Continuous Topological Evolution

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Abstract

A non-statistical theory of continuous, but irreversible evolution can be constructed in terms of the Cartan calculus. The fundamental postulate for an evolutionary theory which admits irreversible processes is that the topology of the initial state will be different from the topology of the final state. Several fundamental theorems of uniformly continuous evolution are established, yielding a set of global conservation laws for reversible and irreversible processes. As examples, a comparison of the evolution of Topological Torsion and Topological Action is made for hydrodynamic and electromagnetic systems. The relationship between the evolution of Topological Torsion and a thermodynamically irreversible process is established.

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1. Introduction

An objective of this article is to develop a theory of topological evolution that may be used to describe the irreversible evolution of dissipative non-conservative physical systems. The ideas will utilize topological concepts for it is postulated that a necessary condition for irreversible evolution involves topological change. The basis for such a postulate follows from the fact that if an evolutionary process is described by a map, Φ, between initial and final states, and if the map is not continuously reversible, then the observable topology of the final state is different from the observable topology of the initial state. Cartan's methods can be used to extend these concepts to the dynamics of physical systems that admit description in terms of exterior differential forms. It is remarkable that the mathematical development leads to recognizable thermodynamic features which permit the determination of classes of processes which are reversible or irreversible. For example, all Hamiltonian processes are thermodynamically reversible. An essential feature of irreversible processes is that they involve the evolution of what has been defined as Topological Torsion.

The observation of topological change, with the production and destruction of defects and holes, lines of self-intersection and other obstructions, will be the signature of topological irreversible evolution. Topological change can occur discontinuously as in a cutting process, or continuously, as in a pasting process. Such continuous but irreversible processes can be used to study the decay of turbulence, but not its creation. The production of disconnected components will be the signature of those discontinuous processes which are necessary to describe the creation and evolution of chaotic but perhaps reversible evolution, or turbulent, irreversible evolution. In this article, emphasis will be place upon those processes which are continuous, but not reversible.

Processes or maps that preserve topology are technically described as homeomorphisms [1]. Homeomorphisms are both continuous and reversible. Homeomorphic reversibility means that the inverse function, Φ^{-1}, must exist and must be continuous. Topological properties, such as orientability, compactness, connectivity, hole count, lines of self-intersection, pinch points, and Pfaff dimension are invariants of homeomorphisms, but geometrical properties such as size and shape are not necessarily invariants of homeomorphic deformations. In fact an elementary method of recognizing topological properties is to observe those properties that stay the same under continuous deformations that do not preserve size and shape.

The theory of Continuous Topological Evolution is developed herein in terms of physical systems that undergo certain thermodynamic processes. The physical system is assumed to be modeled in terms of the topological features inherent in Cartan's theory of exterior differential systems. The thermodynamic process will be defined in terms of a vector field, V, and its effect on the differential forms that make up the exterior differential system. The action of the process will be defined in terms of the Lie differential with respect to V acting on the differential forms that make up the exterior differential system, and which in turn approximate the physical system. The methods lead to concepts that are coordinate free and are well behaved in any reference system. A precise non-statistical definition of thermodynamic irreversibility will be stated, and a cohomological equivalent of the first law of thermodynamics will be derived and studied relative to the single constraint of continuous but irreversible topological evolution. Remarkably, many intuitive thermodynamic concepts can be stated precisely, without the use of statistics, in terms of the theory of continuous topological evolution based on the Cartan topology.

Given a topology on the final state and a map from an initial state to the final state it is always possible to define a topology on the initial state such that the given transformation, or even a given set of transformations, is continuous. However, the topologies of the initial and final states need not be the same; hence the map need not be reversible. Recall that with respect to a discrete topology all maps from the initial to final state are continuous, while relative to the concrete topology, only the constant functions are continuous [2]. A first problem of a theory of topological evolution is to devise a rule for constructing a topology that is physically useful and yet is neither too coarse nor too fine. Such a rule is necessary for the concept of continuity of an evolutionary transformation is defined relative to the topologies of the initial and final states. In this article the topological rules will be made by the specification of an exterior differential system that will model the physical system of interest. Many physical systems appear to be adequately modeled by 1-form of Action.

Physical exhibitions of continuous and discontinuous transformations can be achieved through the deformations of a soap film attached to a wire frame. For example, a soap film attached to a single closed, but double, loop of wire can be deformed from a non-orientable surface into an orientable surface continuously (the topological property of orientability is changed). That is, the soap film can be transformed continously from a Moebius band into a cylindrical strip. As another example, consider an initial state where a soap film is attached to two slightly separated but concentric circular wire loops. The resulting surface is a minimal surface of a single component. As the separation distance of the concentric rings forming the boundary of the soap film is slowly increased, the minimal surface is stretched until a critical separation is reached. Then, without further displacement, the surface spontaneously continues to deform to form a "two sheeted" cone connected at a singular vertex point. The surface separates at the conical singularity, and the two separate sheets of the cone continue to collapse to form a minimal surface of two components. The final state consists of two flat films attached, one each, to each ring. The originally connected minimal surface undergoes a topological (phase) change to where it becomes two disconnected

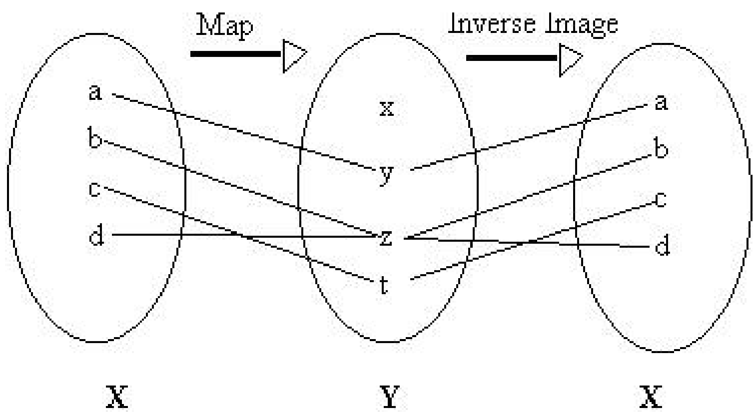

Figure 2.1:

(still minimal) surfaces. An example of this topological transition in the surface of null helicity density has been described in conjunction with the parametric saddle node Hopf bifurcation of a Navier-Stokes flow [3].

In this article the fundamental set, X, will be the points $\{x, y, z, t, ...\}$ that make up an N-dimensional space. Upon this fundamental set will be constructed arbitrary subsets, such as functions, tensor fields and differential forms. Many different topologies may be constructed on the fundamental set in terms of special classes of subsets that obey certain rules of logical closure. In fact the very existence of subsets can be used to define a course topology on X in terms of a topological base. The topological base consists of those subsets whose unions form a special collection of all possible subsets that is closed under logical union and intersection. This special collection of subsets will be defined as the open sets of a topology. The topological base can be used to define a topological structure. A space is said to have a topological structure if it is possible to determine if a transformation on the space is continuous [4].

2. Continuity

The classic definition [5] of a continuous transformation between a set X with topology $T1$ to a set Y with a topology $T2$ states that the transformation is continuous if and only if the inverse image of open sets of $T2$ are open sets of $T1$. This definition can be made transparent by use of a simple point set example.

Consider two sets of 4 points, an initial state, $\{a, b, c, d\}$ and a final state $\{x, y, z, t\}$. Define an open set topology on the initial state $T1 = [X, \emptyset, a, ab, abc]$ and a open set topology on the final state $T2 = [X, \emptyset, x, y, xy, yzt]$. The transformation considered is exemplified by the Figure 1.

The open set (y) has a preimage (a) which is open. The open set (yzt) has a preimage (abc) which is open. Hence the Map is continuous. (The open sets that involve x are not included as the map does involve x.) However, the Inverse Image mapping is not continuous for the open set (ab) has a preimage as (yz) but (yz) is not an open set of Y. The point set example demonstrates the idea of a continuous but not homeomorphic mapping. The objective herein is to examine such maps in terms of exterior differential systems.

There exists another more useful method of defining continuity which does not depend explicitly on being able to define open sets and their inverse images. This second method of defining continuity is based on the concept of closure. The closure of a set can be defined in (at least) two ways:

1. The closure of a set is the union of the interior and the boundary of a subset.
2. The closure of a set is the union of the set and its limit points.

The first definition of closure is perhaps the most common, and is often exploited in geometric situations, where a metric has been defined and a boundary can be computed easily. The second definition of closure is independent from metric and is the method of choice in this article, both for defining continuity and establishing a topological structure. In terms of the concept of closure, a transformation is continuous if and only if for every subset, the image of the closure of the initial subset is included in the closure of the image of that subset [5]. Another way of stating this idea is

3. A map is continuous iff the limit points of every subset in the domain permute into the closure of the subsets in the range.

If a method for constructing a closure operator (a Kuratowski closure operator K of a subset relative to a topology) can be defined, then a strong version of continuity would imply that the Kuratowski closure operator commutes with those transformations which are continuous. The test for continuity would be to construct the closure of an arbitrary subset on the initial state, and then to propagate the elements of the closure to the final state by means of a transformation. If this result is the same as the result obtained by first propagating the subset to the final state by means of the transformation, and then constructing its closure on the final state, then the map is continuous. Note that such a procedure has defined a topological structure which will be exploited in this article, for the subsets of interest will be defined as a Cartan system of exterior differential forms, Σ, on X. The topological base defined by this class of sets is too course to be of interest. Hence the Cartan exterior derivative will be used to generate additional sets of forms, $d\Sigma$, which when adjoined to the initial system of forms defines the Kuratowski closure of the Cartan system as the system of forms, $K(\Sigma) = \{\Sigma \cup d\Sigma\}$.

