neq 0$, then the ordering of the limit points of the 1-form of Action is perturbed, as the continuity is not uniform. If $dQ\hat{}dA \neq 0$, then the ordering of the limit points of the 3-form of topological torsion is perturbed, as the continuity is not uniform. It is conjectured that a change of ordering of the limit sets of the 1-form of Action is related to an "entropy" of translation (folding process), while the change of ordering of the limit sets of the 3-form of topological torsion is related to an "entropy" of rotation.

It should be remarked that if the 1-form of Action, A, is completely integrable in the sense of Frobenius, then the 3-form $A\hat{}F$ is evanescent, and the evolutionary equation for $H = A\hat{}F$ has no applicability. Such evolutionary processes ($H = 0$) are the equivalent to laminar flows in fluid dynamics and completely integrable, non-chaotic, Hamiltonian processes. It is known that if a Lagrangian system is not chaotic, then the action, A, is reducible to two variables (or less), and the 3-form H is necessarily zero. However when there exists a sense of helicity in the evolutionary process, or chaos is present, then the formula for H describes the appropriate topological evolution.

The first expression (5.54) may be put into correspondence with the evolution of energy, while the second fundamental equation (5.55) may be described as the evolution of complexity, or perhaps better as the evolution of defects, links, knots, or in abstract terms, the evolution of an entropic concept. If the heat 1-form Q is zero, then the evolutionary process is adiabatic, and topology is preserved. However, as the Cartan topology is not connected when $H \neq 0$, then continuous evolution of H can be accomplished only between connected subsets. The transition from a connected topology with $H = 0$ to a disconnected topology with $H \neq 0$ can only take place via a discontinuous transformation. The idea is that the continuous rate of change of H is definite (and arbitrarily taken to be positive). This feature is one of the key properties of entropy. Entropy can never change its sign. The creation of topological torsion, H, is a discontinuous process from a state of zero topological torsion, but once created, the growth (or decay) of H can be described by a continuous process (relative to the Cartan topology). These entropic features of the topological torsion 3-form will be useful in the description of the transition to turbulence.

5.6.4 The Hamiltonian Extremal Sub-Category

It should be remarked, that Cartan has proved, on a domain of dimension 2n+1, that if

$$i(\mathbf{V})F = W = 0, \qquad Q = dU \tag{5.56}$$

for any reparameterization, ρ, then $\mathbf{V}$ generates a Hamiltonian system, and visa versa [Cartan 1958 (1922)]. This remarkable result indicates that Hamiltonian flows are not only continuous, but preserve many topological properties. The 1-form Q must be exact for Hamiltonian flows. Hence the observable holes and handles are topological invariants of Hamiltonian flows, as the ρ terms vanish. However, the fact that Q is exact for Hamiltonian flows does not completely establish a proof that Hamiltonian systems preserve all topological properties of the Cartan topology.

In the calculus of variations, vector fields that satisfy [Klein 1962] are defined as extremal vector fields. Characteristic vector fields are a subclass of extremal fields that satisfy the equations

$$L_{(\mathbf{V})}A = 0 \ \textit{and} \tag{5.57}$$

$$L_{(\mathbf{V})}F = 0. \tag{5.58}$$

In other words, continuous characteristics preserve the Cartan topology ($Q = 0$ and $dQ = 0$). Characteristic Hamiltonian vector fields generate waves in systems that can be endowed with the additional structure of a metric.

5.6.5 The Bernoulli-Euler subcategory

The Bernoulli-Euler category is not quite Hamiltonian. W is not zero, but must be a perfect differential, $W = d\Theta$. However, this perfect differential must be a first integral in order to satisfy the transversality condition, $i(\rho\mathbf{V})W = 0$. The 1 form Q is not necessarily so constrained. The abstract flows of this category are to be compared with the equations of motion of a compressible Eulerian fluid in which there may be stratification. If the pressure, P, is a function of the density, ρ, alone, then the Eulerian flow can be reduced to a Hamiltonian system [**?**]. If the exists some anisotropy due to stratification, then the Hamiltonian reduction is not perfect. Note that the first integral, Θ, acts as a Bernoulli constant along a given streamline, but the constant can vary from streamline to streamline because the function is transversal.

5.6.6 The Stokes subcategory

The Stokes category admits topological evolution in the sense that the harmonic contributions to W are not null, and therefore the "hole and handle" count of the Cartan topology is changing in an evolutionary manner. Such closed flows are not reversible. Note that all closed flows preserve topological vorticity and topological parity, and so if the flow is without vorticity in the initial state, then the flow is without vorticity in the final state. The Pfaff topological dimension remains less than 2. However, if the initial state has vorticity, that vorticity will be preserved, but the Topological Torsion 3-form can change. In fact the Topological Torsion 3-form could be non-zero in the initial state, and zero in the final state, for the decay rate of topological torsion is proportional to $Q\hat{}F$. Both the 1-form of action and its hole count, and the 3-form of Topological Torsion, and its twisted handle count, are not necessarily invariants of a Stokes flow.

A method of distinguishing between "holes and twisted handles" is of some interest. Note that physically a handle can be constructed by deforming the rims of two holes in a surface into tubes and pasting the tubular ends together. If the rims are twisted by half integer or integer multiples of pi before the ends are glued together, then the handles have torsion. Note that a handle cannot be constructed in the plane, so it is an intrinsically 3-dimensional thing. If the 3-form H vanishes, then there are no handles in the initial state, and as the Hamiltonian evolution produces no more new holes, there can be no more new handles in a Hamiltonian flow. However, existing handles may become twisted or knotted, because $Q\hat{}F \neq 0$, even for Hamiltonian flows. These facts correspond to the physical result that Hamiltonian systems are not dissipative and preserve energy, but that does not mean that entropy must be conserved.

It should be noted that for all uniformly continuous flows, $dW = 0$. It follows that the transversality condition $i(\rho\mathbf{V})W = 0$ implies that the 1-form of virtual work W is an absolute invariant of the flow :

$$Uniformly\ continuous\ Flows: \quad L_{(\rho\mathbf{V})}W = 0. \tag{5.59}$$

5.6.7 Non-Uniformly Continuous Processes

To be written

5.7 Global Conservation Laws

5.7.1 First Variation

Extremal (or Hamiltonian) flows and Eulerian flows induce a set of global conservation laws in the sense that the closed integrals of all odd dimensional Pfaff classes of the fundamental forms are relative integral invariants of uniformly continuous evolution. The result follows from the fact that the evolutionary rates, Q and $Q\hat{}F$ respect to such flows are zero. Integrals of exact forms evaluated over closed cycles, whether the cycle ($z1d$ *or* $z3d$) is a boundary or not, vanish. Hence all closed integrals of odd dimensional sets, $\int_{z1d} A$ and $\int_{z3d} H$, are evolutionary invariants of Hamiltonian and Eulerian flows.

For the closed flows of the Stokes category, the evolutionary rates of all odd Pfaff classes are closed, but not necessarily exact. That is,

$$dQ = 0, \;\; and \;\; d(Q\hat{}F) = 0, \tag{5.60}$$

implying closure, but Q and $Q\hat{}F$ are not exact. The deRham classes are not empty and are not flow invariants. Topology changes during such evolutionary processes.

Hence a global set of conservation laws in terms of closed integrals of A and H can be devised only for those closed chains that satisfy Stokes theorem, and those chains must be boundaries (of support). Arbitrary closed integrals are not evolutionary invariants. This lack of relative integral invariance [RMK 1974] for $\int_{z3d} H$ corresponds to the production or destruction of 3 dimensional defects, and these new defects are indications of changing topology and changing inhomogeneity. Formally, a closed integral over a closed form is a period integral whose value, by Brouwer's theorem [RMK 1977], is an integer multiple of some smallest value. A variation of a period integral signals a change in a Betti number and hence a change in topology. Such flows can produce three dimensional defects.

These results point out the limitations of Moffatt's and Gaffet's claims [Gaffet 1985] that the volume integral of helicity density, $\mathbf{v}\bullet curl\mathbf{v}$, is an evolutionary invariant. Helicity is NOT necessarily an invariant of a continuous flow. Moreover, open or closed integrals of Helicity are not necessarily integral invariants of continuous evolution. In particular, the closed volume integral of helicity density, the fourth component of the Helicity four current, is not an invariant of continuous flows for which there is a torsion current .

A theorem depending on only the first variation can be stated for the continuous evolution of flows restricted to Hamiltonian or Eulerian flows:

Theorem 26 ***III***: *The (uniformly) continuous evolution of all odd dimensional Pfaff classes of the Cartan base with respect to Hamiltonian or Eulerian flows (*$dQ = 0$*,* Q

exact) are exact. Hence, the closed integrals of A and $H = A\hat{}dA$ over closed cycles or boundaries are relative integral invariants with respect to Hamiltonian or Eulerian flows.

The proof of the theorem is as follows:

Proof. $L_{(\mathbf{V})}A = i(\mathbf{V})dA + d(i(\mathbf{V}))A) = d[P + i(\mathbf{V}))A] = Q$ and is exact.
Therefore $L_{(\mathbf{V})}\int_{z1} A = \int_{z1} Q = \int_{z1} d[P + i(\mathbf{V}))A] = 0 \supset$ invariance of $\int_{z1} A$.
Similarly,
$L_{(\mathbf{V})}H = L_{(\mathbf{V})}(A\hat{}F) = (L_{(\mathbf{V})}A)\hat{}F = Q\hat{}F = d[P + i(\mathbf{V}))A]\hat{}F$ is exact such that
$L_{(\mathbf{V})}\int_{z3} H = \int_{z3} d[P + i(\mathbf{V}))A]\hat{}F \supset$ invariance of $\int_{z3} H$. Q.E.D. ■

In the hydrodynamic case of a compressible Eulerian fluid, this theorem is the generalization of the "invariance of Helicity theorem" often stated for a barotropic domain or isentropic constraints. Closed flows therefore exhibit global conservation laws based on relative integral invariants of A and H, as well as absolute integral invariants of F and K. As will be demonstrated below, the integral of the 3-form of topological torsion, not the helicity density, over a boundary is an invariant of all flows that satisfy the Navier-Stokes equations and for which the vorticity vector field satisfies the Frobenius complete integrability conditions. This result is independent from the magnitude of the viscosity coefficient. On the other hand, the continuous destruction of 3-dimensional defects can be associated with closed flows of the Stokes category. Helicity is NOT necessarily a relative integral invariant of Stokes flows. Remarkably, such flows also admit a set of relative integral invariants, but these are determined only in terms of a second variational process.

5.7.2 Second Variation

It should be noted that the second Lie differential of the odd dimensional Pfaff classes (represented by A and H) does produce a set of global conservation laws for uniformly continuous processes. The result follows from the fact that the second Lie differential of the Action with respect to closed flows is exact, where the first Lie differential is closed!

The fundamental theorem is then:

Theorem 27 ***IV****: The (uniformly) continuous evolution of all odd dimensional Pfaff classes of the Cartan base with respect to closed flows ($dQ = 0$) are closed , but not necessarily exact. The second Lie differential is always exact so that $\int_{z1} Q$ and $\int_{z3} Q\hat{}F$ are relative integral invariants of (uniformly) continuous ($dQ = 0$) evolution.*

The proof of the fundamental theorem is as follows:

Proof.

$$L_{(\rho\mathbf{V})}A = i(\rho\mathbf{V})dA + d(i(\rho\mathbf{V}))A) = Q. \tag{5.61}$$

$$L_{(\rho\mathbf{V})}L_{(\rho\mathbf{V})}A = L_{(\rho\mathbf{V})}Q = R \tag{5.62}$$

$$= i(\rho\mathbf{V})d(i(\rho\mathbf{V})dA) + di(\rho\mathbf{V})di(\rho\mathbf{V})A \tag{5.63}$$

$$= i(\rho\mathbf{V})d(Q) + di(\rho\mathbf{V})di(\rho\mathbf{V})A = 0 + d(\Lambda) \tag{5.64}$$

which is exact. Similarly,

$$L_{(\rho\mathbf{V})}L_{(\rho\mathbf{V})}A\hat{}dA = L_{(\rho\mathbf{V})}Q\hat{}F = d(\Delta\ F) \tag{5.65}$$

which is exact. It follows that

$$L_{(\rho\mathbf{V})}\int_{z3} Q\hat{}F = \int_{z3} d(\Delta\ F) = 0 \tag{5.66}$$

such that $\int_{z3} Q\hat{}F$ is a relative integral invariant. Q.E.D. ■

Uniform continuity requires that $d(L_{(\rho\mathbf{V})}A) = L_{(\rho\mathbf{V})}dA = dQ = 0$, which insures that Q and $Q\hat{}F$ are closed. Hence closed integrals of the odd dimensional p-forms of Q and $Q\hat{}F$ (and not necessarily A and H) are relative integral invariants of uniformly continuous evolution. The integrals $\int_{z1} Q$ and $\int_{z3} Q\hat{}F$ generate global conservation laws for uniformly continuous processes in which $dQ = 0$. In elementary terms, on a space time variety, the fundamental theorem of uniformly continuous evolution states that the Lorentz force has zero curl, and the torsion defect production rate has zero divergence ($K = 0$), whether the system is dissipative or not.

The successive Lie derivations with respect to a uniformly continuous vector field $J = \rho V$ produces an exact sequence, starting from the concept of action-angular momentum, A, evolving to a closed set, Q, which under continued Lie derivation evolves to an exact kernel of radiation-power, R [RMK 1977]. A similar exact sequence can be constructed for all odd dimensional Pfaff classes, $A, A\hat{}dA, A\hat{}dA\hat{}dA, ...$

5.7.3 Continuity and the Integers

A most remarkable feature of the fundamental theorem of uniformly continuous evolution is that the integral of any radiation 1-form, R, through a container which is a maximal cycle is in relation to the integers. This concept is another application of the Brouwer degree of a map theorem, that says that all period integrals are integer multiples of some smallest value. The maximal cycle is a closed set that is not a boundary but can contain a system with internal defects, hence the name, the "container". As a simple example consider a disc with several internal holes; the maximal cycle is the cycle which would be the boundary if the disc had no holes. The global conservation laws stated above imply that radiation through the maximal cycle must be compensated by a change in the cohomology class, or the production of a defect of inhomogeneity in the interior. Radiation defects ("holes and torsion handles") are quantized, for it is impossible to create half a hole.

It would appear from the above argument that Planck's hypothesis of quantized radiation oscillators may be considered a consequence of theorem IV and Uniformly Continuous evolution as defined by equation 5.59.

5.8 Pfaff's Problem, Characteristics, and the Torsion Current.

Closely related to the concept of topological torsion is the Pfaff problem that asks about the solubility of the system of differential equations defined by setting each element of the Cartan closure to zero. The problem is equivalent to finding characteristic vector fields which, if continuous, generate an evolutionary flow that preserves the Cartan topology. The key idea of Pfaff's problem is to find maps from spaces of q dimensions into the variety, X, such that when these maps and their differentials are substituted into the system of forms that make up the Cartan closure, then the new forms are equal to zero. In this sense, the pullback of the forms of the Cartan closure to the spaces of dimension q are zero. In the case of usual interest to physics, the maps are of a single parameter which almost always is associated with the concept of time. However, they may exist higher dimensional solutions of say two parameters or more.

The question arises as to the largest dimension of such a "solution" and is determined in terms of the "characters" and "genus" of the Pfaff system [Slebodzinsky 1970]. It is the objective of this section to demonstrate that the genus of the Pfaff system built from a single 1-form of action is 3 if the Torsion current, $\mathbf{T}$, vanishes, and can be 2 only if $\mathbf{T} \neq 0$. The genus is an arithmetic invariant and a topological property. A change of genus implies topological evolution. However for the special Pfaff system described, the characters are such that only 1-parameter solutions are possible, when $\mathbf{T} = 0$, and a unique 2 parameter solution is admissible only when $\mathbf{T} \neq 0$. In other words the Pfaff problem admits a "string" solution (a two parameter solution) only when the Torsion current is not zero.

Consider an electromagnetic format. For the electromagnetic case, the Cartan 1-form may be defined in terms of the vector and scalar potentials,

$$A = \mathbf{A} \bullet d\mathbf{r} - \varphi dt. \tag{5.67}$$

Using the classical notation of Sommerfeld, define the $\mathbf{E}$ and $\mathbf{B}$ field intensities as

$$\mathbf{B} = curl\mathbf{A}, \quad \mathbf{E} = -\partial\mathbf{A}/\partial t - grad\varphi. \tag{5.68}$$

Then the components of the Darboux-Cartan-Maxwell field, $F_{\mu\nu}$, may be written as an anti-symmetric matrix (or as a Sommerfeld six-vector) of components :

$$F_{12} = B_z,\ F_{13} = -B_y,\ F_{23} = B_x,\ F_{14} = E_x,\ F_{24} = E_y,\ F_{34} = E_z \tag{5.69}$$

such that the components of $dA = F = F_{\mu\nu} dx^{\mu}\hat{}dx^{\nu}$

The Topological torsion, H, becomes

$$H = A\hat{}dA = -i\{\mathbf{E} \times \mathbf{A} + \varphi\mathbf{B}, \mathbf{A} \bullet \mathbf{B}\}dx\hat{}dy\hat{}dz\hat{}dt. \tag{5.70}$$

with the torsion current defined as,

$$\mathbf{T} = -[\mathbf{E} \times \mathbf{A} + \varphi\mathbf{B}] \tag{5.71}$$

and the helicity density,

$$h = -\mathbf{A} \bullet \mathbf{B}. \tag{5.72}$$

The Topological Parity 4-form becomes the global top Pfaffian on the 4 dimensional space-time variety, and is equal to

$$K = dA\hat{}dA = -2\mathbf{E} \bullet \mathbf{B} dx\hat{}dy\hat{}dz\hat{}dt. \tag{5.73}$$

Note that

$$div\mathbf{T} + \partial h/\partial t = 2\mathbf{E} \bullet \mathbf{B} \tag{5.74}$$

. The 3-form of axial current, H, is NOT conserved when $K \neq 0$. This result has been observed by Berger [Berger 1984]. Following Chern, the Euler index on a compact manifold would be the integral

$$\chi = \int_{z4} 2\mathbf{E} \bullet \mathbf{B} dx\hat{}dy\hat{}dz\hat{}dt. \tag{5.75}$$

Now the Pfaff problem is determined by the equations

$$A = 0, \quad F = 0. \tag{5.76}$$

Following Slebodzinsky, as there is only one 1-form in the Pfaff system, the first character, $s0$, of the Pfaff system is equal to 1. Multiply F by φ, and use $A = 0$ to eliminate φdt in the equation $F = 0$. The result is given by the equation,

$$\{\mathbf{E} \times \mathbf{A} + \varphi \mathbf{B}\}_{\mu\nu} dx^{\mu}\hat{}dx^{\nu} = \{\mathbf{T}\}_{\mu\nu} dx^{\mu}\hat{}dx^{\nu} = 0, \tag{5.77}$$

which is an expression that does not contain dt. The polar system of these resultant equations determines the genus of the Pfaff system. In particular, if $\mathbf{T}$, the torsion current vanishes, then (5.77) vanishes, the second character, $s1$ is zero and the genus of the Pfaff system is 3. All higher characters vanish, so the Pfaff system is special. Only 1-parameter homeomorphic evolutionary solutions are possible for the Pfaff system in 4 dimensions, when $\mathbf{T} = 0$.

On the other hand, for any arbitrary vector field, $\mathbf{V}$, such that the two 1-forms $\{\mathbf{T} \times \mathbf{V}\}_{\mu} dx^{\mu}$ and A, are linearly independent, then the second character, $s1$, equals 1, and the genus is 2. There then exists a two parameter characteristic evolutionary system (a string). In other words, the presence of the torsion current is necessary for the existence of a two parameter solution to the Pfaff problem. There are no 3 parameter solutions to this Pfaff problem in 4-dimensions. This extraordinary connection between the concept of the Torsion current and the solubility of Pfaff's problem serves to further emphasize the content of the often neglected quantity of topological torsion.

5.8.1 The Euler index

The coefficients of the Action 1-form globally define a covariant vector field on the variety. This vector field need not be a section without singularities. As mentioned in section 13 Arnold has shown how the singular points (zeros) of the Action 1-form, A, can be used to define the Euler index of the topology induced on the variety. Another method for evaluating this key topological property has been devised by Chern [Greub 1973] [Chern 1988]. Following Chern, the Euler index becomes the integral

$$\chi = \int_{z4} K = \int_{z4} 2\mathbf{E} \bullet \mathbf{B} dx\hat{}dy\hat{}dz\hat{}dt. \tag{5.78}$$

In Lagrangian field theories, a non-zero value for K implies that the second Chern class is not empty and signals the demise of time reversal and parity symmetry [Callan 1976] (hence, the name Topological Parity 4-form). It should be remarked that K is the exterior derivative of the 3-form of topological torsion, H, and that this 3-form, H, can be put into correspondence with Pfaffs problem and the Pfaff topological dimension, or class, of the 1-form A. The class of a 1-form and its relationship with Lagrangian field theory dates back to Forsyth [**?**], Vanderkulk [Schouten 1949], and Post [Post 1983] . In effect the evolutionary law for the 3-form of Topological Torsion given by 5.70 is a Lagrangian field theory built on Pfaff's problem. Much later than these Pfaff problem methods, Chern and Simons developed an action principle based upon the connection matrix of 1-forms, and its exterior differential of this matrix. The concepts became know as Chern Simons theory, but is different from the Pfaff problem and topological torsion, which does not require a connection.

When the electric field is orthogonal to the magnetic field, then the Euler index is zero. The idea that this Poincare invariant might have deeper meaning led Eddington [Eddington 1963 (1923)] to state: "It is somewhat curious that the scalar-product of the electric and magnetic forces is of so little importance in classical theory, for ..5.78 .. would seem to be the most fundamental invariant of the field. Apart from the fact that it vanishes for electromagnetic waves propagated in the absence of any bound electric field (i.e., remote from electrons), this invariant seems to have no significant properties. Perhaps it may turn out to have greater importance when the study of electron-structure is more advanced."

A non-zero value of the Topological Parity 4-form, K, implies that the divergence of $\mathbf{T}$ is not zero. Therefore, torsion lines can stop or start within the variety even though the evolution is C2 continuous. The torsion current is not necessarily conserved and 3-dimensional defects can be produced internally. String theorists describe this effect as an anomaly of the axial (Torsion) current. In the same sense that the closed but not exact 1-form leads to a complex representation involving ordered pair of variables, a closed but not exact 3-form leads to a quaternionic representation.

The concept of a domain of non-null Euler index ($K \neq 0$) now appears to be useful to the theory of magnetic reconnection in the electromagnetic case [Otto 1990] and to vortex reconnection [Metcalf 1990] in the hydrodynamic case. The correspondence between the bridging and rib structures produced in numerical simulations of turbulent fluid flows and the 4-string interaction of superstring theory is remarkable [Kaku 1988]. The concept ($K \neq 0$) appears to be applicable to the understanding of the stretching of lines and surfaces in turbulent flows where time-reversal symmetry is violated [Drummond 1990]. The appearance of large scale structures in certain flows has been associated with the lack of parity invariance [Sulem 1989]. The concepts of macroscopic violations of P and T symmetries appear to have application to the theory of the quantum Hall effect [Wen 1989].

With regards to hydrodynamic systems, the evolution of a flow from a laminar flow to a turbulent flow involves topological evolution. For the Navier-Stokes system, the Euler index depends upon the viscosity and the lack of Frobenius integrability of the vorticity field [RMK 1992 a]. Such a term yields a local source for the creation of Torsion currents. The lack of reversibility of such flows, and the irreducible time dependent, 3 dimensional features of such flows, implies that K can not be zero for the turbulent state. It is conjectured that the Euler index of the flow (the integral of K over the domain) is not zero during the transition to turbulence. That is, K is not a last multiplier of the spatial volume element, $dx\hat{}dy\hat{}dz$ for the flow describing the continuous (relative to the Cartan C2 topology) transition to turbulence. If $dQ\hat{}F = 0$ then the function K defines an integrating actor in the sense of a mass density such that

$$div(K\mathbf{V}) + \partial K/\partial t = 0. \tag{5.79}$$

If K were a mass density, this equation is often called the "equation of continuity", but it is more accurately described as the "conservation of mass". Relative to the Cartan topology all C2 vector fields are continuous. The transition to the turbulent state, however, must be discontinuous, for the Cartan topology in the turbulent state is disconnected.

5.8.2 Evolution of Topological Torsion

In summary, a topology has been constructed on a variety in terms of the elements of closure of a Cartan system of C2 differential forms and their intersections. The associated topological structure indicates that all processes generated by the Lie convective derivative (relative to a C2 vector field, $\mathbf{V}$) are continuous relative to the Cartan topology. However, the processes so generated are not necessarily homeomorphisms for they need not be reversible; i.e., the topology of the initial state can evolve continuously into a different topology on the final state. The method for constructing the Cartan topology is the same on both the initial and the final state, but, for example, the "hole and handle" count on the initial state can be different from the "hole and handle" count in the final state.

In terms of a single 1-form of Action, A, a Cartan topological base was constructed in terms of a set of distinct elements, defined as a Pfaff sequence, and their closures. The fundamental laws of evolution of each of the elements of the topological base was formulated relative to an arbitrary vector field. It was determined that there are two categories of continuous flows, those which are "uniformly continuous" and those which are "non-uniformly continuous". A special sub-category of closed flows describe a Hamiltonian evolution, an evolutionary process which preserves the number of "holes and handles".

Relative to the uniformly continuous category of continuous processes, all even dimension elements of the Cartan topological base are evolutionary invariants. For uniformly continuous flows, topological evolution takes place only in terms of the odd elements of the topological base. The first odd element of the topological base is the Action, and its law of evolution is equivalent to the evolution of energy. The next odd element (and the only other odd element on space-time) of the Cartan topological base is formulated as the novel 3-form of Topological Torsion. The evolution of this 3-form is studied, for although it does not necessarily satisfy a local conservation law, the anomalous source term, defined as topological parity, can be computed. It is a source of system evolutionary defects. However, it is still possible to establish a set of global conservation laws for the category of non-uniformly continuous and irreversible evolutionary flows. Although the evolution of topological torsion may be described by a continuous process, the *creation* of topological torsion from a state without topological torsion is not described by a continuous process. As the Cartan topology is not connected, the creation of topological torsion must involve discontinuous processes or shocks. The decay of topological torsion can be described by a continuous process.

The fundamental equation of topological evolution, $L_{(\rho\mathbf{V})}A = Q$, is equivalent to cohomological format of the first law of thermodynamics, $W + dU = Q$. The heat 1-form Q may be used to form a Pfaff sequence whose Pfaff dimension may be used to further classify evolutionary flows. For example, if the Pfaff dimension of Q is 2 or less, then Q can be written in the equilibrium format, $Q = TdS$. An example of a non-uniformly continuous system of flows (defined as $dQ \neq 0$) is presented in Chapter 4 in terms of the Navier-Stokes equations, for which the anomalous source term, can be computed. In effect it was demonstrated that C2 irreversible flows are among the solution set to the Navier-Stokes system. An abstract example was also given for an electromagnetic Action in Chapter 2, in which the concept of time reversal and parity symmetry breaking was associated with a non-null Euler characteristic of the Cartan topology.

5.8.3 Thermodynamic processes

In thermodynamics, a reversible process is defined as a process for which the 1-form of heat, Q, admits and integrating factor, and an irreversible process is a process for which the 1-form of heat does not admit an integrating factor (of reciprocal temperature). [Morse 1964]. This definition may be made precise in terms of Cartan's magic formula and the Frobenius theorem, for if the 1-form of heat, Q, does not admit an integrating factor then the three form, $Q\hat{}dQ$, does not vanish. However, for a given physical system defined in terms of a 1-form of Action, A, and its Pfaff sequence, those processes, $\rho\mathbf{V}$, that satisfy the equation $L_{(\rho\mathbf{V})}A\ \hat{}\ L_{(\rho\mathbf{V})}dA = 0$ are reversible.

$$\text{Definition of an reversible process, } \mathbf{V}: \qquad L_{(\rho\mathbf{V})}A\hat{}L_{(\rho\mathbf{V})}dA = Q\hat{}dQ = 0 \qquad (5.80)$$

This precise definition of thermodynamic reversibility will be subsumed, and the cohomological equivalent of the first law of thermodynamics will be studied relative to the constraint of continuous reversible or continuous irreversible topological evolution. Many intuitive thermodynamic concepts can be stated precisely in terms of the theory of continuous topological evolution based on the Cartan topology. For example, those processes, $\rho\mathbf{V}_{LA}$, for which $i(\rho\mathbf{V_{LA}})L_{(\rho\mathbf{V})}A = i(\rho\mathbf{V_{LA}})Q = 0$ are locally adiabatic.

$$\text{Local adiabatic process, } \mathbf{V}_{LA}: \qquad L_{(\rho\mathbf{V})}i(\rho\mathbf{V}_{LA})A = i(\rho\mathbf{V}_{LA})Q = 0. \qquad (5.81)$$

As must be the case in thermodynamics, there is a fundamental difference between the 1-form W and the 1-form Q. From the definition $W = i(\rho\mathbf{V})dA$, it follows that

$$i(\rho\mathbf{V})W = i(\rho\mathbf{V})i(\rho\mathbf{V})dA \Rightarrow 0\ (transversality) \qquad (5.82)$$

This fact implies that the 1-form W must be constructed from first integrals, ϕ, of the flow V, or from transversal fluctuations in the kinematics:

$$W = d\phi + \mathbf{f} \circ (d\mathbf{x} - \mathbf{v}dt). \qquad (5.83)$$

Although W can be included in the concept of Q, there are parts of Q that are not transformable into W. A precise difference between the 1-form of (virtual) work and the 1-form of heat can be established: the 1-form of work is necessarily transversal to the process, while the 1 form of heat is not. This issue is at the heart of the second law of thermodynamics. The argument is pleasing for it gives formal substance to the intuitive differences between the thermodynamic concepts of heat and work.

5.8.4 The Kinematic Topological Base

For continuous evolution in space-time, the key idea is that the exterior differential system consists of a Pfaff sequence constructed from a single 1-form of Action A, plus (perhaps) some additional constraints defining a domain of support and its boundary. The work of Arnold (and others) [Arnold 1981] has established that the singular points (zero's) of a global 1-form carry topological information. This idea is to be extended to the singular points of all elements of the Pfaff sequence, or topological base. In the Appendix, the idea of how a global 1-form of Action, A, existing on a space of dimension N+1 can be put into correspondence with a line bundle on a variety of dimension N is worked out in detail. The key features are that the Jacobian matrix of the projectivized 1-form of Action carries most of the information about the subspace. The trace and determinant of the Jacobian matrix determine the mean and Gaussian curvature of the subspace. The anti-symmetric components of the Jacobian are the functions that make up the 2-form, $F = dA$. The polynomial powers of F form the Chern classes for the line bundle [See the appendix].

Chapter 6

HOMOGENEOUS P- FORMS, FRACTALS AND QUANTIZED PERIOD INTEGRALS.

6.1 Introduction

Historical About 25 years ago (1977), the present author published an article with a title similar to the heading of this chapter. The article was entitled, "Periods on Manifolds, Quantization and Gauge" [RMK 1977]. At that time, it had become apparent that at least some of the quantum mechanical features of measurables with rational ratios (the quantum numbers) could be interpreted in terms of topological period integrals. Further motivation for the original publication was based on the idea that the Sommerfeld integrals (which could be interpreted as one-dimensional period integrals) might be used to explain the details of that Copenhagen mystery whereby the quantum jump, or radiative transition from one quantum state (initial state period integral value) to another quantum state (final state period integral value), is described as a "miracle". What was, and still is, needed was a method of describing the dynamics of topological evolution. A period integral is a topological invariant of a homeomorphism, and to described the change of a period integral would require a process that is not a homeomorphism. It was apparent from Cartan's work [Cartan 1958 (1922)] that all Hamiltonian processes preserve the Sommerfeld integrals (closed 1-forms of mechanical action), and could not describe the dynamics of topological evolution, much less the dynamics of a radiative transition. Prior work had indicated that a modification of the Hamiltonian method using Cartan techniques might be used to explain topological evolution [RMK 1974]. Part of the presentation in this article will be the demonstration of Cartan techniques that can be used to describe continuous topological evolution and thermodynamic irreversibility.

The major part of this article, however, is to give examples and methods of construction of closed p-forms, which may serve as the integrand of period integrals. The basic idea stems from the recognition that the integrands of topological period integrals can be mapped to homogeneous p-forms of degree zero. Homogeneous p-forms of degree zero are independent from scale changes, not only at a point, but globally. The most common of such objects is to be found in projective geometry, where the fractional linear, or Moebius transformation, is used to deduce the important invariants which are functions of cross ratios. The concept of "gauge invariance" as introduced by Weyl relates to changes of scale of the independent variables used as arguments of functions. Weyl asked if the under the constraints of parallel transport along a closed path, was it possible that not only the orientation, but also the "length" of a vector could change. Orientation defects in the tangent plane of the starting point were known to be related to curvature, and orientation defects orthogonal to the tangent plane had been related to the concepts of torsion. Gauge invariance implies that the scale of the vector stays the same. Herein, the concept of relative gauge invariance of functional form is related to homogeneous functions of degree D, and absolute invariance to homogeneous functions of degree D equal to zero. These concepts are then extended to scale or gauge invariance of homogeneous differential forms.

One of the principle results of the first cited article was the presentation and utilization of three period integrals, of dimension 1, 2, and 3, which have dominant physical significance. A period integral is defined as a closed p-form, ω, with $d\omega = 0$, integrated over a (closed) cycle of dimension p, z_p. In this article another 3-dimensional period integral is added to the list originally presented. The format chosen will emphasize, for purposes of more rapid comprehension, an electromagnetic application, but the basic ideas apply to many other areas of physical speciality, such as hydrodynamics and thermodynamics.

The four topological period integrals presented in electromagnetic format are:

1. The Flux quantum $= \int_{z1} A$. The integrand A is a pair 1-form, and the cycle is a 1-dimensional closed integration chain, z_1. In electromagnetic format the physical unit of the flux quantum period integral is h/e.

2. The Topological Torsion or Polarization quantum $= \int\int\int_{z3} A\hat{} F$. The integrand $A\hat{} F$ is a pair 3-form, and the closed cycle is 3-dimensional, z_3. In electromagnetic format the physical unit of the Topological Spin quantum period integral is $(h/e)^2$.

3. The Charge quantum $= \int\int_{z2} G$. The integrand G is an impair 2-form, and the closed cycle is 2-dimensional, z_2. In electromagnetic format the physical unit of the charge quantum period integral is e.

4. The Topological Spin quantum $= \int\int\int_{z3} A\hat{} G$. The integrand $A\hat{} G$ is an impair 3-form, and the cycle is 3-dimensional, z_3. In electromagnetic format the physical unit of the Topological Spin quantum period integral is h.

As the integration cycles are in domains where the exterior derivatives of the integrands vanish, then the values of the integrals have rational ratios, a fact which leads to the idea of topological "quantization" and the term "period integral". The integrands for the Flux quantum and Topological Torsion quantum behave as scalars with respect to transformations of the independent variables in their arguments. Such scalars are, in the language of invariant theory, called "absolute" invariants. The

Charge quantum and the Topological Spin quantum, are densities, and therefor depend upon the determinant of the transformation. Such objects are, in the language of invariant theory, called "relative" invariants [Turnbull 1960].

The Flux quantum and the Charge quantum were more or less well known in 1977, but the concept that these period integrals were independent from any metrical constraints was not so well known. Even now (2002) the fact of metrical independence is not fully appreciated. The fact that these objects depend upon the determinant of differentiable transformations is almost completely ignored. In 1977 the third period integral, the Topological Spin quantum was somewhat novel, having been discovered just a few years before in a somewhat different context [RMK 1969]. About the same time [RMK1976], the second 3 dimensional period integral of Topological Torsion was created to study the topological transition from turbulence to the streamline state in a fluid. It took some 10 to 20 years before it was appreciated that the non-zero closure of the pair 3-form, $A\hat{}F$, defined domains that could be put into correspondence with thermodynamic irreversibility. Hence in a hydrodynamic context, a turbulent flow must be irreducibly 4 dimensional: $d(A\hat{}F) \neq 0$.

Although each of these period integrals appear to have application to the microphysical world, they also should have applicability to the macrophysical world. After all, period integrals are topological objects independent from metric constraints of size and shape. When written in terms of a homogeneous degree zero format, all components of a homogeneous p-form can be multiplied by a factor, λ, and the p-form, like a cross-ratio in projective geometry does not change. Size and shape are not important to these continuous deformation invariants. This fact initially posed an ontological conflict, for experience (or prejudice) seems to indicate that quantum features are artifacts of the microphysical world, alone.

E. J. Post became interested in this predicament, and now champions the idea that Quantum Mechanics of the microworld should be developed in terms of metric free ideas [Post (1962)]. On the other hand, the physics of gravity, constitutive relations, and the synergetic aggregates of the macrophysical world should have metric dependent topological features, as well as geometric dependent features. In order to examine metric-based topological features, Post recommends the use of general diffeomorphic invariance principle be used to determine metrical based topological features. That is, the diffeomorphic maps should not be restricted to some particular geometrical group, such as is presumed in gauge theories. The problem with the use of diffeomorphic maps is that they miss the discrete symmetry breaking features of handedness and to-fro evolution. Perhaps a better method to discover metric independent features is to choose a metric arbitrarily, and then show (as did Hodge) that certain topological invariants arise which do not depend upon the choice of metric. Such invariants include those invariants which are "gauge" invariant, in the sense that they are independent from metric based scales. At what physical level a metric-based topology evaporates into a non-metric based topology is still unknown. At what level a non-metric based topology condenses into a metric based topology is intuitively at the level of forming coherent quantum macro states, such as those that appear in superconductivity. It is conjectured that such a process occurs when the closed, but not exact, homogeneous differential forms used to construct period integrals become harmonic.

Closed differential forms Period integrals are constructed by integrating exterior differential p-forms over integration chains in regions where the exterior differential of the p-form vanishes, but the form itself is not zero. When the exterior differential of a p-form is zero, the p-form is said to be closed. Closed p-forms can be of two types: exact p-forms, $d\Theta$, which are the exterior differential of a single p-1 form, and closed p-forms, γ, that can be mapped to p-forms that are homogeneous of degree zero, but cannot be represented (globally) by the exterior differential of a unique p-1 form. A useful theorem is that closed p-forms that are homogeneous of degree k $\neq$ 0 are exact [Liebermann 1987].

When written in terms of non-homogeneous variables, closed p-forms, γ, can not be constructed, usually, in terms of the exterior differential of a single unique p-1 form. In the language of 3D vectors, a vector field can be decomposed into three parts, a gradient field, $d\theta$, a closed but not exact part, γ, and a part Z that is not closed. The covariant vorticity of a hydrodynamic vector flow field* is due to the exterior derivative of the part, Z. The circulation of the flow field is due to the closed integral of the closed (but not a gradient) part, γ. The part γ fails to exist on a simply connected euclidean component that contains the identity. In the 1-dimensional case, each hole in a surface behaves as a topological defect, and there is a different closed but not exact p-form for each hole. A surface with two oppositely oriented, closed but not exact, 1-forms can be deformed continuously to form a surface with a handle.

Period integrals have the remarkable property that the ratios of integrals of given closed, but not exact, p-forms relative to different cyclic (closed) integration chains have rational values [deRham 1960]. The integral of an exact p-form over any closed cycle or over a boundary vanishes. The integral of a closed p-form over a bounding cycle is also zero. However, the cyclic integral over a closed cycle which is not a boundary, of a closed but not exact p-form, γ, is not necessarily zero. The domain may contain topological obstructions in the sense that the p-form has regions of singularity.

The classic example is given by the closed, but not exact, 1-form, γ :

$$\gamma = \Gamma(ydx - xdy)/(x^2 + y^2) \tag{6.1}$$

$$d\gamma = 0. \tag{6.2}$$

The 1-form γ is well defined everywhere except at the origin, which is a "topological obstruction". A closed integration chain that encircles the origin can not be "shrunk

*The covariant hydrodynamic flow field, v, is not the same as the contravariant kinematic velocity field, V, unless a Euclidean metric structure is imposed on the system.

to zero". Integration over any closed chain that does not encircle the origin yields zero, but integration of γ over a closed 1 dimensional cycle, $z1$, that encircles the origin {x=0,y=0) once will yield the value $\Gamma \cdot (2\pi)$. If the integration chain encircles the origin n times, then the value of the integral is $n \cdot \Gamma \cdot 2\pi$. These statements can be demonstrated by considering the map from $\{r,\ \theta\}$ to $\{x, y\}$:

$$\begin{aligned} \{r,\ \theta\} &\Rightarrow \{x, y\} = \{r\cos\theta,\ r\sin\theta\} && (6.3)\\ \{dr,\ d\theta\} &\Rightarrow \{dx, dy\} = \{dr\cos\theta - r\sin\theta d\theta,\ dr\sin\theta + r\cos\theta\, d\theta\}. && (6.4)\end{aligned}$$

Substitution for $\{x, y, dx, dy\}$ in terms of $\{r,\ \theta, dr,\ d\theta\}$ yields

$$\begin{aligned} \int_{z1}\gamma &= \Gamma(ydx - xdy)/(x^2+y^2) && (6.5)\\ &= \int_{z1}\Gamma d\theta = \Gamma\cdot(2\pi), && (6.6)\end{aligned}$$

where the cycle $z1 \approx x^2+y^2 = r^2 = 1$ (a constant) encloses the topological obstruction r⇒0. This metric free topological counting mechanism is valid for domains where a metric can be defined, and for domains where a metric is not defined. It should be noted that the closed but not exact integrand in terms of non-homogeneous variables, (x, y), is homogeneous of degree zero, and therefor independent of scales:

$$\begin{aligned} \gamma &= \Gamma(ydx - xdy)/(x^2+y^2) && (6.7)\\ &= \Gamma((\eta y)d(\eta x) - (\eta x)d(\eta y))/((\eta x)^2 + (\eta y)^2) && (6.8)\\ \eta &= \text{constant}. && (6.9)\end{aligned}$$

The 1-form, $d\theta$, appears to be an exact differential, but the statement is not valid globally. Note that $d\theta$ is mapped to a closed 1-form that is homogeneous of degree zero.

The integration chains which are used to evaluate the period integrals are limited to closed cycles. Closed cycles can be of two types: cycles which are boundaries and cycles which are not. Boundaries are composed of cycles. As an example, consider the punctured disc. It has an inner cycle, and an outer cycle. The boundary of the interior of the punctured disc is composed of the two cycles, the inner one (say clockwise) and the outer one (say anti-clockwise). The integration of closed form, γ, whose exterior derivative vanishes on the interior of the disc, over the inner cycle is a finite number. The value of the integral over the outer cycle is also a finite number, and is equal and opposite in sign to the first period integral. The integral over the boundary, being the sum of the two cycles, vanishes. Note that the Stokes theorem cannot be applied directly to the evaluation of a period integral when the closed integration chain is not a boundary.

It is often not appreciated that the classic example given above is in effect a canonical representation for many 1-D period integrals. The concept of differential closure can be satisfied by any pair of functions, $\alpha(x^m)$ and $\beta(x^m)$ on a space of N independent variables. For example the 1-form defined below in terms of 2 scalar functions, $\alpha(x^m)$ and $\beta(x^m)$, is closed:

$$\text{Assume } A = \Gamma \cdot (\alpha(x^m)d\beta(x^m) - \beta(x^m)d\alpha(x^m))/(\alpha(x^m)^2 + \beta(x^m)^2), \quad (6.10)$$
$$\text{then } dA = 0 \text{ (the vector of coefficients has zero "curl" in 3D)} \quad (6.11)$$

What is even more remarkable is that the 1-form denominator can be any form of the Holder norm [6].

$$\text{If } A = \Gamma \cdot (\alpha(x^m)d\beta(x^m) - \beta(x^m)d\alpha(x^m))/\lambda, \quad (6.12)$$
$$\text{and } \lambda = \{a \cdot \alpha(x^m)^p + b \cdot \beta(x^m)^p\}^{n/p} \text{ a Holder norm,} \quad (6.13)$$
$$\text{then } dA = 0 \text{ Any constant a, any constant b, any p, } n = 2. \quad (6.14)$$

The Holder norm has the exponent $n = 2$, which makes the 1-form, A, homogeneous of degree zero in terms of the two functions, $\alpha(x^m)$ and $\beta(x^m)$.

As another ubiquitous example, note that if $\alpha(x^m)$ is a complex function, $\Psi = u(x^m) + iv(x^m)$ and $\beta(x^m)$ is its complex conjugate, $\Psi^* = u(x^m) - iv(x^m)$, then the period integral becomes related to the "probability current" of Copenhagen quantum mechanics:

$$\text{Probability current Period Integral } P = \int_{z1} J = \int_{z1} \{\Psi^* d\Psi - \Psi d\Psi^*\}/(\Psi^*\Psi) \quad (6.15)$$
$$dJ = 0. \quad (6.16)$$

In this article, these results will be generalized to produce examples of closed p-forms for any p.

In physics, a closed 1-form has zero "curl". A closed N-1 form has zero "divergence" and is called a "current". Physical theories recognize that vector direction fields with zero divergence are the basis of many conservation laws. Hence, the formulation of period integrals based upon p-forms where p=N-1 are of particular interest. For the physical arena of space time, where N=4, integrals of closed 3-forms will represent topological properties invariant with respect to continuous differentiable processes. Simply said, when the divergence of a vector is zero, that which goes in equals that which comes out. There is no accumulation in the interior.

In both hydrodynamics and electromagnetism there is interest in the possibility that such divergence-free direction field lines (often interpreted as frozen in lines of vorticity or magnetic field) are linked or knotted. The evolution, or the creation, of such a topological state is an unsolved problem. In general, the concept of a contravariant direction field in the Cartan calculus is represented by a N-1 form on a space of N dimensions. If the N-1 form is closed (which is related to the idea that the direction field is divergence free) then the closed integrals of such N-1 forms are deformation invariants (hence represent topological properties) of all evolutionary processes that can be represented by a single parameter semi-group. If the N-1 form is closed, then either the lines that represent the direction field begin and terminate on boundary points of the domain , or are cyclic and close upon themselves. A closed (divergence free) direction field never stops or starts in the topological interior. The lines which stop and start on a boundary are of two types: those that stop and start on the same boundary component, and those the stop and start on a different boundary components. The fundamental idea starts with the concept of divergence free vector fields. However, the arguments of deformation of closed integrals extends to p-forms which are closed.

Herein, the discussion will be at first directed to producing an algorithm for generating closed p-forms on any space of N independent base variables. The method employs projective maps from the base space to vector spaces of dimension $M \leq N$. The closed p-forms generated by the algorithm will be of degree p = M-1. Special attention will be paid to closed 1- forms, and closed N-1 forms, and to all closed forms on base spaces of dimension 4. Closed 2-forms in 3D and closed 3-forms in 4D will be related to Linking and Braid integrals.

Homogeneous differential forms A homogeneous function is encoded by Gauss's formula:

$$f(\beta\xi^k) = \beta^D f(\xi^k). \tag{6.17}$$

Such functions of $0 < k \leq M$ variables ξ^k are said to be homogeneous of degree D. Changing the scale of each of the independent variables by the same factor β replicates the same homogeneous function, times a factor. The homogeneous function is self similar upon change of scale, β.

A homogeneous function, $f(\xi^k)$, satisfies Euler's equation

$$\xi^k \partial f(\xi^k)/\partial \xi^k = D \cdot f(\xi^k). \tag{6.18}$$

Usually the homogeneity degree D has been interpreted as an integer. However, since the popularization of fractals it is to be recognized that D does not need to be an integer.

1. The factor β can be interpreted as a scaling parameter. It need not be a constant!.

2. The factor D can be interpreted as the homogeneity degree, or the fractal dimension.

3. The factor β^D could be interpreted as the number of copies, N (of $f(\xi^k)$).

These interpretations lead to the formula

$$N = \beta^D, \tag{6.19}$$

and the "Hausdorff" dimension,

$$D = \ln N / \ln \beta, \tag{6.20}$$

Consider a function $\rho(V^m)$ that depends upon the variables V^m of a vector direction field of dimension M. Then Euler's equation can be encoded by Cartan's Magic formula: First for functions (zero forms),

$$L_{(V^m)}\rho(V^m) = i(V^m)\, d\rho + d(i(V^m)\,\rho) = i(V^m)\, d\rho + 0 = D \cdot \rho \Rightarrow \tag{6.21}$$

$$i(V^m)\, d\rho = V^m \partial\rho(V^m)/\partial V^m = D \cdot \rho, \tag{6.22}$$

and then for p=M forms of the type $\Omega_M(V^m) = dV^1\hat{}dV^2\hat{}...\hat{}dV^M$,

$$L_{(V^m)}\Omega_M(V^m) = i(V^m)\, d\Omega_M(V^m) + d(i(V^m)\,\Omega_M(V^m)) = 0 + d(i(V^m)\,\Omega_M(V^m)) \tag{6.23}$$

$$= M \cdot \Omega_M(V^m). \tag{6.24}$$

Note that every monomial p-form of the type $\Omega_M(V^m)$ is homogeneous of degree M, and is closed and exact (as M is an integer not equal to zero.)

Next, combine the density and the M-form to form the composite "mass-density" M form, m,

$$m = \rho(V^m)\,\Omega_M(V^m), \tag{6.25}$$

with the closure and homogeneous properties:

$$dm = 0, \tag{6.26}$$

$$L_{(V^m)}m = L_{(V^m)}\{\rho(V^m)\,\Omega_M(V^m)\} = (D+M)\rho(V^m)\,\Omega_M(V^m) = (D+M)m \tag{6.27}$$

$$m = dj \quad \text{(m is exact when D+M} \neq 0\text{)}, \tag{6.28}$$

$$\text{where } j = i(V^m)\{\rho(V^m)\,\Omega_M(V^m)\} = i(\rho V^m)\Omega_M(V^m) \tag{6.29}$$

$$dj = 0. \tag{6.30}$$

First note that the mass-density M form need not be homogeneous of integer degree; it can be a fractal. Second, note that the homogeneity coefficient of the density is can vanish when $D = -M$. When the homogeneity degree of the coefficient density is chosen to be $-M$ then the $m - form$ is invariant with respect to V^m.

A useful homogeneous function is given by the Holder norm λ which is homogeneous of degree D for any polynomial exponent, σ, and any set of constants a_k:

$$\begin{aligned}\text{Holder Norm} \quad &: \quad \lambda(\xi^k) = \{a_1(\xi^1)^\sigma + a_2(\xi^2)^\sigma + ... + a_n(\xi^n)^\sigma\}^{D/\sigma} &(6.31)\\ \lambda(\beta\xi^k) \quad &= \quad \beta^D\lambda(\xi^k). &(6.32)\end{aligned}$$

The a_k are usually treated as constants that determine the isotropy and signature of the Norm. Note that the length of a vector defined as the square root of a quadratic (Euclidean) inner product is a homogeneous function of degree $D = 1$, isotropic in its components, and of signature zero (no minus signs). In terms of the vector components, V^k,

$$\begin{aligned}\lambda(V^k) \quad &= \quad \langle\mathbf{V}| \circ |\mathbf{V}\rangle = \{(V^1)^2 + (V^2)^2 + (V^3)^2\}^{1/2}, &(6.33)\\ \sigma \quad &= \quad 2,\ D = 1,\ a1 = a2 = ... = ak = 1. &(6.34)\end{aligned}$$

6.2 Closed p-forms

The fundamental theorem utilizes differentiable projections, π, from a base space $\{x^k\}$ of dimension N to vector subspaces $\{V^m(x^k)\}$ of dimension M,with $M \leq N$.

$$\begin{aligned}\pi \quad &: \quad \{x^k\} \Rightarrow \{V^m(x^k)\}, &(6.35)\\ d\pi \quad &: \quad \left|dx^k\right\rangle \Rightarrow \left|dV^m(x^k)\right\rangle = [\partial V^m(x^k)/\partial x^j] \circ \left|dx^j\right\rangle. &(6.36)\end{aligned}$$

These maps are NOT diffeomorphisms when $M < N$, for the Jacobian matrix does not have an inverse. A differential "volume" element $\Omega(V^m)$ of dimension M is constructed on the M dimensional subspace in terms of the differentials of the components of vector direction field, $\mathbf{V} = [V^m(x^k)] = [U, V, W..]$:

$$\Omega(V^m) = dV^1\hat{}dV^2\hat{}...\hat{}dV^M = dU\hat{}dV\hat{}dW... \tag{6.37}$$

The components of the direction field $[V^a(x^k)]$ can be used to construct a Holder Norm, λ, of the form,

$$\lambda = (a(U)^p + b(V)^p + c(W)^p...)^{M/p} = \left(\sum_{m=1}^{m=M} a_m \cdot (V^m)^p\right)^{n/p}.$$

The Holder norm has what can be called as "signature" and/or "anisotropy" constants, $\{a, b, c, ...\}$, an arbitrary polynomial index, p, and a "homogeneity" index, n.

Although the exponent p can be any integer, special focus is given to the quadratic form† that is generated when $p = 2$.

First note that the volume element $\Omega(V^m)$ is homogeneous of degree M. Substitute $\beta(V^n)V^m$ for V^m to show that

$$\Omega(\beta V^m) = d\beta V^1\hat{}d\beta V^2\hat{}...\hat{}d\beta V^M = \beta^M dV^1\hat{}dV^2\hat{}...\hat{}dV^M = \beta^M \Omega(V^m). \quad (6.38)$$

The vector direction field $\mathbf{V}$ can be renormalized by division with respect to the Holder norm, and the result can be used to construct an $M-1$ form current, J :

$$J = i(\mathbf{V}/\lambda)\Omega(V^m) = \{UdV\hat{}dW... - VdU\hat{}dW... + WdU\hat{}dV...\}/\lambda(U,V,W). \quad (6.39)$$

If the homogeneity index is selected such that $n = M$, then the $M-1$ form J is homogeneous of degree zero. The reciprocal of the Holder Norm plays the role of a "mass" density function, ρ, on the M dimensional projection, creating the "currrent" M-1 form, $J = i(\rho\mathbf{V})\Omega(V^m)$. These constructions lead to the fundamental theorem:

Theorem 28 *If for* $0 < m \leq M$ *functions,* $\mathbf{V}(x^k)$*, of* N *variables,* $\{x^k\}$ *such that*

$$\begin{aligned} J &= i(\mathbf{V}/\lambda)\Omega(V^m) = i(\rho\mathbf{V})\Omega(V^m) \\ and\ \lambda &= \{aU^p + bV^p + cW^p...\}^{n/p} \\ &\quad then\quad for\ any\ p\ and\ any\ \{a,b,c...\}\ and \\ n &= M, \\ dJ &= 0. \end{aligned}$$

Proof: With n not fixed, by direct computation, the closure (divergence) of the current J with respect to the coordinates $\{U, V, W..\}$ on M is

$$dJ = \{(M-n)/\lambda\}\Omega_N \quad (6.40)$$

Hence, $dJ = 0$ for $n = M$, any p, and any signature for the anisotropic constants $\{a, b, c...\}$. In other words, the Holder type divisor, λ, not only acts as a reciprocal density function, but also acts as an integrating factor for the vector direction field, when $n = M$, any p, any $a, b, c...$ The "excluded" points are the zero sets of λ. Recall that the Lie differential of the mass density satisfies the homogeneity equation,

$$L_{(V^m)}m = (-D + M)m \Rightarrow 0, \quad (6.41)$$

which vanishes indicating that m is an invariant with respect to the direction field $\mathbf{V}$. The Lie differential of the current, J also vanishes,

†When the signature constants are unity, the polynomial index p=1, and the homgeneity index n=1, the Holder renormalization yields the Gauss map, such that $\langle\mathbf{V}/\lambda| \circ |\mathbf{V}/\lambda\rangle = 1$.

$$L_{(V^m)}J = i\,(V^m)\,d\{i\rho\,(V^m)\,\Omega_M\,(V^m)\} = i\,(V^m)\,d\{J\} = 0, \tag{6.42}$$

indicating that it too is an invariant with respect to the direction field **V**. The projective mapping can be used by functional substitution to pull back the closed $M-1$ form $J(V^m, dV^m)$ on M to a closed $M-1$ form $J(x^k, dx^k)$ on N.

6.2.1 Closed 1-forms in 3D and 4D

In this section, the method will be applied to construct a closed 1–form on a base space of 3. First, consider a projective map from $\{x, y, z\}$ or $\{x, y, z, s\}$ to an ordered array of two independent functions, $U(x, y, z, s)$ and $V(x, y, z, s)$, such that

$$\mathbf{V} = [V^m(x^k)] = [U, V]. \tag{6.43}$$

The $M = 2$ dimensional volume element is

$$\Omega(V^m) = dU\hat{}dV. \tag{6.44}$$

The $M - 1 = 2 - 1 = 1$-form Φ becomes

$$\Phi = (UdV - VdU)/\lambda, \tag{6.45}$$
$$\text{where } \lambda = \{aU^p + bV^p\}^{2/p}. \tag{6.46}$$

Then for any choice of the constants a,b,p,

$$dG = 0. \tag{6.47}$$

As the 1-form is closed, Φ is a candidate for the integrand of a 1-dimensional period integral. The zero sets of λ form topological obstructions. When the closed 1-form Φ is pulled back to a 3D base space (where s is considered to be a parametric constant) with the symbol γ, it can be used to represent a hydrodynamic flow on the 3D base space which is without vorticity, but for which the circulation period integral is not zero and is equal to:

$$\int_{z1} \gamma = \int_{z1} \{(U\partial V/\partial x - V\partial U/\partial x)dx + \{(U\partial V/\partial y - V\partial U/\partial y)dy + \{(U\partial V/\partial z - V\partial U/\partial z)dz\} \tag{6.48}$$
$$= \int_{z1} 1/\lambda(x, y, z)\{u(x, y, z, s)dx + v(x, y, z, s)dy + w(x, y, z, s)dz\}. \tag{6.49}$$

Note that the integration cycle is on N, not M. The value of the circulation integral is a deformation invariant, a fact that permits the deformation (and solution) of a circle into the shape of an airfoil. From the Joukowski analysis it is this circulation without vorticity that generates lift on a wing. The pull back to a 4D space would lift the restriction that s is a constant, and merely add a term involving ds in the expression for the circulation integral.

The factor $1/\lambda(x, y, z)$ acts as a "closure" factor. It is not the same as an integrating factor, which reduces the 1-form to an exact differential. In an electromagnetic context, γ does not produce a **B** field, but there exists a finite non-zero value for the flux quantum generated by the period integral, $\int_{z1}\gamma$.

Example : Consider the projective map ϕ from (x, y, z, s) to $[U, V]$:

$$U(x, y, z) = \sqrt{x^2 + y^2 - s^2}, \tag{6.50}$$
$$V(x, y, z) = z \tag{6.51}$$

with differentials $d\phi$:

$$dU(x, y, z) = \{xdx + ydy + sds)/\sqrt{x^2 + y^2 - s^2}, \tag{6.52}$$
$$dV(x, y, z) = dz \tag{6.53}$$

and (one of many) closure factor(s):

$$\lambda = (U^2 + V^2)^{2/2} \tag{6.54}$$
$$= x^2 + y^2 + z^2 - s^2 \tag{6.55}$$

to yield the pullback 1-form on x,y,z:

$$\phi^*\Phi = \gamma = ((x^2+y^2-s^2)dz - zxdx - zydy - zsds)/\{\sqrt{x^2 + y^2 - s^2}(x^2+y^2+z^2-s^2)\}. \tag{6.56}$$

This 1-form Φ is closed, implying that the co-vector field on $\{x, y, z, s\}$ with components

$$\mathbf{A} = [-zx, -zy, (x^2 + y^2 - s^2), -zs]/\{\sqrt{x^2 + y^2 - s^2}(x^2 + y^2 + z^2 - s^2)\}, \tag{6.57}$$

has zero exterior derivative. From electromagnetic theory, in 4D, the exterior derivative of a 1-form of electromagnetic potentials has 6 components, three of which are **B** like and three of which are **E** like. As the 1 form generated above is closed, the 1-form of potentials defines a region where both the **B** field and the **E** field are excluded. Yet there is a finite period integral, which can be interpreted as the flux quantum. In 3D, the variable s could be considered to be a constant. The resulting 1-form has coefficients that could represent a stationary fluid flow, without vorticity, but with finite Kelvin circulation.

6.2.2 Closed 2-forms in 3D or in 4D

Next consider three ordered functions (which could be considered as the components of a vector field on the base space), $U(x, y, z), V(x, y, z)$ and $W(x, y, z)$. Then construct the 2-form

$$G = (UdV\char`^dW - VdU\char`^dW + WdU\char`^dV)/\lambda \quad (6.58)$$
$$\text{where } \lambda = \{aU^p + bV^p + cW^p\}^{n/p} \quad (6.59)$$

Then for any choice of the constants a, b, c, p,

$$dD = 0 \quad (6.60)$$
$$\text{when } n = 3. \quad (6.61)$$

The closure factor $1/\lambda$ is again in the format of a Holder norm, forces the vector field $[U/\lambda, V/\lambda, W/\lambda]$ to have zero divergence. There are many choices of a, b, c, p that will transform the original vector field into a vector field with zero divergence. For physics, this means that it is possible to construct many different conservation laws.

Example: Consider the projective map ϕ from (x, y, z, t) to $[U, V, W]$:

$$U(x, y, z) = \sqrt{x^2 + y^2}, \quad (6.62)$$
$$V(x, y, z) = z, \quad (6.63)$$
$$W(x, y, z) = t, \quad (6.64)$$

with

$$dU(x, y, z) = \{xdx + ydy)/\sqrt{x^2 + y^2} \quad (6.65)$$
$$dV(x, y, z) = dz \quad (6.66)$$
$$dW = dt. \quad (6.67)$$

Construct the Holder norm, $\lambda = (aU^p + bV^p + eW^p)^{n/p}$, specializing to

$$\lambda = (x^2 + y^2 + z^2 + et^2)^{3/2}. \quad (6.68)$$

The pull back of the form G becomes the closed 2-form on $\{x, y, z, t\}$:

$$\phi^*G = \frac{\{(x^2 + y^2)dz\char`^dt - z\,xdx\char`^dt + txdx\char`^dz + tydy\char`^dz - zydy\char`^dt\}}{\sqrt{x^2 + y^2}(x^2 + y^2 + z^2 + et^2)^{3/2}} \quad (6.69)$$
$$d\phi^*G = 0 \quad (6.70)$$

6.2.3 Closed 3-forms in 4D

The extension to 4 dimensions follows from the assumption that the four components of the vector field are functions of $\{x, y, z, t\}$. From a given array of N = four functions, $U(x, y, z, t), V(x, y, z, t)$, $W(x, y, z, t), S(x, y, z, t)$ construct the N-1=3-form

$$T = (UdV\hat{}dW\hat{}dS - VdU\hat{}dW\hat{}dS + WdU\hat{}dV\hat{}dS - SdU\hat{}dV\hat{}dW)/\lambda \quad (6.71)$$
$$\text{where } \lambda = \{aU^p + bV^p + cW^p + eS^p\}^{n/p}. \quad (6.72)$$

Then for any choice of the constants a,b,c,e,p,

$$dT = 0 \quad (6.73)$$
$$\text{when } n = 4. \quad (6.74)$$

By choosing the closure factor in the form of a Holder norm and with proper selection of the homogeneity index, the methods above permit the construction of globally closed 3-forms given an arbitrary direction field in 4D. The construction also focuses attention on the idea that there can be subspaces of 4D upon which the 3-form in 4D is closed. In particular, consider the direction field given by the map

$$\{x, y, z, t\} \Rightarrow [U, V, W, S] \quad (6.75)$$
$$= [U, V, Z, (x^2 + y^2 + z^2 - c^2t^2)^2]. \quad (6.76)$$

For any Holder norm of the form, ($e \neq 0$)

$$\lambda = \{aU^p + bV^p + cW^p + e(x^2 + y^2 + z^2 - c^2t^2)^{2p}\}^{n/p}. \quad (6.77)$$

it may be demonstrated by direct substitution that the 3-form

$$T = (UdV\hat{}dW\hat{}dS - VdU\hat{}dW\hat{}dS + WdU\hat{}dV\hat{}dS - SdU\hat{}dV\hat{}dW)/\lambda \quad (6.78)$$

has an exterior derivative such that

$$dT = m(x, y, z, t)(x^2 + y^2 + z^2 - c^2t^2)(n - 4)dx\hat{}dy\hat{}dz\hat{}dt. \quad (6.79)$$

The factor $m(x, y, z, t)$ depends upon the functions that compose the direction field. The curious result is that the 3-form T is closed globally when n=4, but is also closed on the submanifold $(x^2 + y^2 + z^2 - c^2t^2) = 0$ (the lightcone) for any n. This result has applicability to electromagnetism. For example, when the constants are $a = b = c = 1, e = \pm r_0^{-2}$ and $p = 2$, the closure renormalization denominator is

$$\lambda = \{U^2 + V^2 + W^2 \pm r_0^{-2}(x^2 + y^2 + z^2 - c^2t^2)^4\}^{n/2}. \quad (6.80)$$

The closure factor looks like a euclidean norm on the lightcone. Note that the scale factor r_0 is arbitrary. (Could it be equal to the Hubble radius?)

Direction Fields with Zero Divergence.

The algorithms presented above for construction of closed p-forms can also be viewed as the projective construction of an M-1 current on an M dimensional vector space (range) pulled back to the N dimensional domain of base variables $\{x^k\}$. When M is less than N, the Jacobian matrix of the projective mapping is not square, and will have an obvious zero determinant. When the range has the same dimension as the domain, then the zero determinant property for the projective mapping is not so obvious. However, given a N dimensional vector field, $\mathbf{X}$, if the components of the vector field are divided by a Holder norm with homogeneity index $n = 1$,

$$\text{Renormalized } \mathbf{t} = \mathbf{X}/\lambda(n = 1), \tag{6.81}$$

then the Jacobian matrix, $[Jac(\mathbf{t})]$, of the rescaled (or in physics - renormalized) vector field, $\mathbf{t}$, will have a zero determinant globally:

$$\det[Jac(\mathbf{t})] = 0. \tag{6.82}$$

If the rescaled vector field is multiplied by the adjoint of the said Jacobian matrix, then the resulting vector field, $\mathbf{j}$, has zero divergence:

$$\begin{aligned} \text{if } \mathbf{j} &= [J(\mathbf{t})]^{adjoint} \circ |\mathbf{t}\rangle\,, && (6.83)\\ div\,\mathbf{j} &= 0 && (6.84) \end{aligned}$$

By contraction with the N dimensional volume element, the construction produces a closed N-1 form, or current:

$$\begin{aligned} j &= i(\mathbf{j})dx^1\hat{}dx^2..dx^N = i(\mathbf{j})\Omega_N && (6.85)\\ d(j) &= 0. && (6.86) \end{aligned}$$

The closed N-1 form computed by this method agrees to within a factor N with the closed N-1 form computed from algorithmic equation (6.71).

These results have obvious consequence to the Cartan - Frenet theory of the moving Frame. For then, the Holder norm is explicitly given as euclidean norm, and implies that the identity matrix is a metric for the space:

$$\begin{aligned} \lambda &= \{aU^p + bV^p + cW^p + eS^p\}^{n/p} && (6.87)\\ \text{with } a &= b = ... = 1,\ p = 2,\ n = 1, && (6.88)\\ &\supset \lambda_{Frenet} = (\textstyle\sum (X^k)^2)^{1/2} = \sqrt{\langle \mathbf{X} \circ \mathbf{X}\rangle}, && (6.89)\\ \mathbf{t} &= \mathbf{X}/\lambda_{Frenet} = \mathbf{X}/\sqrt{\langle \mathbf{X} \circ \mathbf{X}\rangle} = \text{"unit tangent vector"}, && (6.90)\\ \mathbf{t} \circ \mathbf{t} &= 1 && (6.91) \end{aligned}$$

The differential of $\mathbf{t}$ with respect to the arclength s, is equivalent to

$$d\mathbf{t}/ds = [J(\mathbf{t})] \circ |[\mathbf{t}\rangle = \kappa\,|\mathbf{n}\rangle \tag{6.92}$$

As $\det[J(\mathbf{t})] = 0$, it follows that $\mathbf{t}$ is orthogonal to $\mathbf{n}$ with respect to the Euclidean version of the Holder norm,

$$\langle \mathbf{t}| \circ |d\mathbf{t}/ds\rangle = \langle \mathbf{t}| \circ [J(\mathbf{t})] \circ |[\mathbf{t}\rangle = \kappa\,\langle \mathbf{t}| \circ |\mathbf{n}\rangle = 0. \tag{6.93}$$

The concept can be extended to a constant anisotropic metric $[\eta_{jk}]$ with arbitrary signature :

$$\begin{aligned}
\lambda_{Frenet} &= \sqrt{\langle \mathbf{X}^k \circ [\eta_{jk}] \circ \mathbf{X}^k \rangle}\; p = 2, n = 1 && (6.94)\\
\text{metric } \eta_{jk} &= 0,\; j \neq k,\;\; \eta_{kk} = a_k, && (6.95)\\
\mathbf{t} &= \mathbf{X}/\lambda_{Frenet} && (6.96)\\
\mathbf{t} \circ \mathbf{t} &= \langle \mathbf{X}^k \circ [\eta_{jk}] \circ \mathbf{X}^k \rangle / \lambda^2_{Frenet} = 1 && (6.97)
\end{aligned}$$

If the renormalized vector is use as the coefficients of a 1-form, then

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note that there is a difference between vector fields that are explicitly time dependent and those that are not.

6.3 The Gauss Integrals (2-forms)

The basic issue is that not all divergence free fields (differential forms) are exact. It is true that all divergence free fields in a 3 dimensional *euclidean topology* are exact, but that is precisely where the topological features enter into the picture. A euclidean topology is simply connected and without obstructions.

Even in 3-dimensions (and with euclidean dogma) there are still two species of 3 component fields. Every one learns from Gibbs vector analysis that the 3 vector of angular momentum $\mathbf{L} = \mathbf{r} \times \mathbf{p}$ is never to be added to the 3 vector of momentum, $\mathbf{p}$. The angular momentum and the linear momentum are "two different species" of vectors (direction fields). However, with regard to non-euclidean topological domains there is also another concept, defined as a vector density. There is a topological difference, even in 3D, between and a covariant tensor and a contravariant tensor density, but not detectable if volume deforming processes are excluded (the typical non-dissipative case).

To demonstrate these ideas, first consider ordinary 3 dimensional vector fields and a C1 map of a 3-space onto a euclidean domain of 3 dimensions:

$$\phi : \{x, y, z\} \Rightarrow \{U, V, W\} = \phi^j(x^k). \tag{6.98}$$

This map defines a vector field

$$\mathbf{Z} = [U(x,y,z), V(x,y,z), W(x,y,z)]. \tag{6.99}$$

The (square) Jacobian matrix is defined by the equations,

$$d\phi : \{dx, dy, dz\} \Rightarrow \{dU, dV, dW\} = [\mathbf{J}] \circ |d\mathbf{r}\rangle = \left[\partial\phi^j/\partial x^k\right] \left|dx^k\right\rangle \tag{6.100}$$

Next construct the volume element generated by these three functions.

$$\Omega_z = dU\hat{}dV\hat{}dW = \det[\mathbf{J}]dx\hat{}dy\hat{}dz, \tag{6.101}$$

and the $N-1=2$ form

$$\begin{aligned} G &= i(\mathbf{Z})dU\hat{}dV\hat{}dW = UdV\hat{}dW - VdU\hat{}dW + WdU\hat{}dV & (6.102)\\ &= D^x dy\hat{}dz - D^y dx\hat{}dz + D^z dx\hat{}dy. & (6.103)\end{aligned}$$

The Vector $\mathbf{Z}$ induces a preimage, $\mathbf{D}$, on $\{x,y,z\}$. Formally, the vector , $\mathbf{D}$, is defined in terms of the adjoint mapping, by the matrix equation:

$$Contra{-}variant\ tensor\ density \quad \left|\mathbf{D}^k(x^m)\right\rangle_{pb} = \left[\partial\phi^j/\partial x^k\right]^{adjoint} \circ \left|\mathbf{Z}(x^j)\right\rangle, \tag{6.104}$$

The functional substitution and pullback (pb) construction works even though the Jacobian map does not have an inverse. In this respect the retrodictive process resembles the pull back of a 1-form, where the covariant tensor field is functionally well behaved with respect to the transpose mapping:

$$Co-variant\ tensor \quad |\mathbf{A}_k(x^m)\rangle_{pb} = \left[\partial\phi^j/\partial x^k\right]^{transpose} \circ \left|\mathbf{Z}(x^j)\right\rangle, \tag{6.105}$$

Now, the extraordinary result is that if $\mathbf{Z}$ is rescaled by the divisor

$$\lambda(U,V,W) = \{aU^p + bV^p + cW^p\}^{3/p} \tag{6.106}$$

then the 2-form

$$\begin{aligned} \widehat{G} &= i(\mathbf{Z}/\lambda)dU\hat{}dV\hat{}dW & (6.107)\\ &= i(\mathbf{D}/\lambda)dx\hat{}dy\hat{}dz, \qquad d\widehat{G} = 0 & (6.108)\end{aligned}$$

is closed. This result implies that the rescaled vector field, $\widehat{\mathbf{D}} = \mathbf{D}/\lambda$, has zero divergence.