The Cartan exterior product may be used as a convenient intersection operator between sets of differential forms. Starting from the system, $\{\Sigma\}$, the Cartan topology is then determined by the construction of the Cartan-Pfaff sequence, which consists of all possible intersections that may be constructed from the subsets of the closure of the differential system:

$$Pfaff\ Sequence: \ \{\Sigma, d\Sigma, \Sigma\hat{}d\Sigma, d\Sigma\hat{}d\Sigma, ...\}. \tag{2.1}$$

The subsets of the Cartan topological space consist of all possible unions of the subsets that make up the Pfaff sequence. The Cartan topology will be constructed from a topological basis which consists of the odd elements of the Pfaff sequence, and their closures:

$$the\ Cartan\ topological\ base: \{\Sigma, K(\Sigma), \Sigma\hat{}d\Sigma, K(\Sigma\hat{}d\Sigma), ...\}. \tag{2.2}$$

With respect to a topological base constructed from a single 1-form of Action it has been shown [34] that the Cartan exterior derivative may be viewed as a closure or limit point operator. Given any subset of the Cartan topological space, the exterior derivative of that subset generates its limit points, if any. This is a remarkable result, for as will be demonstrated below, all C2 vector fields acting through the concept of the Lie differential on a set of differential forms, with C2 coefficients, generate continuous transformations with respect to the Cartan topology. Moreover, the Cartan topology is disconnected if $\Sigma\hat{}d\Sigma \neq 0$ is not zero.

3. The evolutionary process

An arbitrary evolutionary process, $X \Rightarrow Y$, is defined by a map Φ. The map, Φ, may be viewed as a propagator that takes the initial state, X, into the final state, Y. In this article the evolutionary processes to be studied are asserted to be generated by vector fields, $\mathbf{V}$. However, evolutionary vector fields need not be topologically constrained such that they are generators of a single paramenter group. In other words, kinematics without fluctuations is not imposed a priori. The local trajectories defined by the vector fields may be viewed as propagators that carry domains into ranges in the manner of a convective fluid flow. The evolutionary propagator of interest to this article is the Lie differential with respect to a vector field ,$\mathbf{V}$, acting on differential forms, Σ [6]. The Lie differential has a number of interesting and useful properties.

1. The Lie differential does not depend upon a metric or a connection.
2. The Lie differential has a simple action on differential forms producing a resultant form that is decomposed into a transversal and an exact part:

   $$L_{(\mathbf{V})}\Sigma = i(\mathbf{V})d\Sigma + di(\mathbf{V})\Sigma. \tag{3.1}$$

   Marsden [43] calls this Cartan's Magic Formula (see below).
3. The Lie differential may be used to describe deformations and topological evolution.
4. If the Lie differential of Σ is zero, then Σ is a (Bernouilli type) invariant along the flow trajectories generated by $\mathbf{V}$.
5. With respect to vector fields and forms constructed over C2 functions, the Lie differential commutes with the Kuratowski closure operator. Hence, the Lie differential generates transformations on differential forms which are continuous with respect to the Cartan topology.

For example, the action of the Lie differential on a 0-form (scalar function) is the same as the directional derivative of ordinary calculus,

$$L_{(\mathbf{V})}\varphi = i(\mathbf{V})d\varphi + 0 \Rightarrow \mathbf{V}\cdot grad\varphi. \tag{3.2}$$

3.1. The Covariant derivative vs. the Lie differential.

The covariant derivative of tensor analysis, and as used in General Relativity, is often defined in terms of isometric diffeomorphic processes (that preserve the differential line element) and can be used to describe rigid body motions and isometric bendings, but not deformations and shear processes associated with convective fluid flow. Another definition of the covariant derivative is based on the concept of a connection, such that the differential process acting on a tensor produces a tensor. The definition of the covariant derivative usually depends upon the additional structure (or constraint) of a metric or a connection placed on a given variety, while the Lie differential does not. As the Lie differential is not so constrained, it may be used to describe non-diffeomorphic processes for which the topology changes continuously. The covariant derivative is avoided in this article

In the examples given below, it will be demonstrated that the action of the Lie differential on a 1-form of Action typically will generate hydrodynamic equations of motion. As mentioned above, the Lie differential is not the same as the classic metric dependent covariant derivative (based upon Christoffel symbols), or generalizations of the metric connection used in certain gauge or fiber bundle theories. The abstract reason is that the Lie differential satisfies the equations

$$L_{(f\mathbf{V})}\Sigma = f \cdot L_{(\mathbf{V})}\Sigma + df \, \hat{} \, i(\mathbf{V})\Sigma, \tag{3.3}$$

while the covariant derivative, $\mathcal{D}$, and its generalizations are constrained [12] such that the second term on the right vanishes:

$$\mathcal{D}_{(f\mathbf{V})}\Sigma = f \cdot \mathcal{D}_{(\mathbf{V})}\Sigma. \tag{3.4}$$

This latter equation is often interpreted by saying that f represents the action of some "group", and the covariant derivative is defined such that it commutes with the action of the group. The Lie differential is not limited to the constraint of a specified group. However, there may exist a special sub-class of vector fields relative to a specific differential form, Σ, that permit the Lie differential to be identified with a covariant derivative. This special class of vector fields are called associated vectors (relative to the exterior differential form Σ), and are defined by the equation,

$$\text{Class of associated vectors}: \quad i(\mathbf{V})\Sigma = 0. \tag{3.5}$$

Those vector fields that satisfy $i(\mathbf{v})d\Sigma = 0$ are defined as *extremal* vector fields relative to Σ, a term that comes from the calculus of variations and its close correspondence to evolution defined by the Lie differential.

$$\text{Class of extremal vectors}: \quad i(\mathbf{V})d\Sigma = 0. \tag{3.6}$$

Vector fields that are both extremal and associated are defined as characteristic vector fields.

$$\text{Class of characterisic vectors}: \quad i(\mathbf{V})\Sigma = 0 \quad \text{and} \quad i(\mathbf{V})d\Sigma = 0 \tag{3.7}$$

Characteristic vector fields admit propagating discontinuities, which form the precise definition of a signal in electromagnetism [Fock Luneberg]

3.2. The Lie differential and continuity

The first four properties of the Lie differential appear in the literature, but the extraordinary property that all C2 vector fields that propagate C2 differential forms in the manner of a convective flow (Lie differential) are continuous relative to the Cartan topology requires proof: Given Σ, first construct the closure, $\Sigma \cup d\Sigma$. Next propagate Σ and $d\Sigma$ by means of the Lie differential to produce the decremental or residue forms, say Q and Z,

$$L_{(\mathbf{V})}\Sigma = Q \quad and \quad L_{(\mathbf{V})}d\Sigma = Z. \tag{3.8}$$

Now compute the contributions to the closure of the final state as given by $Q \cup dQ$. If $Z = dQ$, then the closure of the initial state is propagated into the closure of the final state, and the evolutionary process defined by $\mathbf{V}$ is continuous. However,

$$dQ = dL_{(\mathbf{V})}\Sigma = di(\mathbf{V})d\Sigma + dd(i(\mathbf{V})\Sigma) \tag{3.9}$$

and

$$Z = L_{(\mathbf{V})}d\Sigma = (i(\mathbf{V})dd\Sigma) + di(\mathbf{V})d\Sigma. \tag{3.10}$$

The difference becomes

$$Z - dQ = (i(\mathbf{V})dd\Sigma) - dd(i(\mathbf{V})\Sigma). \tag{3.11}$$

The concept of continuity requires that $Z - dQ \Rightarrow 0,$ forming an exterior differential system. For vector fields and differential forms with coefficient functions that are twice differentiable, the continuity condition is always satisfied relative to the Cartan topology (the Poincare lemma states that $dd\omega = 0$ where ω is any differential p-form with C2 coefficients). Therefore subject to the constraint of C2 differentiability, every vector field, $\mathbf{V}$, generates a continuous evolutionary process relative to the Cartan topology. The set $\{\Sigma, d\Sigma\}$ forms a differential ideal (closure) which is permuted into the differential ideal $\{Q, dQ\}$ by the action of the Lie differential with respect to $\mathbf{V}$. QED.

The Lie differential also can be used to make some sense out of certain classes of discontinuous evolutionary processes (which are not C2). For example, consider a vector field $\mathbf{V} = \rho\mathbf{v}$ where the support function, ρ, is not C2. Then, the action of the Lie differential produces the discontinuity or excess function,

$$Z - dQ = -dd(i(\rho\mathbf{v})\Sigma) = d(d\rho\hat{}(i(\mathbf{v})\Sigma) + d\rho\hat{}d(i(\mathbf{v})\Sigma). \tag{3.12}$$

This equation is of use in the study of shock waves and other discontinuous processes.

Note that special situation arise when $(i(\mathbf{v})\Sigma) = 0$. Such special vector fields were defined above to be *associated* vector fields, and have the properties that the Lie differential has the same abstract form as the covariant derivative. It can be shown that for even dimensional symplectic manifolds, there is a unique vector direction field that satisfies $i(\mathbf{T})\Sigma = 0$ and $L_{(\mathbf{V})}\Sigma = \Gamma\ \Sigma$. This direction field will generate thermodynamically irreversible evolution, and is continuous if C0.

4. Topological Evolution

4.1. Evolutionary Invariants.

If the flow field generated by $\mathbf{V}$ acting on a Cartan system of forms satisfies the equations

$$L_{(\mathbf{V})}\Sigma = 0 \quad \mathit{and} \tag{4.1}$$

$$L_{(\mathbf{V})}d\Sigma = 0. \tag{4.2}$$

then, with respect to such evolutionary processes, the forms of the closure are said to be absolute invariants. It follows that each element that makes up the Cartan topological base is also invariant, such that the whole Cartan topology is invariant. As $\mathbf{V}$ is continuous, and the topology is preserved, those vector fields, $\mathbf{V}$, that satisfy the equations above must be homeomorphisms, and are reversible. In other words, $Q = 0$ and $dQ = 0$ are sufficient conditions that $\mathbf{V}$ be reversible.