The notation above is deliberate, for in 4 dimensions it distinguishes the electromagnetic Intensities, $\mathbf{E}, \mathbf{B}$, (as components of a covariant tensor deduced from

$\mathbf{A}_k$) from the electromagnetic Quantities (or excitations) $\mathbf{D}, \mathbf{H}$ (as components of a contravariant tensor density).

The assumption of a euclidean domain masks these topological features. The topological closure of $|\mathbf{D}\rangle$ is the concept of zero divergence; the topological closure of $|\mathbf{A}\rangle$ is a zero curl concept. In 3D, for c2 differentiable fields where $|\mathbf{B}\rangle = curl\ \ |\mathbf{A}\rangle$, it follows that the closure of the 2-form generated from the components of $|\mathbf{B}\rangle$ is always empty, in a global manner! However, the closure of $|\mathbf{D}\rangle$ need not be globally empty!

On a space of three dimensions there are 2-forms of three components that are exact, and there are 2-forms of three components that are not exact. Although the 2-form with covariant components $|\mathbf{B}\rangle$ constructed from the curl of a vector potential $\mathbf{A}$ is closed and exact, the 2-form with tensor density components $|\mathbf{D}/\lambda\rangle$ is closed, BUT NOT NECESSARILY EXACT. The fundamental idea is that for a non-bounding closed cycle (*nbcc*) (such as formed by a closed twisted ribbon),

$$\iint_{nbcc} \mathbf{B} \circ d(Area) = 0, \qquad but \quad \iint_{nbcc} \mathbf{D} \circ d(Area) \neq 0 \tag{6.109}$$

where for a boundary (such as toroidal surface)

$$\iint_{boundary} \mathbf{B} \circ d(Area) = 0, \qquad and \quad \iint_{boundary} \mathbf{D} \circ d(Area) = 0. \tag{6.110}$$

If the integration chain is a closed in the sense of cycle, and is not a boundary, then there must exist points of the integration domain which must be excluded. These points form the topological defects (the point charges in EM theory or "topological holes") or the topological obstructions that are of interest to the theory of Links and Braids. In particular, the theory of links depends upon such obstructions and is represented by integrals of the form:

$$Lk = \underset{nbcc}{\iint} (D^z dy\hat{}dz - D^y dx\hat{}dz + D^z dx\hat{}dy)/\lambda = \underset{nbcc}{\iint} G \neq 0, \quad dG = 0. \tag{6.111}$$

and should have nothing to do with magnetic flux,

$$\Phi_m = \underset{nbcc}{\iint} (B_x dy\hat{}dz - B_y dx\hat{}dz + B_z dx\hat{}dy) = \underset{nbcc}{\iint} F = 0, \quad dF = 0, \tag{6.112}$$

which has no obstructions, as the integrand is globally exact. If F was to have obstructions, the pre-images global postulate of potentials $F - dA = 0$ must fail, and the conservation of flux would not be true. Such a failure implies the existence of magnetic monopoles (the obstructions to F being globally exact). The authors personal view (along with E.J.Post and many others) subsumes that the failure to detect magnetic monopoles is proof that classical electromagnetism is defined by the postulate of potentials; i.e., $F - dA = 0$, globally. On the other hand, the 2-form of field excitations, G, is not exact.

6.3.1 Example 1. The Gauss Link Integral

The first application is to the divergence free vector field on 3 dimensions which is not exact, but is closed, and requires three functions for its description. The generic form for the integral of interest is given by the expression

$$Lk = \iint i(Z)\Omega_z = \iint_{closed} (UdV\hat{}dW - VdU\hat{}dW + WdU\hat{}dV)/\lambda \qquad (6.113)$$

As an example of the Gauss integral, Lk, consider the case where the displacement vector is the difference of two position vectors to two separate space curves. Define

$$\mathbf{Z} = (\mathbf{R}_2 - \mathbf{R}_1) \quad \mathbf{R}_2 = [x_2, y_2, z_2] \quad \mathbf{R}_1 = [x_1, y_1, z_1] \qquad (6.114)$$

$$\lambda = (a(x_2 - x_1)^p + b(y_2 - y_1)^p - +c(z_2 - z_1)^p)^{3/p} \qquad (6.115)$$

where $\mathbf{R}_1$ defines the position vector to one field of space curves, and $\mathbf{R}_2$ defines the position vector to a second field of space curves. Space curves from different families can have different parameterizations. Hence, the vector $\mathbf{Z}$ represents the vector difference of points on two different space curves which cannot be synchronized parametrically. Next assume that the displacements of interest are constrained by two parametric curves given by the exterior differential system

$$d\mathbf{R}_1 - \mathbf{V}_1 dt = 0 \quad and \quad d\mathbf{R}_2 - \mathbf{V}_2 dt' = 0, \qquad (6.116)$$

where the parameters dt and $dt\prime$ are not functionally related, such that

$$dt\hat{}dt\prime \neq 0, \; but \; dt\hat{}dt = 0 \; and \; dt'\hat{}dt' = 0. \qquad (6.117)$$

The vector $\mathbf{D}$ can be interpreted as the displacement vector between points on the space curve C1 parametrized by t, and the points on another space curve C2 parametrized by $t\prime$. The integral to be evaluated is

$$\begin{aligned} Lk &= \iint_{closed} \Gamma = \iint_{closed} i(\mathbf{Z}/\lambda) d(x_2 - x_1)\hat{}d(y_2 - y_1)\hat{}d(z_2 - z_1) \\ &= \iint_{closed} (1/\lambda)(V_{21} - V_{11})\hat{}(V_{22}dt' - V_{12}dt)\hat{}(V_{23}dt' - V_{13}dt) + ... \\ &= \iint_{closed} (1/\lambda)(V_{21} - V_{11})\hat{}(V_{22}V_{13} - V_{12}V_{23}) dt\hat{}dt' + ... \end{aligned} \qquad (6.118)$$

using $dt\hat{}dt' \neq 0$, *but* $dt\hat{}dt = 0$ and $dt\prime\hat{}dt\prime = 0$. Rewriting the formula using the isotropic Gauss format, a=b=c=1,p=2 leads to the classic Gauss Linkage formula,

$$Lk = \iint_{closed} G = \oint_t \oint_{t'} \{(\mathbf{R}_2 - \mathbf{R}_1)\circ\mathbf{V}_1 \times \mathbf{V}_2\} dt\hat{}dt'/\lambda \tag{6.119}$$

$$\lambda = (\mathbf{R}_1 \circ \mathbf{R}_1 - 2\mathbf{R}_1 \circ \mathbf{R}_2 + \mathbf{R}_2 \circ \mathbf{R}_2)^{3/2}. \tag{6.120}$$

However, the zero divergence formula works for the anisotropic case, for any a,b,c and for any exponent p.

From Stokes theorem, if the closed 2 dimensional integration domain is a boundary of a 3 dimensional domain, then the Link integral vanishes. However, if a particular integration chain is a closed cycle (not a boundary of a 3 dimensional domain) then the linking integral has values with rational ratios. These closed integrals are deRham period integrals in two dimensions. Points where $\mathbf{D}$ vanishes are excluded.

When the two curves are distinct, the integration is over the two bounding cycles of a closed ribbon. The ribbon surface is closed but it is not a boundary of any volume. Then the two non-intersecting cycles (that form the boundary of the ribbon area) are defined by the two distinct parameters, dt, and dt'. When integrations are computed along these closed curves whose tangent vectors are $\mathbf{V}_1$ and $\mathbf{V}_2$, then the integer values of the closed integral may be interpreted as how many times the two curves are linked. The interpretation of the closed surface integral as a orientable ribbon works if the triple product divided by lambda does not change sign as t and t' are varied. If the integrand changes sign, then the ribbon in non-orientable.

The constraint that $dt\hat{}dt' \neq 0$ implies that the "motion" along the curve generated by $\mathbf{R}_1$ is independent of the "motion" along the curve generated by $\mathbf{R}_2$. If the curve generated by $\mathbf{R}_1$ is a conic in the xy plane and the curve generated by $\mathbf{R}_2$ is a conic in the xz plane, then the surface swept out by the vector $\mathbf{D}$ is a Dupin cyclide. Such surfaces have application to the propagation of waves in electromagnetic systems.

6.3.2 Example 2: Flat tangential developables

From another point of view, consider the ruled surface defined by the vector field of two parameters, $\{t, \mu\}$ (isotropic, a=b=c=1 , p=2). The ruled surface will be defined by the position vector $\mathbf{R}(t)$ to a space curve and a ruling parameter μ times the tangent Velocity vector to the space curve, $\mathbf{V}(t)$.

Use the general methods above to create the doubly parametrized divergence free vector field:

$$\mathbf{Z}(\mu, t) = \{\mathbf{R}(t) \pm \mu\mathbf{V}(t)\ \} \tag{6.121}$$

$$\lambda(\mu, t) = (\mathbf{R}(t) \circ \mathbf{R}(t) \pm 2\mu\mathbf{R}(t) \circ \mathbf{V}(t) + \mu\mathbf{V}(t) \circ \mu\mathbf{V}(t))^{3/2}. \tag{6.122}$$

Vector fields of this type are primitive examples of "strings" for fixed values of the parameter, t, and string parameter, μ. Direct substitution of the physical constraints, $d\mathbf{R}-\mathbf{V}dt=0$, and $d\mathbf{V}-\mathbf{A}dt=0$, such that $d\mathbf{Z}=d\{\mathbf{R}(t)\pm\mu\mathbf{V}(t)\ \}$ into the definition of the linking integral

$$\iint_{closed\,on\,N} i(\mathbf{Z}/\lambda)dZ^1\hat{}dZ^2\hat{}dZ^3 \tag{6.123}$$

leads to yet another realization and interpretation of the Gauss formula:

$$\begin{aligned} Q &= \iint_{closed} G = \iint_{closed\,on\,\mu t} \{\mathbf{R}\circ\mu\mathbf{V}\times\mathbf{A}\}dt\hat{}d\mu/\lambda \\ &= \iint_{closed} \{\mathbf{A}\circ\mathbf{R}\times\mu\mathbf{V}\}dt\hat{}d\mu/(\mathbf{R}\circ\mathbf{R}\pm 2\mu\mathbf{R}\circ\mathbf{V}+\mu\mathbf{V}\circ\mu\mathbf{V})^{3/2}. \end{aligned} \tag{6.124}$$

It is apparent that the interaction of the "angular" momentum, $\mathbf{L}=\mathbf{R}\times\mu\mathbf{V}$, and the acceleration, $\mathbf{A}$, produces a topological invariant whose values are "quantized" (in the sense that the ratios of the closed integrals are rational). Note that the triple vector product of the integrand numerator is proportional to the Frenet torsion of the orbit. For an orbit that is planar the Frenet torsion is zero everywhere, and the Gauss integral vanishes.

Recall that if a the space curve is an edge of regression, then the ruled surfaces associated with to the forward and backward motions (the $\pm$ signs in the formula) are not same to second order. Such a result demonstrates an obvious distinction between forward and backward motion that breaks time reversal symmetry. Linear rulings in one direction are on 1 sheet of the ruled surface, and rulings in the opposite direction are on the other surface. The two surfaces meet at an edge of regression. Similar time reversal symmetry breaking effects have been observed macroscopically in dual polarized ring lasers.

6.3.3 Example 3. Scrolls

The two parameter surface described above is closely related to the ruled surface known as the tangential developable. Such ruled surfaces (parametrized by arc length s rather than time, t, and with the directrix of the ruling in the direction of the unit tangent vector, and multiplied by μ) have zero Gauss curvature. Though bent, such surfaces can be rolled out flat. By constructing the ruled surfaces in terms of the normal and/or binormal to a space curve, other forms of ruled surfaces yield negative values for the Gauss curvature of the surface, and are not "flat". They are defined as Scrolls.

Of particular interest to physics are those ruled surfaces of negative Gauss curvature, which are also minimal surfaces. They have application in describing hydrodynamic wakes. These surfaces can be viewed as double edged ribbons for given values of μ. The equations for the ruled surface of a scroll, with a directrix in the direction of the binormal, $\mathbf{b}(s)$, are :

$$\mathbf{D}(\mu, s) = \{\mathbf{R}(s) \pm \mu\mathbf{b}(s)\}/\lambda \tag{6.125}$$
$$\lambda = (\mathbf{R}(s) \circ \mathbf{R}(s) \pm 2\mu\mathbf{R}(s) \circ \mathbf{b}(s) + \mu\mathbf{b}(s) \circ \mu\mathbf{b}(s))^{3/2}. \tag{6.126}$$

When the parameter μ takes on the constant values $\mu = \kappa/\tau^3$, (with $\kappa =$ the Frenet curvature, and $\tau =$ the torsion of the space curve) then the ruled surface is a minimal surface, and the binormal field twists about the space curve generated by $\mathbf{R}(s)$.

Another interesting scroll is that generated by the Darboux vector.

$$\mathbf{D}(\theta, s) = \{\mathbf{R}(s) \pm (\mathbf{n}(s)\cos(\theta) + \mathbf{b}(s)\sin(\theta)\}/\lambda \tag{6.127}$$
$$\lambda = (\mathbf{D} \circ \mathbf{D})^{3/2}. \tag{6.128}$$

which seems to be of interest to Longcope.

6.4 Braids, Spin and Torsion_Helicity (3-forms)

6.4.1 Chaos and the Unknot

Much interest of late has been shown in knot theory and its application to an understanding of the trajectories of dynamical systems. The conjecture is that somehow an understanding of knot theory will give a better understanding of chaos. Counter intuitively is the idea that chaos is to be related to the unknot. Of particular interest will be those cases where lines of vorticity have an oscillatory Frenet torsion with a period equal to 2/3 of the fundamental period of closure. The topological Gauss integral will average to zero for such systems; but these systems can be created by continuous deformations of folding and twisting a closed loop of vorticity, producing a period 3 system which is known to be related to chaos [Yorke 1975]. In the non-deformed circular state, tubular neighborhoods guided by the vortex lines can continuously evolve into domains without stagnation points or tangential singularities, or knots, or twists. However, when the closed vortex line is in the deformed period 3 configuration, tangential (hyperbolic) singularities are created by the flow lines of the velocity field, and the evolution becomes highly convoluted and chaotic. See Figure 1.

These topological features may be demonstrated visually by taking a long strip of paper and wrapping the strip three times around your fingers. Close the strip by going under one strand and over the next before pasting together. The strip is of obvious period three. Now slide the closed strip from the fingers and note that it can be deformed 9continuously into a cylindrical strip without twists or knots (Spin 0). If the same procedure is used, except that a double over or a double under crossing is used before pasting the strip ends together, the resulting closed loop will have a continuously irreducible 4π twist (Spin 2). Both the Spin 2 and the Spin 0 strips have a zero Euler characteristic. However, the Spin 2 strip can be continuously deformed into a Klein bottle, or a double lapped Mobius band, and is not homeomorphic to the spin zero strip [24].

If a model of the Spin0 and Spin 2 systems (deformed to their period 3 configurations) is made from a copper tube, and if flexible bands are created to link any pair of neighboring tubular strands, then it is readily observed that the paired domain twists and folds as it is propagated unidirectionally along the vortex lines. For the spin 2 system the flexible bands will return to their original state in 3 revolutions. However, the paired domain continues to twist and fold, becoming ever more complicated as it follows the evolution around the Spin 0 configuration. The folded spin 0 system has chaotic neighborhoods. This result indicates that the source of chaos in dynamical systems may be due to the unknot, and not the knot! The Cartan theory thereby predicts that the source of chaos in turbulent systems does not require a discontinuous cut and connect process, but may be induced by vortex lines that continuously evolve by twisting and folding into a closed, spin 0, period three configuration.

6.4.2 The Torsion 3-form and the Braid integral

For n = 4 the same procedures used above can be used to produce a period integral over a closed 3-dimensional domain. In fact, the same vector field that is used to define the Cartan 1-form of Action may be used to construct a dual N-1 form that is closed. The algorithm is to substitute for the functions of the vector field, V, the functions that make up the covariant 1-form of Action, A. This construction is equivalent to constructing the Jacobian matrix of the original vector field on the N-dimensional velocity space, computing its cofactor matrix, multiplying the original vector by the cofactor matrix, and then dividing by the quadratic form, λ. When these operations are completed, functional substitution will lead to an conserved axial vector current density on (x,y,z,t). Another form of the topological integral invariant is constructed in the following way. First, for the classic Cartan action, $A = P_k dx^k - Edt/c$, construct the N-volume, $\Omega = -dP_x \hat{} dP_y \hat{} dP_z \hat{} dE/c$. Next contract Ω with the vector, $(Px, Py, Pz, -E/c)$, and then divide by $\lambda = \{\pm P \circ P \pm (E/c)^2\}^2$. For sake of simplicity, assume that E/c is a constant such the $dE = 0$. Then the closed 3-form or current becomes equivalent to

$$J = (E/c) dP_x \hat{} dP_y \hat{} dP_z / \lambda \qquad with \qquad dJ = 0 \tag{6.129}$$

Now invoke the same Cartan trick of individual parametrization as uses above. Consider a total momentum vector composed of three individual vector components, $\mathbf{P} = \mathbf{p_1} + \mathbf{p_2} + \mathbf{p_3}$. Assume that the Cartan topology is constrained in such a way that for each vector component a Newtonian kinematic law of parametrization is maintained such that

$$d\mathbf{p}_1 - \mathbf{f}_1 dt = 0, \quad d\mathbf{p}_2 - \mathbf{f}_2 dt' = 0, \quad d\mathbf{p}_3 - \mathbf{f}_3 dt'' = 0. \tag{6.130}$$

Also note that $dt\hat{}dt'\hat{}dt'' \neq 0$; that is, the parameters used in the Newtonian kinematic descriptions are not sychronizeable. If they were functionally related the value of J must be zero. Substitute these expressions into the equation for the closed current J and integrate over a closed 3 dimensional chain to yield a triple Braid integral,

$$\begin{aligned} Braid &= \oint_3 J = \oint_3 (E/c) dP_x \hat{} dP_y \hat{} dP_z / \lambda \\ &= \oint_3 (E/c)\{\mathbf{f}_1 \circ (\mathbf{f}_2 \times \mathbf{f}_3)\}\, dt\hat{}dt'\hat{}dt'' / \{\pm \mathbf{P} \circ \mathbf{P} \pm (E/c)^2\}^2 \end{aligned} \tag{6.131}$$

The integrations are now over three closed curves whose tangents are the Newtonian forces, **f**, on three "particles". Where in the two dimensional Gauss integral, of the previous section, the evaluation was along the closed curves of two particles that formed the ends of a string, in this case the integrations are along the closed trajectories of three "particles" which form the vertices of a triangle. In every case, the trajectories are the trajectories of a system of limit points.

The idea that three "lines" are used to form the integral (whose values form rational ratios) is the reason that this topological integral in the format given above is defined as the braid integral. Of course the three form of topological torsion is a variant of the braid integral, but applies to those topologies where the system is not reducible to three factors $dt, dt\prime$ and dt" (such systems are said to have torsion cycles). An example of a period 3 braid with Braid integral zero (chaotic) and Braid integral 2 (non-chaotic) is given in Figure 1

The equivalent to this Figure, and the fact that there are two distinct period 3 configurations, one chaotic and one non-chaotic, was brought to the attention of the present authors during a stimulating lecture given by J. Los at the August, 1991, Pedagogical Workshop on Topological Fluid Mechanics held at the Institute for Theoretical Physics, Santa Barbara UCSB.

It is to be noted that the 3-form of topological torsion is related to the braid integral, a three dimensional thing in four dimensions, and not the Gauss linkage integral, which is a two dimensional thing in three dimensions. The literature of helicity is sometimes confused on this point, and often attempts to relate the helicity integral to the linkage integral.

6.4.3 Braids

For n = 4 the same procedures described above may be used to produce a period integral over a closed 3-dimensional domain. The technique is to define a 4 dimensional vector field, $\mathbf{Z} = [Z_1, Z_2, Z_3, Z_4]$. Use the general renormalization function,

$$\lambda = \{\alpha Z_1^p + \beta Z_2^p + \gamma Z_3^p + \epsilon Z_4^p\}^{n/p} \tag{6.132}$$

and set n=4, for zero four divergence. Construct the closed 3 -form,

$$\Gamma = i(\mathbf{Z}/\lambda)dZ_1\hat{}dZ_2\hat{}dZ_3\hat{}dZ_4 \tag{6.133}$$

Assume the 4 component vector has a realization as $\mathbf{Z} = \mathbf{P_1}+\mathbf{P_2}+\mathbf{P_3}$, where the three independent fields $\mathbf{P}$ represent three space-time curves that obey the kinematic constraints:

$$d\mathbf{P_1}-\mathbf{f_1}ds = 0, \quad d\mathbf{P_2}-\mathbf{f_2}ds' = 0, \quad d\mathbf{P_3}-\mathbf{f_3}ds'' = 0. \tag{6.134}$$

Substitute for each of the differentials in Γ (and further assume that the domain {x,y,z,t} of interest is further constrained such that $dt = 0$) to yield the three form

$$G = \{\, \mathbf{f}_1 \circ (\mathbf{f}_2 \times \mathbf{f}_3)\}ds\hat{}ds'\hat{}ds''/\lambda \tag{6.135}$$

$$\lambda = \{\alpha Z_1^p + \beta Z_2^p + \gamma Z_3^p + \epsilon Z_4^p\}^{4/p} \tag{6.136}$$

The spatial braid integral becomes equal to

$$Br := \oint_t \oint_{t'} \oint_{t''} \mathbf{f}_1 \circ (\mathbf{f}_2 \times \mathbf{f}_3)\}ds\hat{}ds'\hat{}ds''/\lambda \tag{6.137}$$

The integrations are now over three closed curves whose tangents are the "Newtonian forces", $\mathbf{f}$, on three "particles". Where in the two dimensional Gauss integral, of the previous section, the evaluation was along the closed curves of a ribbon, in this case the integrations are along the closed trajectories of three "particles" which form the vertices of a triangle.

The idea that three "lines" are used to form the integral (whose values form rational ratios) is the reason that this topological integral in the format given above is defined as the braid integral. Of course the three form of topological torsion is a variant of the braid integral, but applies to those topologies where the system is not reducible to three factors $dt, dt\prime$ and dt". An example of a period 3 braid with Braid integral zero (chaotic) and Braid integral 2 (non-chaotic) is given in Figure 5. It is illuminating to construct the two braids by wrapping a long flat ribbon of paper smoothly around the palm of your hand. Close the ribbon surface by pasting the ends together. Then make another example, where this time thread the loose end underneath the middle wrap, rather than over the middle wrap, before gluing the ends together. Take the two examples from your hand and note that one is continuously deformable into a closed cylinder (Tw = zero) while the other has a 4 pi twist (Tw =2). What is surprising is that it is the Tw =0 configuration that has a chaotic neighborhood, while the Tw = 2 structure is not chaotic. To test for chaos construct the equivalent of the closed braid from copper tubes. Then link any pair of tubes with a large loop of elastic or thread. Push the looping thread around the period three copper tube, and note that for a Tw=2 configuration, the looping thread becomes untangled after 6 pi revolutions about the central axis. For the Tw

= 0 configuration, the looping thread never unwinds, but becomes more and more twisted and complex.

The equivalent to Figure 5, and the fact that there are two distinct period 3 configurations, one chaotic and one non-chaotic, was brought to the author's attention during a stimulating lecture given by J. Los at the August, 1991, Pedagogical Workshop on Topological Fluid Mechanics held at the Institute for Theoretical Physics, Santa Barbara UCSB.

It is to be noted that the 3-form of topological torsion is related to the braid integral, a three dimensional thing in four dimensions, and not the Gauss linkage integral, which is a two dimensional thing in three dimensions.

6.5 Electrodynamic Applications

6.5.1 Phase-Polarization and Orientability in 4D

The physical experiments yielding states of matter which are superconducting, and states of matter that emit laser coherent radiation indicate that these properties are deformation invariants and therefor should have a topological basis. In the superconductor, it has been suggested that the electrons form a boson pair with spin zero, such that orientability feature of Fermion spin up or spin down has been lost (or averaged to zero). In the laser, the radiating electrons form a phase-polarization coherent state, such that concept of phase-polarization differences (between electrons) is "averaged to zero". What (different?) topological features are to be associated with orientability and phase?

Topological Spin and the Spin quantum - a closed 3-form density

A topological basis for Electromagnetism [RMK 1999 b] is based upon the differential system

$$F - dA = 0 \qquad J - dG = 0. \tag{6.138}$$

The system admits 3 fundamental 3-forms:

$$\text{The charge-current density 3-form,} \quad J = dG \tag{6.139}$$
$$\text{The Topological Torsion Helicity 3-form,} \quad H = A\hat{}F \tag{6.140}$$
$$\text{The Topological Spin 3-form,} \quad S = A\hat{}G \tag{6.141}$$

The first 3-form is closed globally, but the second and third 3-forms do not closed necessarily. However, in a space of 4 dimensions, the latter two 3-forms admit integrating factors that will cause the 3-forms to be closed and homogeneous of degree zero.

On space time of 4 dimensions, starting from an electromagnetic Action 1-form, there are two important 3-forms: the 3-form of Topological Torsion and the

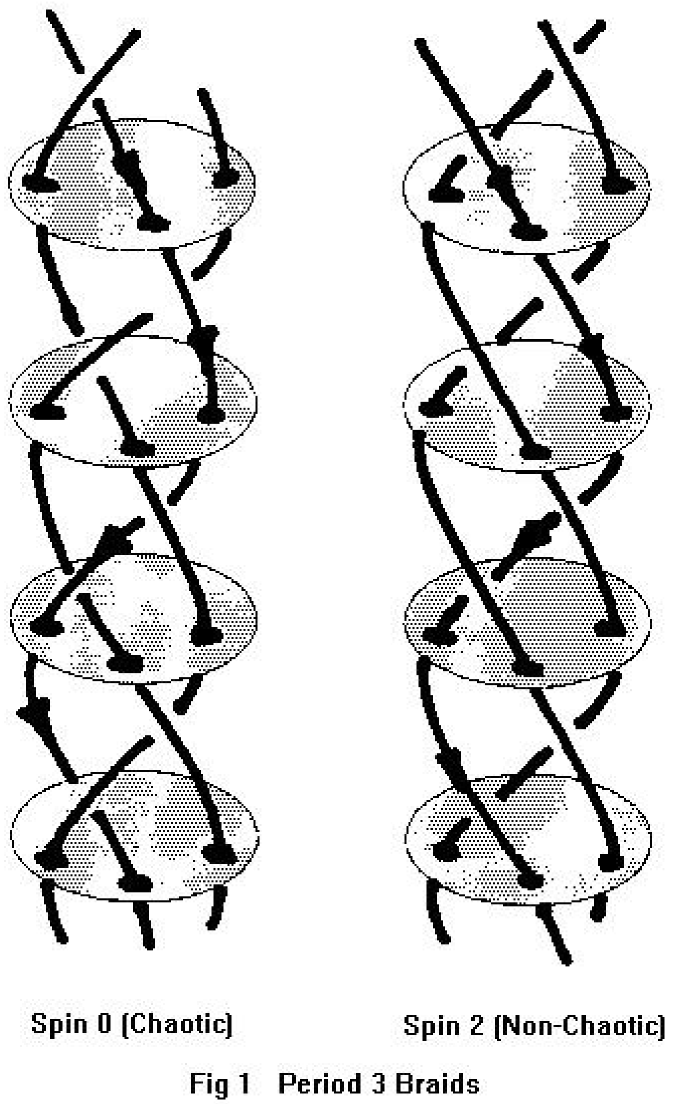

Figure 1

3-form density of Topological Spin . Each of these 3-forms are represented by a 4D direction field, and involve the 1-form of Action as a factor in the sense of an exterior product. To relate the above definition of a 3-form of Helicity with the six dimensional formulation involving the Biot-Savart substitution (to me) is an extraordinary constraint on the topology of the domain. The substitution effectively mixes a tensor and a tensor density, where definition 1 above mixes a tensor with a tensor, $A\hat{}F$. There is however, another well defined electromagnetic 3-form, $A\hat{}G$, which mixes a tensor (A) and a tensor density (G) . I call $A\hat{}G$ –with physical dimensions of angular-momentum– the Spin 3-form. (The 3-form $A\hat{}F$ has physical dimensions of Angular momentum divided by Ohms.)

In electrical engineering notation,

$$Spin : \mathbf{S}_4 = [\mathbf{A} \times \mathbf{H} + \phi\mathbf{D}, \mathbf{A} \circ \mathbf{D}]$$

$$Torsion\_Helicity : \mathbf{T}_4 = [\mathbf{E} \times \mathbf{A} + \phi\mathbf{B}, \mathbf{A} \circ \mathbf{B}]$$

The closure of $A\hat{}G$ defines a measure known as the first Poincare invariant,

$$P1 := (\mathbf{B} \circ \mathbf{H} - \mathbf{D} \circ \mathbf{E}) - (\mathbf{A} \circ \mathbf{J} - \rho\phi) \equiv d(A\hat{}G)$$

while the closure of $A\hat{}F$ yields the second Poincare invariant.

$$P2 := 2(\mathbf{E} \circ \mathbf{B}) \equiv d(A\hat{}F).$$

When these measures vanish (the divergences of the 4-vectors vanish), then there exist separate topological conservation laws (of Spin and Helicity).

The question arises, can the 1-form of Action be multiplied by a closure factor, not for the 1-form A, but for the 3-form $A\hat{}F$ or the 3-form density $A\hat{}G$. The answer is obviously yes, and the appropriate closure factor will have the functional form of the appropriate Holder norm. For example, the Topological Torsion direction field is generated by the exterior product $A_0\hat{}F_0 = A_0\hat{}dA_0$. If the 1-form A_0 is divided by the renormalization factor, β, then the Topological Torsion direction field A^dA is proportional to the direction field $A_0\hat{}dA_0$.

$$A\hat{}dA = (A_0/\beta)\hat{}d(A_0/\beta) = (1/\beta^2)A_0\hat{}dA_0. \qquad (6.142)$$

As

$$A_0\hat{}dA_0 = T^x dy\hat{}dz\hat{}dt - T^y dx\hat{}dz\hat{}dt + T^y dx\hat{}dy\hat{}dt - T^t dx\hat{}dy\hat{}dz, \qquad (6.143)$$

can be rewritten as

$$A_0\hat{\ }dA_0 = Z^x dZ^y\hat{\ }dZ^z\hat{\ }dZ^t - Z^y dZ^x\hat{\ }dZ^z\hat{\ }dZ^t + Z^z dZ^x\hat{\ }dZ^y\hat{\ }dZ^t - Z^t dZ^x dZ^y\hat{\ }dZ^z, \tag{6.144}$$

where the components Z^k may be computed as

$$|Z^n(x,y,z,t)\rangle = [\partial Z^m/\partial x^n]^{-1} \circ |T^m(x,y,z,t)\rangle\,. \tag{6.145}$$

The renormalization factor then can be computed from the algorithms given above, such that

$$\lambda = \beta^2 = \{a(Z^x)^p + b(Z^y)^p + c(Z^z)^p + e(Z^t)^p\}^{4/p}. \tag{6.146}$$

Hence,

$$\beta = \pm\{a(Z^x)^p + b(Z^y)^p + c(Z^z)^p + e(Z^t)^p\}^{2/p} \tag{6.147}$$

is a closure factor for the 3-form $A_0\hat{\ }dA_0$, such that $d(1/\beta^2)A_0\hat{\ }dA_0 = 0$. It is important to note that β need not be a closure factor for the 1-form, A_0; that is, $d(A_0/\beta) \neq 0$.

Example:

Consider the (Hopf) 1-form

$$\begin{aligned}
A_0 &= ydx - xdy + zdt - tdz, & (6.148)\\
dA_0 &= 2dy\hat{\ }dx + 2dz\hat{\ }dt, & (6.149)\\
A_0\hat{\ }dA_0 &= -\{+xdy\hat{\ }dz\hat{\ }dt - ydx\hat{\ }dz\hat{\ }dt + zdx\hat{\ }dy\hat{\ }dt - tdx\hat{\ }dy\hat{\ }dz\}, & (6.150)\\
dA_0\hat{\ }dA_0 &= -4dx\hat{\ }dy\hat{\ }dz\hat{\ }dt. & (6.151)
\end{aligned}$$

The Topological Torsion vector is

$$T = -[x,y,z,t], \tag{6.152}$$

which leads to

$$Z = -[x,y,z,t], \tag{6.153}$$

such that the closure factor, β, becomes

$$\beta = \pm\{a(x)^p + b(y)^p + c(z)^p + e(t)^p\}^{2/p}. \tag{6.154}$$

The renormalized Action which produces a closed topological Torsion 3-form is of the format

$$A_0 = \{ydx - xdy + zdt - tdz\}/\pm\{a(x)^p + b(y)^p + c(z)^p + e(t)^p\}^{2/p}. \tag{6.155}$$

Although $d(A_0/\beta)\hat{}d(A_0/\beta) = 0$, $d(A_0/\beta) \neq 0$, hence the 3-form of topological torsion is closed globally, but the 2-form $F = d(A_0/\beta) \neq 0$. The implication is that there are domains for which the 3-form can be used as a period integral, but the domain supports finite **E** and **B** fields. The components of the Topological Torsion vector are proportional to the position vector from the origin.

The closure of these two 3-forms is not necessarily zero, for

$$d(A\hat{}F) = F\hat{}F = 2(\mathbf{E} \circ \mathbf{B})dx\hat{}dy\hat{}dz\hat{}dt, \tag{6.156}$$

and

$$d(A\hat{}G) = F\hat{}G - A\hat{}J = \{(\mathbf{B} \circ \mathbf{H} - \mathbf{D} \circ \mathbf{E}) - (\mathbf{A} \circ \mathbf{J} - \rho\phi)\}dx\hat{}dy\hat{}dz\hat{}dt. \tag{6.157}$$

Each of these equations can be interpreted in terms of a 4-component vector field, $\mathbf{Z}_4$, and each 3-form admits an integrating factor such that the closure ("divergence") is zero. In such situations, the 3-form is homogenous of degree zero in terms of the component functions of

6.5.2 Topological Torsion and the Polarization quantum - a closed 3-form

Perhaps it is more important that the theorem of Topological deformation invariance of the closed integrals of the 3-form is also valid in **any** dimension for which $F\hat{}F$ is zero. In higher dimensions the integral must be evaluated over 3-sub_manifolds that need not be space-like. For 4 dimensions the 3-form $A\hat{}F$ has 4 components and can be constructed as

$$\text{Topological Torsion-Helicity:} \quad A\hat{}F = i(\mathbf{T}_4)dx\hat{}dy\hat{}dz\hat{}dt \tag{6.158}$$

where

$$\mathbf{T}_4 = -[\mathbf{E} \times \mathbf{A} + \phi\mathbf{B}, \mathbf{A} \circ \mathbf{B}]$$

From this formulation it must be remembered that the components of this "vector" transform as a third rank covariant tensor field; $\mathbf{A} \circ \mathbf{B}$ is merely the fourth component. In the 4-dimensional literature such objects are often described as pseudo vectors. They are not pseudo vector densities. If the 3-form $A\hat{}F$ is to be an evolutionary invariant, then

$$L_{(\beta\mathbf{V})} \iiint_{closed} A\hat{}F = \iiint_{closed} di(\beta\mathbf{V})A\hat{}F + i(\beta\mathbf{V})d(A\hat{}F) \tag{6.159}$$

$$= \iiint_{closed} i(\beta\mathbf{V})F\hat{}F \tag{6.160}$$

$$= \iiint_{closed} 2 \cdot (\mathbf{E} \circ \mathbf{B})i(\beta\mathbf{V})dx\hat{}dy\hat{}dz\hat{}dt. \tag{6.161}$$

If the closed integral of the Topological Torsion, $\iiint_{closed} A\hat{}F$, is to be an evolutionary deformation (hence topological) invariant for ANY process, $\mathbf{V}$, then $\mathbf{E}\circ\mathbf{B} = 0$. This requirement, which is equivalent to $F\hat{}F = 0$, implies that the Topological Parity must vanish. The condition is sufficient, but not necessary, for topological invariance of the Torsion- Helicity integral, even when the fields are explicitly time dependent. The relative integral invariant in 4D is:

$$H = \iiint\limits_{closed} A\hat{}F = \iiint\limits_{closed} \mathbf{T}^x dy\hat{}dz\hat{}dt - \mathbf{T}^y dx\hat{}dz\hat{}dt + \mathbf{T}^z dx\hat{}dy\hat{}dt - \mathbf{A}\circ\mathbf{B}dx\hat{}dy\hat{}dz \tag{6.162}$$

In all cases the 3 divergence of $\mathbf{B}$ vanishes, and the 2-form F is closed, for it is exact, $dF = 0$. For isochronous domains, $dt = 0$, and the integral reduces to the standard spatial format of plasma physics. However, there are example electromagnetic fields for which $\mathbf{A}\circ\mathbf{B} = 0$, and yet $\mathbf{T}_4$ is not zero.