However, for continuous transformations on the elements of the C2 Cartan topology the general equations of topological evolution become,

$$L_{(\mathbf{V})}\Sigma = Q \tag{4.3}$$

and

$$L_{(\mathbf{V})}d\Sigma = dQ, \tag{4.4}$$

from which it follows that

$$L_{(\mathbf{V})}\Sigma\hat{}d\Sigma = Q\hat{}d\Sigma + \Sigma\hat{}dQ \tag{4.5}$$

and

$$L_{(\mathbf{V})}d\Sigma\hat{}d\Sigma = 2dQ\hat{}d\Sigma. \tag{4.6}$$

As these equations of continuous topological evolution imply that the elements of the topological base may not be constant, then specific tests must be made to determine what features of the topology are changing, if any. For if it can be determined that the topology is indeed modified by the evolutionary process, then the process generated by this class of vector fields, $\mathbf{V}$, is continuous, but need not be reversible.

When $dQ \neq 0$, the limit points are not invariants, and it would be natural to expect that the topology is not constant. However, even if Q is closed, such that $dQ = 0$, it may be true that Q contains harmonic components, such that DeRham cohomological classes of Σ are not evolutionary invariants. Even though the topology of the initial state is not the same as the topology of the final state (for the "hole" count of the initial state is not the same as the hole count of the final state) it is not necessarily true that such continuous processes are thermodynamically irreversible.

4.2. Deformation Invariants.

Consider the flow lines tangent to a given vector direction field, $\mathbf{V}(x, y, z, t...)$ that generates a dynamical system, $d\mathbf{r} - \mathbf{V}d\tau = 0$. By reparameterization, $\mathbf{V} \Rightarrow \beta(x, y, z, t...)\mathbf{V}$, the "speed" at which points move down the lines of flow can be changed, but the points that start on a particular flow line, remain upon the same flow line. Next consider a closed curve, Z1, intersecting the flow lines transversely for say $\tau = 0$. The flow lines that intersect Z1 form a "tube of trajectories" As τ increases to some value, say $\tau = 1$, the points of the closed curve appear to flow down the "tube of trajectories". The result of this convective evolution is to produce a new closed curve, Z2 Now choose another parameterizarion function β‘, which is equal to the original β at $\tau = 0$. The points that make up the closed curve Z1 now flow down the same tube of trajectories, but at $\tau = 1$ form a new closed curve $deformed\ Z2$ that may be considered as a deformation of the closed curve Z2.

Next consider the propagation by means of the Lie differential relative to the direction field, $\beta\mathbf{V}$, of the closed integral of a 1-form, $\int_{z1} A$. The integration

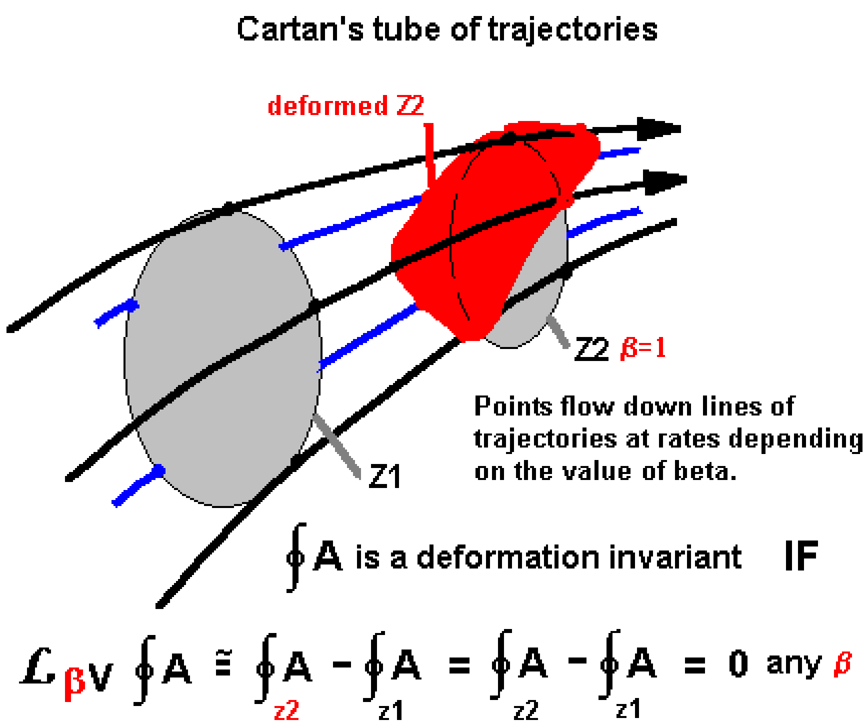
Cartan's tube of trajectories
deformed Z2
Z2 β=1
Z1
Points flow down lines of trajectories at rates depending on the value of beta.
$\oint A$ is a deformation invariant IF
$\mathcal{L}_{\beta V} \oint A \cong \oint_{z2} A - \oint_{z1} A = \oint_{z2} A - \oint_{z1} A = 0$ any β

Figure 4.1:

chain $z1$ is defined as a 1 dimensional cycle, or a closed curve of points. The action of the Lie differential on the closed integral of action can be written, for arbitrary parameterization, as:

$$L_{(\beta\mathbf{V})}\int_{z1}A=\int_{z1}i(\beta\mathbf{V})dA+\int_{z1}di(\beta\mathbf{V})A=\int_{z1}i(\beta\mathbf{V})dA+0=\int_{z1}\beta i(\mathbf{V})dA, \tag{4.7}$$

It follows that if the term, $\beta i(\mathbf{V})dA$, is closed, such that $d(\beta i(\mathbf{V})dA)=0$, then the Lie differential of the closed integral vanishes, and does not depend upon the choice of β. In such special cases, the closed integral may be viewed as a deformation invariant, and becomes a topological invariant property of the evolution.

The same arguments may be used to deduce topological properties of arbitrary p-forms. For example consider the 2-form $F=dA$. Then the Lie derivative of the closed integral of F becomes

$$L_{(\beta\mathbf{V})}\int_{z2}F=\int_{z2}i(\beta\mathbf{V})dF+\int_{z1}di(\beta\mathbf{V})F=\int_{z1}i(\beta\mathbf{V})ddA+0=0. \tag{4.8}$$

The result (for C2 functions) is zero for any evolutionary vector field acting on the closed integral of a closed p-form (in this case, the 2-form F is exact.). Hence the closed integrals of closed p-forms are deformation invariants, or topological properties, of the evolutionary process generated by $\beta\mathbf{V}$. The values of the closed integrals (deRham period integrals) depend upon the integration chains, and have ratios which are rational. Cartan developed these methods to prove that the necessary and sufficient condition that a vector field have a Hamiltonian generator, was that the closed integral of the Action 1-form was a deformation invariant. [14]

5. Simple Systems

5.1. The Action 1-form and its Pfaff Sequence

Consider an arbitrary 1-form, A, on an n dimensional variety of independent functions. The exterior derivative of A produces a 2-form of closure points, $F=dA$, whose components are given by the expression, $F_{\mu\nu}dx^{\mu}\,\hat{}\,dx^{\nu}$. The combined set $\{A,F\}$ forms the closure of the set $\{A\}$. All possible intersections of the closure, $\{A,F,A\hat{}F,F\hat{}F...\}$, form what is defined herein as the Pfaff sequence for the domain $\{x,y,z,t\}$. In this article (for a 4 dimensional variety) these elements are defined as

$$Topo\log ical\ \ ACTION : A = A_\mu dx^\mu \tag{5.1}$$

$$Topo\log ical\ \ VORTICITY : F = dA = F_{\mu\nu} dx^\mu\,\hat{}\,dx^\nu \tag{5.2}$$

$$Topo\log ical\ \ TORSION : H = A\hat{}dA = H_{\mu\nu\sigma} dx^\mu\,\hat{}\,dx^\nu\hat{}dx^\sigma \tag{5.3}$$

$$Topo\log ical\ \ PARITY : K = dA\hat{}dA = K_{\mu\nu\sigma\tau} dx^\mu\,\hat{}\,dx^\nu\hat{}dx^\sigma\hat{}dx^\tau. \tag{5.4}$$

The union of all elements of the Pfaff sequence and their closures forms the elements of the Cartan topological base:

$$\{A, A \cup F, H, H \cup K...\}. \tag{5.5}$$

In order to take into account projective (and certain discontinuous) features, the vector fields of interest often will be scaled by a support function, ρ, such that $\mathbf{J} = \rho\mathbf{V}$. The fundamental equations of continuous evolution become

$$L_{(\rho\mathbf{V})}A = Q \tag{5.6}$$

$$L_{(\rho\mathbf{V})}F = dQ \tag{5.7}$$

$$\begin{aligned} L_{(\rho\mathbf{V})}H &= Q\hat{}F + A\hat{}Q \\ L_{(\rho\mathbf{V})}K &= 2(dQ\hat{}F) = 2d(Q\hat{}F) \end{aligned} \tag{5.8}$$

Note that for the even dimensional elements of the Pfaff sequence, (F and K), the action of the Lie differential is to produce an exact form: dQ, for the Lie differential of F, and $2d(Q\hat{}F)$ for the Lie differential of K. As integrals of exact forms over closed cycles or boundaries of support vanish, then it is possible to formulate the first theorem.