Example: Define

$$\mathbf{A} = [0, 0, (x^2 + y^2)/2], \quad \phi = z. \tag{6.163}$$

Then

$$\begin{aligned}
\mathbf{B} &= [-y, x, 0] && (6.164)\\
\mathbf{E} &= [0, 0, -1] && (6.165)\\
\mathbf{A}\circ\mathbf{B} &= 0, \quad \mathbf{E}\circ\mathbf{B} = 0, && (6.166)\\
\mathbf{T}_4 &= -[z\mathbf{B}, 0] && (6.167)
\end{aligned}$$

Note that in 4 dimensions (that admit time dependent fields), the frozen in lines are the line of the Torsion vector, which can be dominated by the $\mathbf{B}$ field, can also have a component due to $\mathbf{E}\times\mathbf{A}$. The lines of $\mathbf{T}_4$ can continuously evolve, with most regions in a plasma dominated by $\phi\mathbf{B} \neq 0$, $\mathbf{E}\times\mathbf{A} \to 0$. Then as $\mathbf{B}$ lines approach one another, induction causes $\mathbf{E}\times\mathbf{A} \neq 0$, $\phi\mathbf{B} \to 0$. The "**B** lines" terminate on a null (a boundary point), break apart, and then possibly reconnect different segments, after which $\phi\mathbf{B} \neq 0$, $\mathbf{E}\times\mathbf{A} \to 0$. (This mechanism appears to be at the foundation of the numeric technique investigated by Horning.

Special situations become evident when the integration domain is compact with a boundary, for then Stokes law may be applied, and deformation invariance requires that $i(\beta\mathbf{V})A\hat{}F$=0 *on the boundary.* These are interesting but special cases which are invariants of only a special choice of boundary conditions. For example, if $\beta(x, y, z) = 0$ defines the boundary, then for deformation invariance of the integral it must be true that the function β also must be an evolutionary invariant, such that $L_{(\mathbf{V})}d\beta = i(\mathbf{V})d\beta = 0$. Classically the function β which is used to define the boundary, is not arbitrary, but must be a first integral of the evolutionary vector field.

For such special cases, the field on the boundary need not be tangential! There are other special situations as well.

For integration domains which are open, the criteria for absolute integral invariance is much more severe, and requires that $d(i(\beta\mathbf{V})A\hat{}F) = 0$. This constraint is to be recognized as the criteria that the evolutionary vector field $\beta\mathbf{V}$ be an element of the symplectic group. I have demonstrated that all such evolutionary processes are thermodynamically reversible.

6.5.3 The Flux or Circulation Integral 1-form

For the Cartan topology constructed from a fundamental 1-form of Action and a fundamental N-1 form of Current, several period integrals of closed forms integrated over closed chains appear in a natural manner. In particular on an N=4 dimensional domain, the four period integrals of most interest are the period integrals of flux (circulation), charge, spin and torsion. The fundamental period integral over a closed 1-form will be defined as the "Circulation" or "flux" integral.

When the Pfaff Topological dimension is 2, there exists a submersive map to two dimensions, and the vector fields on this domain will have two irreducible components, say $[\Phi(x, y, z, t), \Psi(x, y, z, t)]$. Following the procedure of the preceding section, construct the 2-dimensional volume element defined as $\Omega = \rho d\Phi\hat{}d\Psi$, and the $n - 1 = 2 - 1 = 1$ form $A = (\Phi d\Psi - \Psi d\Phi)/\{\pm a\Phi^p \pm b\Psi^p\}^{2/p}$. The exterior differential of such a 1-form is exactly zero for all point sets that exclude the null set of the denominator. The classic choice is for $p = 2$,and $a = 1$, $b = 1$, (+,+) signature. The closed integrals of these closed 1-forms then can be expressed as

$$\text{Circulation } \Gamma = \oint_1 A = \oint_1 (\Phi d\Psi - \Psi d\Phi)/\{\Phi^2 + \Psi^2\} \tag{6.168}$$

By substituting the functional forms in terms of (x,y,z,t) the circulation integral can be written in terms of functions on (x,y,z,t) and their differentials, $\{dx,\ dy,\ dz,\ dt...\}$

As an example, suppose that the domain is three dimensional, N=3. Then the zero sets of $\Phi(x, y, z) = 0$ and $\Psi(x, y, z) = 0$, represent two 2 dimensional surfaces which may or may not have one or more lines of intersection. If the surfaces intersect, then

$$Intersection = d\Phi\hat{}d\Psi \neq 0. \tag{6.169}$$

If the closed integration paths cannot be contracted to a point, because they encircle these lines of intersection, the values of the integrals have rational ratios depending on how many lines are encircled and how many times the integration path encircles a line. The lines of intersection must have zero divergence (and therefore must stop or start on boundary points, or are closed on themselves). Otherwise the integration chains can be deformed and then contracted to a point. The classic example is given by the 1-form, $A = (ydx - xdy)/(+x^2 + y^2)$ in three dimensions. For integration contours that encircle the z axis, the value of $\Gamma = \oint_1 A = 2\pi$. In hydrodynamics, this vector field is called a potential (or Abrikosov) "vortex", even though the vorticity $\boldsymbol{\omega} = curl\mathbf{v} = 0$. Stokes theorem does not apply as the closed integration chain is a cycle that is not a boundary.

6.5.4 The Charge Integral 2-form

Many different options exist for construction of these invariant topological structures from closed p-forms. The idea is to find a formulation for a closed form on a domain, and then to specify a closed and compatible integration chain. The integration chain need not be a boundary, but only a closed cycle. For example, from the components of the specified vector, A_μ , the Jacobian matrix, $[\partial A_\mu/\partial x^\nu]$ can be constructed. The rows or columns of the matrix of cofactors of the Jacobian (the adjoint matrix) forms a set of vector fields that have zero divergence [Turnbull 1960], and therefore these vectors could be used to construct relative integral invariants. In every case there exists an algebraic construction which produces a vector that is divergence free and whose line of action is uniquely related to original vector that was used to construct the Cartan topology. That vector may be constructed by multiplying the original vector A_μ by the matrix of cofactors and then dividing by the function λ defined above. The construction replicates the previous procedure. As an application for n = 3, p=2, consider the vector that represents the difference between two space curves, $\mathbf{z} = \mathbf{R}_2 - \mathbf{R}_1$. Then compute the two form $G(z)$ from the "volume" element $\Omega = dz^1\hat{}dz^2\hat{}dz^3/\lambda$, to give

$$G_{n=3} = \{z^1 dz^2\hat{}dz^3 - z^2 dz^3\hat{}dz^1 + z^3 dz^1\hat{}dz^2\}/\lambda \tag{6.170}$$

where

$$\lambda = (\pm(z^1)^2 \pm (z^2)^2 \pm (z^3)^2)^{3/2}. \tag{6.171}$$

Next assert that the displacements of interest are constrained by two parametric curves given by

$$d\mathbf{R}_1 = \mathbf{V}_1 dt \quad and \quad d\mathbf{R}_2 = \mathbf{V}_2 dt', \tag{6.172}$$

where the parameters dt and $dt\prime$ are not functionally related (which would imply that $dt\hat{}dt\prime = 0$).

It is important to realize that kinematic constraints are topological constraints that refine the Cartan topology, a topology based solely upon the specified 1-form of action, A. From a physical point of view, these constraints can be interpreted as constraints of null fluctuations and in certain circumstances can be associated physically with the limit of zero temperature. To demonstrate the utility of such constraints, substitute these differential expressions into the expression for the 2-form G of "current" in N=3 dimensions, and carry out the exterior products, using $dt\hat{}dt' \neq 0$, *but* $dt\hat{}dt = 0$ and $dt\prime\hat{}dt\prime = 0$. The result is the vector triple product representation for the Gauss integral,

$$Q = \oint_2 G = \oint_2 \{\mathbf{z} \circ \mathbf{V}_1 \times \mathbf{V}_2\} dt\hat{\ } dt' / (\mathbf{R}_1 \circ \mathbf{R}_1 - 2\mathbf{R}_1 \circ \mathbf{R}_{21} + \mathbf{R}_2 \circ \mathbf{R}_2)^{3/2}. \quad (6.173)$$

The integration domain is the closed "2-dimensional area" formed by the displacements along the non-intersecting curves defined by the two distinct parameters, dt, and dt'. This double integral is to be recognized as the Gauss linking integral of Knot Theory [7]. (Without the kinematic substitutions, it may also be interpreted as the charge integral of electromagnetic theory.) When integrations are computed along closed curves whose tangent vectors are $\mathbf{V}_1$ and $\mathbf{V}_2$, then the integer values of the closed integral may be interpreted as how many times the two curves are linked. Note that the same integer result is obtained when the vector $\mathbf{z}$ is interpreted as the sum of the two vectors, $\mathbf{z} = \mathbf{R}_2 + \mathbf{R}_1$, although the values of the integrals have different scales.

The constraint that $dt\hat{\ } dt' \neq 0$ implies that the "motion" along the curve generated by $\mathbf{R}_1$ is independent of the "motion" along the curve generated by $\mathbf{R}_2$. If the curve generated by $\mathbf{R}_1$ is a conic in the xy plane and the curve generated by $\mathbf{R}_2$ is a conic in the xz plane, then the surface swept out by the vector $\mathbf{z}$ is a Dupin cyclide. Such surfaces have application to the propagation of waves in electromagnetic systems.

From another point of view, consider the ruled surface [Struik1961] defined by the vector field of two parameters,

$$\mathbf{z}(\mu, t) = \mathbf{R}(t) \pm \mu \mathbf{V}(t). \quad (6.174)$$

Vector fields of this type are primitive types of "strings" for fixed values of the parameter, t, and string parameter, μ. Direct substitution of the physical constraints, $d\mathbf{R} - \mathbf{V}dt = 0$, and $d(\mathbf{V}) - \mathbf{A}dt = 0$ leads to the topological Gauss integral,

$$Q = \oint_2 G = \oint_2 \{\mathbf{R} \circ \mu \mathbf{V} \times \mathbf{A}\} / \lambda =$$
$$\oint_2 \{\mathbf{A} \circ \mathbf{R} \times \mu \mathbf{V}\} dt\hat{\ } d\mu / (\mathbf{R} \circ \mathbf{R} \pm 2\mu \mathbf{R} \circ \mathbf{V} + \mu \mathbf{V} \circ \mu \mathbf{V})^{3/2}. \quad (6.175)$$

It is apparent that the interaction of the "angular" momentum, $\mathbf{L} = \mathbf{R} \times \mu \mathbf{V}$, and the acceleration, $\mathbf{A}$, produces a topological invariant whose values are "quantized" (in the sense that the ratios of the integrals are rational). Note that for the classical central field problem where the force (acceleration) and the angular momentum are orthogonal, the orbits are in a plane and the Gauss–linking number is zero. Further note that the triple vector product of the integrand is proportional to the Frenet torsion of the orbit. An orbit that is planar has Frenet torsion zero everywhere. The Gauss linking integral is a special case of the Gauss two dimensional period integral of electromagnetic theory when the integration domains can be factored into independent products, $dt\hat{\ } dt\prime \neq 0$.

6.5.5 Amperian Links

You might also be interested in something that I did after the Chapman conference stimulated me. I think I have found a *raison d'etre* for the formation of an accretion disk in a otherwise central field problem. Using Maple, I have found a time dependent solution to Maxwell's equations that makes the z=0 plane of a rotating plasma a chiral attractor. The force of attraction from the top and the bottom is due to a Lorentz $\mathbf{J} \times \mathbf{B}$ term. This the first time I have ever seen a simple mechanism that might explain the formation of accretion disks around stars and planets. The Solution is just a model field, but exhibits both Helicity and Spin effects.

6.6 Thermodynamic Singularities

6.6.1 Folds

6.6.2 C0,C1, and C2 processes

Chapter 7

APPENDIX TOPOLOGY AND THE CARTAN CALCULUS

7.1 Why differential forms?

7.1.1 Pair and Impair exterior differential forms.

Exterior differential p-forms should not be confused with p-tensors. All exterior differential forms (with respect to diffeomorphic transformations) are invariant scalars to within a factor. When the factor is not unity, such scalars have been defined classically as "pseudo-scalars". A particularly useful choice of the pseudo-scalar factor is related to the determinant of the Jacobian of the diffeomorphic transformation, $\Delta = \det([\mathbb{J}])$. Explicitly, if the factor is related to Δ^{-k}, then such a pseudo-scalar has been defined as a density of weight, k.

Exterior differential forms are well behaved with respect to diffeomorphic mappings of the independent variables. This means that the functional form of the coefficients can be pushed forward from given initial data to predicted final data, or pulled back from given final data to retrodicted initial data. However, exterior differential forms are also well behaved with respect to a wider class of differentiable mappings. The larger class of differentiable mappings includes those mappings that are not reversible, as well as diffeomorphisms which are reversible. The mapping functions for such transformations are C1-differentiable, and produce a Jacobian matrix of partial derivatives, $[\mathbb{J}]$, but the Jacobian need not be invertible. Such non-invertible differentiable transformations will be called "C1/d" (C1 differentiable mod diffeomorphisms) transformations in order to distinguish them from invertible C1 "diffeomorphisms". The "C1/d" transformations can be used to describe topological evolution, while the subset of diffeomorphisms cannot.

Relative to maps of the independent variables, it is useful for applications to specialize the theory of exterior differential forms to two species: Pair and Impair differential forms.

- Pair exterior differential forms are scalar invariants with respect to diffeomorphic transformations of the independent variables. Pair exterior differential forms are well defined pull_back scalars with respect to C1/d transformations, where the pull_back linear mappings of the coefficients are determined by the *transpose* of the Jacobian matrix. When applied to the theory of electromagnetism, it will become evident that the Pair forms are related to concepts of Forces and field intensities, E and B.

- Impair exterior differential forms are pseudo-scalar invariants; that is, Impair forms are scalars to within a factor Δ^{-1}. Impair exterior differential forms are well defined pull_back pseudo scalars with respect to C1/d transformations, where the pull_back linear mappings are determined by the *adjoint* of the Jacobian matrix. When applied to the theory of electromagnetism, it will become evident that the Impair forms are related to concepts of Sources, and field quantities, D and H.

- Note that relative to the constraint of special or proper unimodular groups of transformations, where $\Delta^{-1} = +1$, the two species of differential forms are not distinguishable. It is also remarkable that, relative to C1/d transformations, the pull_back is well defined, but the push_forward is not. Unique prediction fails, but retrodiction is deterministic. An "arrow of time" is built into the logic of differential forms.

The Constraints of Tensor Analysis

In tensor analysis, there exist two important categories of tensors, known as covariant tensors and contravariant tensors. These sets are distinguished by their transformation properties with respect to invertible diffeomorphisms. By classical convention:

- The functional components of Covariant vectors are pushed forward from the initial state to the final state by means of the linear transformations induced by the Inverse of the Jacobian matrix of the diffeomorphism between independent variables. It is usually not emphasized that Covariant vectors pulled back by means of maps induced by the transpose of the Jacobian matrix.

- The functional components of Contravariant vectors are pushed forward from the initial state to the final state by means of the linear transformations induced by the Jacobian matrix of the diffeomorphism between independent variables. It is usually not emphasized that Contravaraint are pulled back by means of maps induced by the Inverse of the Jacobian matrix.

A preferred notation for the two species of differential forms (using ordered or collective indices) is demonstrated by a Pair 1-form, A, and an Impair N-1 form, J.

$$A = a\ \text{"}pair\text{"}\ exterior\ differential\ 1-form \tag{7.1}$$

$$A = A_k(x)dx^k \tag{7.2}$$

$$F = a \text{ "}pair\text{" } exterior\ differential\ 2 - form : F \ = F_{jk}(x) dx^j \hat{} dx^k \tag{7.3}$$

$$J \ = \ an \text{ "Im}pair\text{" } exterior\ differential\ N-1\ form\ density \tag{7.4}$$
$$J \ = \ J^k(x)\{dx^1 \hat{} dx^2 ... \widehat{dx}^k ... dx^N\} \tag{7.5}$$

$$G \ = \ an \text{ "Im}pair\text{" } exterior\ differential\ N-2\ form\ density \tag{7.6}$$
$$G \ = \ G^{jk}(x)\{dx^1 \hat{} dx^2 ... \widehat{dx}^j ... \widehat{dx}^k ... dx^N\} \tag{7.7}$$

The hatted symbol $\widehat{dx}^k$ means that that term dx^k is left out of the factors that form the volume element. Note that the coefficients of the Impair forms is defined with an upper index which is a collective compliment of "left out" factors in the volume element.

7.1.2 Functional Substitution and the pull_back

Cartan's theory of exterior differential forms is NOT just another notational system of fancy. The important fact, often ignored, is that a differential form is not necessarily a tensor. In short, a differential form is a mathematical object that is well behaved with respect to differentiable maps, $y^k \Rightarrow x^\mu = \phi^\mu(y^k)$, from an initial state $\{y^k\}$ of independent variables to a final state $\{x^\mu\}$ of independent variables. Such maps need not be homeomorphisms. Hence the topology of the final state need not be the same as the topology of the initial state.

A tensor is an object which is well behaved if and only if the differentiable map ϕ^μ from initial to final state has a differentiable inverse. Such maps are defined to be diffeomorphisms, a subset of homeomorphisms, and as such the topology of the final state and the initial state must be the same.

Differential forms can be used to understand and study topological evolution, while tensors cannot.

Contravariant and covariant tensors

A tensor is a restricted class of mathematical objects that transform in a multilinear manner relative to the Jacobian matrix $[\mathbb{J}^\mu_k] = \left[\partial\phi^\mu(y)/\partial y^k\right]$ of the differentiable map, and its inverse. The motivation for the definitions is most transparent when one considers the matrix inner product of two vectors, $\langle A_k(y)| \circ \left|V^k(y)\right\rangle$, on the initial state. Suppose that the Identity matrix can be written in terms of two factors, $[\mathbb{I}] = [\mathbb{J}]^{-1} \circ [\mathbb{J}]$. Then the equation,

$$\langle A_k| \circ \left|V^k\right\rangle \Rightarrow \langle A_k| \circ [\mathbb{J}^\mu_k]^{-1} \circ [\mathbb{J}^\mu_k] \circ \left|V^k\right\rangle = \left\langle \overline{A}_\mu(y)\right| \circ |\overline{V}^\mu(y)\rangle, \tag{7.8}$$

can be interpreted in terms of a matrix transformation, where

$$|\overline{V}^{\mu}(y)\rangle = [\mathbb{J}_k^{\mu}(y)] \circ \left|V^k(y)\right\rangle \tag{7.9}$$

and

$$\langle A_k(y)| \Rightarrow \left\langle \overline{A}_{\mu}(y)\right| = \langle A_k| \circ [\mathbb{J}_k^{\mu}(y)]^{-1} \tag{7.10}$$

in such a way that the value of the matrix row-column vector product before the transformation and after the transformation has the same value. Note that the vector with components indexed by μ on the final state has arguments in terms of the independent variables on the initial state. IN order to produce a vector array of functions on the final state with arguments in terms of the variables x on the final state requires knowledge of the inverse mapping that expresses y in terms of x. The column vector $|V^{\mu}(y)\rangle$ is the epitome of a contra-variant vector. The row vector $\langle A_{\mu}|$ is the epitome of a covariant vector. The matrix $[\mathbb{J}_k^{\mu}(y)]$ is the Jacobian matrix of the transformation from initial to final state. The matrix transformation rules above give primitive meaning to the concepts of covariance and contra-variance. These rules in tensor analysis are extended to multi-linear objects.

The classical definitions consider a tensor to be a mathematical object whose functional form given on the initial state is well defined on the final state. Note that the tensor operations are a "push_forward" from the initial to the final state. The transformation rules are of two types are defined as:

$$\textit{The Contravariant Rule} \quad : \quad (pushforward) \tag{7.11}$$

$$\left|V^k(y)\right\rangle \Rightarrow |V^{\mu}(y)\rangle = \left[\partial x^{\mu}(y)/\partial y^k\right]\left|V^k(y)\right\rangle \tag{7.12}$$

$$\Rightarrow \left|\overline{V}^{\mu}(x)\right\rangle = |V^{\mu}(y(x))\rangle \tag{7.13}$$

$$\textit{The Covariant Rule} \quad : \quad (push\ forward) \tag{7.14}$$

$$A_k(y) \Rightarrow A_{\mu}(y) = A_k(y)\left[\partial y^k(x)/\partial x^{\mu}\right] \tag{7.15}$$

In both cases an inverse mapping ($x = \phi^{-1}(y)$) is required if the arguments of the functions created on the final state are to be expressed in terms of the independent variables, x, of the final state.

When the tensor rule for covariant transformations is multiplied on both sides by the Jacobian matrix, and using the constraint inherent in the tensor definitions that an inverse Jacobian exists, a more general formula for covariant transformations can be obtained. If the functions that make up the covariant object are given on the final state, then the functions on the initial state are well defined in terms of the independent variables on the initial state. This more general situation works when the map is differentiable, but no inverse map exists. The process is defined as the covariant pull_back. For more detail, see the subsection on the Pull_back examples, below.

$$The\ Covariant\ Rule\ (pull\_back) : |A_k(y)\rangle = A_\mu(x(y)) \left[\partial x^\mu(y)/\partial y^k\right] \Leftarrow A_\mu(x) \tag{7.16}$$

Retrodiction versus Prediction

It has been shown [RMK 1976 b] that with respect to maps without differentiable inverses, the classic rules of tensor transformations do not permit the unique prediction of the functional form on the final state of either contravariant or covariant tensors, given the functional form of these tensors on the initial state. Numeric values on the final state sometimes can be predicted, but the functional form (defining neighborhoods) on the final state cannot be uniquely predicted. It also has been shown relative to maps without differentiable inverses that a given the functional form of a contra-tensor on the final state does not permit a unique retrodiction of the functional form on the initial state.

However, there are two situations where unique well defined retrodiction is possible in a functional sense. This retrodiction process is defined as the "Pull-Back" The first situation involves covariant tensors on the final state. The second situation involves contravariant antisymmetric tensor densities on the final state. Given a mapping and its Jacobian, $[J^\mu_k(y)]$, the two forms of pull_back may be written as:

$$pull\_back\ Covariant\ Rule : \overline{A}_k(y) \Leftarrow A_\mu(x(y))\,[J^\mu_k(y)] \tag{7.17}$$

$$pull\_back\ Contravariant\ density\ Rule : \overline{C}^k(y) \Leftarrow \left[AdJ^k_\mu(y)\right] C^\mu(x(y)). \tag{7.18}$$

In the notation above, $\left[AdJ^k_\mu(y)\right]$, is the adjoint of the Jacobian matrix (matrix of cofactors transposed) and does not depend upon the determinant of the transformation. The adjoint matrix may be algebraically determined even when the determinant of the matrix is zero. Hence the pull_back is well defined, even if an inverse does not exist. The contravariant density, or current, $C^\mu(x)$, is like the charge current density of electromagnetism. It is suggested herein that these two rules should be used as foundations for transformations of objects which would be considered to be tensors, if the mappings are constrained to be diffeomorphisms.

7.1.3 Summary: Pair forms and Impair forms

The bottom line is that retrodiction and functional substitution leads to two well defined species of differential form objects: Pair (even) forms with antisymmetric coefficients, which pull_back relative to the Transpose of the Jacobian of the mapping, and Impair (odd) forms with antisymmetric coefficients objects that pull_back via Adjoint of the Jacobian matrix. These latter objects are sensitive to orientations, where the former are not. Both objects are not dependent upon an invertible differential mapping, hence can be used to study continuous topological evolution. However, when the maps from an initial state of independent variables to a final state of independent variables are unimodular det =1 diffeomorphisms, then the coefficients of the pair forms behave as covariant tensors, and the coefficients of the impair forms behave as contravariant tensors. It the diffeomorphisms are not such that det of the Jacobian matrix is 1, then the coefficients transform as contravariant tensor densities.

- The 2-form F of electromagnetic field intensities is a pair differential form. The integrals of exterior differential forms are invariants of diffeomorphisms which may change the sign of the N-volume (a change of orientation). Such integrals are scalars.
- The N-2 form G is an impair 2 form density in a space of 4 dimensions. The integrals of impair differential form densities change sign with respect to diffeomorphisms that change the N-volume orientation. Such integrals are pseudoscalars and can lead to enantiomorphic pairs.

7.2 Why Topology?

The purpose of this monograph is to sensitize the reader to the topological aspects of physical systems. The motivation is to develop a better understanding of the real world of irreversible processes, which if they take place continuously, must involve changing topology. However, before it is possible to make practical use of the concepts of changing topology, it first is necessary to understand what topology itself is all about. It is important to be able to identify topological properties, such that when these topological properties change, under the action of continuous but irreversible processes, the changes will be recognized.

Right up front, memorize the following definitions. The implications of these definitions will be clarified in that which follows.

1. A topological property is an invariant of a homeomorphism.

2. A homeomorphism is a map describing a process from an initial to a final state which is both continuous and reversible. Reversible means that the inverse map exists and is also continuous. A homeomorphism need not be differentiable.

3. A topological structure is the specification of enough topological properties on both the initial and final state to permit a determination to be made if a map, or a process, is continuous from the initial to final state, even though it may not be reversible, and the topology of the initial and final state are not the same!

4. A process or a map is continuous if the limit points of the initial state, relative to the topology of the initial state permute into the limit points of the final state, relative to the topology of the final state.

5. A limit point p of a subset, A, is a point such that every open set of the given topology that contains p contains another point b, of the subset A, but not equal to p. Note that p is not necessarily a point in A.

6. If a homeomorphism and its inverse are both continuous and differentiable, then the map is defined to be a diffeomorphism. Tensor analysis is usually restricted to the study of diffeomorphisms. Simple geometric properties are properties which are invariant with respect to the two special diffeomorphisms of translation and rotation. In this Klein sense, size and shape are geometric properties.

7. Open sets, or their compliments which are closed sets, may be used to define a topology. A given collection of subsets defines an a open set topology if the intersection of any pair of elements in the subset collection is also member of the subset collection, and if every union, infinite or not, of elements in the subset collection is also an element of the given subset collection. In essence, topology is based upon the idea of closure under logical union and intersection.

8. A given collection of sets can support many different topologies, just by choosing subset collections that obey the closure rules described above. These concepts are best explained by examples to be found in the section below entitled Point Set Topology.

9. The bulk of the work herein is directed to the study of twice differentiable maps which are continuous but not homeomorphic.

There are several ways to approach the concepts of topology. The mathematicians of 1800 to 1900 discovered a number of "global" results and features of mathematical systems that were deformation invariants, but the bulk of the work in topology as we know it now began after 1900. It would appear that the name "topology" became accepted by mathematicians about 1925, although there were then two disjoint schools of topology (point set topology and combinatorial algebraic topology). In the early days these disjoint schools were assumed to be studies of different mathematical topics. It is now known that these different "ways" of getting at topological information are equivalent. Most of the classical approaches of teaching topology, however, are much too stilted to have acceptance by applied scientists and engineers.

However, E. Cartan in the early 1900's developed an extraordinary set of ideas based on two simple extensions of the ordinary calculus. He exploited the notion of the Grassmann exterior product, and developed the formalism of exterior algebra. To this (closed) algebraic system he added the concept of the exterior derivative. From the notion of the exterior product and the exterior derivative almost all of the useful features of differential topology can be constructed: the mathematical science is called the theory of exterior differential forms.

Topology is intuitively a non-local, or "global", idea of how interior things, perhaps consisting of many parts, behave synergistically with their environment or exterior, and with their boundary. A topological property may be viewed intuitively as an invariant of a continuous deformation. The concept of physical coherence is intuitively a topological idea in that it corresponds to the notion of a synergetic interaction between parts not at the same point. The classic understanding of the coherence of a crystal is due to a very specialized and refined topology induced on space time by the presence of matter. The fact that a cylindrical wave guide has a low frequency cut-off, while the coaxial cable can transmit DC current is a topological idea demonstrating the interaction of a system and its boundary.

Note that during the discussions above no mention is made of how big or how small the system to be studied is; there has been no *geometric* definition of scales. This notion that topological properties are independent from scales is the hardest concept to absorb for a scientist non-sensitized to topological thinking. After all, scientists are trained to "measure" something; they want to measure a size or shape. These geometrical hangups must be removed from the topological perspective, for if the system is too small, just stretch it out; if the system is to big, just compress it. Physical topology is the study of those properties that are independent from some legislated scale marks on a ruler, or on the number of time ticks of a clock. Physical topology deals with ideas that are in a sense the same on both the scale of the galaxy and on the scale of the atom. This is not to say that the world of microphysics is identical to the world of macrophysics, for when a physical system is constrained to yield geometric invariants, certain topological features will dominate over other topological features, depending on the scales chosen. The coaxial waveguide always has a low-frequency cut-off (a topological idea), but the actual wavelength depends upon the geometric idea of scales.

In this book the emphasis will be on the use of Cartan's theory of differential forms to describe topological features that are independent of scales. Note for those readers with some exposure to the exterior calculus, this means that the concepts of coordinate representations, or metric, or connection, or fiber bundles (with their more geometrical content) are going to be suppressed in favor of the Cartan theory, which does not require such constraints on the base variety of space and time. A natural question arises as to how a "differential form" can be used to describe global information not retrievable from descriptions at a point.

The paradox is resolved when it is realized that the differential form is to be considered as an entity "before" a limiting process has taken place. Recall that the limiting process of differential calculus requires some neighborhood constraint to be specified before the derivative is computed. In fact the limiting process depends upon the specification of a topology. The usual topology assumed is the homogeneous connected euclidean topology of R^N which consists of open sets defined in terms of open balls of domain less than r. Physical systems will require for their description topologies that are not equivalent to R^N, although in a small neighborhood in which certain "singularities" have been removed, it is generally assumed that the physical systems can be approximated by R^N.

Cartan's differential forms have amazing properties not contained in other mathematical entities:

1. Differential forms on the final state are well behaved with respect to functional substitution of differentiable maps from the initial to the final state, even though these maps are irreversible! This fact is known as the "pull-back" and is the cornerstone of the investigations about irreversible processes. By functional substitution, not only is it possible to compute values of a differential form on the initial state from values given on the final state, but also it is possible to compute the *functional form* of the co-tensor fields on the initial state in terms of the functional forms of the co-tensor fields used to define the differential forms on the final state. Differential forms are defined on the co-tangent space of tensors over the base of space time, and pullback via the transpose of the Jacobian of the map from the initial to final state. Contra-variant tensor fields do not enjoy this unique functional behavior under the "pullback", unless the inverse map is well defined. However contravariant tensor *densities* do enjoy the pullback property even though the inverse map does not exist. They are retrodictive relative to the adjoint of the Jacobian of the map from initial to final state. Both the transpose and the adjoint of the Jacobian exist even thought the inverse does not.

2. Differential forms can carry information about singularities, and these singularities dictate much of the topological content of the differential form. The singularity to be studied is not of the type that blows up to infinity, necessarily, but instead is the more innocuous zero set. It is the zero points of functions and vector fields under the action of continuous maps that is of predominant interest herein. The zeros of a map become the infinities of the inverse map, but in this monograph the emphasis is on how much can be determined about processes that need not have an inverse. The singular infinities of the Dirac delta function are avoided in this book, if at all possible, but the "singularities" of the zeros are crucial.

3. Differential forms may have preimages which are not unique. It is precisely this multi-valuedness of an integral preimage that allows differential forms to single out the physically interesting characteristics, or wave-fronts, or shock-fronts, or defects of dynamical systems. These topological singularities or defects are point sets (which may or may not be stationary) upon which a unique solution to some system of partial differential equations describing the physical system cannot be analytically continued from a nearby neighborhood. As will be determined later, dimension and the number of components are topological properties, and the generation of a defect will correspond to a topological change of dimension, or the creation of multiple components, over some domain.

A list is displayed below of a number of physical effects which - to first order - do not depend explicitly on size or shape. Therefore, these phenomena must have a topological basis for their explanation. Key topological properties are the number of parts (connectivity), the dimension, and orientability. Concepts of size and shape, metric, or connection can not enter into the explanation of such phenomena, except in an auxiliary way. As the monograph progresses, the details of these topological features will be explained

Planck's Radiation Formula.
The distribution law is independent from the size and shape of the hot body.

Coaxial Wave-Guide Propagation
A hollow wave guide is a high-pass filter, but a non-simply connected co-axial cable can pass DC.

Chaos does not occur in Dimension 2
A system of ODE's has a PDE equivalent. If the PDE satisfies the Frobenius integrability theorem, according to Darboux there is always a representation in terms of 2 functions.

Bohm-Aharanov effect and the Flux quantum (based on 1-forms)
Flux quanta in superconductors come in integer multiples of h/2e.
Gauss Law and the charge quantum (based on 2-forms)
The integral of the $\iint_{closed} \mathbf{D} \circ \mathbf{dS}$ over a bounding surface only depends upon the number, n, of electrons of charge e= 1.6 x $10^{-19}C$ in the interior of the closed surface, no matter what the size or shape of the closed surface may be.

Topological Torsion and Topological Spin (The Poincare quanta based on 3-forms)
The Bohm-Aharonov effect is a 1-Dimensional period integral. Gauss's Law is a 2-Dimensional period integral. There are two 3-Dimensional period integrals that have been little studied in science, but appear to have significance in non-equilibrium hydrodynamics and plasmas as robust coherent vortex-like helical structures - insensitive to deformations in space time.

Quantum Transition Probability. (Cross ratios are independent from scale)

Fermi's Golden Rule demonstrates that the transition probability is a cross-ratio projective invariant, independent from scale.

Thermodynamic Irreversibility. (The Heat 1-form, Q, does not admit an integrating factor.)

The idea that thermodynamic irreversibility is defined by the failure of the Frobenius Theorem for the 1 form of work implies that the topological dimension of the Action 1-form must be 4. In other words, thermodynamic irreversibility is an artifact of four dimensions.

Thermodynamic Phases

The difference between a vapor and a liquid is that the liquid is apparently connected and the vapor consists of many disconnected parts. Then number of components is a topological property. Condensation implies a change in topology takes place. Condensation is a gluing or pasting process which often can be described continuously. Vaporization, of the other hand is a discontinuous or cutting process, and does not usually permit a continuous description.

The Law of Corresponding States

In chemistry there is the law of corresponding states, which demonstrates the universality of thermodynamics, independent form the size and configuration of the molecules under consideration.

7.2.1 Geometry and Physics

Euclidean Geometry - Invariant Size and Shape - Rigid Body motion

Historically, fundamental physical theories have been based on geometrical models, geometrical relationships and geometrical properties of physical objects. In fact the very idea of a physical measurement is associated with the concepts of how big, how far, or how long. These ideas of measurement are intuitively geometrical ideas, and all involve a comparison of something of immediate interest to some legislated standard. However the idea of a comparison is not restricted to pure geometrical concepts of size and shape. An abstract comparison may be viewed as a mapping from a range to a domain, or a transformation from an initial state to a final state. The mapping or transformation may be an element of an equivalence class of mappings, and that class is determined by its invariant properties. Recall that, according to Felix Klein, a (euclidean) geometrical property is defined to be an invariant of a translation or rotation. A simple observation demonstrates that size and shape are such geometrical properties, and these properties are the fundamental invariants of that branch of physics that deals with rigid body motion.

Isometric Geometry - Invariant distance - Bending

With the advent of tensor calculus, mathematicians relaxed the constraints of pure (euclidean) geometry a bit in order to include the concept of bending without compression or shear. The fundamental physical invariant of interest became the distance between a pair of points: size is considered is an invariant in such geometries, but shape is not (necessarily) an invariant of the equivalence class of transformations that describe bending processes. The equivalence class of transformations based on the property of invariant distance are called isometries. The pure geometrical constraint of invariant shape is relaxed to include the possibility of bending. The idea of invariant distance has dominated physical theories since the turn of the century. In fact a new derivative concept, the covariant derivative, was defined in a such a manner that it preserves the concept of distance as an invariant. Displacements via a "covariant derivative" are constrained such the distance between a pair of points is preserved as an invariant! However, be warned that the prescription of a covariant derivative has in its foundations the impossibility of intrinsically describing compressions or shears!