Theorem 5.1. *All even dimensional Pfaff classes of p-forms,* $dA = F, dA\hat{}dA = K$ *... are relative integral deformation invariants of continuous evolutionary processes relative to the Cartan topology.*

The closed integrals of F, K, ... are invariants of a continuous process as each integrand is exact, and the integral of an exact form over a closed domain vanishes. Hence if the functions are twice differentiable,

$$L_{(\rho \mathbf{V})} \int_{z2} F = \int_{z2} \{ i(\rho \mathbf{V}) dF + di(\rho \mathbf{V}) F \} = \int_{z2} dQ \Rightarrow 0. \tag{5.9}$$

The closed integrals of $F, K, ...$ are invariants of any process generated by $\rho \mathbf{V}$ for integration domains, $z2$, that are boundaries or cycles.

This theorem is an extension of Poincare's theorem for even dimensional p-forms which are absolute integral invariants (the integration domain is not necessarily closed) with respect to the restricted set of Hamiltonian processes. It is important to realize that the theorem expresses the existence of (relative) integral deformation invariants (topologocal properties) with respect to processess that may be thermodynamically reversible or irreversible. It should be noted that the domains of support of the even dimensional Pfaff classes can not be compact without boundary.

5.2. The Action 1-form and fluctuations

For purposes of expose, the Cartan system, Σ, will be limited to a single 1-form of action, A, and perhaps a single pseudoscalar field, or N form density, ρ. The 1-form of Action, A, can be written in several equivalent formats known as the Cartan-Hilbert action:

$$A = A_\mu dx^\mu = \mathbf{p} \cdot d\mathbf{x} - \mathcal{H}(\mathbf{x}, \mathbf{v}, \mathbf{p}, t) dt = \mathcal{L}(\mathbf{x}, \mathbf{v}, t) dt + \mathbf{p} \cdot (d\mathbf{x} - \mathbf{v} dt) \tag{5.10}$$

The last representation indicates that the Action may be viewed abstractly in terms a Lagrangian function, $\mathcal{L}(\mathbf{x}, \mathbf{v}, t)$, and the kinematic fluctuations in position,

$$\Delta \mathbf{x} = (d\mathbf{x} - \mathbf{v} dt). \tag{5.11}$$

It is to be noted that the usual assumption for physical systems is to assume that there are zero kinematic fluctuations. In this sense, kinematic perfection prevails:

$$\Delta \mathbf{x} = (d\mathbf{x} - \mathbf{v} dt) \Rightarrow 0. \tag{5.12}$$

It is rarely appreciated that kinematic perfection is equivalent to an exterior differential system which imposes topological restrictions on the variety. For this example, the fluctuations, $\Delta \mathbf{x}$,are not presumed to be zero.

A simple count of the independent variables that are used to define the Cartan-Hilbert 1-form of action indicates that the "fluctuation" space is a variety of 3n+1⇒10 dimensions $(t, \mathbf{x}, \mathbf{v}, \mathbf{p})$. (For simplicity, the "particle" index n has been chosen to be unity). The coefficients, $\mathbf{p}$, act as Lagrange multipliers for the fluctuations, $\Delta\mathbf{x}$. However, it can be determined that the maximum Pfaff dimension of the sequence $\{A, dA, A\hat{}dA, dA\hat{}dA...)$ is of dimension 2n+2⇒8 and not dimension 10. Hence the 10 dimensional space is redundant, and an 8 dimensional space is adequate to describe the physical system in terms of a 1-form of Action. The given 1-form of Action therefore generates a non-compact symplectic manifold of dimension 8.

If the Lagrange multipliers $\mathbf{p}$ of the kinematic fluctuations $(d\mathbf{x} - \mathbf{v}dt)$ are restricted to be the canonical momenta, as defined by the ubiquitous formula, $\mathbf{p} = \partial\mathcal{L}/\partial\mathbf{v}$, the maximum Pfaff dimension is 7, forming a contact manifold historically defined as state space. If the Lagrange function $\mathcal{L}(\mathbf{x}, \mathbf{v}, t)$ is homogeneous of degree 1 in $\mathbf{v}$, then the maximal Pfaff dimension is 6, forming a symplectic Finsler manifold of dimension 6, the phase space of classical mechanics. This manifold cannot be compact without boundary.

If the contact manifold of dimension 7 is constrained by the equations of kinematic closure,

$$d(\Delta\mathbf{x}) = d(d\mathbf{x} - \mathbf{v}dt) \Rightarrow 0, \tag{5.13}$$

then the space of interest becomes the configuration space of 4 dimensions, a submanifold of the original symplectic structure of 8 dimensions. The constraints of kinematic closure imply that the velocity field is expressible as functions of a single variable, t; $\mathbf{v} \Rightarrow \mathbf{v}(t)$. Note that the more severe constraint of kinematic perfection, $\Delta\mathbf{x} = (d\mathbf{x} - \mathbf{v}dt) \Rightarrow 0$, implies that the maximal Pfaff dimension is 2, as in this case $A\hat{}dA = \mathcal{L}(\mathbf{x}, \mathbf{v}, t)dt\hat{}d\mathcal{L}(\mathbf{x}, \mathbf{v}, t)\hat{}dt = 0$. The Action defines a completely integrable 2 dimensional submanifold that, in this circumstance, is not compact without boundary. These concepts will be exploited in other examples given below.

6. Cohomology and the Evolution of Energy

6.1. Cartan's Magic Formula and the first law.

The evolutionary processes considered in this section are limited to processes defined by vector fields, $\rho\mathbf{V}$, and physical systems that are adequately modeld in terms of a 1-form of Action, A. The evolutionary equation(s) is defined in terms of Cartan's magic formula, which employs the Lie differential relative to $\rho\mathbf{V}$ acting on the 1-form A to produce a 1-form Q :

$$L_{(\rho\mathbf{V})}A = i(\rho\mathbf{V})dA + di(\rho\mathbf{V})A \Rightarrow Q \tag{6.1}$$

Define $i(\rho\mathbf{V})A$ as the function, U, and W as the inexact 1-form $i(\rho\mathbf{V})dA$:

$$\text{"internal energy"}: U = i(\rho\mathbf{V})A \tag{6.2}$$

and

$$\text{(Virtual) Work:} \quad W = i(\rho\mathbf{V})dA. \tag{6.3}$$

Then, formally, the Cartan magic formula becomes equivalent to the statement of cohomology: the difference between the inexact 1-form Q and the inexact 1-form W is a perfect differential, dU.

$$L_{(\rho\mathbf{V})}A = W + dU \Rightarrow Q. \tag{6.4}$$

Cartan's Magic formula, expressing the propagation of the 1-form of Action down the tube of trajectories generated by the vector field $\rho\mathbf{V}$, becomes the dynamical equivalent of the first law of thermodynamics, when the inexact 1-forms Q and W are interpreted as heat = Q and work = W, respectively. These definitions are neither accidental nor whimisical, for it will be demonstrated below that they have utilization in many of the familiar formulas of classical physics.

Fundamentally, the Cartan magic formula is a topological law describing the evolution of energy. It is remarkable that the first law follows, without axiomatization, from the single and simple constraint that the 1-form of action, A, undergoes continuous topological evolution in terms of a dynamical system. It is also intuitively pleasing to see that the inexact 1-forms, Q and W, are defined in terms of a process. Elementary discussions of heat and work often emphasize the energy content of the first law, rather than the engineering idea that heat and work are related to processes.

Other authors have emphasized the topological foundations of thermodynamics [7], and from the time of Caratheodory have noted the connection to Pfaff systems [8]. However, these authors did not have access to, or did not utilize, the Cartan topology and DeRham cohomology. A remark by Tisza, "... the main content of thermostatic phase theory is to derive the topological properties of the sets of singular points in Gibbs phase space" [9], greatly stimulated the early developments of the theory presented in this article.

6.2. Thermodynamic processes

In thermodynamics, a reversible process is defined as a process for which the 1-form of heat, Q, admits and integrating factor, and an irreversible process is a process for which the 1-form of heat does not admit an integrating factor (of reciprocal temperature). [7]. This definition may be made precise in terms of Cartan's magic formula and the Frobenius theorem, for if the 1-form of heat, Q, does not admit an integrating factor then the three form, $Q\hat{}dQ$, does not vanish. However, for a given physical system defined in terms of a 1-form of Action, A, and its Pfaff sequence, those processes, $\mathbf{V}$, that satisfy the equation $L_{(\mathbf{V})}A\ \hat{}\ L_{(\mathbf{V})}dA = 0$ are reversible.

$$\text{Definition of an reversible process, } \mathbf{V}: \qquad L_{(\mathbf{V})}A\hat{}L_{(\mathbf{V})}dA = Q\hat{}dQ = 0 \quad (6.5)$$

This precise definition of thermodynamic reversibility will be subsumed, and the cohomological equivalent of the first law of thermodynamics will be studied relative to the constraint of continuous reversible or continuous irreversible topological evolution. Many intuitive thermodynamic concepts can be stated precisely in terms of the theory of continuous topological evolution based on the Cartan topology. For example, those processes for which $L_{(\mathbf{V})}A = Q = 0$ are adiabatic.

$$\text{Definition of a local adiabatic process, } \mathbf{V}: \qquad L_{(\mathbf{V})}A = Q = 0. \qquad (6.6)$$

As must be the case in thermodynamics, there is a fundamental difference between the 1-form W and the 1-form Q. From the definition $W = i(\rho\mathbf{V})dA$, it follows that

$$i(\rho\mathbf{V})W = i(\rho\mathbf{V})i(\rho\mathbf{V})dA \Rightarrow 0 \ (transversality) \qquad (6.7)$$