Conformal Geometry - Invariant Phase - Twisting

rotational parallelism - rotational shear - a second torsion concept.

Affine Geometry - Invariant parallel planes - Shear

Linear parallelism affine torsion

Homothetic Geometry - Pressure

If the deformation is confined to compressions or expansions, then the distance between a pair of points is no longer invariant, and as both size and shape are not necessarily invariants, the covariant derivative concept becomes obsolete. The question arises as to what invariants might be used to classify such transformations. Such transformations are defined as conformal transformations that leave invariant the angle between a pair of lines.

The next class of transformations are those that are continuous reversible deformations that admit translations, rotations, bending, expansion, and finally shears. The invariants of such transformations are defined as topological properties.

7.2.2 Topological Physics

Indeed the geometric method has served physics well, but there are many things in nature that obey rules that are independent from size and shape. For example, the Planck blackbody radiation distribution in frequency is independent from the size, shape, and even chemical makeup of the hot body that is radiating. The simply connected space of a hollow wave guide of any finite size or shape will always have a low frequency cut-off, but the co-axial cable, which is topologically not simply connected, can support DC currents. A physical system is conservative in a thermodynamic sense if the cyclic work vanishes, independent from the length (size and shape) of the process path. The closed surface integral of Gauss' law does not depend upon the shape or size of the surface but only on the number of charges contained in its interior. A flowing fluid can be in a laminar streamline state which can evolve into a chaotic turbulent state. A measurement with a finite compact apparatus of a infinite or non-compact property will always have an uncertainty associated with the measurement. These qualities of nature that do not seem to depend upon size and shape, and seem to be independent of continuous reversible deformations, can be defined and studied as topological properties.

The idea is that basic physical properties exist that do not depend upon legislated standards, but are absolute in the sense that they are answers to questions such as:

How many?, or

Is it possible?, or more technically,

If a solution exists, is the solution unique?

If the solution is not unique, how many solutions are there?

A great deal of engineering and physical theories are built around the deterministic geometrical dogma which supposes that, given initial data, the name of the game is to be able to predict the outcome, and predict it uniquely. From a more topological perspective, it will become apparent that unique prediction may become impossible, but deterministic retrodiction can be achieved.

Observables as invariants of transformations

Concepts that do not depend upon size and shape can still be invariants of an equivalence class of transformations. Again, these invariants, which are not pure geometric invariants, may be used to define an equivalence class of transformations. The issue is how to define and observe these qualities of nature that do not depend upon size and shape. Consider a piece of notebook paper made out of flexible rubber material. The sheet has 3 holes along one side, and can be marked as 1,2,3,4 at its corners, in a prescribed sequence or orientation. Translate the sheet, and ask what are the invariants of the transformation. The answer is: the size, the shape, the number of holes and the orientation sequence, 1,2,3,4. are all invariant properties of the translation.

Now rotate the sheet; what stays the same? Again, size, shape, hole count, and orientation stay the same. However, in the case of rotation there exists one other invariant that is not in the class of translations. This additional rotational invariant is the fixed point that defines where the axis of rotation intersects the sheet. Translations are said to be transitive because there is no fixed point, while rotations are intransitive because there must be one fixed point. Recall that by Klein's definition, the four properties of size,shape, hole count and orientation are geometric properties.

Now take the this rubber sheet and deform it by pulling and stretching the sheet. What stays the same? The answer is not the size and not the shape, but the hole count (distorted holes, of course, in the deformed case) and the orientation sequence 1,2,3,4 do stay the same under the deformation. Those properties that stay the same under continuous and reversible deformation are defined to be topological properties. Note that topological properties are included in the class of geometrical properties, but the class of geometrical transformations are included in the class of topological transformations. Pure geometric properties will be defined as those properties which are invariant under translations and rotations only.

A topological property is defined as an invariant of a homeomorphism, or in more simple terms, a topological property is an invariant of a continuous and reversible deformation, while pure geometrical properties are not. Pure geometrical properties such as size and shape can evolve with respect to homeomorphisms. In this monograph, the process of studying invariants of transformations will be taken one step further, for of physical interest to dissipative systems are those processes that are continuous but not reversible. Pure topological properties are not invariants of continuous but irreversible transformations. As pure geometrical properties evolve with respect to continuous and reversible transformations, pure topological properties evolve with respect to continuous but irreversible processes. For example, if the rim of one of the holes in the rubber sheet was grasped and pulled out of the sheet into the shape of a long trumpet with the rim becoming smaller and smaller until it collapsed to a point that could be glued together, then the topological property of hole count in the rubber sheet would have been changed from three to two during the deformation and gluing process. Note that it was the absolute number of holes that changed during this process of topological evolution which effectively collapsed one of the holes. It is important to note the topological change is quantized, for you can never have half a hole. The question of how many holes is absolute, for it is in relation to the integers.

What are the invariants of the equivalence class of continuous, but irreversible transformations? Examples of such invariant properties are connectivity, compactness, and most important to this monograph, the concept of closure. Rather than carrying the words "continuous but irreversible" throughout the monograph, a biological concept will be used to define such processes: A continuous but irreversible process will be defined as an element of an equivalence class of transformations, and will be defined as an aging process. Like all transformations, the equivalence class of aging processes will be defined in terms of its invariants. The ability to develop a physical understanding of the aging process must be built upon the observable invariants of such processes, and the dynamical theory of those topological invariants that can change during such processes. This dynamical theory will be called the theory of topological evolution.

Physical laws as topological statements

The ultimate goal of this monograph is to establish methods of distinguishing topological effects in physics from geometrical ones, to establish laws describing topological properties of matter, and in particular to establish the laws of physical topological evolution. Note that the first step is to go beyond the constraints of geometry and study strictly topological properties and the evolution of geometrical properties. The second step is to go beyond the constraints of topology to study the evolution of topological properties. The reader may not realize that he or she has often worked with topological concepts without knowing anything about topology, per se. For example, it will be demonstrated herein that the Maxwell theory of electromagnetism, without the geometrical constraint of a Lorentz symmetry group, is a statement about topological properties of space-time. It also will be demonstrated that the first law of thermodynamics is a topological statement of cohomology. The flow of a Navier-Stokes fluid can admit solutions which are examples of an irreversible but continuous topological evolution.

7.3 Cartan's Exterior Calculus

7.3.1 Introduction

After the basic concepts of topology are presented, the next step is to develop a thorough understanding of Cartan's theory of Exterior Calculus. Cartan developed his exterior calculus long before the word Topology became fashionable, but the key feature of Cartan's theory is that it transcends the geometrical constraints of tensor calculus and is truly a theory of topology and topological evolution. It was mentioned above that a topological property was an invariant of a homeomorphism. Technically, a homeomorphism is a map from an initial to a final state that has two qualities: 1) it must be continuous, and 2) it must be reversible in the sense that the inverse exists and is continuous. If topological evolution is to take place, then one or both of these qualities must not be true. Of particular interest to the developments in this monograph are those evolutionary processes which are continuous but not reversible. However continuity is not a geometrical idea; it is a concept that does not depend upon size and shape. A major goal will be the development of a useful topological structure, such that it can be decided whether or not a particular process is continuous, or not. Fortunately, the concept of a topological structure can be developed in terms of the Cartan calculus, such that a decision can be made if a process is continuous or not. If the process is determined to be continuous, and if it can be shown that the topological properties change during the process, then the process is an aging process. That is, the process is continuous but irreversible.

7.3.2 The exterior product and the exterior differential

Cartan's methods utilize (what might be unfamiliar) techniques that are described as the "exterior product" with an algebraic symbol, ^, and the "exterior differential" with a differential symbol, d, acting on objects, ω^p, defined as exterior differential forms of degree (not power) p. These basic concepts will be discussed below, briefly, but are best studied in detail from texts such as that by Harley Flanders ("Exterior Differential Forms"), Bamberg and Sternberg ("A course in Advanced Calculus ", Vols 1 and 2) and by Gockeler and Schucker ("Differential Geometry, Gauge theories, and Gravity"). I believe the best way to learn about these "new" operations and the objects upon they act is to try a few examples. Flanders is the best text with which to start. About the only thing missing in the Flander's presentation is a discussion of the Lie differential acting on p-forms.

There are a number of other texts available that discuss exterior differential forms, but many are a bit pedantic or too pompous to be useful at the applied engineering level. Most applications that have appeared in the literature are in the field of general relativity or super-symmetry or super-gravity or string types of theories. Very few texts are to be found which present the Cartan methods as applied to hydrodynamics, electrodynamics, thermodynamics, and other engineering sciences. (domage!) To reiterate previous statements, recall that the methods of exterior differential forms are important because they carry topological information, and can be used to study topological evolution. My interest in the Cartan methods was cemented when I realized how natural it was to write Maxwell's equations in terms of differential forms. Moreover it became apparent the Maxwell theory was independent from metric or connection constraints. From this point of view (perhaps initiated by VanDantzig) electromagnetism is not a geometrical theory, but instead is a topological theory. The PDE's of the Maxwell - Faraday induction equations, form a nested set in every dimension greater than 3. Experience with electromagnetic theory is very useful, for you can use EM theory to check your developing skills with the Cartan methods. If your application of the Cartan techniques does not replicate well known results in electromagnetism, you have made a mistake. These concepts will be detailed in that which follows.

There are two other important operations, besides the exterior product and the exterior differential, which act on differential forms, but these other operations require the specification of a vector field, V, in addition to the differential form, ω. From a physical point of view, differential form(s) may be used to define a physical system, and the vector field may be used to define a thermodynamic evolutionary process. What is remarkable is that this point of view can be used to justify the topological basis of thermodynamics, and to give a non-statistical description of irreversible processes. The two additional operations (with respect to V) are called the "interior product" with the symbol $i(V)$, and the Lie differential, $L_{(V)} = i(V)d + di(V)$, which combines two operations, the interior product and the exterior derivative. The Lie differential is an alternative to the "covariant" differential of tensor analysis, and, like the covariant derivative, will produce tensors from other tensors by means of a differential process. Examples and definitions will be given below. The Lie differential will become the most important tool from a topological perspective, for it permits computations to be made which will distinguish those objects which are topological invariants of a process and those objects which are not. It is remarkable that the Lie differential operating on a 1-form of Action is equivalent to the cohomological statement that defines the first law of thermodynamics.

The exterior differential forms (differential forms for short) are objects that are built from functions defined on a vector bundle. What this means is that starting from the assumption that there exists an n dimensional variety of independent variables, $\{y^a\}$, often called coordinates, it is possible to construct two other vector spaces. These two vector spaces consist of a 1 dimensional vector space $\Lambda^0\{1\}$ and an m dimensional vector space $\Lambda^1\{\sigma^k\}$. The one dimensional vector space $\Lambda^0\{1\}$ consists of all functions that can be constructed from $\{y^a\}$, and has the unit 1 as a basis element. The m dimensional vector space $\Lambda^1\{\sigma^k\}$ is endowed with a "differential" basis elements, σ^k, $k = \{1, 2, ...m\}$ The basis set σ^k is presumed to be linearly related to the differentials dy^a of the independent variables via the formula

$$|dy^a\rangle \Rightarrow \left|\sigma^k\right\rangle = \left[F_a^k\right] \circ |dy^a\rangle . \tag{7.19}$$

(The dimension of the set σ^k may be different from the dimension of $\{y^a\}$.) An alternative point of view is that a linear combination of the differentials on the initial state $|\sigma^a\rangle$ such that the linear mapping $\left[F_a^k\right]$ acting on the $|\sigma^a\rangle$ produces perfect differentials $\left|dx^k\right\rangle$ on the final state (This point of view is assumed in the Flander's book).

$$|\varpi^a\rangle \Rightarrow \left|dx^k\right\rangle = \left[F_a^k\right] \circ |\varpi^a\rangle . \tag{7.20}$$

The two vector sets of linear combinations of differentials, $|\varpi^a\rangle$ and $\left|\sigma^k\right\rangle$ on the initial state are not the same, even when the dimension of the initial and final states are the same. Given the initial state $|dy^a\rangle$, the $\left|\sigma^k\right\rangle$ are determined by the linear map $\left[F_a^k\right]$. Given the final state $\left|dx^k\right\rangle$, the $|\varpi^a\rangle$ are determined by the inverse of the linear map, $\left[F_a^k\right]^{-1} = \left[G_k^a\right]$. If the two spaces are not of the same dimension then the inverse of the linear map need not exist. It is important for generalizations that the concept of the initial state or domain (and its coordinate functions, y^a) be kept distinct from the final state or range (and its coordinate functions, x^k). The format $\left|\sigma^k\right\rangle = \left[F_a^k(y)\right] \ |dy^a\rangle$ will be emphasized herein.

All of this had its origins in the theory of differentiable coordinate mappings from an initial to a final state, where, given a (more than likely) non-linear map ϕ from a domain $\{y^a\}$ to a range $\{x^k\}$, a linear map $d\phi$ between differentials could be generated to establish the (linear) differential vector space ideas. In symbols

$$Nonlinear \quad \phi : \{y^a\} \Rightarrow \{x^k\} = \phi^k(y) \tag{7.21}$$

$$Linear \quad d\phi : |dy^a\rangle \Rightarrow \left|dx^k\right\rangle = \left[\partial\phi^k(y)/\partial y^a\right] \ |dy^a\rangle = \left[F_a^k(y)\right] \ |dy^a\rangle . \tag{7.22}$$

The Linear mapping $\left[F_a^k(y)\right]$ so generated is the Jacobian matrix of partial differentials of the coordinate (or vector space) mappings. As an example, review the concept of spherical or cylindrical coordinates mapped into Cartesian space. (Maple programs have been developed giving most of the details in terms of symbolic mathematics. See http://www22.pair.com/csdc/pdf/mtpertu5.pdf)

At first, in this presentation, the dimension of the vector space $|\sigma^k\rangle = |dx^k\rangle$ will be assumed to be n, the same dimension as the space of independent variables. Then the ranges of the index a is: $a = \{1...n\}$, and the range of the index k is : $k = \{1...n\}$. This restriction will be relaxed later during the development of a more general theory. The $n\,x\,n$ Jacobian matrix of the n mapping functions establishes the vector space ideas as a linear mapping, and gives the primitive realization of what is to be known as a Frame matrix (of functions on the initial state).

Suppose that another function, say $\Phi(y^a)$, is given in terms of the initial variety, $\{y^a\}$. Then its total differential is given by the expression,

$$d\Phi(y^a) = \{\partial\Phi(y^a)/\partial y^b\}\, dy^b = \sum_b A_b(y^a) dy^b = \langle A_b(y^a)| \circ |dy^b\rangle \,. \tag{7.23}$$

The object on the right is an example of an exterior differential 1-form, ω^1, with coefficient functions $A_b(y^a)$ and basis elements, dy^b. (From here on the sum convention on up-down symbols - the index b in the formula above - will be presumed, without the use of the $\sum$ symbol). The coefficients, by construction in this example,

$$A_b(y^a) = \{\partial\Phi(y^a)/\partial y^b\}, \tag{7.24}$$

form the components of a covariant gradient vector field.

In the Cartan theory of differential forms these concepts are extended to situations where the differential basis elements σ^k, of the vector space $\Lambda^1\{\sigma^k\}$ can be written in terms of some arbitrary matrix (of functions on the initial state) acting on the differentials of the independent variables in a linear way:

$$|\sigma^k\rangle = \left[F^k_a(y)\right]\, |dy^a\rangle \,. \tag{7.25}$$

In other words, in the Cartan extension, it is not assumed that the linear map $\left[F^k_a(y)\right]$ is necessarily a Jacobian matrix of some non-linear coordinate mapping, nor is it assumed that the matrix is even similar to the Jacobian matrix of a coordinate mapping. This more general matrix of functions, $\left[F^k_a(y)\right]$, will be defined as a basis Frame and is the cornerstone of Cartan's development of the Repere Mobile. Given such a matrix of functions a key question revolves about the determination of the solubility of the Frame. Given a Frame, does there exist a unique set of mapping functions ϕ from which the Frame is determined to be the Jacobian matrix $d\phi$ of the mapping? If not, is it possible that there exists a non-unique solution set to the problem? The question of non-unique integrability of the Frame matrix is the basis of what is called Affine and Topological Torsion. Torsion appears when the basis Frame (or its equivalence class) is NOT integrable.

For applications, how the Frame may be related to specific physical problems is of key importance. The early development of the Frenet-Serret-Cartan Frame for a point moving along a space curve indicates that it is possible to construct the Frame from differentials of the mapping function with respect to a parameter along a space curve. That is, the velocity, acceleration and the rate of change of acceleration can be used to build a Frame matrix of a point moving along a space curve in three dimensions. These things are physical, measurable, and applicable quantities. The same idea can be generated for continuous media such as a fluid. The velocity field, the vorticity field, and the helicity field of the fluid become the analogs of the Frenet - Serret differentiations.

For those ubiquitous cases (or better said, on those restricted domains) where the Frame has an inverse, then the Frame matrix is an element of the General Linear group. Often in particular applications the Frame matrix is constrained to be an element of an equivalence class of "admissible" Frames by assuming the Frame belongs to some sub-group of the GL group. In the Frenet-Serret case, a usual restriction constrains the 3D Frame matrix elements such that the Frame is a member of the special orthogonal (orthonormal) group. The columns of the basis Frame matrix are orthogonal unit vectors. This constraint is used to create the concepts of arc length, curvature, and torsion of the 3D space curve. These "intrinsic" properties of the space curve are the similarity invariants of all equivalent Frames (that is, all Frames that are members of the orthonormal group). These intrinsic (often called invariant) properties of the equivalence class are computed by means of the coefficients of the Cayley-Hamilton theorem.

From a physics point of view, all observers who may use different elements or representations of the orthonormal group for reference systems will be able to express their views in terms of a common set of qualities, the similarity invariants. All equivalent observers will agree that the values of the similarity invariants are the same. Restrictions to particular subgroups are often called "gauge theories". It is important to note that certain (normal) subgroups (such as the orthonormal subgroup) cannot distinguish between left and right handedness (chirality) , but other equivalence classes of subgroups can. It would seem that this ability to distinguish a chiral property is of value to the study of biological systems, where most biological molecules appear to be left or right handed. The moral (or warning) of this paragraph is that the common orthonormal system of basis vectors $(\mathbf{i}, \mathbf{j}, \mathbf{k})$ of engineering practice must be modified to handle chiral distinctions.

It is important to be reminded of the idea of a similarity transformation. Given a matrix $[M]$ and a transformation matrix $[F]$, the matrix $[N]$ is said to be similar to $[M]$ if

$$[M] \Rightarrow [N] = [F]^{-1} \circ [M] \circ [F]\,. \tag{7.26}$$

When the Cayley-Hamilton polynomial is constructed for $[M]$ and $[N]$ the coefficients of the polynomials are the same (if the matrices are "similar"). Two of the important similarity invariants are the trace of $[M]$ and its determinant. In differential geometry, these ideas will be used to define curvature properties of manifolds. In the Frenet-Serret-Cartan theory of the orthonormal subgroup, the similarity invariants lead to the concepts of arc length, curvature and torsion. The zero sets of the similarity invariants have particular physical importance. In the thermodynamics of a van der Waals gas, the Cayley-Hamilton polynomial based upon the Gibbs function is a cubic polynomial with the surface shape of a swallow-tail. The critical point is where all three similarity invariants vanish. The spinodal line of phase instability is where the quadratic similarity coefficient (the Gauss curvature of the swallow-tail surface) vanishes.

The similarity equation can be rewritten in a manner that does not require the immediate computation of an inverse:

$$[F] \circ [N] = [M] \circ [F]. \tag{7.27}$$

This equation can be used to test if $[N]$ is similar to $[M]$. A special situation occurs if the matrix $[N]$ is the same as $[M]$. This situation places a constraint on the equivalence class of matrices that can be used for the transformations $[F]$. Suppose that $[N] = [M] \mp d[M]$, then the differential similarity equation becomes

$$d[M] = [M] \circ [F] \pm [F] \circ [M], \tag{7.28}$$

and is suggestive of the Heisenberg matrix operator format (the transformation matrix $[F]$ plays the role of the "Hamiltonian" operator). These similarity formats will reappear below when the matrix of connection 1-forms is discussed.

Note that the column vector array of differential basis elements, $|dy^a\rangle$, transforms as a contravariant tensor in the Jacobian case, where a coordinate mapping is available. This property can be extended if the Frame matrix of functions has a non-zero determinant, for then the columns of the Frame matrix can be used as a basis set for contravariant vectors in the initial space (domain or state). This basis argument does not depend upon the fact that matrix elements $\left[F_a^k(y)\right]$ form a Jacobian (i.e., integrable) system. It will be demonstrated that it is this lack of unique integrability for the $\left|\sigma^k\right\rangle = \left[F_a^k(y)\right] |dy^a\rangle$ that leads to the concepts of affine Torsion and Topological Torsion , two topics that will be discussed in great detail in subsequent sections.

At present, given a basis of 1-forms, construct arbitrary exterior differential 1- forms from the matrix product of arbitrary coefficient functions arranged as a row vector, $\left\langle \widehat{A}_k(y^a) \right|$ and the basis set arranged as a column vector, $\left|\sigma^k\right\rangle$

$$\omega^1 = \left\langle \widehat{A}_k(y^a) \right| \circ \left|\sigma^k\right\rangle = \widehat{A}_k(y)\, \sigma^k. \tag{7.29}$$

Note that if the coefficient functions are chosen to be a covariant vector array (and that is why the index is a lower index on the $A_k(y)$), then the differential 1-form ω is a scalar invariant of "coordinate transformations". The coefficient functions, however, do not have to be a gradient array. The covariant constraint implies that if (as assumed)

$$\left|\sigma^k\right\rangle = \left[F_a^k(y)\right]\ \left|dy^a\right\rangle \tag{7.30}$$

then

$$\left\langle \widehat{A}_k(y^a)\right| = \left\langle A_b\right| \circ \left[G_k^b\right] \tag{7.31}$$

where $\left[G_k^b(y)\right]$ is the inverse matrix of functions to the frame matrix, $\left[F_a^k(y)\right]$. These are the rules of classical tensor analysis defining what is meant by contravariant and covariant vectors of ordered sets of components with respect to special transformations (defined as diffeomorphisms).

The differential form so constructed in terms of tensor coefficients is then independent from a "choice of coordinate system".

$$\omega^1 = \left\langle \widehat{A}_k(y^a)\right| \circ \left|\sigma^k\right\rangle = \left\langle A_b\right| \circ \left[G_k^b\right] \circ \left[F_a^k\right] \circ \left|dy^a\right\rangle . \tag{7.32}$$

For physicists and engineers what this implies is that laws of physics written in terms of differential forms are independent of the observer's choice of a "reference system". A "reference system" is defined as an element of an equivalence class of differentiable mappings. The most common equivalence class usually accepted is the class of diffeomorphisms, which implies that the mapping, ϕ, and the linear mapping, $d\phi$,have inverses, and the inverse mapping is differentiable. Such diffeomorphic mappings are constrained subsets of other mappings known as homeomorphisms. Homeomorphisms (and therefor diffeomorphisms) preserve topology from initial to final state, and therefor cannot be used to describe topological evolution. (Bummer.) Sometimes the equivalence class of reference systems is even further constrained. For example, the acceptable class of reference systems known as inertial frames of reference in the physics of special relativity is constrained to be the Lorentz equivalence class. Sometimes such constraints throw the baby out with the wash. For example, General Relativity is designed to admit all diffeomorphisms as the equivalence class of frames of reference; Special Relativity admits only elements of the Lorentz equivalence class, which is a subset of all diffeomorphisms.

The Lorentz equivalence class consists of those matrices, $[L]$, for which the Minkowski line metric is preserved. That is

$$\begin{bmatrix} -1 & & & \\ & -1 & & \\ & & -1 & \\ & & & 1 \end{bmatrix} = [\eta] = \left[L^{-1}\right] \circ [\eta] \circ [L] \tag{7.33}$$

There is a further subclass of Lorentz matrices, with matrix elements which are constants, and which are use in special relativistic (non-accelerated) applications. (This special subclass turns out to be "affine" torsion free, so that left handed and right handed chirality species evolve in the same way). However, there are Lorentz matrices that preserve the Minkowski metric that are not composed of constant elements. Such matrices admit accelerations, and also admit Affine Torsion coefficients. (See http://www22.pair.com/csdc/pdf/lorentz.pdf). That is, the system of 1-forms generated by a Lorentz transformation of non-constant elements is not necessarily uniquely integrable, and therefor admit different behavior for chiral systems. (This difference in behavior distinguishes Optical Activity from Faraday Rotation in electromagnetic systems.)

. A major thrust of the work that appears on Cartan's Corner is that the reference systems are extended to include topological change, so that non-diffeomorphic transformations will be investigated. A closer look a Cartan's concepts yields the result that, unlike tensors which are well behaved with respect to diffeomorphisms, exterior differential forms are well behaved in a functional sense with respect to a class of transformations even wider than the class of diffeomorphisms – in fact, wider than the class of homeomorphisms! Hence differential forms are useful to the study of topological evolution, which is the main theme of Cartan's Corner and these lectures. (See "Retrodictive Determinism" http://www22.pair.com/csdc/pdf/retrodic.pdf).

It should be realized that differential forms have the tensor like property that if the differential form is zero in one coordinate system of reference, then it is zero in all other diffeomorphically equivalent systems, no matter what constraints are applied to limit the elements of the equivalence class of diffeomorphisms. In addition, if a differential form is zero on the final state, then its pullback to the initial state is also zero with respect to continuous but not homeomorphic, and therefor not diffeomorphic maps. (See http://www22.pair.com/csdc/ed3/ed3fre1.htm)

7.3.3 The exterior algebra

The exterior algebra of Cartan is based upon an associative, but not commutative, multiplication rule defined as the exterior (wedge or hook) product of objects defined as exterior differential forms. The symbol for the product is ˆ. The structure of the algebra can be built starting from 1-forms on the n-dimensional vector space $\Lambda^1\{\sigma^a\}$, in terms of a basis of 1-forms denoted by $\{\sigma^a\}$. The arbitrary 1-form is constructed from the basis elements according to the formula given above, $\omega^1 = A_k(y)\ \sigma^k$. The addition rule of the algebra is that of vector space addition: add the coefficients of the same basis elements.

$$\begin{aligned} \omega 1 &= A_k(y)\ \sigma^k \\ \omega 2 &= B_k(y)\ \sigma^k \\ \omega 1 + \omega 2 &= \{A_k(y) + B_k(y)\}\ \sigma^k. \end{aligned} \tag{7.34}$$

Example 29 *Add* $(3x\,dx + 4xz\,dy)$ *and* $(2y\,dy + 17y\,dz)$:

$$basis = (dx, dy, dz)$$

$$(3x\,dx + 4xz\,dy) + (2y\,dy + 17y\,dz) = 3x\,dx + (4xy + 2y)\,dy + 17y\,dz$$

It is the multiplication rule that is perhaps unfamiliar. The multiplication rules are defined in terms of elements of the basis set.

$$\begin{aligned} \sigma^a\hat{}\sigma^b &= -\sigma^b\hat{}\sigma^a \\ \sigma^b\hat{}\sigma^b &= 0 \end{aligned} \tag{7.35}$$

$$\begin{aligned} dy^a\hat{}dy^b &= -dy^b\hat{}dy^a \\ dy^b\hat{}dy^b &= 0 \end{aligned} \tag{7.36}$$

These rules are similar to the cross product of Gibbs 3D vector analysis, but the difference is that the exterior product rule extends to n dimensions (the Gibbs cross product does not) and is associative (Gibbs product is not). Associative means $(\sigma^a\hat{}\sigma^b)\hat{}\sigma^c = \sigma^a\hat{}(\sigma^b\hat{}\sigma^c)$. In 3D, the Gibbs cross product yields $\mathbf{A}\times(\mathbf{B}\times\mathbf{C}) \neq (\mathbf{A}\times\mathbf{B})\times\mathbf{C}$

Example 30 *Multiply* $A = \sum A_k\sigma^k$ *times* $B = \sum B_m\sigma^m$ $\qquad A\hat{}B = C$

$$\begin{aligned} A\hat{}B &= \{A_k\sigma^k\}\hat{}\{B_m\sigma^m\} \\ &= ...A_kB_k\sigma^k\hat{}\sigma^k + ...A_kB_m\sigma^k\hat{}\sigma^m + ...A_mB_k\sigma^m\hat{}\sigma^k \\ &= ...\quad 0 \qquad + ...A_kB_m\sigma^k\hat{}\sigma^m - ...A_mB_k\sigma^k\hat{}\sigma^m \\ &= ...\ \{A_kB_m - A_mB_k\}\sigma^k\hat{}\sigma^m... \\ &= ...C_{[km]}\sigma^k\hat{}\sigma^m.. \\ &= ...C_{[km]}\sigma^{[km]}... \\ &= ...C_H\sigma^H \end{aligned} \tag{7.37}$$

Note that $A\hat{}B \neq B\hat{}A$ and that the exterior product of two elements of the vector space $\Lambda^1(\sigma^k)$ produce a linear combination of paired basis elements of the form $(\sigma^k\hat{}\sigma^m)$. Moreover the coefficients of these elements are always anti-symmetric under interchange of the paired indices. The anti-symmetry and the rules of 1-form multiplication permit the writing of the product of two 1-forms in terms of another vector space, whose basis elements are the anti-symmetric pairs $(\sigma^k\hat{}\sigma^m)$. This object is defined as a 2-form. It is conventional to rewrite the 2-form so constructed without repeating basis element pairs that are of different sign. That is, $\Lambda^2(\sigma^k\hat{}\sigma^m) \Rightarrow \Lambda^2(\sigma^{[km]}) = \Lambda^2(\sigma^{[H]})$.

The symbol $H = [km]$ stands for all order pairs where $k < m$, and is equivalent to the set $(12, 13, 14.....23, 24.....34...)$. It is apparent that the number of ordered singlet basis elements in 3 dimensions is the same as the number of ordered anti-symmetric pairs; but this is only true in 3D. For 4D, the number of singlet basis elements is 4 and the number of ordered anti-symmetric pairs is 6. The result of the exterior product is to produce from elements of one vector space of dimension n, another element of a different vector space of dimension $n(n-1)/2$. In this limited sense the exterior product is not closed. The exterior multiplicative combination of two objects of the same type (1-forms) does not produce an object of the same type, but instead produces a 2-form. The process of exterior multiplication can be repeated where 2 -forms are multiplied by 1-forms to produce 3-forms, and 3-forms are multiplied by 1-forms to produce 4-forms, ultimately building a "closed" algebra. The elements of the closed algebra will consist of classes, or vector spaces with basis element doublets, $\sigma^a\hat{}\sigma^b$, classes of triplets, $\sigma^a\hat{}\sigma^b\hat{}\sigma^c$, ... and even n-tuplets of basis vectors, $\Omega = \sigma^a\hat{}\sigma^b\hat{}\sigma^c\hat{}...\hat{}\sigma^n$. However, the rules of multiplication are such that the exterior multiplicative combination of more that n basis vectors must vanish. Hence any element of the algebra times another element of the algebra is an element of the algebra, or zero. In this sense, the exterior algebra is closed.

The doublets are called 2-forms, the triplets are called 3-forms, and the n-tuplets are called n-forms. The multiplication rules demonstrate that each of the p-tuplets has a number of linearly independent elements equal to the possible combinations of n things take p at a time. Each of the p-tuplets forms a vector space basis of dimension equal to the appropriate combinatorial number of Pascal's triangle.

$$
\text{Pascal's Triangle:}\quad
\begin{array}{lccccccccc}
n=1: & & & & & 1 & & & & \\
n=2: & & & & 1 & & 2 & & 1 & \\
n=3: & & & 1 & & 3 & & 3 & & 1 \\
n=4: & 1 & & 4 & & 6 & & 4 & & 1
\end{array}
$$

The vector subspace dimension of the 0-forms, ω^0, is 1; the dimension of the 1-forms, ω^1, is n, the dimension of the 2-forms, ω^2, is equal to the number of combinations of n things taken 2 at a time (n(n-1)/2 is equal to 3 for n =3, equal to 6 for $n = 4$, etc.); the dimension of the n -1 forms, ω^{n-1}, is n, the dimension of the n-forms, ω^n, is 1. The dimension of the exterior algebra is the sum of the dimensions of all vector spaces produced by the exterior product; this dimension is equal to 2^n.

As an example consider the exterior algebras up to $n = 4$. The elements of Pascal's triangle yield a 1-dimensional (scalar) vector space Λ^0 for the 0-forms, a 4 dimensional vector space Λ^1 for the 1-forms, a 6 dimensional space Λ^2 for the 2-forms, a 4 dimensional vector space Λ^3 for the 3-forms, and a 1 dimensional vector space Λ^4 for the n=4 forms.

From geometrical studies in 3 dimensions, n=3, the elements of the 4 different vector spaces are called points, lines, surfaces and volumes. From applications to 3D mechanics, the position vector and the momentum vector are from the vector subspace, Λ^1, and their exterior (cross) product is an element of the vector subspace,

Λ^2. The fact that angular momentum is an element of a vector space Λ^2 is why one never sees angular momentum added to linear momentum (which is an element of a different vector space Λ^1) in the elementary mechanics text books. From applications to special relativistic physics in 4 dimensions, n=4, the coefficients of the elements of these five different vector subspaces of the exterior algebra are known as scalars, vectors, tensors, pseudo-vectors, and pseudo scalars.

$p-forms\ in\ 4D$	0	1	2	3	4
$\#\ of\ basis\ elements$	1	4	6	4	1
$name$	Scalars	Vectors	Tensors	Pseudo-Vectors	Pseudo-Scalars

(7.38)

The Cartan-Grassmann exterior algebra in consists of a vector space of 2^n (= 16 in 4D) components with n+1 (= 5 in 4D) different vector subspaces. The algebra is technically called a graded algebra. The exterior algebra is closed with respect to multiplication, for all possible products of the algebra reside within the algebra of 2^n dimensions (or are zero). The exterior product of a p form and a q-form produces a p+q form, or zero if p+q > n. In every case, the higher p-forms can be constructed from sums of products of the singlets. All elements of the algebra can be constructed from linear combinations of the primitive n basis elements, σ^k and their products.

A nice feature of the exterior algebra (besides being closed) is that the definitions of symbolic operations can be described entirely in terms of 0-forms and 1-forms, when the collective index is used. Every p-form can be rewritten in terms of symbolic coefficients (0-forms) and basis elements (p-forms from vector spaces of different dimensionality of course) with a format similar to that of a 1-form. For example, for n = 4, there are 6 elements of the vector subspace of two forms. That is, there are 6 independent non-zero pairs of the 4 basis 1-forms σ^k from Λ^1 that can be used as the basis elements of Λ^2 : namely the set $\{\sigma^1\hat{}\sigma^2, \sigma^1\hat{}\sigma^3, \sigma^1\hat{}\sigma^4, \sigma^2\hat{}\sigma^3, \sigma^2\hat{}\sigma^4, \sigma^3\hat{}\sigma^4\}$. If these basis pairs are given a new symbolism as $\{\sigma^{12}, \sigma^{13}, \sigma^{14}, \sigma^{23}, \sigma^{24}, \sigma^{34}\}$,then the general 2-form (for n=4) can have the expansion coefficient - basis representation given by the formula:

$$F = A_{12}\sigma^{12} + A_{13}\sigma^{13} + ... + A_{34}\sigma^{34} = A_H\sigma^H, \tag{7.39}$$

where H is the collective index described above for ordered pairs. This formula for the 2-form F in 4 dimensions looks like a vector formula for a 1-form, but in space of 6, not 4, dimensions; it is just that the index labels are different. All of the distinct basis combinations will be completely antisymmetric in their indices ($p > 1$). For example, H could be the set of triples $[i1, i2, i3]$ with $i1 < i2 < i3,$ for the vector space of 3 forms, H could be the set of quadruples $[i1, i2, i3, i4]$ with $i1 < i2 < i3 < i4,$ for the vector space of 4 forms; etc. With this collective index notation, combinatorial rules of multiplication and differentiation developed for 1 forms can be applied directly to higher order p-forms.

This technique, where a 2-form expansion in 4D was used as a 6 - vector, was applied (intuitively?) more than 60 years ago by Arnold Sommerfeld in his studies of electromagnetic systems. The 3 components of **E** and the 3 components of **B** formed the components of the 6-vector. (See his volumes on Lectures on Theoretical Physics.) A similar but little used 6 vector composed of the acceleration and vorticity can be developed for a fluid. It is not clear whether Sommerfeld knew the theory of exterior differential systems at the time of his 6 - vector development.