This fact implies that the 1-form W must be constructed from first integrals, ϕ, of the flow V, or from transversal fluctuations in the kinematics:

$$W = d\phi + \mathbf{f} \circ (d\mathbf{x} - \mathbf{v}dt). \qquad (6.8)$$

Although W can be included in the concept of Q, there are parts of Q that are not transformable into W. A precise difference between the 1-form of (virtual) work and the 1-form of heat can be established: the 1-form of work is necessarly transversal to the process, while the 1 form of heat is not. This issue is at the heart of the second law of thermodynamics. The argument is pleasing for it gives formal substance to the intuitive differences between the thermo-dynamic concepts of heat and work.

$$i(\mathbf{V})W = i(\mathbf{V})i(\mathbf{V})dA = 0 \qquad but \qquad i(\mathbf{V})Q = -dU \neq 0. \tag{6.9}$$

Continuous processes on isolated systems satisfy the (extremal) equations

$$W = i(\mathbf{V})dA = 0\,. \tag{6.10}$$

Continuous processes on closed but not isolated systems satisfy the (Helmholtz or symplectic) equations

$$dW = di(\mathbf{V})dA = 0\,. \tag{6.11}$$

Continuous processes on open systems satisfy the equations

$$W\hat{}dW \neq 0. \tag{6.12}$$

It should be noted that if a process is such that the closed integral of the Action 1-form is a deformation invariant (a topological property that is preserved) then $\beta W = \beta i(\mathbf{V})dA$ must be closed. Hence the process acting on the physical system must be such that the work 1-form satifies the integrability conditon, $W\hat{}dW = 0$. In general, a hierarchy of processes will be defined by the sequence Pfaff equivalence classes constructed from the 1-form opf Work, W:

$$\{W = i(\mathbf{V})dA = 0, \quad dW = di(\mathbf{V})dA = 0, \quad i(\mathbf{V})dW = 0, \quad di(\mathbf{V})dW = 0, ...\}. \tag{6.13}$$

All continuous processes may be put into equivalence classes as determined by the vector fields, V, that generate the flow. For example, for the 1-form, A, those vector fields that satisfy the transversal equation,

$$Associated : i(\rho\mathbf{V})A = 0 \tag{6.14}$$

are said to be elements of the associated class of vector fields relative to the form A. For such processes, the internal energy is zero.

Those vectors that satisfy the equations,

$$Extremal : i(\rho\mathbf{V})dA = 0 \tag{6.15}$$

are said to be elements of the extremal class of vector fields. For such processes, the virtual work vanishes, $W = 0$. It should be noted that the 2-form dA admits a unique extremal vector only on spaces of odd dimensions, a 2n+1 dimensional state space which is defined as a contact manifold. If the Pfaff dimension of the 1-form A is 4, then a unique extremal vector does not exist. The domain is a symplectic manifold of even dimension. However, on the symplectic manifold it then follows that there does exists a unique vector field, the Torsion vector described above, but no extremal vector.

Vectors which are both extremal and associated are said to be elements of the characteristic class of vector fields [10].

$$Characteristic : i(\rho\mathbf{V})A = 0 \quad and \quad i(\rho\mathbf{V})dA = 0 \tag{6.16}$$

Note that characteristic flow lines generated by V of the Characteristic class preserve the Cartan topology, for each form of the Cartan topological base is invariant with respect to the action of the Lie derivative relative to characteristic flows. Characteristics are often associated with wave phenomena.

6.3. Thermodynamic Irreversibility and the Torsion vector.

It is important to realize that Q represents the inexact 1-form of heat, and its integral is the measurable quantity. . When $Q\hat{}dQ \neq 0$, then the heat 1-form is said to be non-integrable. The implication is that there does not exist an integrating factor for Q. Recall that classical thermodynamics states that a process that creates a heat 1-form which does not admit an integrating factor is thermodynamically irreversible. Hence, given a physical system described in terms of a 1-form of Action, A, it is possible, for a given process, to compute Q and dQ. If $Q\hat{}dQ \neq 0$, then that process $\mathbf{V}$ is thermodynamically irreversible. It is also possible to solve for those processes V that are thermodynamically irreversible when applied to a specified physical system.

$$\text{Definition of an irreversible process, } \mathbf{V}: \qquad L_{(\mathbf{V})}A\hat{}L_{(\mathbf{V})}dA = Q\hat{}dQ \neq 0 \tag{6.17}$$

Given an Action 1-form in 4D, construct the 3 form of Topological Torsion, $H = A\hat{}dA$. Then there exists a vector field $\mathbf{T}$ such that $i(\mathbf{T})dx\hat{}dy\hat{}dz\hat{}dt = A\hat{}dA$. This vector field $\mathbf{T}$ is defined as the Topological Torsion vector. The properties of the Topological Torsion vector are such that

$$i(\mathbf{T})dA = \Gamma A, \quad and \quad i(\mathbf{T})A = 0. \tag{6.18}$$

Such vector fields are said to be homogeneous and associated relative to the 1-form A. The evolution of the physical system defined by A, in the direction of $\mathbf{T}$, yields the formula

$$L_{(\mathbf{T})}A = \Gamma A = Q, \quad and \quad L_{(\mathbf{T})}dA = dQ = d\Gamma\hat{}A + \Gamma dA \tag{6.19}$$

The heat 3-form becomes

$$Q\hat{}dQ = L_{(\mathbf{T})}A\hat{}L_{(\mathbf{T})}dA = \Gamma^2 A\hat{}dA \tag{6.20}$$

If the physical system admits Topological torsion, then $A\hat{}dA$ is not zero. Hence if the coefficient Γ^2 is not zero then the Heat 3-form $Q\hat{}dQ$ is not integrable and the process is thermodynamically irreversible. Examples will show that Γ is equal to the coefficient of the Topological Parity 4 form, $dA\hat{}dA$. When $dA\hat{}dA$ is not zero, such that Γ is not zero, the Torsion vector is uniquely defined, for then the coefficients of the 2-form $F = dA$ form an anti-symmetric matrix with an inverse. The physical system is said to define a symplectic 4D manifold. The conclusion is that thermodynamic irreversibility is an artifact of 4 dimensions [42].

It is important to realize that Q represents the inexact 1-form of heat, and its integral is the measurable quantity. When $L(V)A = Q = 0$ then the topological evolution process is defined to be an adiabatic process. When $L(V)Q = R \neq 0$, the process is defined to be radiative. When $Q\hat{}dQ \neq 0$, then the heat 1-form is said to be non-integrable. The implication is that there does not exist an integrating factor for Q. However, classical thermodynamics states that a process that creates a heat 1-form which does not admit an integrating factor is thermodynamically irreversible. Hence, given a physical system described in terms of a 1-form of Action, A, it is possible for a given process to compute Q and dQ. If $Q\hat{}dQ = 0$ then the process acting on the specified physical system is thermodynamically reversible. If $Q\hat{}dQ \neq 0$, then that process is thermodynamically irreversible. Given a specific physical system, it is also possible to solve for those processes V that are thermodynamically irreversible. Note that the same process acting on a different physical system need not be irreversible. Examples of this idea and its expression in terms of the Topological Torsion 3-form will be given below.

7. Continuous Processes

7.1. Closed Continuous Processes.

The continuous processes are naturally divided into two main categories: those for which $dQ = 0$ (closed processes) and those for which $dQ \neq 0$ (open processes). Closed flows also will be defined as uniformly continuous flows, to distinguish them from open flows, which are also continuous relative to the C2 constraint: Closed processes also will be defined as uniformly continuous processes, to distinguish them from open flows, relative to the C2 constraint. Therefore, relative to the Cartan Topology,

$$Closed\,process : L_{(\mathbf{V})} dA = dQ = 0 \tag{7.1}$$

defines a uniformly continuous closed process, while

$$Open\,process : L_{(\mathbf{V})} dA = dQ \neq 0 \tag{7.2}$$

defines an open process. Flow in the direction of the Torsion vector is an open flow.

Uniform continuity implies that the limit sets are invariant. Continuity only requires that the limit points permute amongst themselves. For example a fold into pleats which are then pasted together is a processes that rearranges the limit points and is not therefor uniformly continuous. Hence uniform continuity is a more constrained situation. When $dQ = 0$, it is possible to formulate immediately the following theorem (Poincare) for closed flows:

Theorem 7.1. *All even dimensional Pfaff classes of p-forms, $dA = F, dA\hat{}dA = K, \ldots$ are invariants of evolutionary processes that satisfy $L_{(\mathbf{V})}(dA) = dQ = 0$ relative to the Cartan topology. The forms $F, K, \ldots$ form a set of absolute integral invariants with respect to uniformly continuous processes.*

The difference between Theorem 1 and Theorem 2 is that in Theorem 2, the integration chains need not be closed. The proof of the theorem follows immediately by application of the Leibniz rule, using the constraint, $dQ = 0$:

$$L_{(\mathbf{V})}(dA\hat{}dA\hat{}dA..\hat{}dA) = \text{integer} \times \{L_{(\mathbf{V})}(dA)\}\hat{}dA\hat{}dA..\hat{}dA) = 0. \tag{7.3}$$

The integrands of the selected integrals are local invariants and so are their convected integrals.

The first application of theorem II gives,

$$L_{(\mathbf{V})}(dA) = L_{(\mathbf{V})}F = 0 \tag{7.4}$$

which is the equivalent of Helmholtz' theorem [14]. The theorem often is interpreted as the local conservation of angular momentum per unit moment of inertia, or the conservation of Topological Vorticity.

The second application of theorem II gives:

$$L_{(\mathbf{V})}(dA\hat{}dA) = L_{(\mathbf{V})}F\hat{}F = L_{(\mathbf{V})}K = 0 \tag{7.5}$$

which leads to the local conservation of Topological Parity, with respect to uniformally continuous flows.