Example 31 *In 3D, exterior multiply the 1-form, A, times 2-form, B to produce the 3-form $C = A\hat{}B$.*

$$\begin{aligned} & (A_x dx + A_y dy + A_z dz)\hat{}(B_x dy\hat{}dz + B_y dz\hat{}dx + B_z dx\hat{}dy) \\ = & (A_x B_x + A_y B_y + A_z B_z) dx\hat{}dy\hat{}dz = C \end{aligned}$$

Note that this product of a 1-form and n-1=3-1=2 - form produces a n - form with a coefficient that looks the same as the euclidean inner product of two ordinary vectors $\mathbf{A} \circ \mathbf{B}$. Recall that the 1-form has n components and the n-1 form has n components. The euclidean inner product result is valid in all dimensions, n.

Example 32 *In 3D, exterior multiply the 1-form, A, times 1-form, B to produce the 2-form $D = A\hat{}B$.*

$$\begin{aligned} D &= (A_x dx + A_y dy + A_z dz)\hat{}(B_x dx + B_y dy + B_z dz) \\ &= (A_x B_y - A_y B_x) dx\hat{}dy \\ &\quad +(A_y B_z - A_z B_y) dy\hat{}dz \\ &\quad +(A_z B_x - A_x B_z) dz\hat{}dx \end{aligned}$$

Note that the result has coefficients equivalent to the Gibbs cross product of two vectors, $\mathbf{A} \times \mathbf{B}$.

7.3.4 The Exterior Differential.

The exterior differential is a definition of a differential process acting on p-forms, ω^p. The operation takes the p-form into a p+1 form. Hence, like the exterior product, the exterior differential generates a vector in a different vector subspace of the exterior algebra.

$$d(\omega^p) \Rightarrow \omega^{p+1}. \tag{7.40}$$

Other properties of the exterior differential will be described by the rules for distributing the operator over a product of 1-forms (note the order of factors and the minus sign modification of the Liebniz rule for the differential of a product of scalars)

$$d(A\hat{}B) = dA\hat{}B - A\hat{}dB, \tag{7.41}$$

and

$$d(d(\omega^p)) \Rightarrow 0. \tag{7.42}$$

For a product of a p-form and a q-form, it follows that

$$d(\omega^p\hat{}\omega^q) \ = d\omega^p\hat{}\omega^q - (-1)^p\omega^p\hat{}d\omega^q, \tag{7.43}$$

The epitome of the exterior differential is the concept of the total differential of a scalar function, which is a familiar operation that takes the 0-form, or function $\omega^0 = \theta(y^a)$, into the 1-form, $\omega^1 = A_b\, dy^b$.

$$d(\omega^0) = d\{\theta(y^a)\} \Rightarrow \{\partial\theta(y^a)/\partial y^b\}dy^b = A_b\, dy^b = \omega^1, \tag{7.44}$$

The function could be constrained such that the set $\theta(y^a)$ = constant defines an implicit surface. It follows that in the constrained case, the total differential is also zero: $d\{\theta(y^a)\} = 0$. An object that has zero for the value of its exterior differential is said to be closed (in an exterior differential - not algebraic - sense). For consistency reasons note that the differential basis elements symbolized by dy^b are defined to be closed. That is, $d(dy^b) = 0$. The exterior derivatives of an arbitrary basis, σ^k, are NOT necessarily zero, but $d(\sigma^k)$ is closed for $dd(\sigma^k) = 0$.

The exterior differential of a 1-form is defined as

$$\begin{aligned} d\omega^1 &= d(A_b\, dy^b) = (dA_b)\hat{}dy^b + A_b\, d(dy^b) \\ &= (\partial A_b/\partial y^e\, dy^e)\hat{}dy^b + 0 \\ &= (\partial A_b/\partial y^e - \partial A_e/\partial y^b)\, dy^e\ \hat{}\ dy^b \\ &= F_{[eb]}\, dy^{[eb]} = F_{[H]} dy^{[H]}. \end{aligned} \tag{7.45}$$

The collective index notation permits the formula defining exterior differentiation to be generalized:

$$d\omega^p = d(A_H\, dy^H) = (dA_H)\hat{}dy^H \tag{7.46}$$

So to compute the exterior derivative of any p -form, first compute the exterior (= total differential) of the scalar coefficients dA_H and then exterior multiply the result into the remaining base elements of the form dy^H, component by component.

Also note that the special 1-form with gradient coefficients, $\omega^1 = d\{\theta(y^a)\}$, has an exterior differential equal to

$$\begin{aligned} d\omega &= d(d\{\theta(y^a)\}) \\ &= ...\{\partial^2\theta(y^a)/\partial y^b\partial y^c\}dy^c\ \hat{}\ dy^b + ... + \{\partial^2\theta(y^a)/\partial y^c\partial y^b\}dy^b\ \hat{}\ dy^c \\ &= ...\{\partial^2\theta(y^a)/\partial y^b\partial y^c - \partial^2\theta(y^a)/\partial y^c\partial y^b\}dy^c\ \hat{}\ dy^b \\ &= 0, \end{aligned} \tag{7.47}$$

for C2 functions. Hence this special 1-form ω^1 is a closed 1-form, assuming the coefficient functions are twice differentiable. However, as the example 1-form ω^1 has a unique primitive function, $\theta(y^a)$, whose exterior derivative creates $\omega^1 = d\theta$, the 1-form ω is said to be not only closed, but also exact. The same concepts hold for all p forms. A p-form is closed if its exterior differential vanishes, and the p-form is exact if it is constructed by means of the exterior differential operation acting on some p-1 form. There are differential forms that are closed but not exact, and those that are neither exact nor closed. The importance of closed and exact, or closed but not exact, p-forms is that they carry topological information about the domain of definition. For example, in a two dimensional surface every hole is associated with a unique 1-form that is closed, but not exact. The number closed but not exact 1-forms on a domain counts the topological number of holes. This fact is the basis of the Bohm-Aharanov idea in EM theory, and is at the foundation of the theory of flight in terms of the Joukowski transformation.

Suppose that the given exterior differential p-form is expressed in terms of a non-integrable basis set σ^H. Then the exterior differential formula becomes

$$d\omega^p = d(A_H\,\sigma^H) = (dA_H)\hat{}\sigma^H + A_H(d\sigma^H) \tag{7.48}$$

Now it must be recognized that the second term $(d\sigma^H)$ is not necessarily zero. Such complications arise when the Frame matrix generates 1-forms (σ^k) which are not closed. The basis Frame in that case is not uniquely integrable.

Remember that the exterior derivative has to be applied to products of 1-forms in terms of a modified Leibniz rule, that alternates in sign for every other odd factor. For example, the exterior derivative of the product of two 1-forms is

$$d(\sigma^1\hat{}\sigma^2) = d\sigma^1\hat{}\sigma^2 - \sigma^1\hat{}d\sigma^2. \tag{7.49}$$

Example 33 *Compute the exterior differential of the function* $\theta(y^a) = (y1)^2 + (y2)^2 + (y3)^2 - 1.$

$$\omega = d\theta(y^a) = 2(y^1dy^1 + y^2\,dy^2 + y^3\,dy^3)$$

Note that the zero set of the function describes a spherical 2-surface, and the coefficients of the deduced 1-form describe the normal field orthogonal to the tangent vectors on the surface. Direct computation demonstrates that ω is closed, as

$$d\omega = 2(dy^1\hat{}dy^1 + dy^2\hat{}\,dy^2 + dy^3\hat{}\,dy^3) = 0.$$

Example 34 *Compute the exterior differential of* $(A_xdx + A_ydy + A_zdz)\hat{}(B_xdx + B_ydy + B_zdz)$

$$d(A\hat{}B) = dA\hat{}B - A\hat{}dB = d(C_{[mn]}dy^{[mn]}) = d((C_{[mn]})\hat{}dy^{[mn]}$$

The object which is the result of the exterior differentiation of the 2-form constructed by the product is a 3-form with completely antisymmetric indices. The modified Leibniz rule for products is required to make the two ways of computing the resultant 3-form compatible.

The operation of the exterior differential acting on an arbitrary 1-form is defined as

$$\begin{aligned} d(A_k(y^a)\,\sigma^k) &= \{d(A_k(y^a)\}\ \hat{}\ \sigma^k + A_k(y^a)\hat{}\{d\sigma^k\} \\ &= \{\partial A_k(y^a)/\partial y^b\}dy^b\}\ \hat{}\ \sigma^k + A_k(y^a)\hat{}\{d\sigma^k\}. \end{aligned} \tag{7.50}$$

Consider the case where the arbitrary 1-forms are known to be linearly related to the differentials by means of the linear (Frame) formulas, $\left|\sigma^k\right\rangle = \left[F^k_a(y)\right]\ \left|dy^a\right\rangle$. Then

$$\begin{aligned} d(A_k(y^a)\,\sigma^k) &= \{\partial(A_k F^k_a)/\partial y^b\}dy^b\ \hat{}\ dy^k \\ &= \{\partial\widehat{A}_a/\partial y^b - \partial\widehat{A}_b/\partial y^a\}dy^b\ \hat{}\ dy^k \\ &= \widehat{F}_{[ab]}(y)dy^b\ \hat{}\ dy^k, \end{aligned} \tag{7.51}$$

where

$$\widehat{A}_a(y) = (A_k(y)F^k_a(y)) \tag{7.52}$$

This formula for $d(\omega^1) = d(\widehat{A}_a dy^a) = \widehat{F}_{[ab]}(y)dy^b\ \hat{}\ dy^k = \widehat{F}_{[ab]}(y)dy^{[ab]} = \widehat{F}_{[H]}(y)dy^{[H]}$ is valid on the initial state, or variety $\{y^a, dy^a)$, whether the Frame matrix has an inverse or not. The coefficients of the 2-form correspond to the antisymmetric components of a "curl" (when n = 3). However, the exterior differential procedure generalizes to spaces of higher dimension.

For differential forms expanded in terms of non-closed basis 1-forms, the exterior differential has two terms. The first term is just the exterior product of the total differential of the coefficient function(s) and the remaining factor of non-closed basis forms, while the second term is the exterior product of the functions and the exterior differentials of the non-closed basis forms.

The operation of the exterior differential acting on a p-form follows that same formulas, using the collective ordered index, H:

$$d(A_H(y^a)\,\sigma^H) = \{d(A_H(y^a)\}\ \hat{}\ \sigma^H + A_H(y^a)\hat{}\{d\sigma^H\}. \tag{7.53}$$

Lets examine more carefully the situation for the exterior differential of a 1-form expanded in terms on non-closed basis 1-forms. The outcome of the exterior differential process is to produce a 2-form, which can be expanded in terms of products of

1-forms. For any particular basis 1-form, σ^k, the differential is a 2-form, and as such it can be expanded in terms of the paired basis elements, $\sigma^{[mn]}$. That is

$$\begin{aligned} d\sigma^k &= \Lambda^k_{[mn]}(y^e)\sigma^{[mn]} \\ &= \Lambda^k_{[12]}\sigma^{[12]} + ... + \Lambda^k_{[34]}\sigma^{[34]} \\ &= \Lambda^k_{[12]}\sigma^1\hat{}\sigma^2 + ... + \Lambda^k_{[34]}\sigma^3\hat{}\sigma^4 \end{aligned} \tag{7.54}$$

Hence the exterior differential of a 1-form where the basis σ^k is not integrable is given by the formula

$$d(A_k\sigma^k) = dA_k\ \hat{}\ \sigma^k + A_k\Lambda^k_{[mn]}\,\sigma^{[mn]} \tag{7.55}$$

When the basis forms σ^k are closed in a differential sense, then the coefficients $\Lambda^k_{[mn]}$ vanish. How this relates to Affine and Topological torsion will be discussed below, along with the topic of anholonomic coordinates.

The n basis 1-forms must be linearly independent otherwise the dimension of the vector space $\Lambda^1\{\sigma^k\}$ is not n. The implies that the exterior product of the n 1-forms σ^k are such that the n-form so constructed is not zero. For basis 1-forms σ^k constructed from a Frame matrix according to the formula $\left|\sigma^k\right\rangle = \left[F^k_a(y)\right]\,|dy^a\rangle$, the non-zero property for the n-fold product implies that the Frame matrix has a non-zero determinant.

$$\omega^n = \sigma^1\hat{}\sigma^2\hat{}...\hat{}\sigma^n = \det[F]dy^1\hat{}dy^2....\hat{}dy^n \neq 0. \tag{7.56}$$

Such domains are either positive or negative and are therefor said to be orientable.

Compute the exterior differential of the 1-form $A = dz + ydx - xdy$

$$dA = ddz + dy\hat{}dx - dx\hat{}dy = 0 - 2(dx\hat{}dy)$$

Example 35 ***The Gradient:*** *Compute the exterior differential of the general 0-form $\theta(x,y,z)$*

$$d\theta(x,y,z) = \partial\theta/dx\,dx + \partial\theta/dy\,dy + \partial\theta/dz\,dz =$$

The coefficients form the gradient of the scalar function (3D)

Example 36 ***The Curl:*** *Compute the exterior differential of a general 1-form $A = (A_x dx + A_y dy + A_z dz)$*

$$dA = \{\partial A_y/\partial x - \partial A_x/\partial y\}dx\hat{}dy + \{\partial A_z/\partial y - \partial A_y/\partial z\}dy\hat{}dz + \{\partial A_x/\partial z - \partial A_z/\partial x\}dz\hat{}dx$$

The coefficients form the components of the "curl **A**" in 3D.

Example 37 ***The Divergence:*** *Compute the exterior differential of the 2-form* $V = Udy\hat{}dz - Vdz\hat{}dx + Wdx\hat{}dy$

$$dV = dU\hat{}dy\hat{}dz - dV\hat{}dz\hat{}dx + dW\hat{}dx\hat{}dy$$

If $\mathbf{V} = [U(x,y,z), V(x,y,z), W(x,y,z)$, then

$$\begin{aligned} dV &= \partial U/\partial x\, dx\hat{}dy\hat{}dz + \partial U/\partial y\, dy\hat{}dy\hat{}dz + \partial U/\partial z\, dz\hat{}dy\hat{}dz \\ &\quad -\partial V/\partial x\, dx\hat{}dz\hat{}dx - \partial V/\partial y\, dy\hat{}dz\hat{}dx - \partial V/\partial z\, dz\hat{}dz\hat{}dx \\ &\quad +\partial W/\partial x\, dx\hat{}dx\hat{}dy + \partial W/\partial y\, dy\hat{}dx\hat{}dy + \partial W/\partial z\, dz\hat{}dx\hat{}dy \\ &= \partial U/\partial x\, dx\hat{}dy\hat{}dz + 0 + 0 \\ &\quad -0 - \partial V/\partial y\, dy\hat{}dz\hat{}dx - 0 \\ &\quad +0 + 0 + \partial W/\partial z\, dz\hat{}dx\hat{}dy \\ &= \{\partial U/\partial x + \partial V/\partial y + \partial W/\partial z\}dx\hat{}dy\hat{}dz = div(\mathbf{V})dx\hat{}dy\hat{}dz \end{aligned}$$

Note that these algebraic ideas do not depend upon the existence of a norm or a metric.

Example 38 *Derivation of the Maxwell Faraday induction equations*

The Maxwell-Faraday induction equations are a set of partial differential equations that are logically deducible starting with the ordered sequence $[1, 2, 3, 4]$. Next assume the existence of "ordered coordinate" variables given the symbols $[x, y, z, t]$. Next assume the existence of an ordered set of functions of the coordinate variables, with symbols $[A_x, A_y, A_z, \phi]$. From these beginnings the Maxwell - Faraday equations follow as a consequence of the Exterior Calculus of Cartan.

Construct the 1-form from the ordered set of functions and variables:

$$A = A_x dx + A_y dy + A_z dz - \phi dt. \tag{7.57}$$

Next construct the 2-form $F = dA$. Then construct the 3-form ddA which must vanish: $ddA = dF \Rightarrow 0$. In 4D the 3-form has 4 coefficient functions of partial derivatives that must vanish. These PDE's correspond in format to the 4 Maxwell - Faraday equations, with 3D symbols

$$curl\mathbf{E} + \partial\mathbf{B}/\partial t = 0 \quad div\mathbf{B} = 0, \tag{7.58}$$

where the symbols are defined in terms of the coefficient functions of the 1-form (of potentials) as,

$$\mathbf{E} = -grad\,\phi - \partial\mathbf{A}/t \qquad \mathbf{B} = curl\,\mathbf{A}. \tag{7.59}$$

Now the choice of symbol functions and coordinate functions was completely arbitrary, but the format of the PDE's that satisfy $ddA = dF \Rightarrow 0$ are always the same relative to the ordering process. Experimentally, the logical equations of Maxwell - Faraday have been exploited in electromagnetic applications. However, the SAME formulas (different symbols) are applicable to hydrodynamics (as well as other physical systems of interest). Surprisingly, little has been done with the induction equations in hydrodynamics. These are not analogies. These are consequences of the logic of the Exterior Calculus and have universal applicability.

7.3.5 The Interior Product

The interior product is an operation on p forms that requires a direction Vector field, V. The interior product lowers the degree of a p-form, changing a p-form into a p-1 form. The interior product of a Vector direction field and a zero form (function) is defined to be zero. The symbol for the interior product herein is taken to be $i(V)$. The interior product of a Vector field and an exact basis element equal to the differential of a coordinate dy^a is not zero, but is defined to equal to the a^{th} component of V. Hence the fundamental definitions can be written as

$$i(V)\theta(y^a) = 0 \qquad , \qquad i(V)dy^a = V^a. \tag{7.60}$$

It follows that the inner product with respect to the vector field V acting on a 1-form, $A = (A_x dx + A_y dy + A_z dz)$ is given by the expression:

$$i(V)A = i(V)(A_x dx + A_y dy + A_z dz) = (A_x V^x + A_y V^y + A_z V^z) \tag{7.61}$$

An additional rule is required to take care of the anti symmetries of differential forms. That is for the product $A\hat{}B$ of two 1-forms, the interior product with respect to V becomes

$$i(V)\{A\hat{}B\} = (i(V)A)\hat{}B\} - A\hat{}(i(V)B)$$

and

$$i(V)i(V)A = 0 \tag{7.62}$$

similar to the modified Leibniz rule for the exterior differential.. Other expressions can be worked out for higher p-forms can be worked out using these rules,

$$i(V)i(V)\omega^p = 0 \qquad i(V)i(W)\omega^p \neq i(W)i(V)\omega^p$$

Example 39 *Compute the interior product of $J = [J^x, J^y, J^z]$ in 3D with the 3-form vol element $Vol = dx\hat{}dy\hat{}dz$*

$$i(J)Vol = J^x dy\hat{}dz - J^y dx\hat{}dz + J^z dx\hat{}dy.$$

Example 40 *Compute the interior product of* $V = [V^x, V^y, V^z]$ *with the 2-form* $i(J)Vol$

$$\begin{aligned} i(V)\{i(J)Vol\} &= i(V)\{J^x dy\hat{\ }dz - J^y dx\hat{\ }dz + J^z dx\hat{\ }dy\} \\ &= (J^y V^z - J^z V^y)dx + (J^z V^x - J^x V^z)dy + (J^x V^y - J^y V^x)dz. \end{aligned}$$

Note that the construction (in 3D) of the double interior product generates coefficients equal to the cross product of the two different vector fields, J and V, and the double interior product with the same vector is zero.

7.3.6 The Lie Differential

The Lie differential with respect to a vector field generates a p-form ϑ^p from a p-form ω^p. It is constructed from the raising operator d and the lowering operator $i(V)$. The general formula is

$$\omega^p \Rightarrow \vartheta^p : \qquad L_{(V)}\omega^p = i(V)d\omega^p + d(i(V)\omega^p = \vartheta^p. \tag{7.63}$$

Marsden has called this Cartan's Magic formula. The reason is that most of the equations of mechanics can be put into this form or derived from its construction. For example, those processes V which are "Hamiltonian" processes are those V such that $i(V)d\omega^p$ is exact. It is also remarkable that this formula is equivalent to the first law of thermodynamics. Consider a 1-form of Action, A, that presents a physical system (this will be done in detail in later sections). Then consider a vector field V that represents an evolutionary process. Define the 0-form (scalar function) of internal energy as $U = i(V)A$, the 1-form of Work as $W = i(V)dA$, and the output 1-form ϑ^p as Q. Then Cartan's Magic formula becomes

$$L_{(V)}A = i(V)dA + d(i(V)A = W + dU = Q \tag{7.64}$$

which is to be recognized as the first law of thermodynamics for a physical system A undergoing an evolutionary process V. This result will be exploited in later sections.

Example 41 *Compute the Lie differential with respect to* $V = [F, V, 1]$ *acting on the 1–form*
$A = pdq - H(p, q, t)dt$ *in 3 dimensions. The basis elements are* $[dp, dq, dt]$.

$$\begin{aligned} L_{(V)}A &= i(V)dA + d(i(V)A) \\ &= i(V)\{dp\hat{\ }dq - dH\hat{\ }dt\} + d(pV - H) \\ &= i(V)\{dp\hat{\ }dq - \partial H/\partial p\, dp\hat{\ }dt - \partial H/\partial q\, dq\hat{\ }dt\} + d(pV - H) \\ &= Fdq - Vdp - F(\partial H/\partial p)dt - V(\partial H/\partial q)dt + dH + d(pV - H) \\ &= F(dq - \partial H/\partial pdt) - V(dp + \partial H/\partial qdt) + d(pV) \end{aligned}$$

Note that the RHS of the equation above is a perfect differential for all evolutionary vector fields with components $V = [F, V, 1]$, if the two bracket factors vanish. Therefor, vector fields that are generated from the partial derivatives of $H(p, q, t)$ according to the formulas,

$$\begin{aligned} (dq - \partial H/\partial p dt) &\Rightarrow 0 \supset V = \partial H/\partial p \\ (dp - \partial H/\partial q dt) &\Rightarrow 0 \supset F = -\partial H/\partial p, \end{aligned} \tag{7.65}$$

which produce a 1-form of heat Q which is closed, $dQ \Rightarrow 0$. Such processes (vector fields) are defined to be Hamiltonian vector fields (processes). Hamiltonian dynamics is the (constrained) domain of much of theoretical mechanics. The domain is constrained, as the 3-form $Q\hat{}dQ \Rightarrow 0$. Such processes are then always thermodynamically reversible. Later on, irreversible processes for which $Q\hat{}dQ \neq 0$ will be studied.

One notes for Hamiltonian processes,

$$\begin{aligned} L_{(V)}H &= i(V)dH \\ &= F\partial H/\partial p + V\partial H/\partial q + \partial H/\partial t \\ &= FV - VF + \partial H/\partial t \\ &= \partial H/\partial t, \end{aligned} \tag{7.66}$$

so that if H is independent from time, then H is an evolutionary invariant. In mechanics, the function H is typically defined to be equal to be the sum of kinetic and potential energy, $H = p^2/2m + \varphi(x)$, so that time independent Hamiltonian processes "conserve energy". Even if the Hamiltonian is a function of time, Hamiltonian processes are thermodynamically reversible, as $Q\hat{}dQ = 0$.

7.3.7 The Pull_back with examples:

First consider differential forms whose coefficients would be covariant tensor fields, if the Jacobian matrix has an inverse. A typical representation is a 1-form written in terms of the variables on the final state as:

$$A = A_\mu(x^\nu)\, dx^\mu = \langle A_\mu(x^\nu)| \circ |dx^\mu\rangle\,. \tag{7.67}$$

Consider the differentiable non-linear map

$$\phi :\ y^k \Rightarrow x^\mu = \phi^\mu(y^j), \tag{7.68}$$

$$d\phi :\ dy^k \Rightarrow dx^\mu = \left[\partial\phi^\mu(y^j)/\partial y^k\right]\, dy^k = \left[J^\mu_k(y^j)\right] \circ \left| dy^k \right\rangle. \tag{7.69}$$

Substitute these formulas into the expression for the differential 1-form expressed in terms of the independent variables on the final state:

$$A = \langle A_\mu(x^\nu)| \circ |dx^\mu\rangle = \left\langle A_\mu(\phi^\nu(y^j))\right| \circ \left[J^\mu_k(y^j)\right] \circ \left| dy^k \right\rangle \tag{7.70}$$

$$= \left\langle \overline{A}_k(y^j)\right| \circ \left| dy^k \right\rangle. \tag{7.71}$$

The coefficients $\langle \overline{A}_k(y^j) |$ are well defined functions on the initial state, with arguments in terms of the initial state variables. Now if the map from initial to final state is such that the Jacobian is an invertible matrix, then the coefficient variables

$$\langle A_\mu(x^\nu)| \circ \left[J_k^\mu(y^j)\right] = \langle A_\mu(\phi^\nu(y^j))| \circ \left[J_k^\mu(y^j)\right] = \langle \overline{A}_k(y^j)| \tag{7.72}$$

is equivalent to the transformation rules of a covariant tensor field in classic tensor analysis:

$$pull\_back\ Rule : A_\mu \partial x^\mu / \partial y^k = \overline{A}_k \tag{7.73}$$

Note: The pull_back coefficients are not tensor equivalents when the Jacobian matrix is not invertible. However, the pull_back is always well defined in terms of the language which uses the Jacobian matrix transpose. The transpose always exists even though the inverse does not.

Example 1: pull_back of a 1-form

Consider the example map from 3 to 3 dimensions, for which an **inverse Jacobian does not exist.**

$$\phi : \{X, Y, Z\} \Rightarrow \{x, y, z\} = \{XY, Y^3, X\} \tag{7.74}$$

$$d\phi : \{dX, dY, dZ\} \Rightarrow \{dx, dy, dz\} = \{YdX + XdY, 3Y^2dY, dX\} \tag{7.75}$$

$$\begin{aligned} Jacobian\ [J_k^\mu(X)] &= \left[\partial\phi^\mu(x^j)/\partial X^k\right] = \begin{bmatrix} Y & X & 0 \\ 0 & 3Y^2 & 0 \\ 1 & 0 & 0 \end{bmatrix} && (7.76) \\ \det [J_k^\mu(X)] &= 0 && (7.77) \end{aligned}$$

$$\begin{aligned} Given\ on\ final\ state\ \ A &= y\,dx - xdy + dz && (7.78) \\ Substitute\ \phi\ and\ d\phi &= Y^3(YdX + XdY) - XY3Y^2dY + dX \\ pull\_back\ to\ initial\ state &= (Y^4 + 1)dX - (2Y^3X)dY + 0dZ && (7.79) \end{aligned}$$

$$\begin{aligned}
Coefficients\ on\ final\ state\ \ \langle A_\mu| &= [y,\ -x\,,+1] && (7.80)\\
pull_back\ to\ initial\ state\ \left\langle \overline{A}_k \right| &= [(Y^4+1),\ -(2Y^3X),0] && (7.81)\\
&= \langle A_\mu| \circ [J_k^\mu] && (7.82)\\
&= \langle y,\ -x\,,+1| \circ \begin{bmatrix} Y & X & 0\\ 0 & 3Y^2 & 0\\ 1 & 0 & 0 \end{bmatrix} && (7.83)\\
&= \left\langle yY+1\,, yX - x3Y^2,\ 0\right| && (7.84)\\
&= \left\langle (Y^4+1),\ -(2Y^3X),\ 0\right| && (7.85)
\end{aligned}$$

The example demonstrates the fact that the coefficients of the differential forms do not behave as tensors with respect to the non-invertible, but differentiable map. Yet, everything is well defined, functionally, with respect to the pull_back operation.

Example 2: pull_back of a 2-form

For the same map given in the first example, consider the 2-form below

$$\begin{aligned}
& Given\ on\ final\ state\\
F &= F_{xy}dx\hat{}dy + F_{yz}dy\hat{}dz + F_{zx}dz\hat{}dx && (7.86)\\
& Substitute\\
&= F_{xy}(3Y^3dX\hat{}dY) - F_{yz}(3Y^2)dX\hat{}dY + F_{zx}(XdX\hat{}dY) && (7.87)\\
& pull_back\ to\ initial\ state\\
&= \{F_{xy}(3Y^3) - F_{yz}(3Y^2) + F_{zx}(X)\}dX\hat{}dY && (7.88)
\end{aligned}$$

$$\begin{aligned}
& Coefficients\ on\ final\ state\\
\langle F_{\mu\nu}| &= \langle F_{xy},\ F_{yz}\,, F_{zx}| && (7.89)\\
& pull_back\ to\ initial\ state\\
\left\langle \overline{F}_{XY}\right| &= \left\langle \{-F_{yz}(3Y^2) + F_{zx}(X) + F_{xy}(3Y^3)\}, 0, 0\right| && (7.90)
\end{aligned}$$

Example 3. pull_back of a contravariant tensor delta density and the adjoint Jacobian

Now consider the volume element 3-form

$$Vol3 := dx\hat{}dy\hat{}dz = (\det[J])dX\hat{}dY\hat{}dZ \qquad (7.91)$$

and the Current N-1 form delta-density,

$$C = i(C^\mu)Vol3 = C^x dy\hat{}dz - C^y dz\hat{}dx + C^z dx\hat{}dy. \qquad (7.92)$$

The adjoint matrix to the Jacobian (for which no inverse exists) is

$$[AdJ] = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -3Y^2 & X & 3Y^3 \end{bmatrix}. \tag{7.93}$$

Substitution yields

$$\begin{aligned} C &= -C^x(3Y^2)dX\hat{}dY + C^y(XdX\hat{}dY) + C^z 3Y^3 dX\hat{}dY & (7.94) \\ &= \{-(3Y^2)C^x + (X)C^y + 3Y^3 C^z\}dX\hat{}dY & (7.95) \\ &= \overline{C}^Z dX\hat{}dY, & (7.96) \end{aligned}$$

from which it is apparent that

$$\left| \begin{matrix} \overline{C}^X \\ \overline{C}^Y \\ \overline{C}^Z \end{matrix} \right\rangle \Leftarrow [AdJ] \circ \left| \begin{matrix} C^x \\ C^y \\ C^z \end{matrix} \right\rangle. \tag{7.97}$$

Note that these results are valid for the case where the inverse Jacobian does not exist.

For those cases where the inverse matrix does exist, it is apparent that the Current coefficients do not transform as a contravariant tensor, but instead transform as a contravariant tensor delta density: Multiplication of both sides of the preceding equation by the Jacobian matrix yields:

$$[J_k^\mu] \circ |\overline{C}^k(Y^j)\rangle = J_k^\mu \overline{C}^k \Rightarrow \Delta \cdot C^\mu(x) = (\det [J_k^\mu]) \cdot C^\mu(x). \tag{7.98}$$

If the coefficients transformed as a contravariant tensor, then the preceding formula would have been written as

$$J_k^\mu \overline{C}^k \Rightarrow C^\mu(x). \tag{7.99}$$

Tensor delta densities are sensitive to the sign and magnitude of the determinant of the mapping.

7.3.8 Some Topological Features

The concepts of intersections, closure, and limit points are fundamental topological concepts that have a relationships to the Cartan Calculus. In certain situations the exterior product exhibits properties of intersection operator, and the exterior derivative exhibits properties of a limit point operator. More formally, given a domain with two exact 1-forms in 3D, the exterior product of the two exact 1-forms (if not zero) represents the points of intersection of the two implicit surfaces generated by the two functions whose gradient coefficients make up the components of the two exact 1-forms.

Example 42 *The exterior product and the concept of intersection.*

Consider two 1-forms created by applying the exterior differential to two distinct functions $\alpha(x, y, z)$ and $\beta(x, y, z)$. The coefficients of $d\alpha = grad(\alpha) \circ d\mathbf{r}$ form the gradient field $grad(\alpha)$ which is perpendicular to the implicit surface $\alpha(x, y, z) = 0$. Similarly, $d\beta = grad(\beta) \circ d\mathbf{r}$ implies that the gradient coefficients $grad(\beta)$ are perpendicular to the implicit surface $\beta(x, y, z) = 0$. If the two implicit surfaces intersect, then exterior product of the two 1-forms create a 2-form, $J = d\alpha\hat{}d\beta$, which is not zero. The components of the 2 form, J, can be interpreted as a contravariant vector in 3D, which is tangent to the points in common (intersections) that make up the intersection of the two surfaces. For $n = 3$, the number of components of a 2-form are 3, and are in agreement with the 3D cross-product formulas of Gibbs.

Example 43 *The exterior derivative is a limit point generator.*

From another point of view, it is possible to deduce a topological structure from a given 1-form A on the domain. The it is possible to show that the exterior derivative, relative to this Cartan topology, acts as a generator of the limit points of the given topology. This is given further credence from the physical idea that the divergence (an application of the exterior derivative) of the $\mathbf{D}$ field in electromagnetism has finite values that terminate on charges. That is, the Faraday lines of $\mathbf{D}$ come from limit points of positive charge and wind up on limit points of negative charge. However, the concept that the exterior differential d is a limit point operator is more formal, and has a basis in Kuratowski's closure operator.

Example 44 *The Lie differntial can be used to select topological invariants of a process.*

The Lie differntial with respect to a vector field V may be construed as a convective propagator describing the flow of the points of a p-form down the flow lines generated by V. If the p-form is integrated over a domain of such flowing points, then it is possible to ask if the integral is an invariant of the flow. Moreover it is possible to ask if the flowing points are distorted and deformed, does the integral over the deformed points equal the integral over the non-deformed points. If is true that the value of the integral is unchanged by continuous deformation, then the integral must represent a topological property.

To deform the flowing points is easy enough; just multiply the original vector field V by a function of (say) $\lambda(x, y, z, t)$. The function λ does not change the flow lines generated by V, but it does deform the points that make up the flowlines by stretching or compression along the flow lines. Then $V \Rightarrow \lambda V$ and $L_{(V)} \Rightarrow L_{(\lambda V)}$, and the Lie differential becomes a deformation operator. If it can be shown that if

$$L_{(\lambda V)} \oint A = \oint i(\lambda V) dA + \oint d(i(\lambda V)A) = 0 \tag{7.100}$$

for any function λ then the closed integral $\oint A$ is a deformation invariant of the process. Note that the second integral always vanishes, $\oint d(i(\lambda V)A) = 0$, as the integrand is an exact perfect differential. For the first integral to vanish for arbitrary deformation parameter, λ, the integrand must be zero. This leads to the conclusion that

$$\begin{aligned} if \quad i(\lambda V)dA &= \lambda i(V)dA = 0 \quad \text{any } \lambda, \\ then \quad \oint A &= \text{deformation invariant} \end{aligned} \tag{7.101}$$

Hence if the Work 1-form is zero, $W = i(V)dA \Rightarrow 0$ then the closed integral of the Action $\oint A$ is a topological property (of that process). Cartan has shown that a necessary and sufficient condition for a process to be a Hamiltonian process, is that the closed integral of the Action should be a topological invariant of the process.

Example 45 *The first law of thermodynamics is related to a topological statement of Cohomology.*

A non-exact closed p form Q is defined to be Cohomologous to another non-exact closed p-form, W, if the difference between the two p-forms is exact. This means that the integrals of the two different p-forms over any closed integration path (cycle or boundary) are the same. For non-exact, closed, 1-forms of Heat, Q, and Work, W, the cohomological statement is the germ of the First Law:

$$Q - W = dU.$$

It was noted above that the Lie differential with respect to a vector field (process) acting on a physical system described by a 1-form of Action, is essentially a Cohomological statement of the first law. The Lie differential is a Cohomological generator.

Example 46 *Thermodynamic Isolation and Frobenius integrability.*

In thermodynamics, it is recognized that there are isolated, closed, and open systems. These words are also used to describe topological properties. A set is topologically isolated if it has no intersection with its limit points. This result translates to $A\hat{}dA = 0$ for a given 1-form and its induced Cartan topology. The constraint of isolation is also equivalent to the Frobenius idea of unique integrability. That is when $A\hat{}dA = 0,$ there exists a unique function whose gradient (or surface normal) is proportional to the given coefficients of the given 1-form. Caratheodory's statements about inaccessible states is a statement related to the concept of isolation and connectivity to an equilibrium system. When $A\hat{}dA = 0$ (no matter what the dimension of the coordinate space happens to be) there exists a transformation to a domain of two independent functions that will describe the properties of the 1-form. That is, the 1-form can be written as $\phi d\chi$, and its coefficient functions are proportional to a gradient, $d\chi$. The problem becomes essentially a two dimensional problem.

The property of isolation is a topological property, hence if a process causes $A\hat{}dA \neq 0$ to change to $A\hat{}dA = 0$, or from a state where $A\hat{}dA = 0$ to a state where $A\hat{}dA \neq 0$, a topological change has take place. In hydrodynamics, all streamline flows satisfy $A\hat{}dA = 0$. Hence turbulent flows must involve domains where $A\hat{}dA \neq 0$. The transition to (from) turbulence from (to) a state of non-turbulence must involve topological change.

It should be mentioned that with respect to diffeomorphic transformations, or more simply those transformations that preserve pure geometrical properties, the differences between contravariant and covariant concepts cannot be distinguished. With respect to an aging process involving topological change, the behavior of the two concepts is observably different.

7.4 Point Set Topology

In order to establish a foundation for topological evolution, an introduction to topological ideas and definitions is presented in terms of point set methods for which the topological concepts can be exhibited in terms of simple examples. This expose of topology given in this monograph will not be complete, and will not cover all of topological theory. Only those parts of topology that the author feels are necessary and useful for the development of physical and engineering applications will be presented. A conventional introduction to topology often starts with a metric topology, but herein the concept of a metric is purposely avoided, as the idea of a metric is the essence of those geometrical qualities of size and shape. The conventional procedure is to develop the topological ideas in terms of a space with a euclidean or some Riemannian metric. Then the topological concepts are shown to be independent of the choice of metric. However, the notion of a metric is not needed, and the point set approach takes that point of view that the metric is just extra baggage that can often confuse the issues.

Perhaps the best way to learn basic ideas about topology is through the study of point set topology. The concepts and definitions can be illuminated by means of examples over a discrete and small set of elements. The early champions of point set topology were Kuratowski in Poland and Moore at UT-Austin. For a long time Point Set topologists were isolated from the Combinatorial Topologists. In fact the name topology, evidently, was introduced about 1925, about the time that it was recognized that topology had many equivalent expositions.

One of the best books for rapid assimilation of point set topology is the Schaum's Outline Series, "General Topology" by S Lipschutz [Lipschutz 1965]. Review chapter 1, then skip to chapters 5, 6, and 7.

In that which follows, four different topologies will be defined over a set of five elements, {a,b,c,d,e}. Then the topological definitions of Open sets, Closed Sets, Limit Points, Closure, Boundary, Interior, Exterior , and the concept of Continuity will be defined and exemplified for each of the four point set topologies. It should become apparent that the same sets of points can have different topologies imposed as a set of constraints on the same elements. It is the topology that allows the concepts of boundary, closure and limit points to be define. In that which follows, the applications of these ideas will done for systems of differential forms, rather than systems of points.

7.4.1 Closed and Open Sets

Consider a set of elements $\{a, b, c, d, e\}$ and a combinatorial process which is symbolized, for example, by the brackets (ab) or (ade). Construct all possible combinations, and include the null set, 0. Define $X = (abcde)$.

Now from the set of all possible combinations, it is possible to select many subset collections. Certain of these subset collections have the remarkable property of logical closure. As an example, consider the subset collection, or class of subsets, given by:

$$T1(closed) = \{X, 0, a, b, (ab), (bcd), (abcd)\} \tag{7.102}$$

Note that the intersection of (a) with (ab) is a, $(a \cap ab = a)$, which is an element of the collection, and the intersection of (ab) with (bcd) is b which is also an element of the collection. In fact, the intersection of every element of $T1(closed)$ with every other element of $T1(closed)$ produces one of the original seven elements of the collection, $T1closed$. In other words, the process of set intersection acting on any number of elements is closed with respect to $T1(closed)$.

Now also note that the union of any two elements of the set is also contained within the set. The idea that a closed algebra can be built upon the notions of union and intersection, and that this algebra be a division algebra, is at the heart of the theory of logic. This idea of logical closure with respect to arbitrary intersection and finite union is said to define a topology, $T1(closed)$, of closed sets.

Definition: A topology $T1(closed)$ on a set X is a collection or class of subsets that obeys the following axioms:

1. **A1(closed):** X and the null set 0 are elements of the collection.
2. **A2(closed)**: The arbitrary intersection of any number of elements of the collection belongs to the collection.
3. **A3(closed):** The arbitrary union of any pair of elements of the collection belongs to the collection.

The elements of the collection, $T1(closed)$, are defined to be "closed" sets. The compliments of the closed sets are defined as "open" sets. The open sets of the topology are the collection of subsets given by

$$T1(open) = \{0, X, (bcde), (acde), (cde), (ae), e\}. \tag{7.103}$$

It is important to note that the same set of all combinations of subsets can support many topologies. For example, the subsets of the collection,

$$T2(closed) = \{X, 0, (bcde), (cde), (de)\}, \tag{7.104}$$

are closed with respect to both logical intersection and union. Hence $T2(closed)$ is a different topology built on the same set of points, X. The open sets of this topology are

$$T2(open) = \{0, X, a, (ab), (abc)\} \tag{7.105}$$

The compliments of closed sets are defined to be "open" sets and they too can be used to define a topology.

A subset can be both open, or closed, or both, or neither, relative to a specified topology. For example, with respect to the topology given by the closed sets,

$$T3 = \{X, 0, a, (bcde)\}, \tag{7.106}$$

$(bcde)$ is both open and closed, and the set (bc) is neither open nor closed.

The topology of closed sets given by the collection,

$$T4(closed) = \{X, 0, (bcde), (abe), (be), (a)\}, \tag{7.107}$$

has its dual as the topology of open sets

$$T4(open) = \{0, X, a, (cd), (acd), (bcde)\}. \tag{7.108}$$

Note that this topology, $T4$, is a refinement of the topology, $T3$, in that it contains additional closed (or open) sets.

Remarks: In the definition of a topology when the number of elements of the set is not finite, the logical intersection of open sets is restricted to any pair, and the logical union of closed sets to restricted to any pair. There are many other ways to define a topology, but the concepts always come back to the idea of logical closure.

7.4.2 Limit Points

The next idea to be presented is the concept of a limit point. A standard definition states that a point p is a limit point of a subset, A, iff every open set that contains p contains another point of A. Note that p is an element of X and need not be an element of A. Given a subset, A, each point of X must be tested to see if it is a limit point of A relative to the topology specified on the points. If A is a singleton, it can have no limit points, for there are no other points of A. It follows that the limit points of a limit point (a singleton) is the null set. If the limit point of A consists of the singletons or points symbolized by dA, then $d(dA) = 0$. The set of limit points as a collection of singletons, $\{a, b, c..\}$ will be denoted by dA, where the union of all limit points will be denoted by A'. The symbol d may be viewed as a limit point operator; the symbol d when applied to a set, A, means that each point p of the domain is tested against the specified topology to see if another point of A is included in each open set of the topology.

Consider the subset $A = (ab)$ and the topology given by $T1(open)$. Now test each point relative to the collection $T1(open)$:

$$T1(open) = \{0, X, (bcde), (acde), (cde), (ae), e\} \tag{7.109}$$

The point a is not a limit point of (ab) because the open set $(acde)$ which contains a does

not contain b; $dA = 0$ at a.

The point b is not a limit point of (ab) because the open set $(bcde)$ does not contain a;

$dA = 0$ at b.

The point c is not a limit point of (ab) because the open set (cde) does not contain either a

or b; $dA = 0$ at c.

The point d is not a limit point of (ab) because the open set (cde) does not contain either a

or b; $dA = 0$ at d.

The point e is not a limit point of (ab) because the open set (cde) does not contain either a

or b; $dA = 0$ at e.

In other words, the subset (ab) has no limit points in the topology given by $T1$. The limit point set of ab, designated in this monograph as $A‘(ab)$, is given by

$$A‘(ab) = \{0\}, \tag{7.110}$$

the empty set, as $dA = 0$ at all points of X.

Now make the same tests with regard to the same subset $A = (ab)$, but this time relative to the topology given by $T2(open)$.

$$T2(open) = \{0, X, a, (ab), (abc)\} \tag{7.111}$$

The point a is not a limit point of (ab) because the open set (a) is a singleton;

$dA = 0$ at a.

The point b is a limit point of (ab) because both the open sets that contain b also contain a;

$dA \neq 0$ at b.

The point c is a limit point of (ab) because the open set X, the only open set that contains

c, contains a and b which are points of A; $dA \neq 0$ at c.

The point d is a limit point of (ab) because the open set X , the only open set that contains

d, contains another point of A; $dA \neq 0$ at d.

The point e is a limit point of (ab) because the open set X, the only open set that contains

e, another point of A; $dA \neq 0$ at e.

Hence, the points b, c, d and e are limit points of $A = (ab)$ relative to the topology $T2(open)$.

$$\text{The limit set of } A`(ab) = (bcde) \tag{7.112}$$

With respect to the topology of $T4(open)$,

$$T4(open) = \{0, X, a, (cd), (acd), (bcde)\} \tag{7.113}$$

test for limit points of the set (ab):

The point a is not a limit point of (ab) because the open set a is a singleton; $dA = 0$ at a.

The point b is not a limit point of (ab) because the open set $(bcde)$ does not contain a;

$dA = 0$ at b.

The point c is not a limit point of (ab) because the open set (cd) which contains c does not contain a or b; $dA = 0$ at c.

The point d is not a limit point of (ab) because the open set (cd) which contains d does not contain a or b; $dA = 0$ at d.

The point e is a limit point of (ab) because the open set $(bcde)$ which contains e contains another point b of A; $dA \neq 0$ at e.

The limit set of (ab) relative to $T4(open)$ becomes $A` = \{e\}$.

Note that the set of limit points as a collection, or a class of sets, may or may not have limit points. If the limit set is a singleton, then the limit points of the set of limit points is the null set. However, consider the limit set $A` = \{bcde\}$ of the set $(ab)$ relative to the topology $T2$. Then the limit points of $A`$ are the points (b, c, d, e). In other words $dA` \neq 0$ necessarily, but $ddA` = 0$.

7.4.3 Closure

The closure of a set is defined to be the union of the set and its limit points. Note that a closed set contains its limit points, if any exist. In the examples given above the closure of the set $A = (ab)$ relative to the topology $T1$ is equal to the union of $A = (ab)$ and its limit points, which is the null set.

$$\widetilde{A} = A \cup A` = (ab) \tag{7.114}$$

Note that $A = (ab)$ is a closed set, and has no limit points relative to the topology $T1(open)$. However, the closure of A relative to the topology $T2(open)$ is

$$\widetilde{A} = A \cup A` = (ab) \cup (bcde) = (abcde). \tag{7.115}$$

which is the whole set. When the closure of a subset is the whole set X, the subset is said to be dense in X relative to the specified topology.

The closure of (ab) relative to the topology $T4(open)$ is

$$\widetilde{A} = A \cup A' = (ab) \cup (e) = (abe). \tag{7.116}$$

Note that the closure of a subset is equal to the smallest closed set that contains the subset. Every closed set is its own closure. A closed set may or may not have limit points, but if it does have limit points they are contained within the (closed) set.

7.4.4 Continuity

Now comes a major issue of this appendix. Continuity of a transformation is defined relative to the topologies that may exist on the initial and final states. Let the set of points X with topology $T1(open)$ be mapped into the set of points Y with the topology $T2(open)$. Then the map is continuous iff the closure of every subset of the initial state relative to $T1$ is included in the closure of the image of the final state relative to the topology $T2$.

Another test for continuity is given by the statement that the inverse image of every open set of Y relative to $T2$ is an open set of X relative to the topology $T1$.

Later on, the first definition will be used to prove that any topology built on subsets of exterior forms with C2 (twice differentiable) coefficients will be continuously transformed by evolutionary processes that are generated by the Lie convective derivative with respect to C2 vector fields. For the present, the second definition will be used in terms of simple point set topological systems.

As a first example consider the transformations given on X to Y by the following diagram:

Continuous example

Discontinuous example

Homeomorphic example

(equivalent topologies)

7.4.5 Interior

When emphasis is placed on open sets rather that closed sets, other ideas come to the forefront. In particular, the concept dual to the notion of closure is the concept of interior. While closure asks for the smallest closed set that covers any specified subset, the idea of interior asks for the largest open set included in the specified subset. The interior of a set can be empty (for there may be no open sets other than the null set contained within the specified set)!

For example, the subset (ab) has no interior relative to the topology $T1(open)$. However, the interior of (ab) is itself, (ab), relative to the topology $T2(open)$, because (ab) is an open set in this topology! Relative to the topology $T4(open)$, the interior of (ab) is the singleton, (a):

$IntA = \{0\}$ relative to $T1$

$IntA = (ab)$ relative to $T2$
$IntA = (a)$ relative to $T4$.

The set (abe) has an interior (ae) relative to $T1(open)$ and an interior (ab) relative to the topology $T2(open)$.

7.4.6 Exterior

The exterior of a specified set is the interior of the compliment of the specified set. The compliment of (ab) is the set (cde) which has the interior (cde) relative to $T1(open)$ and has no interior relative to the topology $T2(open)$. Relative to the topology $T4(open)$, the exterior of (ab) is the set (cd).

$ExtA = \{cde\}$ relative to $T1$
$ExtA = (0)$ relative to $T2$
$ExtA = (cd)$ relative to $T4$.

7.4.7 The Boundary

The points that make up the boundary of a subset are union of those points that are not included in the interior or the exterior. However, the union of the points that make up the boundary may have subsets that are not connected. Consider a solid disk. The points that make up the rim of the disk forms its boundary. Now punch a hole in the disk. The collection of points that make up the outer rim and the inner hole now form the boundary of the disk. The two sets of boundary points are not connected.

Similar to the limit point operator, d, a boundary operator, ∂ (some books use δ) , may be defined in terms of a procedure, such that when ∂ is applied to the set A, it implies that a test is made at each point p to see if p is an element of the interior or of the exterior of the selected subset A. If the test fails then $\partial A \neq 0$ and the point is a boundary point. It the point p is an element of the exterior or interior of A, then $\partial A = 0$ at the point p. The boundary of A, or bA, of the set A, is defined as the union of all boundary points.

As a first example, consider again the set $A = (ab)$ and the $T1$ topology. The set $A = (ab)$ has no interior, but the exterior of (ab) is the set (cde), and therefore the boundary set, bA, consists of the union of the points (a, b). In this first example, then

$A` = \{0\}$, but $bA = (ab) \subset \widetilde{A}$
It follows that
$A \cap A` = 0$,
$A \cap bA \neq 0$.

The boundary set exists even though the limit set does not!

Relative to $T2(open)$, the set (ab) has an interior set (ab), no exterior set, but a boundary set is $bA = (cde)$. In this case the boundary is included in the closure,

$$A` = (bcde), bA = (cde) \subset \widetilde{A} = (abcde), \tag{7.117}$$

and

$A \cap A` \neq 0,$
$A \cap bA = 0.$

Although all boundary points are limit points, there exist limit points that are not elements of the boundary.

Relative to $T4(open)$, the set (ab) has an interior set (a), and exterior set (cd) and a boundary set (be),

$A` = (e), bA = (be),$
$A \cap A` = 0,$
$A \cap bA \neq 0.$

It is apparent that the boundary points contain limit points, but there are boundary points which are not limit points!.

In all cases, note that the union of the interior and the boundary is equal to the union of the set and its limit points. The boundary is always included in the closure, but the boundary may contain points which are not limit points.

$$\widetilde{A} = IntA \cup bA = A \cup A`. \tag{7.118}$$

These examples point out that there exist certain correspondences between limit points and boundaries, but they are not necessarily the same concept. Much of current physical theory has emphasized the boundary and open set point of view, while in this monograph the emphasis is on the limit points and closure point of view. It will become evident that these concepts are at the heart of the differences between contravariant and covariant concepts in physical theories, an idea that ultimately expresses itself in the differences between the particle or wave perspective of physics. In topology, these notions are at the heart of the differences between Homology and Cohomology (which will be discussed in detail later). In this monograph, the cohomological point of view is emphasized.

If a set has the property that its intersection with its limit set is empty, then the set is said to be isolated. This idea of isolation, whereby $A \cap A` = 0$, can be translated into the Cartan statement, $A\hat{}dA = 0$. The physical significance of the topological concept of isolation will be correlated with the Caratheodory statement of the existence of inaccessible states in a thermodynamic system, and to the notion of Frobenius complete integrability for a laminar, non-chaotic flow. The concept of isolation is a topological property. and its compliment is a necessary condition for chaos. The observation of a flow transforming from a laminar state (isolated) to a turbulent state (non-isolated) is an observation of topological evolution.

It should be mentioned that with respect to diffeomorphic transformations, or more simply those transformations that preserve pure geometrical properties, the differences between contravariant and covariant concepts cannot be distinguished. Further note that the existence of a metric implies that the contravariant concepts can be converted into covariant concepts, and their possible differences are masked into an alias-alibi format; that is, there are no measurable differences between the two concepts. However, with respect to an aging process, the behavior of the two concepts is observably different. The differences between the behavior of contravariant and covariant concepts may be interpreted as the existence of topological evolution.

Note that although all of the symbols used above are familiar in the realm of electromagnetism, the topological results and formulas obtained apply to any set of symbols, representing an arbitrary physical system, for example a fluid. The Faraday Maxwell equations are universal ideas on continuous physical systems of C2 functions.

For a computational examples in Maple, see http://www22.pair.com/csdc/pdf/maxwell.pd

7.5 Distributions and the Adjoint Field

Although the emphasis in this article is on concepts that are independent from the choice of metric or connection*, it is useful to demonstrate how a 1-form of Action, A, may be used to generate a compatible frame field $[\mathbb{F}]$ and a Cartan connection $[\mathbb{C}]$ on the variety. The symmetry features of $[\mathbb{F}]$ lead to metric ideas, and certain anti-symmetry features of $[\mathbb{C}]$ lead to the concept of Affine torsion (which is not the same as Topological torsion, or Frenet torsion). The concept of a differential connection also leads to the famous geometrical structural equations of Cartan, which are different from the topological structure concept utilized in this article. The topological structure concepts uses herein are independent from the choice of connection or metric. The details of the refined topological features of subspaces based upon the constraints of a global Cartan connection or metric will be the topic of another article.

The construction of the Frame field $[\mathbb{F}]$ can be done in such a manner that it admits differential closure of a vector basis over the domain of support ($\det[\mathbb{F}] \neq 0$). That is, the differential of any basis vector (contra-variant columns of functions) of the matrix Frame field creates a displaced vector which can be linearly composed of the vectors of the basis frame, each multiplied by differential 1-forms. The N^2 differential 1-forms that make up the coefficients of the vector differentials can be used to define the Cartan right connection matrix, $[C]$. The differential closure condition can be expressed by the equation,

$$d\,[\mathbb{F}] = [\mathbb{F}] \circ [\mathbb{C}]. \tag{7.119}$$

The differential closure process on a Frame of independent vectors which is based on a connection is not the same as the operation of forming the exterior differential of a p-form. The exterior differential of a p form takes a p form into a p+1 form, where the p-form is an element of one vector subspace, and the p+1 form is an element of a different vector subspace of the . Grassmann algebra of dimension 2^N. The differential process constrained by a connection takes a vector of dimension N into a vector of dimension N. In other words the connection based closure process is a process where the initial and final state are within the same vector subspace.

*The evolutionary processes of primary interest herein are those described by operating on differential forms with the Lie differential with respect to a direction field.

It should be realized from the outset that Frame fields are not uniquely determined by a given 1-form of Action. When a Frame field exists, the differential connections $[C]$ which generate differential closure can be placed into equivalence classes, determined by the group properties of the matrices involved. The investigation of the properties of these various group equivalence classes has become known as the study of "gauge theories", and the method enjoys great popularity at present. However, the choice of a gauge group in physics is often just that, a choice made by guessing, followed by attempts to put the constrained results into correspondence with physical properties and measurements. In this article, the focus is on those topological features that can be put into correspondence with experiment, and yet are independent from a specific choice of connection and/or metric.

Even though a Frame field is not necessarily unique, and goes beyond the primitive topological concepts that do not depend upon metric or connection, an algorithm for producing a Frame field will be discussed in the next section. From a given 1-form, A, there are two important types of procedures that can be used to construct a useful Frame field. One procedure is differential, and is related to parametric surface theory. The second procedure is algebraic, and is more closely related to implicit surface theory. The algebraic procedure will be discussed first.

7.5.1 The implicit algebraic Frame field

To construct the Frame field from a given 1-form, note that at a regular point, $\{x\}$, of an N dimensional space, any given 1-form, A, will admit N-1 linearly independent vector direction fields, $\mathbf{V}(x)$. Each vector direction field has N component functions, V^μ, to be determined algebraically from the following formula:

$$\text{algebraic orthogonality} \quad A_\mu V^\mu = 0. \tag{7.120}$$

The collection of N-1 vectors orthogonal to the 1-form are called elements of a distribution direction field, for multiplication of each vector field $\mathbf{V}$ by any non-zero function $1/\lambda(x^\mu)$ is also a solution to the algebraic orthogonality equation:

$$A_\mu(V^\mu/\lambda(x^\mu)) = 0 \ \text{ if } \ A_\mu V^\mu = 0. \tag{7.121}$$

This independence from scale is typical of projective geometries.

One possible (algebraic) construction, using the given functional coefficients, A_μ, of the 1-form, A, yields a Frame matrix of the form:

$$[F] = \begin{bmatrix} A_n & 0 & ... & A_1/\lambda \\ 0 & A_n & ... & A_2/\lambda \\ ... & ... & ... & ... \\ -A_1 & -A_2 & ... & A_n/\lambda \end{bmatrix}. \tag{7.122}$$

The first N-1 columns satisfy the algebraic orthogonality constraint, which implies that the last column vector is proportional to the adjoint of the matrix $[F]$. The determinant of the Frame field is given by the expression:

$$\det\,[F] = (A_n)^{(n-2)}\{(A_1)^2 + (A_2)^2 + ... + (A_n)^2\}/\lambda^2 \tag{7.123}$$

If the rescaling factor, $1/\lambda$, is chosen such that the determinant is unity over the domain of support of A, then on that domain the Frame field is globally defined and always has an inverse. The N-1 vector (direction fields) which satisfy the orthogonality relations, each of N components, are defined to be a basis of the "associated or horizontal" vectors relative to the given 1-form, A. Note that in the construction above, the coefficient A_n appears to have a privileged position. However, in spaces of odd topological (Pfaff) dimension, a canonical (Darboux [Oneill 1966]) format indicates that there is one coefficient (presumed to be A_n) that is equal to unity. The differential 1-form then has the canonical format, $A = p_\mu dq^\mu + 1ds$. For even topological dimensions, the canonical format is $A = p_\mu dq^\mu + Hdt = Ldt + p_\mu(dq^\mu - V^\mu dt)$, where H is an independent function. Note that the classical Hamiltonian constraint that $H \Rightarrow H(p, q, t)$ reduces the topological dimension 2n+2 to 2n+1.

The Cartan connection matrix for a Frame field constructed in an implicit algebraic manner can admit certain anti-symmetries of subspace that have been defined as Affine translational torsion. The parametric method described below, will not produce a connection with affine translational torsion of subspaces.

7.5.2 The parametric differential Frame field

If a parametric mapping of N functions in terms of N-1 parameters is given,

$$\xi^\alpha \Rightarrow x^k = X^k(\xi^\alpha) \quad (1 \le k \le N) \ (1 \le \alpha \le N-1) \tag{7.124}$$

is given, then the N-1 associated vectors can be defined differentially. That is, the partial derivatives of the N mapping functions with respect to the N-1 parameters can be used to form the first N-1 columns (associated vectors) of the matrix, $[M]$.

$$[M] = \begin{bmatrix} \mathbf{e}_\alpha^1 & \mathbf{e}_\beta^1 & ... & 0 \\ \mathbf{e}_\alpha^2 & \mathbf{e}_\beta^2 & ... & 0 \\ ... & ... & ... & ... \\ \mathbf{e}_\alpha^n & \mathbf{e}_\beta^n & ... & 0 \end{bmatrix} \tag{7.125}$$

$$\mathbf{e}_\alpha^k = \partial X^k(\xi^\beta)/\partial \xi^\alpha \tag{7.126}$$

The adjoint vector direction field to this N-1 system of associated vectors can be interpreted as a "normal or vertical" direction field via the algebraic orthogonality relations. Given the N-1 associated vectors, the adjoint vector, $\mathbf{n}$, can be constructed algebraically by adding a column of zeros to the N by N-1 matrix $[M]$ of contravariant associated vectors, $\mathbf{e}_\alpha^k$. (The component index k ranges from 1 to N and the index α ranges from 1 to N-1) .

$$[M] = \begin{bmatrix} \mathbf{e}_\alpha^1 & \mathbf{e}_\beta^1 & ... & 0 \\ \mathbf{e}_\alpha^2 & \mathbf{e}_\beta^2 & ... & 0 \\ ... & ... & ... & ... \\ \mathbf{e}_\alpha^n & \mathbf{e}_\beta^n & ... & 0 \end{bmatrix} \tag{7.127}$$

The determinant of $[M]$ is zero, but there always exists an adjoint matrix consisting of a column of N-1 by N-1 sub determinants.

$$\text{transpose of the Adjoint of } [M] = \begin{bmatrix} 0 & 0 & ... & \mathbf{n}^1 \\ 0 & 0 & ... & \mathbf{n}^2 \\ ... & ... & ... & ... \\ 0 & 0 & ... & \mathbf{n}^n \end{bmatrix} \tag{7.128}$$

The parametric method permits the creation of the (orthogonal) Adjoint 1-form given the N-1 distribution vectors, while the implicit method permits the creation of the N-1 orthogonal distribution vectors from a given 1-form, that is adjoint to the vectors of the distribution.

The adjoint direction field, $\mathbf{n}$, exists algebraically whether or not the distribution of N-1 vectors, $\mathbf{e}_\alpha^k$, span a simple hypersurface. By construction via the orthogonality constraint, the coefficients of the given 1-form A_μ are in effect proportional to the adjoint direction field. As discussed in the previous subsection, in the more simple situations the coefficients of a differential 1-form, A, can be viewed as a representation of the normal field to a hypersurface. In all cases the coefficients of a differential 1-form can be viewed as being an adjoint direction field.

Perhaps even more remarkably, it is possible to scale the adjoint direction field (hence the differential 1- form) by a function λ such that the determinant of the N x N matrix,

$$[F] = \begin{bmatrix} \mathbf{e}_\alpha^1 & \mathbf{e}_\beta^1 & ... & \mathbf{n}^1/\lambda \\ \mathbf{e}_\alpha^2 & \mathbf{e}_\beta^2 & ... & \mathbf{n}^2/\lambda \\ ... & ... & ... & ... \\ \mathbf{e}_\alpha^n & \mathbf{e}_\beta^n & ... & \mathbf{n}^n/\lambda \end{bmatrix} \tag{7.129}$$

is globally equal to a constant. The procedure thereby defines a Frame of N basis vectors everywhere over the N dimensional domain of support of the 1-form, A. It follows that exterior differentials of each of the basis vectors of the Frame are linear combinations of the set of the basis vectors. That is, the exterior differential process acting on the basis vectors of the Frame is closed. The process of exterior differentiation acting on elements of the set creates objects that remain within the set. Although this parametric procedure is similar to the implicit method described previously, the parametric method never generates a Frame with a connection that supports translational Affine torsion of subspaces.

When acting on p-forms, the exterior derivative carries a p-form from one vector space into a p+1 form in a different vector space. The concept of a connection constrains the differential process to transport a initial vector of one vector space into a final vector in the same vector space. Both vectors have the same basis.

7.5.3 Projective Frames.

In each of the "adjoint" methods given above, the orthogonality conditions are in effect 2(N-1) constraints on the general N^2 variables of a Frame matrix. A determinantal constraint of the type det $[\mathbb{F}] = 1$ adds one more constraint condition. Quadratic (metric) symmetry features implies that symmetric product of the Frame fields constructed by the adjoint procedure above yields a matrix with a fixed point.

$$\widetilde{[\mathbb{F}]} \circ [\mathbb{F}] = \begin{bmatrix} \mathbf{g}_\alpha^1 & \mathbf{g}_\beta^1 & ... & 0 \\ \mathbf{g}_\alpha^2 & \mathbf{g}_\beta^2 & ... & 0 \\ ... & ... & ... & ... \\ 0 & 0 & ... & \det[F]/A_n^{n-2} \end{bmatrix} \tag{7.130}$$

The coefficients of a projective frame would have only one constraint.

The utility of the "adjoint" procedure is that quadratic geometric metric properties of the tangent space can be decoupled from the geometric properties of the "adjoint" or "normal" space with an appropriate choice of $1/\lambda$.

Remarks

In three dimensions, the Gibbs cross product of engineering vector calculus is considered to be a "vector" for it has the same number of components as the gradient. Yet it has different behavior under transformations of the basis, and is therefor called a "pseudovector" or an axial vector. In the exterior calculus, the exterior product of the two 1-forms, with components proportional to covariant tensor of rank 1, creates a 2-form with covariant components of rank 2. Only in constrained geometries, such as euclidean three space, do 2-forms have any resemblance to the Gibbs cross product (a rule which fails in dimension n > 3). The pseudo-vector is an object that behaves like a contravariant tensor density of rank 1. Such objects are usually defined as "currents". In general, there are two species of differential forms (that are often dual to one another and are well behaved with respect to functional substitution and the pullback (p.115 [Marsden 1994])operation: p-forms and N-p form densities or currents. One species pulls back (meaning that the form is well defined with respect to functional substitution) with respect to the Jacobian transpose, while the other pulls back with respect to the Jacobian adjoint. Of course for orthogonal systems, these concepts are degenerate, for the inverse and the adjoint and the transpose of the Jacobian matrix are the same. Recall that at a point it is always possible to define a vector basis in terms of an orthogonal system (use the Gram-Schmidt process), but the possibility of extending, or mapping, the property of orthogonality smoothly and uniquely (without singularities) from one neighborhood to another neighborhood in a global sense requires that the mapping process be constrained to the be an element of the orthogonal group. Such constraints apply nicely to rigid body motion, but fail to describe the deformation of a solid. Hence the reader is advised that the automatic or indiscriminate use of orthonormal basis frames will not yield a complete understanding of nature.

If the neighborhoods can be connected by a singly parameterized vector field, then these concepts are at the basis of the Frenet-Serret moving frame analysis. Cartan extended these ideas to domains that are not so simply connected, and developed the notion of the moving basis Frame, which he called the Repere Mobile. In that which follows, it will be demonstrated how to construct these moving basis frames. There will be two distinct problems. The first problem will be how to construct a matrix frame of basis vectors at some point p of a space. Depending on the constraints inherent in their construction, the basis frames can be elements of an equivalence class. The equivalence class can be refined by imposition of other constraints. The second problem will be how to determine the origin, O, such that the point p can be defined. The intuitive idea is that the origin can be uniquely defined. However, it will soon be discovered that the origin need not be unique, and might even incorporate fluctuations.

7.6 The Cartan 1-form of Action as a Line Bundle

This appendix is used to describe the details of how to compute topological and geometrical properties of generalized spaces using the Cartan exterior methods. The objective is to construct Cartan's connection 1-forms, (Repere Mobile), and call attention to the fact that two types of torsion defects (both rotational and translational) can be generated on a projective manifold of dimension n+1. Although the affine translational torsion has a growing literature, the projective rotational torsion has been ignored. Yet, rotational torsion, intuitively, seems to be of more importance for hydrodynamic situations. The idea is to display the rudiments of Frames, Cartan connections, and matrices of local curvature 2-forms, which can be used in the form of Chern classes [**?**] which in turn are used to evaluate the Euler characteristic on a variety. The methods can be used to display the correspondence between the Euler characteristic and the concept of Topological Parity.

Consider a 1-form of Action on a 2n+2= 4D domain of definition given by the expression,

$$A = \lambda(x, y, z, t)\{v_k(x, y, z, t)dx^k - cdt\}. \tag{7.131}$$

At any point p of the domain, there exists 2n+1=3 vectors $\mathbf{e}_m$ of four components that are orthogonally transversal to the form in the sense that $i(\mathbf{e}_m)A = 0$. These vectors (to within an arbitrary factor) may be used as column vectors of a basis frame at the point p. The coefficient functions of the one form itself (to within an arbitrary factor) form the 2n+2 elements of a basis frame at the point p. A useful but not unique choice for a basis set at the point p is given by the expression,

$$\mathbb{F} = [\mathbf{e}_k, \mathbf{n}] = \mathbb{F} = \begin{bmatrix} 1 & 0 & 0 & -\lambda v_x \\ 0 & 1 & 0 & -\lambda v_y \\ 0 & 0 & 1 & -\lambda v_z \\ v_x/c & v_y/c & v_{/cz} & +\lambda c \end{bmatrix}. \tag{7.132}$$

The determinant of this matrix is equal $\det \mathbb{F} = c\lambda(c^2 + A_x^2 + A_y^2 + A_z^2)$, which is never zero for $\lambda > 0$. Hence this basis frame has an inverse almost everywhere. Note that the components of a 1-form, A, to within a factor, have been used construct a basis frame at almost all points of the manifold. Although the adjoint normal direction field, $\mathbf{n}$, established by the coefficients of the given 1-form, are orthogonal to the other column vectors, $\mathbf{e}_k$, in the frame, it is apparent that the $\mathbf{e}_k$ are not orthogonal to one another. The direction field, $\mathbf{n}$, could be rescaled by appropriate choice of the factor in order to force the Frame matrix to be unimodular. However, this choice, as shown below, is a severe constraint on the geometry.

The existence of the inverse matrix, of course, guarantees that the differential of every basis vector is decomposable into a linear combination of the original elements of the basis set, independent of the partition. The matrix of coefficients of this linear expansion defines the (right) Cartan matrix of connection 1-forms, $\mathbb{C}_r$,

$$d\mathbb{F} = \mathbb{F} \circ \{d\mathbb{F} \circ \mathbb{F}^{-1}\} = \mathbb{F} \circ \{-d\mathbb{F}^{-1} \circ \mathbb{F}\} = \mathbb{F} \circ \mathbb{C}_r \tag{7.133}$$

over the domain of support for the basis frame (where $\mathbb{F}^{-1}$ exists). (An alternate development would use the left Cartan matrix representation, $d\mathbb{F} = \mathbb{C}_l \circ \mathbb{F}$).

It is convenient to partition the (arbitrary) basis frame $\mathbb{F}$ in terms of the *associated* (horizontal, interior, coordinate or transversal) vectors, $\mathbf{e}_k$, and the *adjoint* (normal, exterior, parametric or vertical) field, $\mathbf{n}_p$,

$$\mathbb{F} = [\mathbf{e}_k, \mathbf{n}] = [\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{n}]. \tag{7.134}$$

The corresponding Cartan matrix has the partition,

$$d\mathbb{F} = d\begin{bmatrix} e_1^1 & e_2^1 & ... & n^1 \\ e_1^2 & e_2^2 & ... & n^2 \\ ... & ... & ... & ... \\ e_1^n & e_2^n & ... & n^n \end{bmatrix} = \mathbb{F} \circ \mathbb{C} = \mathbb{F} \circ \begin{bmatrix} \Gamma_1^1 & \Gamma_2^1 & ... & \gamma^1 \\ \Gamma_1^2 & \Gamma_2^2 & ... & \gamma^2 \\ ... & ... & ... & ... \\ h_1 & h_2 & ... & \Omega \end{bmatrix} \tag{7.135}$$

The vector equations for the differentials of the columns of the frame matrix can be written as

$$d\mathbf{e}_k = \mathbf{e}_m \Gamma_k^m + \mathbf{n} h_k \tag{7.136}$$
$$d\mathbf{n} = \mathbf{e}_m \gamma^m + \mathbf{n}\Omega, \tag{7.137}$$

which indicates a closure concept in the sense that the differentials of basis vectors are composed of linear combinations of themselves. However, it is important to note that in general the differentials of the interior vectors are not linear combinations of interior vectors and the differentials of exterior vectors are also not closed among themselves.

The Cartan matrix, $\mathbb{C}$, is a matrix of differential 1-forms which can be evaluated explicitly from the C1 functions that make up the basis frame. Moreover, the differential of the position vector can be expanded in terms of the same basis frame and a set of Pfaffian 1-forms:

$$d\mathbf{R} = \mathbb{I} \circ \left| \begin{array}{c} d\mathbf{x} \\ ... \\ d\mathbf{t} \end{array} \right\rangle = \mathbb{F} \circ \mathbb{F}^{-1} \circ \left| \begin{array}{c} d\mathbf{x} \\ ... \\ d\mathbf{t} \end{array} \right\rangle = \mathbb{F} \circ \left| \begin{array}{c} \boldsymbol{\sigma} \\ ... \\ \boldsymbol{\omega} \end{array} \right\rangle, \tag{7.138}$$

where the vector $\left| \begin{array}{c} \boldsymbol{\sigma} \\ ... \\ \boldsymbol{\omega} \end{array} \right\rangle$ is a (4 component) vector of 1-forms that can be computed explicitly. The 1-form ω need not be closed, and in many interesting cases is not integrable, $\omega\hat{}d\omega \neq 0$.