In general,

$$L_{(\mathbf{V})}(dA\hat{}dA\hat{}...dA) = 0 \tag{7.6}$$

which expresses the invariance of a 2N dimensional area with respect to uniformally continuous flows.

7.2. Continuous Hydrodynamic Processes

Consider the domain of four independent variables of space time, $\{x, y, z, t\}$, and the three form of topological torsion

$$H = A\hat{}dA = A\hat{}F = i(\mathbf{T}_4)dx\hat{}dy\hat{}dz\hat{}dt. \tag{7.7}$$

The continuous evolution of this 3-form is determined relative to an arbitrary process, $\mathbf{V}_4 = [\mathbf{V}, 1]$, by the equation:

$$L_{(\beta\mathbf{V}_4)}H = L_{(\beta\mathbf{V}_4)}(A\hat{}dA) = i(\beta\mathbf{V}_4)dH + di(\beta\mathbf{V}_4)H = Q\hat{}F + A\hat{}dQ \tag{7.8}$$

For local invariance of the 3-form with respect to arbitrary parameterizations, the evolutionary vector $\beta\mathbf{V}_4$ must be collinear with the topological torsion vector

$(\mathbf{T}_4)$ such that the term $i(\beta\mathbf{V}_4)H \Rightarrow 0$. This constraint implies that the three form H then must be of the format:

$$H = A\hat{}F \approx \rho(x,y,z,t)(dx - V^x dt)\hat{}(dy - V^y dt)\hat{}(dz - V^z dt) = \rho \;\; i(\mathbf{V}_4)dx\hat{}dy\hat{}dz\hat{}dt \tag{7.9}$$

The invariance of the 3-form H then requires that a function $\rho(x,y,z,t)$ exist such that $dH \Rightarrow 0$. But this constraint becomes the equivalent of the famous hydrodynamic equation of continuity:

$$dH = \{div_3\, \rho\mathbf{V} + \partial\rho/\partial t\}dx\hat{}dy\hat{}dz\hat{}dt \Rightarrow 0 \tag{7.10}$$

which is interpreted physically as the conservation of mass. The implication is that those vector fields, $\beta\mathbf{V}_4$, that define a continuous hydrodynamic current, need not satisfy necessarily the formulas of topological kinematic constraint, $d\mathbf{x} - \mathbf{V}dt = 0$, but instead must be collinear with the topological torsion vector, $\mathbf{J}_4 = \lambda(x,y,z,t)\mathbf{T}_4$, if it exists. The important idea is that local deformable conservation of mass is to be associated with the conservation of the 3-form of Topological torsion as an absolute evolutionary invariant.

These results are to be compared with the even dimensional Poincare absolute integral invariants [12] for the more restrictive case of Hamiltonian (extremal) evolution of a Hamiltonian action,

$$A = A_\mu dx^\mu = \mathbf{p}\cdot d\mathbf{x} - H(\mathbf{x},\mathbf{p},t)dt \tag{7.11}$$

on a 2N+1 dimensional state space. It is the result (8.4) which is interpreted in statistical mechanics as the invariant area of phase space with respect to extremal, or Hamiltonian, evolution. The fact of the matter is that uniform continuity alone produces a set of absolute integral invariants for any action, in Hamiltonian format or not. Hamiltonian extremal flows satisfy the equation $dQ = 0$, and are therefore uniformly continuous, but they are not the only flows that satisfy this constraint. The invariance of "phase space area" is a consequence of uniform continuity alone, and does not require the additional constraints of constant homogeneity that limit the set of continuous flows to that subset of continuous vector fields which are extremal, and Hamiltonian.

7.3. DeRham categories of Closed Vector Fields

DeRham's cohomology theory [13] may be used to classify p-forms, and such ideas may be applied to the 1-form W defined by $W = i(\rho\mathbf{V})F$. Correspondingly, the vector fields that are used to construct the 1-forms W of virtual work perit processes to be put into the following categories, depending on whether the virtual work, W, is null, exact, closed, or not closed with respect to exterior differentiation. These categories are defined as:

Closed Flows $Categories\ \ for\, Q-W=dU$	$W=i(\rho\mathbf{V})F$	Q	dW	dQ
$Hamiltonian-extremal$	0	dU	0	0
$Bernoulli-Eulerian$	$d\Theta$	$d(U+\Theta)$	0	0
$Helmholtz-Symplectic$	$d\Theta+\gamma$	$d(U+\Theta)+\gamma$	0	0
Open Flows $Navier-Stokes-Torsion$	$arbitrary$	$arbitrary$	$dW\neq 0$	$dQ\neq 0$

(7.12)

The Bernoulli-Casimir functions, Θ, must be first integrals as in general,

$$i(\mathbf{V})W=i(\mathbf{V})d\Theta=0. \tag{7.13}$$

For closed flows the first law insures that the 1-form W is closed, $dW=dQ=0$, but W need not be exact and may contain harmonic components. That is, the 1-form W is not necessarily representable over the variety x,y,z,t in terms of the gradient of a single scalar function. The classic example of a non-exact 1-form is given by the expression,

$$\Gamma=\sigma_z(ydx-xdy)/(x^2+y^2) \tag{7.14}$$

for which $d\Gamma=0, but\ \int_{z1}\Gamma=2\pi\sigma_z$. The coefficient σ_z is assumed to be a constant. Such forms, Γ, generate period integrals and the DeRham cohomology classes. The number of independent forms of the type given by equation (20) determine the Betti numbers of a variety for which the singular point (at the origin in the example) has been excised. The Betti numbers can be interpreted as a method for counting the number of holes or handles in the variety. It is these contributions to the general differential form that carry topological information about the domain of support. The duals to these forms are also closed, leading to the definition, harmonic forms.

From the first law the harmonic contributions to W are equal to the harmonic contributions to Q. If the harmonic contributions to Q are not zero, then the number of "holes and handles" in the Cartan topology of the final state is different from the number of holes and handles in the Cartan topology of the initial state, and the evolutionary process is continuous but not reversible.

In order to make (20) transversal, use the Cartan trick of substituting $dx^i - V^i dt$ for each dx^i. The transversal harmonic form becomes

$$\Gamma = \sigma_z\{ydx - xdy + (\mathbf{r} \times \mathbf{V})_z dt\}/(x^2 + y^2) \tag{7.15}$$

which demonstrates the close relationship to transversal harmonic forms and angular momentum. The format may be extended to a spin vector of components

$$\boldsymbol{\sigma} = [\boldsymbol{\sigma}_1, \boldsymbol{\sigma}_2, \boldsymbol{\sigma}_3] = [\sigma_x/(y^2 + z^2), \sigma_y/(z^2 + x^2), \sigma_z/(x^2 + y^2)] \tag{7.16}$$

such that the harmonic form becomes

$$\Gamma = \boldsymbol{\sigma}_1(zdy - ydz) + \boldsymbol{\sigma}_2(xdz - zdx) + \boldsymbol{\sigma}_3(ydx - xdy) + (\boldsymbol{\sigma} \circ \mathbf{r} \times \mathbf{V})dt. \tag{7.17}$$

The last term is recognized as a "spin orbit" coupling term. The idea of harmonic contributions to a 1-form is closely related to the concept of a complex number or ordered pair representation; i.e., the form cannot be represented by a map to a space of 1 dimension. Other formats for harmonic 1-forms are given by the expressions:

$$\Gamma = \{\phi d\chi - \chi d\phi\}/(a\phi^p + b\chi^p)^{2/p}, \tag{7.18}$$

where ϕ and χ are arbitrary functions on the base space, and for the complex function, ψ,

$$\Gamma = \{\psi d\psi^* - \psi^* d\psi\}/(\psi^* \psi). \tag{7.19}$$

The last representation of a harmonic form is in the format of the "probability current" of quantum mechanics, and gives a clue as how to adapt the formalism of this article to quantum systems. Such a development is deferred to a later article.

For closed flows on space time, the fundamental equations of evolution are given by the expressions for the odd 1-form and the odd 3-form. The even forms are invariant. The two fundamental equations of uniformly continuous evolution are:

$$\begin{aligned} L_{(\rho\mathbf{V})}A &= Q \ \ and && (7.20)\\ L_{(\rho\mathbf{V})}H &= Q\hat{}F && (7.21) \end{aligned}$$

It should be remarked that if the 1-form of Action, A, is completely integrable in the sense of Frobenius, then the 3-form H is evanescent, and the evolutionary equation for H has no applicability. Such evolutionary processes ($H = 0$) are the equivalent to laminar flows in fluid dynamics and completely integrable, non-chaotic, Hamiltonian systems. It is known that if a Lagrangian system is not chaotic, then the action, A, is reducible to two variables (or less), and the 3-form H is necessarily zero. However when there exists a sense of helicity in the evolutionary process, or chaos is present, then the formula for H describes the appropriate topological evolution.

The first expression (7.20) may be put into correspondence with the evolution of energy, while the second fundamental equation (7.21) may be described as the evolution of complexity, or perhaps better as the evolution of defects, links, knots, or in abstract terms, the evolution of an entropic concept. If the heat 1-form Q is zero, then the evolutionary process is adiabatic, and topology is preserved. However, as the Cartan topology is not connected when $H \neq 0$, then continuous evolution of H can be accomplished only between connected subsets. The transition from a connected topology with $H = 0$ to a disconnected topology with $H \neq 0$ can only take place via a discontinuous transformation. The idea is that the continuous rate of change of H is definite (and arbitrarily taken to be positive). This feature is one of the key properties of entropy. Entropy can never change its sign. The creation of topological torsion, H, is a discontinuous process from a state of zero topological torsion, but once created, the growth (or decay) of H can be described by a continuous process (relative to the Cartan topology). These entropic features of the topological torsion 3-form will be useful in the description of the transition to turbulence.

7.4. The Hamiltonian Sub-Category

It should be remarked, that Cartan has proved, on a domain of dimension 2n+1, that if

$$i(\mathbf{V})F = W = 0, \qquad Q = dU \tag{7.22}$$

for any reparametrization, ρ, then $\mathbf{V}$ generates a Hamiltonian system, and visa versa [14]. This remarkable result indicates that Hamiltonian flows are not only continuous, but preserve many topological properties. The 1-form Q must be exact for Hamiltonian flows. Hence the observable holes and handles are topological invariants of Hamiltonian flows, as the ρ terms vanish. However, the fact that Q is exact for Hamiltonian flows does not completely establish a proof that Hamiltonian systems preserve all topological properties of the Cartan topology.

In the calculus of variations, vector fields that satisfy (21) are defined as extremal vector fields. Characteristic vector fields are a subclass of extremal fields that satisfy the equations

$$L_{(\mathbf{V})}A = 0 \; and \tag{7.23}$$

$$L_{(\mathbf{V})}F = 0. \tag{7.24}$$

In other words, continuous characteristics preserve the Cartan topology ($Q = 0$ and $dQ = 0$). Characteristic Hamiltonian vector fields generate waves in systems that can be endowed with the additional structure of a metric.

7.5. The Bernoulli-Euler subcategory

The Bernoulli-Euler category is not quite Hamiltonian. W is not zero, but must be a perfect differential, $W = d\Theta$. However, this perfect differential must be a first integral in order to satisfy the transversality condition, $i(\rho\mathbf{V})W = 0$. The 1 form Q is not necessarily so constrained. The abstract flows of this category are to be compared with the equations of motion of a compressible Eulerian fluid in which there may be stratification. If the pressure, P, is a function of the density, ρ, alone, then the Eulerian flow can be reduced to a Hamiltonian system [15]. If the exists some anisotropy due to stratification, then the Hamiltonian reduction is not perfect. Note that the first integral, Θ, acts as a Bernoulli constant along a given streamline, but the constant can vary from streamline to streamline because the function is transversal.

7.6. The Stokes subcategory

The Stokes category admits topological evolution in the sense that the harmonic contributions to W are not null, and therefore the "hole and handle" count of the Cartan topology is changing in an evolutionary manner. Such closed flows are not reversible. Note that all closed flows preserve topological vorticity and topological parity, and so if the flow is without vorticity in the initial state, then the flow is without vorticity in the final state. The Pfaff dimension [16] remains less than 2. However, if the initial state has vorticity, that vorticity will be preserved, but the Topological Torsion 3-form can change. In fact the Topological Torsion 3-form could be non-zero in the initial state, and zero in the final state, for the decay rate of topological torsion is proportional to $Q\hat{}F$ (See Figure 6). Both the 1-form of action and its hole count, and the 3-form of Topological Torsion, and its twisted handle count, are not necessarily invariants of a Stokes flow.

A method of distinguishing between "holes and twisted handles" is of some interest. Note that physically a handle can be constructed by deforming the rims of two holes in a surface into tubes and pasting the tubular ends together. If the rims are twisted by half integer or integer multiples of pi before the ends are glued together, then the handles have torsion (see Figure 7). Note that a handle cannot be constructed in the plane, so it is an intrinsically 3-dimensional thing. If the 3-form H vanishes, then there are no handles in the initial state, and as the Hamiltonian evolution produces no more new holes, there can be no more new handles in a Hamiltonian flow. However, existing handles may become twisted or knotted, because $Q\hat{}F \neq 0$, even for Hamiltonian flows. These facts correspond to the physical result that Hamiltonian systems are not dissipative and preserve energy, but that does not mean that entropy must be conserved.

It should be noted that for all closed flows, $dW = 0$. It follows that for closed flows, the transversality condition $i(\rho\mathbf{V})W = 0$ implies that the 1-form of virtual work W is an absolute invariant of the flow :

$$Closed\ Flows: \quad L_{(\rho\mathbf{V})}W = 0. \tag{7.25}$$

7.7. The Navier-Stokes category of open flows

It should be noted that the 1-form Q may be use to construct the Pfaff sequence, $\{Q, dQ, Q\hat{}dQ, dQ\hat{}dQ\}$,and generates another Pfaff dimension depending upon the rank or class of the elements of the Pfaff sequence for Q. For closed flows, $dQ = 0$ and the Pfaff dimension generated by Q is 1. The bulk of this article is devoted to closed flows. For open flows, $dQ \neq 0$, but the Pfaff sequence demonstrates that the topological features of open flows can have various levels of complexity. For example, the criteria that the Pfaff dimension of Q be 2 or less is equivalent to the Frobenious integrability constraint, $Q\hat{}dQ = 0$. This is precisely the Caratheodory condition that there exist "inaccessible paths" [17], and that (on a simply connected neighborhood) the 1-form of heat be representable as, $Q = TdS$. The topological evolution theory presented herein permits an analysis to be made for non-equilibrium processes, where the heat 1-form is not of the equilibrium monomial format, $Q \neq TdS$.

For the Navier-Stokes flow, the key feature is that $dQ = dW \neq 0$, but it still must be true that W is transversal. Therefore the 1-form W must be constructed from fluctuations, in the format,

$$W = f(dx - \mathbf{V}dt) + \text{closed additions transversal to} \mathbf{V}. \tag{7.26}$$

For open flows W is no longer a flow invariant. In the examples below, a particular choice is made for f which will generate the Navier-Stokes equations, which may have equilibrium or non-equilibrium solutions.

7.8. The Kinematic Topological Base

For continuous evolution in space-time, the key idea is that the exterior differential system consists of a Pfaff sequence constructed from a single 1-form of Action A, plus (perhaps) some additional constraints defining a domain of support and its boundary. The work of Arnold (and others) [18] has established that the singular points (zero's) of a global 1-form carry topological information. This idea is to be extended to the singular points of all elements of the Pfaff sequence, or topological base. In Appendix A, the idea of how a global 1-form of Action, A, existing on a space of dimension N+1 can be put into correspondence with a line bundle on a variety of dimension N is worked out in detail. The key features are that the Jacobian matrix of the projectivized 1-form of Action carries most of the information about the subspace. The trace and determinant of the Jacobian matrix determine the mean and Gaussian curvature of the subspace. The anti-symmetric components of the Jacobian are the functions that make up the 2-form, $F = dA$. The polynomial powers of F form the Chern classes for the line bundle.

For continuous transformations on a variety of $\{x, y, z, t\}$, the Cartan Action, A, can be defined kinematically as:

$$A = \sum_1^3 \mathbf{v}_k dx^k - \mathcal{H}dt, \tag{7.27}$$

where the "Hamiltonian" function, $\mathcal{H}$, is defined as,

$$\mathcal{H} = \mathbf{v} \bullet \mathbf{v}/2 + \int dP/\rho \tag{7.28}$$

Substitute this 1-form into the constraint equation given by **??**. Carry out the indicated operations of exterior differentiation and exterior multiplication to yield a system of necessary partial differential equations yields of the form,

$$\partial\mathbf{v}/\partial t + grad(\mathbf{v} \bullet \mathbf{v}/2) - \mathbf{v} \times curl\mathbf{v} = -gradP/\rho. \tag{7.29}$$

These equations are exactly the Euler partial differential equations for the evolution of a perfect fluid.

By direct computation, the 2-form $F = dA$ has components,

$$F = dA = \omega_z dx\hat{}dy + \omega_x dy\hat{}dz + \omega_y dz\hat{}dx + a_x dx\hat{}dt + a_y dy\hat{}dt + a_z dz\hat{}dt, \quad (7.30)$$

where by definition

$$\boldsymbol{\omega} = curl\ \mathbf{v}, \quad \mathbf{a} = -\partial\mathbf{v}/\partial t - grad\mathcal{H} \quad (7.31)$$

These vector fields always satisfy the Poincare-Faraday induction equations, $dF = ddA = 0$ for C2 functions, or,

$$curl\ \mathbf{a} - \partial\boldsymbol{\omega}/\partial t = 0, \quad div\boldsymbol{\omega} = 0. \quad (7.32)$$

The 3-form of Helicity or Topological Torsion, H, is constructed from the exterior product of A and dA as,

$$\begin{aligned} H &= A\hat{}dA = H_{ijk} dxi\hat{}dxj\hat{}dxk & (7.33) \\ &= -\mathbf{T}_x dy\hat{}dz\hat{}dt - \mathbf{T}_y dz\hat{}dx\hat{}dt - \mathbf{T}_z dx\hat{}dy\hat{}dt + h dx\hat{}dy\hat{}dz, & (7.34) \end{aligned}$$

where $\mathbf{T}$ is the fluidic Torsion axial vector current, and h is the torsion (helicity) density:

$$\mathbf{T} = \mathbf{a} \times \mathbf{v} + \mathcal{H}\boldsymbol{\omega}, \quad h = \mathbf{v} \bullet \boldsymbol{\omega} \quad (7.35)$$

The Torsion current, $\mathbf{T}$, consists of two parts. The first term represents the shear of translational accelerations, and the second part represents the shear of rotational accelerations. The topological torsion tensor, H_{ijk} , is a third rank completely anti-symmetric covariant tensor field, with four components on the variety $\{x, y, z, t\}$.