By the Poincare lemma, it follows that

$$dd\mathbb{F} = d\mathbb{F}\hat{}\mathbb{C} + \mathbb{F}\hat{}d\mathbb{C} = \mathbb{F} \circ \{\mathbb{C}\hat{}\mathbb{C}+d\mathbb{C}\} = 0, \tag{7.139}$$

and

$$dd\mathbf{R} = d\mathbb{F}\hat{} \left| \begin{array}{c} \boldsymbol{\sigma} \\ \boldsymbol{\omega} \end{array} \right\rangle + \mathbb{F} \circ \left| \begin{array}{c} d\boldsymbol{\sigma} \\ d\boldsymbol{\omega} \end{array} \right\rangle = \mathbb{F} \circ \{\mathbb{C}\hat{} \left| \begin{array}{c} \boldsymbol{\sigma} \\ \boldsymbol{\omega} \end{array} \right\rangle + \left| \begin{array}{c} d\boldsymbol{\sigma} \\ d\boldsymbol{\omega} \end{array} \right\rangle\} = 0. \tag{7.140}$$

The bracket factor

$$[\text{Cartan Curvature 2-forms}] \ = \{\mathbb{C}\hat{}\mathbb{C}+d\mathbb{C}\}, \tag{7.141}$$

defines what is called the (global) matrix of Cartan Curvature 2-forms, and the bracket factor

$$|\text{Cartan Torsion 2-forms}\rangle = \{\mathbb{C}\hat{} \left| \begin{array}{c} \boldsymbol{\sigma} \\ \boldsymbol{\omega} \end{array} \right\rangle + \left| \begin{array}{c} d\boldsymbol{\sigma} \\ d\boldsymbol{\omega} \end{array} \right\rangle\} \tag{7.142}$$

defines what is called the (global) vector Cartan Torsion 2-forms. For any Frame matrix with inverse, the global Cartan Torsion 2-forms, and the global Cartan Curvature 2-forms must vanish. However, as shown below the vectors of Cartan torsion 2-forms and the Cartan curvature 2-forms associated the interior subspace are not necessarily zero. The Torsion 2-forms , and can be decomposed into interior dislocations and exterior disclinations.

As the Frame matrix and the Cartan matrix are partitioned relative to the tangent (or interior) vectors $\mathbf{e}$ and the normal (or exterior) vectors, $\mathbf{n}$, the Poincare lemma breaks up into linearly independent factors, each of which must vanish. The results are:

$$dd\mathbf{R} = \mathbf{e}\{d\,|\boldsymbol{\sigma}\rangle + [\boldsymbol{\Gamma}]\hat{}\,|\boldsymbol{\sigma}\rangle - \omega\hat{}\,|\boldsymbol{\gamma}\rangle\} + \mathbf{n}\{d\omega + \Omega\hat{}\omega + \langle\mathbf{h}|\,\hat{}\,|\boldsymbol{\sigma}\rangle\} = 0 \tag{7.143}$$

$$dd\mathbf{e} = \mathbf{e}\{d[\mathbf{\Gamma}] + [\mathbf{\Gamma}]\hat{}[\mathbf{\Gamma}] + |\boldsymbol{\gamma}\rangle \ \hat{} \ \langle\mathbf{h}|\} + \mathbf{n}\{d\langle\mathbf{h}| + \Omega\hat{}\langle\mathbf{h}| + \langle\mathbf{h}|\hat{}[\mathbf{\Gamma}]\} = 0 \quad (7.144)$$

$$dd\mathbf{n} = \mathbf{e}\{d|\boldsymbol{\gamma}\rangle + [\mathbf{\Gamma}]\hat{}|\boldsymbol{\gamma}\rangle - \Omega\hat{}|\boldsymbol{\gamma}\rangle\} + \mathbf{n}\{d\Omega + \Omega\hat{}\Omega + \langle\mathbf{h}|\hat{}|\boldsymbol{\gamma}\rangle\} = 0 \quad (7.145)$$

By reasons of linear independence, each of the curly bracket factors must vanish, leading to the results on the interior domain (coefficients of $\mathbf{e}_k$):

$$d|\boldsymbol{\sigma}\rangle + [\mathbf{\Gamma}]\hat{}|\boldsymbol{\sigma}\rangle = \omega\hat{}|\boldsymbol{\gamma}\rangle \equiv |\mathbf{\Sigma}\rangle = \left| \begin{array}{l} \omega\hat{}\gamma^1 \\ \omega\hat{}\gamma^2 \\ \omega\hat{}\gamma^3 \end{array} \right\rangle \quad (7.146)$$

with $|\mathbf{\Sigma}\rangle$ = the interior torsion vector of dislocation 2-forms.

$$d[\mathbf{\Gamma}] + [\mathbf{\Gamma}]\hat{}[\mathbf{\Gamma}] = -|\boldsymbol{\gamma}\rangle \ \hat{} \ \langle\mathbf{h}| \equiv [\mathbf{\Theta}] = \begin{bmatrix} \gamma^1\hat{}h_1 & \gamma^1\hat{}h_2 & \gamma^1\hat{}h_3 \\ \gamma^2\hat{}h_1 & \gamma^2\hat{}h_2 & \gamma^2\hat{}h_3 \\ \gamma^3\hat{}h_1 & \gamma^3\hat{}h_2 & \gamma^3\hat{}h_3 \end{bmatrix} \quad (7.147)$$

with $[\mathbf{\Theta}]$ = the matrix of interior curvature 2-forms.

$$d|\boldsymbol{\gamma}\rangle + [\mathbf{\Gamma}]\hat{}|\boldsymbol{\gamma}\rangle = \Omega\hat{}|\boldsymbol{\gamma}\rangle \equiv |\mathbf{\Psi}\rangle = \left| \begin{array}{l} \Omega\hat{}\gamma^1 \\ \Omega\hat{}\gamma^2 \\ \Omega\hat{}\gamma^3 \end{array} \right\rangle \quad (7.148)$$

with $|\mathbf{\Psi}\rangle$ = the exterior torsion vector of disclination 2-forms.

$|\mathbf{\Psi}\rangle$ physically seems to represent a different kind of "torsion" when compared to the torsion 2-forms represented by $|\mathbf{\Sigma}\rangle$. The $|\mathbf{\Psi}\rangle$ components depend upon $|\boldsymbol{\gamma}\rangle$ and Ω, while the $|\mathbf{\Sigma}\rangle$ components depend upon upon $|\boldsymbol{\gamma}\rangle$ and ω. Kondo [**?**] developed the theory of dislocation defects based on $|\mathbf{\Sigma}\rangle$, while it appears that $|\mathbf{\Psi}\rangle$ can represent disclination defects.

The first two equations 7.146, 7.147 are precisely Cartan's equations of structure (on an affine domain). It is the last equation 7.148 of exterior disclination 2-forms, $d|\boldsymbol{\gamma}\rangle + [\mathbf{\Gamma}]\hat{}|\boldsymbol{\gamma}\rangle = \Omega\hat{}|\boldsymbol{\gamma}\rangle = |\mathbf{\Psi}\rangle$, that appears to be a new equation of structure valid on a projective domain, when $\Omega \neq 0$. The 1-form Ω, defined as the abnormality 1-form, can be interpreted in terms of a combined expansion and rotation. The components $|\boldsymbol{\gamma}\rangle$ can be interpreted in terms of rotations, while the components $\langle\mathbf{h}|$ can be interpreted in terms of translations.

Additional constraints can be imposed upon the Frame matrix, limiting the generality and application. When the normalization factor, λ, is chosen in such a way as to force the determinant of the transformation to be unity (or a constant), the abnormality 1-form Ω becomes zero. This single constraint on the determinant can be interpreted as reducing the general Cartan connection matrix to a projective Cartan matrix. In such cases, the disclination 2-forms, $|\mathbf{\Psi}\rangle$, vanish. If the arbitrary Frame matrix is locally constrained such that the Cartan connection matrix is an element of the orthogonal structure group, then Ω vanishes, and the Cartan matrix, becomes antisymmetric, with $\langle\mathbf{h}| = -|\boldsymbol{\gamma}\rangle$. There are two types of Affine Cartan matrices. The first type is an element of the matrix group where $|\boldsymbol{\gamma}\rangle = 0$. The second type of affine transformation is an element of the matrix group where $\langle\mathbf{h}| = 0$. Often the structural group is chosen as a Lie group.

A purpose of this section was to prove constructively the existence of $|\mathbf{\Psi}\rangle$, a vector of "exterior" torsion 2-forms which, it is suggested herein, should be put into correspondence with disclination defects, rotational shears and coherent structures in hydrodynamics. This vector is zero on euclidean orthonormal or affine manifolds. Another purpose was to focus attention on the Cartan matrix of curvature 2-forms.

The matrix of interior curvature two forms, $[\mathbf{\Theta}]$, can be constructed from the knowledge of connection coefficients, $d[\mathbf{\Gamma}] + [\mathbf{\Gamma}]\hat{}[\mathbf{\Gamma}]$, (which requires the use of a second differential process), or algebraically from the "outer-exterior" product of $-|\boldsymbol{\gamma}\rangle\ \hat{}\ \langle\mathbf{h}|$.

$|\mathbf{\Psi}\rangle$ physically seems to represent a different kind of "torsion" which I am trying to put into correspondence with disclination defects. Recall that Kondo has developed the theory of dislocation defects based on $|\mathbf{\Sigma}\rangle$.

There are also three equations of structure on the exterior domain (coefficients of $\mathbf{n}$) which are given by the constructions:

$$d\omega + \Omega\hat{}\omega = -\langle\mathbf{h}|\ \hat{}\ |\boldsymbol{\sigma}\rangle, \tag{7.149}$$

$$d\langle\mathbf{h}| + \Omega\hat{}\ \langle\mathbf{h}| = -\langle\mathbf{h}|\ \hat{}[\mathbf{\Gamma}], \tag{7.150}$$

$$d\Omega + \Omega\hat{}\Omega = \theta = -\langle\mathbf{h}|\ \hat{}\ |\boldsymbol{\gamma}\rangle, \tag{7.151}$$

where θ represents the exterior curvature 2-forms

A remarkable result of this construction is the fact that the matrix of interior curvature 2-forms, $[\mathbf{\Theta}]$, can be constructed in two ways. The classical method utilizes differential processes $\{d[\mathbf{\Gamma}] + [\mathbf{\Gamma}]\hat{}[\mathbf{\Gamma}]\}$, while the second method is purely algebraic $\{-|\boldsymbol{\gamma}\rangle\ \hat{}\ \langle\mathbf{h}|\}$. The order of partial derivatives contained in the algebraic (exterior) expression for the interior curvature $\{-|\boldsymbol{\gamma}\rangle\ \hat{}\ \langle\mathbf{h}|\}$ is one less than the classic expression built on the connection coefficients, $\{d[\mathbf{\Gamma}] + [\mathbf{\Gamma}]\hat{}[\mathbf{\Gamma}]\}$.

Exterior differentiation of the matrix of interior curvature 2-forms yields:

$$d[\mathbf{\Theta}] = -d\,|\boldsymbol{\gamma}\rangle\ \hat{}\ \langle\mathbf{h}| = (-\,|d\boldsymbol{\gamma}\rangle\ \hat{}\ \langle\mathbf{h}|) + (|\boldsymbol{\gamma}\rangle\ \hat{}\ \langle d\mathbf{h}|) =$$

$$([\mathbf{\Gamma}]\hat{}\ |\boldsymbol{\gamma}\rangle\ \hat{}\ \langle\mathbf{h}|) - (\Omega\hat{}\ |\boldsymbol{\gamma}\rangle\ \hat{}\ \langle\mathbf{h}|) - (|\boldsymbol{\gamma}\rangle\ \hat{}\ \Omega\hat{}\ \langle\mathbf{h}|) - (|\boldsymbol{\gamma}\rangle\ \hat{}\ \langle\mathbf{h}|\ \hat{}[\mathbf{\Gamma}]) = 0$$

The fundamental result is that the matrix of 2-forms that forms the interior curvature matrix is closed! It is this fact that leads ultimately to the idea relating the Euler characteristic and the components of the curvature 2-forms.

It is important to note that due to the partition, the exterior curvature is a closed (in this example a scalar valued) 2-form $\theta = -\langle \mathbf{h}| \hat{} |\gamma\rangle$ with

$$d\theta = -\langle d\mathbf{h}| \hat{} |\gamma\rangle + \langle \mathbf{h}| \hat{} |d\gamma\rangle = +\Omega\hat{} \langle \mathbf{h}| \hat{} |\gamma\rangle + \langle \mathbf{h}| \hat{} [\mathbf{\Gamma}]\hat{} |\gamma\rangle - \langle \mathbf{h}| \hat{} [\mathbf{\Gamma}]\hat{} |\gamma\rangle + \langle \mathbf{h}| \hat{} \Omega\hat{} |\gamma\rangle = 0,$$

is closed. As Ω is a 1-form for a single exterior vector, , then the 2-form θ is exact. The exterior exterior curvature 2-forms can generate a Maxwell-Faraday system of PDE's See Appendix B).

Both the exterior and the interior curvature 2-forms can be matrix valued depending upon the partition of the Frame. Each curvature matrix exhibits a set of similarity invariants deduced from the coefficients of the Cayley-Hamilton characteristic polynomial. It would appear therefore that their are two species of Chern characteristic classes that can be constructed from the Cayley-Hamilton polynomial similarity invariants.

If (in the example) the projective Cartan matrix is constrained to be euclidean, then $\Omega = 1$, and both $\mathbf{h} = 0$,and $\boldsymbol{\gamma} = 0$. Hence both the interior and the exterior curvature vanish. Indeed, then both types of torsion 2-forms vanish.

On the otherhand, if the Cartan matrix is anti-symmetric (as it must be for an orthonormal frame matrix) then $\Omega = 0$, and $\boldsymbol{\gamma} = -\mathbf{h}$. Hence, the exterior curvature vanishes, and $|\mathbf{\Psi}\rangle = 0$, but the domain could support interior curvature and dislocation torsion 2-forms, $|\mathbf{\Sigma}\rangle \neq 0$. If the Cartan matrix is left affine, then $\mathbf{h} = 0, \Omega = 1$. The interior and exterior domains are flat, but the structure could admit both forms of torsion 2-forms.

The moral of this appendix is that there usually is more than one way to do things. In the case of a projectivized line bundle over a variety, the Cartan method of computing the significant quantities is equivalent to the methods of fiber bundle theory, but it is much simpler to use and easier to interpret in physically useful ways for engineering applications

onelli (1993)] Antonelli, P. L., Ingardern, R.S., and Matsumoto,. M. , (1989) "The Theory of Sprays and Finsler Spaces with applications to Biology and Physics", Kluwer, Dordrecht, Nd

1. Arnold, V.I., (1989) "Mathematical Methods of Classical Mechanics", p.198, Springer Verlag, NY

old 1981] Arnold, V.I., (1981) "London Math. Soc. Lect. Note Series 53", Cambridge University Press p. 225.

lwin 1991] Baldwin, P. and Kiehn, R. M. (1991) “Cartan's Topological Structure“, Poster presented at the summer workshop, “Topology in Fluid Dynamics“, Inst. for Th. Phy. UCSB, August 1991. arXiv math-ph/0101033, http://www22.pair.com/csdc/pdf/topstru4.pdf ıberg 1992] Bamberg, P. and Sternberg, P. (1992) "A course in mathematics for students of physics, 2", Cambridge University press, Cambridg, p. 775- 780

eman 1914] Bateman, H. (1914-1955) "Electrical and Optical Wave Motion", Dover,

ger 1984] M. A. Berger, M. A. and Field, (G. B 1984). J Fluid Mech. 147, 133-148 N.Y., p.12

op 1968] Bishop, R. L. and Goldberg, S. I. (1968) "Tensor Analysis on Manifolds" Dover, N. Y. p. 199.

hke]

inov 1990] A. V. Bolsinov, S. V. Mateveev and A. T. Fomenko, Russian Math Surveys 45:2 (1990) p59-94

nov 1995] A.V. Bolsinov, A.T.Fomenko, and K. Chang, Proceedings of the Steklov Institute of Mathematics, 1995, Issue 4, p,. 29

1994] Bott. R. and Tu, L. W. (1994) "Differential Forms in Algebraic Topology", Springer Verlag, N.Y.

ouin 1964] Brillouin, L. (1964) "Tensors in Mechanics and Elasticity", Academic Press, NY p 93.

ınt 1991] Bryant, R.L.,Chern, S.S., Gardner, R.B.,Goldschmidt, H.L., and Griffiths, P. A. (1991) "Exterior Differential Systems", Springer, Verlag.

an 1976] Callan, Jr, C. G., ., Gilkey, P. B. and Hanson, A. J. (1976) Phys. Lett 63B 334. Note the phrase "This $E \cdot B \neq 0$ in the Lagrangian) clearly breaks P and T invariance.. "

ıevale 1990] Carnevale, G. F. and Vallis, G. K. (1990) Isovortical Relaxation to Stable Two-Dimensional Flows in: "Topological Fluid Mechanics", H. K. Moffatt and T. S. Tsinober eds, Cambridge University Press, Cambridgep 449-458. Also see Carnevale, G. F. and Vallis, G. K. (1989) J. Fluid Mech., 207 p.133-152

an 1899] E. Cartan "Sur certaines expressions differentielles et le systeme de Pfaff" Ann Ec. Norm. 16 329 (1899)

an 1922] E. Cartan, "Systems Differentials Exterieurs et leurs Applications Geometriques" (Hermann, Paris, 1922).

an 1937] E. Cartan, "La Theorie des Spaces a Connexion Projective", (Hermann, Paris, 1937)

an 1958 (1922)] Cartan, E., (1958) "Lecons sur les invariants integrauxs", Hermann, Paris .

an 1966] Cartan, E. (1966) "The Theory of Spinors", Dover, New York, p. 39.

mir 1948] H. B. G. Casimir, Proc. Kon. Ned. Akad. Wetensch. B51, 793 (1948)

rn 1944] Chern, S .S. (1944), Annals of Math. 45, 747- 752 .

rn 1948] Chern, S.S. (1948) Science Reports Nat. Tsing Hua Univ. 5, 95-121.

rn 1988] Chern, S.S. (1988) "Historical Remarks on Gauss-Bonnet", MSRI 04808-88 Berkeley, CA. Also see ... (1944) "A Simple intrinsic proof of the Gauss-Bonnet formula for Closed Riemannian Manifolds", Annals of Math. 45, 747- 752.

nmond 1990] I. T. Drummond, and W. H. P. Munch, "Turbulent Stretching of Lines and Surfaces", in "Topological Fluid Mechanics" H. K. Moffatt and A. Tsinober, editors, (Cambridge University Press, 1990), p. 628-636.

ington 1963 (1923)] [26] A. S. Eddington,. "The Mathematical Theory of Relativity" (Cambridge University Press 1963), p. 194.

.996] Eu, B.C. and L.S.Garcia-Colin, Phys Rev E 54,3 (1996)

ders 1963] Flanders, H. (1963) "Differential Forms", Academic Press, New York, p.126.

roptics]

: 1964] Fock, V. (1964), "Space, time and gravitation", Pergamon-MacMillan, New York.

yth (1890) 1959] Forsyth, A.R. (1890-1959), "Theory of differential equations V1 and V2", Dover, N.Y.

et 1985] B. Gaffet, J. Fluid Mech. 156, 141 (1985).

ert 1977] W. Gellert, et.al. Editors "The VNR Concise Encyclopedia of Mathematics" , (Van Nostrand, New York 1977), p.686."

ıb 1973] W. Greub, S.Halperin and R. Vanstone (Academic Press, New York , 1973).

nov]

lberg 1982] Goldberg, S. I., (1982) "Curvature and Homology" Dover, N. Y.

lenblatt 1962] Goldenblat, I.I. (1962), "Some problems of the mechanics of continuous media", Noordhoff, Holland, p.193. "The quantity of heat in a reversible process always has an integrating factor"

nann 1968] Hermann, R. (1968), "Differential Geometry and the Calculus of Variations", Academic Press, New York.

king 1961] Hocking, J. G. (1961) "Topology" Addison Wesley, N. Y. p.2.

finger]

ıig 1996] Hornig, G. and Schindler, K. (1996), Magnetic topology and the problem of its invaraint definition, Physics of Plasmas, 3, p.646

nig 1998] Hornig, G and Rastatter, L. (1998), The Magnetic Structure of B $\neq$ 0 Reconnection, Physica Scripta, Vol T74, p.34-39

inor.pdf] http://www22.pair.com/csdc/pdf/implinor.pdf and http://www22.pair.com/csdc/pdf/parametric

u 1988] Kaku, M. (1988) "Introduction to Superstrings" Springer-Verlag, New York p.386.

in 1966] Kestin, J. (1966) "A Course in Thermodynamics" Blaisdel, Waltham, Mass.

n 1962] Klein, J. (1962), Ann. Inst. AIFUB Fourier, Grenoble 12, 1.

e 1965] Kline, M. and Kay, I.W. (1965), "Electromagnetic Theory and Geometrical Optics", Interscience, N.Y.

ıine 1964] Kochin, N.E., Kibel, I.A., Roze, N. N., (1964) "Theoretical Hydrodynamics" Interscience NY p.157

ov 1995] Kozlov, V. (1995) Hydrodynamics of Noncommutative Integration of Hamiltonian Systems. Amer Math Soc Transl. (2) 168 p. 227-238

ıtowski 1948] Kuratowski, K. (1948) "Topology" (Warsaw, , Vol. I.

b 1945] Lamb, H.(1945) "Hydrodynamics" Dover, New York, p.245.

lau 1958] Landau, L. D. and Lifshitz, E. M., (1958), "Statistical Physics" Pergamon Press, London p. 369.

lau 1960] Landau, L. D. and Lifshitz, E. M. (1960), "Electrodynamics of Continuous Media", Pergamon Press, London, p. 97.

dsberg 1961] Landsberg, P. T. (1961), ”Thermodynamics” Interscience NY

ermann 1987] Libermann P. and Marle, C-M. (1987), "Symplectic Geometry and Analytical Mechanics" Riedel, Dordrecht/Boston p.65

ermann 1987] Libermann P. and Marle, C-M. (1987), "Symplectic Geometry and Analytical Mechanics" Riedel, Dordrecht/Boston p.307

chutz 1965] Lipschutz, S. (1965) "General Topology", Schaum, New York, p. 88.

le EM 2003] http://www22.pair.com/csdc/maple/MapleEM.pdf

le Fresnel 2003] http://www22.pair.com/csdc/maple/MapleFR.pdf

tinet (1970)] Martinet, J. (1970), On the Singularities of Differential Forms, Ann. Inst. Fourier, Grenoble, 20, 1 , 95-178

sden 1994] Marsden, J.E. and Riatu, T. S. (1994) "Introduction to Mechanics and Symmetry", Springer-Verlag, p.122

)uff 1995] McDuff and Salamon ”Introduction to Symplectic Topology”

erve 1893] Meserve, B. E. (1983) "Projective Geometry", Dover, New York Oxford University Press (1995)

calf 1990] R. W. Metcalfe and F. Hussain, (1990), Topology of Coherent structures and Flame Sheets in Reacting Mixing Layers, in ”Topological Fluid Mechanics” H. K. Moffatt and A. Tsinober, editors, Cambridge University Press, 659-678. Also see: M. V. Melander, and F. Hussain, Topological Aspects of Vortex Reconnection , in: loc. cit. 485-499.

fat 1969] H. K. Moffatt, J. Fluid Mech. 35,117 (1969).

tgomery 1989] D. Montgomery and L. Phillips, (1989) MHD Turbulence: relaxation processes and variational principles, in ”Advances in Fluid Turbulence” G. Doolen, et. al. editors North Holland Amsterdam, p.215

se 1964] Morse, P. M. (1964) "Thermal Physics", Benjamin, NY, p. 60

1 1989] C. Nash and S. Sen, ”Topology and Geometry for Physicists” , (Academic Press, San Diego 1989).

ill 1966] O’Neill, B. (1966) "Elementary Differential Geometry", Academic Press, New York, p.189...

rman 1986] Osserman, R. (1986) "A Survey of Minimal Surfaces", Dover, N.Y. 1986)

› 1990] A. Otto, M. Hesse, and K. Schindler (1990) General Magnetic Reconnection in 3D Systems, in ”Topological Fluid Mechanics”, H. K. Moffatt and A. Tsinober, editors, Cambridge University Press, p. 225-234.

rose 1999] Penrose, R. (1999) The central programme of Twistor Theory, Chaos, Solitons, and Fractals, 10, 2-3, p.581-611.

sics Today May 2000] Physics Today, May 2000, p.17

ie 1976] Pittie, H.V. (1976) "Characteristic classes of Foliations", Res Notes in Math., Ditman, London

(1962)] E.J. Post (1997) "Formal Structure of Electromagnetics", Dover, N.Y.

1983] E.J. Post (1983) Phys Lett. A, 94 p. 343

depudi 1998] Kondepudi, D. and Prigogine, I., "Modern Thermodynamics", Wiley, New York.

ham 1960] deRham, G. (1960) "Varietes Differentables", Hermann, Paris.

K 1969] Kiehn, R.M. and Pierce, J. F., (1969), An Intrinsic Transport Theorem, Phys. Fluids, 12, p.1971. (http://www22.pair.com/csdc/pdf/inttrans.pdf)

K 1974] Kiehn, R.M. (1974), Extensions of Hamilton's Principle to Include Dissipative Systems, J. Math Phys. 5, p. 9

K 1975 a] Kiehn, R.M. (1975), Intrinsic hydrodynamics with applications to space-time fluids, Int. J. of Eng. Sci. 13, 941. (http://www22.pair.com/csdc/pdf/inthydro.pdf) arXiv/math-ph/0101032

K 1975 b] Kiehn, R. M., (1975), Geometrodynamics and Spin", Lett. al Nuovo Cimento 12, p. 300

K 1975 c] Kiehn, R. M. (1975), Submersive equivalence classes for metric fields, Lett. al Nuovo Cimento 14, p. 308

K 1975 d] Kiehn, R.M. (1975), Conformal Invariance and Hamilton Jacobi theory for dissipative systems, J. Math Phys. 16, 1032

K1976] Kiehn, R.M. (1976), NASA AMES NCA-2-OR-295-502 (http://www22.pair.com/csdc/pdf/rmkames.pdf)

K 1976 b] Kiehn, R.M. (1976), Retrodictive Determinism, Int. J. of Eng. Sci. 14, 749 . (http://www22.pair.com/csdc/pdf/retrodic.pdf)

K 1977] Kiehn, R.M. (1977), Periods on manifolds, quantization and gauge, J. Math Phy. 18, 614. (http://www22.pair.com/csdc/pdf/periods.pdf)

K 1977 b] Sanders, V. E. and Kiehn, R.M. (1977), Dual Polarized Ring Lasers, IEEE J Quant Elec QE-13, p. 739

K 1979] Schultz, A., Kiehn, R. M., et al. (1979), Lifting of the Four-fold EM Degeneracy and PT Symmetry, Phys Lett 74A, 384.

K 1986] Kiehn, R. M., (1987) The Falaco Effect as a topological defect was first noticed by the present author in the swimming pool of an old MIT friend, during a visit in Rio de Janeiro, at the time of Halley's comet, March 1986. The concept was presented at the Austin Meeting of Dynamic Days in Austin, January 1987, and caused some interest among the resident topologists. The easily reproduced experiment added to the credence of topological defects in fluids. It is now perceived that this topological phenomena is universal, and will appear at all levels from the microscopic to the galactic. LANL arXiv/gr-qc/0101098 (http://www22.pair.com/csdc/pdf/falaco97.pdf)., see http://www22.pair.com/csdc/pdf/topturb.pdf

K 1987 b] Sterling . et.al. (1987), Why are these disks dark? Phys Fluids 30 p.11

K 1987] Kiehn, R. M. (1987), Falaco Solitons, http://www22.pair.com/csdc/pdf/falaco01.pdf

K 1989] Kiehn, R. M., (1989), Irreversible Topological Evolution in Fluid Mechanics, ASME 89-WA/FE-5. Invited panelist "Unanswered questions in Fluid Mechanics" at the SAF meeting of the ASME on

K 1990] Kiehn, R.M. (1990), Topological Torsion, Pfaff Dimension and Coherent Structures, in: "Topological Fluid Mechanics", H. K. Moffatt and T. S. Tsinober eds, Cambridge

University Press p. 449-458. (http://www22.pair.com/csdc/pdf/camb89.pdf)

K 1991 a] Kiehn, R. M. (1991), Compact Dissipative Flow Structures with Topological Coherence Embedded in Eulerian Environments, in: "Non-linear Dynamics of Structures", edited by R.Z. Sagdeev, U. Frisch, F. Hussain, S. S. Moiseev and N. S. Erokhin, p.139-164, World Scientific Press, Singapore

K1991 b] Kiehn, R.M. (1991) “Continuous Topological Evolution“, LANL arXiv/math-ph/0101032 (http://www22.pair.com/csdc/pdf/contevol3.pdf)

K 1991 c] Kiehn, R. M. (1991), Are there three kinds of superconductivity, Int. Journ. of Modern Physics, vol5 #10 p. 1779

K 1991 d] Kiehn, R. M. , Kiehn, G. P., and Roberds, (1991), Parity and time-reversal symmetry breaking, singular solutions and Fresnel surfaces, Phys. Rev A 43, pp. 5165-5671. (http://www22.pair.com/csdc/pdf/timerev.pdf)

K 1991 e] Kiehn, R.M. (1991) Some Closed Form Solutions to the Navier Stokes Equations, LANL arXiv/physics/0102002 (http://www22.pair.com/csdc/pdf/nvsol.pdf)

K 1991 f] Kiehn, R.M. (1991), ”Dissipation, Irreversibility and Symplectic Lagrangian systems on Thermodynamic space of dimension 2n+2”, http://www22.pair.com/csdc/pdf/irrev1.pdf

K 1992 a] Kiehn, R. M., Topological Parity and the Turbulent State, unpublished.

K 1992 d] Kiehn, R. M. (1992), Topological Defects, Coherent Structures and Turbulence in Terms of Cartan’s Theory of Differential Topology, in "Developments in Theoretical and Applied Mathematics, Proceedings of the SECTAM XVI conference", B. N. Antar, R. Engels, A.A. Prinaris and T. H. Moulden, Editors, The University of Tennessee Space Institute, Tullahoma, TN 37388 USA.

K 1993 a] Kiehn, R. M. (1993), Instability Patterns, Wakes and Topological Limit Sets, in ”Eddy Structure Identification in Free Turbulent Shear Flows”, J.P.Bonnet and M.N. Glauser, (eds), Kluwer Academic Publishers p. 363.

K 1993 b] Kiehn, R. M. (1993) Talk presented at the EUROMECH 308 meeting at Cortona, Italy. Unpublished. (http://www22.pair.com/csdc/pdf/italy.pdf)

K 1995] Kiehn, R. M. (1995), Hydrodynamic Wakes and Minimal Surfaces with Fractal Boundaries, in "Mixing in Geophysical Flows", J. M. Redondo and O. Metais, Editors CIMNE , Barcelona ERCOFTAC (meeting in 1992) p. 52-63.

K 1999 a] Kiehn, R. M., (1999), Coherent Structures in Fluids are Topological Torsion Defects, in J, ”IUTAM Symposium on Simulation and Identification of Organized Structures in Flows”, .N. Sørensen, et al., eds., Kluwer Academic Publishers, Dordrecht, . See (http://www22.pair.com/csdc/pdf/copen5.pdf). Presented at the IUTAM-SIMFLO Conference at DTU, Denmark, May 25-29, (1997).

K 1999 b] Kiehn, R.M. (1999), Topological evolution of classical electromagnetic fields and the photon, in, "Photon and Poincaré Group", V. Dvoeglazov (ed.), Nova Science Publishers, Inc., Commack, New York, p.246-262. ISBN 1-56072-718-7. Also see (http://www22.pair.com/csdc/pdf/photon5.pdf)

K 2000] Kiehn, R. M. (2000), 2D turbulence is a Myth, (invited speaker IUTAM conf) http://www22.pair.com/csdc/pdf/hague6.pdf

K 2001] Kiehn, R.M. , (2001) Topological-Torsion and Topological-Spin as coherent structures in plasmas", arXiv physics /0102001. Also see http://www22.pair.com/csdc/pdf/plasmas.pdf, and ..holder3d.pdf)

K 2002 a] Kiehn, R. M. (2002), Curvature and torsion of implicit hypersurfaces and the origin of charge, Annales de la Foundation Louis de Broglie, vol 27, p.411. LANL arXiv /gr-qc /0101109 (http://www22.pair.com/csdc/pdf/contevol3.pdf)

K 2002 b] Kiehn, R.M. (2002), The Photon Spin and other Topological Features of Classical Electromagnetism, in "Gravitation and Cosmology, From the Hubble Radius to the Planck Scale", Amoroso, R., et al., eds., Kluwer, Dordrecht, Netherlands, p.197-206. Vigier 3 conference in 2000. (http://www22.pair.com/csdc/pdf/vig2000.pdf)

K 2003] Kiehn, R. M. (2003), Thermodynamic Irreversibility and the Arrow of Time, in "The Nature of Time: Geometry, Physics and Perception", R. Bucher et al. (eds.), Kluwer, Dordrecht, Netherlands, p.243-250. (http://www22.pair.com/csdc/pdf/arwfinal.pdf)

K 2004] Kiehn, R. M. (2004?), A topological perspective of Electromagnetism, to be published in "Classical electrodynamics: new horizons", Editors Andrew E. Chubykalo and Roman Smirnov-Rueda, Rinton Press, Inc. USA. Also see http://www22.pair.com/csdc/pdf/rmktop.pdf

inson 1996] Robinson, D.C and Shadduck, W. F., (1996) Fields Institute Communications, Vol 7, 1996 p189.

uten 1949] Schouten, J. A. and Van der Kulk, W. (1949) "Pfaff's Problem and its Generalizations", Oxford Clarendon Press.

odzinsky 1970] W. Slebodzinsky, (1970) "Exterior Forms and their Applications", PWN, Warsaw.

ster 1984] Schuster, H, G. (1984), "Deterministic Chaos", Physics Verlag

merfeld 1952] Sommerfeld, A. (1952), "Electrodynamics", Academic Press, New York. Sommerfeld carefully distingushes between intensities and excitations on thermodynamic grounds.

tton 1941] Stratton, J.A., (1941) Electromagnetic Theory McGraw Hill N.Y.

ik1961] Struik, D. (1961) "Differential Geometry", Addison Wesley, Reading, Mass

e 1993] Stuke, B. (1993), "Towards a Fundamental Structure of Continuum mechanics", Z. Naturforsch. 48 a 883-894

990] Su, W., Liu W., and Liu Z. (1990), Topological Structures of Separated Flows, in: "Topological Fluid Mechanics", H. K. Moffatt and T. S.. Tsinober eds, Cambridge University Press, 395-407.

m 1989] P. L. Sulem, Z. S. She, H. Scholl, and U. Frisch, J. Fluid Mech 205 341 (1989).

a 1966] L. Tisza, L. (1966) "Generalized Thermodynamics", MIT press, Cambridge, p .125.

kvist 1997] Tornkvist, O. and Schroeder, E., (1997) Vortex dynamics in Dissipative Systems, Phys Rev. Lett. 78, 10 , p. 1908-1911.

bull 1960] H. W. Turnbull, H. W. (1960) "The Theory of determinants, matrices and invariants" Dover, New York (1934)

1991] Tur, A. V. and Yanovsky, V. V. (1991) "Invariants in dissipationless hydrodynamic media“, Accepted for publication in J. Fluid Mech.

lantzig 1934] Van Dantzig. D. (1934), Proc. Cambridge Philos. Soc. 30, 421. Also see: von Dantzig, D., (1934), Electromagnetism Independent of metrical geometry, Proc. Kon. Ned. Akad. v. Wet. 37

vik 2003] Volovik, G. E., (2003) "The Universe in a Helium droplet", Claredon Press, Oxford, 2003

nhold 1975] Weinhold,(1975), Metric geometry of Thermodynamics I-IV, J. Chem. Phys. 63 2479,2484,2488,2496.

1989] Wen, X.G., Wilczek, F., and Zee, A., Phys Rev B 39 11413 (1989).

ttaker 1944] Whittaker, E.T. (1944) "Analytical Dynamics", Dover, New York .

ke 1975] Yorke, J. (1975), American Mathematical Monthly, 82, 985

znik 1976] Zeleznik, F. J. (1976) Thermodynamics, J. Math Phys. 17, 8. p.1579-1610

omirski (1991)] Zhitomirski, M. "Typical Singularities and Differential 1-forms and Pfaffian Equations", Translations of Mathematical Monographs, 113, AMS Rhode Island 1991