The Topological Parity becomes

$$K = dH = dA\hat{}dA = -2(\mathbf{a} \bullet \boldsymbol{\omega}) dx\hat{}dy\hat{}dz\hat{}dt. \quad (7.36)$$

This equation is in the form of a divergence when expressed on $\{x, y, z, t\}$,

$$div\mathbf{T} + \partial h/\partial t = -2(\mathbf{a} \bullet \boldsymbol{\omega}), \quad (7.37)$$

and yields the helicity-torsion current conservation law if the anomaly, $-2(\mathbf{a} \bullet \boldsymbol{\omega})$, on the RHS vanishes. It is to be observed that when $K = 0$, the integral of K vanishes, which implies that the Euler index, χ, is zero. It follows that the integral of H over a boundary of support vanishes by Stokes theorem. This idea is the generalization of the conservation of the integral of helicity density in an Eulerian flow. Note the result is independent from viscosity, subject to the constraint of zero Euler index, $\chi = 0$.

The torsion vector, $\mathbf{T}$, consists of two parts. The first term represents the shear of translational accelerations, and the second part represents the shear of rotational accelerations. The pseudo scalar function, K, acts as the source for the divergence of the torsion vector, T, and the torsion or helicity density, h. When $K = 0$, the evolutionary "lines" associated with the torsion tensor never cross, implying that the system is free of defects in space time. If K is positive or negative, the defects in the system are either growing or decaying. Equation (7.37) is the fundamental new law of topological physics that governs the specific realizations of controlled processes that minimize or maximize defect evolution.

Recall that if $H = A\hat{}dA = 0$, the 1-form of action satisfies the complete integrability condition of Frobenius. Similar to the Caratheodory equilibrium result for Q, the flow can be described then in terms of two variables; i.e., the flow is laminar. Turbulent flow is not laminar, and the transition from the laminar to the turbulent state must involve the topological evolution of H. It was the evolution of the 3-form of topological torsion as displayed in Figure 6 that galvanized the author's interest in topological evolution. The 3-form, H, and its evolution is intuitively related to the thermodynamic property of entropy. The fact that the Cartan topology is disconnected if the topological torsion, H, is not zero implies that the turbulent state cannot be created from the laminar state by means of a continuous transformation. Turbulence must be created by a discontinuous process. However, the decay of turbulence can be described by means of continuous process.

8. Global Conservation Laws

8.1. First Variation

Extremal (or Hamiltonian) flows and Eulerian flows induce a set of global conservation laws in the sense that the closed integrals of all odd dimensional Pfaff classes of the fundamental forms are relative integral invariants of uniformly continuous evolution. The result follows from the fact that the evolutionary rates, Q and $Q\hat{}F$ respect to such flows are zero. Integrals of exact forms evaluated over closed cycles, whether the cycle ($Z1$ *or* $Z3$) is a boundary or not, vanish. Hence all closed integrals of odd dimensional sets, $\int_{z1} A$ and $\int_{z3} H$, are evolutionary invariants of Hamiltonian and Eulerian flows.

For the closed flows of the Stokes category, the evolutionary rates of all odd Pfaff classes are closed, but not necessarily exact. That is,

$$dQ = 0, \ \ and \ \ d(Q\hat{}F) = 0, \tag{8.1}$$

implying closure, but Q and $Q\hat{}F$ are not exact. The DeRham classes are not empty and are not flow invariants. Topology changes during such evolutionary processes.

Hence a global set of conservation laws in terms of closed integrals of A and H can be devised only for those closed chains that satisfy Stokes theorem, and those chains must be boundaries (of support). Arbitrary closed integrals are not evolutionary invariants. This lack of relative integral invariance [19] for $\int_{z3} H$ corresponds to the production or destruction of 3 dimensional defects, and these new defects are indications of changing topology and changing inhomogeneity. Formally, a closed integral over a closed form is a period integral whose value, by Brouwer's theorem [20], is an integer multiple of some smallest value. A variation of a period integral signals a change in a Betti number and hence a change in topology. Such flows can produce three dimensional defects.

These results point out the limitations of Moffatt's and Gaffet's claims [21] that the volume integral of helicity density, $\mathbf{v} \bullet curl\mathbf{v}$, is an evolutionary invariant. Helicity is NOT necessarily an invariant of a continuous flow. Moreover, open or closed integrals of Helicity are not necessarily integral invariants of continuous evolution. In particular, the closed volume integral of helicity density, the fourth component of the Helicity four current, is not an invariant of continuous flows for which there is a torsion current .

A theorem depending on only the first variation can be stated for the continuous evolution of flows restricted to Hamiltonian or Eulerian flows:

Theorem 8.1. ***III***: *The (uniformly) continuous evolution of all odd dimensional Pfaff classes of the Cartan base with respect to Hamiltonian or Eulerian flows ($dQ = 0$, Q exact) are exact. Hence, the closed integrals of A and $H = A\hat{}dA$ over closed cycles or boundaries are relative integral invariants with respect to Hamiltonian or Eulerian flows.*

The proof of the theorem is as follows:

$L_{(\mathbf{V})}A = i(\mathbf{V})dA + d(i(\mathbf{V}))A) = d[P + i(\mathbf{V}))A] \;= Q$ and is exact.

Therefore $L_{(\mathbf{V})} \int_{z1} A = \int_{z1} Q = \int_{z1} d[P + i(\mathbf{V}))A] = 0 \supset$ invariance of $\int_{z1} A$.

Similarly,

$L_{(\mathbf{V})}H = L_{(\mathbf{V})}(A\hat{}F) = (L_{(\mathbf{V})}A)\hat{}F = Q\hat{}F = d[P + i(\mathbf{V}))A]\hat{}F$ is exact such that

$L_{(\mathbf{V})} \int_{z3} H = \int_{z3} d[P + i(\mathbf{V}))A]\hat{}F \supset$ invariance of $\int_{z3} H$. Q.E.D.

In the hydrodynamic case of a compressible Eulerian fluid, this theorem is the generalization of the "invariance of Helicity theorem" often stated for a barotropic domain or isentropic constraints. Closed flows therefore exhibit global conservation laws based on relative integral invariants of A and H, as well as absolute integral invariants of F and K. As will be demonstrated below, the integral of the 3-form of topological torsion, not the helicity density, over a boundary is an invariant of all flows that satisfy the Navier-Stokes equations and for which the vorticity vector field satisfies the Frobenius complete integrability conditions. This result is independent from the magnitude of the viscosity coefficient. On the other hand, the continuous destruction of 3-dimensional defects can be associated with closed flows of the Stokes category. Helicity is NOT necessarily a relative integral invariant of Stokes flows. Remarkably, such flows also admit a set of relative integral invariants, but these are determined only in terms of a second variational process.

8.2. Second Variation

It should be noted that the second Lie differential of the odd dimensional Pfaff classes (represented by A and H) does produce a set of global conservation laws for uniformly continuous processes. The result follows from the fact that the second Lie differential of the Action with respect to closed flows is exact, where the first Lie differential is closed!

The fundamental theorem is then:

Theorem 8.2. ***IV****: The (uniformly) continuous evolution of all odd dimensional Pfaff classes of the Cartan base with respect to closed flows ($dQ = 0$) are closed , but not necessarily exact. The second Lie differential is always exact so that $\int_{z1} Q$ and $\int_{z3} Q\hat{}F$ are relative integral invariants of (uniformly) continuous ($dQ = 0$) evolution.*

The proof of the fundamental theorem is as follows:

$L_{(\rho\mathbf{V})}A = i(\rho\mathbf{V})dA + d(i(\rho\mathbf{V}))A) = Q$

$L_{(\rho\mathbf{V})}L_{(\rho\mathbf{V})}A = L_{(\rho\mathbf{V})}Q = R$

$= i(\rho\mathbf{V})d(i(\rho\mathbf{V})dA) + di(\rho\mathbf{V})di(\rho\mathbf{V})A = i(\rho\mathbf{V})d(Q) + di(\rho\mathbf{V})di(\rho\mathbf{V})A = 0 + d(\Lambda)$

which is exact.

Similarly,

$L_{(\rho\mathbf{V})}L_{(\rho\mathbf{V})}A\hat{}dA = L_{(\rho\mathbf{V})}Q\hat{}F = d(\Delta\ F)$

which is exact. It follows that

$L_{(\rho\mathbf{V})}\int_{z3} Q\hat{}F = \int_{z3} d(\Delta\ F) = 0$

such that $\int_{z3} Q\hat{}F$ is a relative integral invariant. Q.E.D.

Uniform continuity requires that $d(L_{(\rho\mathbf{V})}A) = L_{(\rho\mathbf{V})}dA = dQ = 0$, which insures that Q and $Q\hat{}F$ are closed. Hence closed integrals of the odd dimensional p-forms of Q and $Q\hat{}F$ (and not necessarily A and H) are relative integral invariants of uniformly continuous evolution. The integrals $\int_{z1} Q$ and $\int_{z3} Q\hat{}F$ generate global conservation laws for uniformly continuous processes in which $dQ = 0$. In elementary terms, on a space time variety, the fundamental theorem of uniformly continuous evolution states that the Lorentz force has zero curl, and the torsion defect production rate has zero divergence ($K = 0$), whether the system is dissipative or not.

The successive Lie derivation with respect to a uniformly continuous vector field $J = \rho V$ produces an exact sequence, starting from the concept of action-angular momentum, A, evolving to a closed set, Q, which under continued Lie derivation evolves to an exact kernel of radiation-power, R [20]. A similar exact sequence can be constructed for all odd dimensional Pfaff classes, $A, A\hat{}dA, A\hat{}dA\hat{}dA, ...$

8.3. Continuity and the Integers

A most remarkable feature of the fundamental theorem of uniformly continuous evolution is that the integral of any radiation 1-form, R, through a container which is a maximal cycle is in relation to the integers. This concept is another application of the Brouwer degree of a map theorem, that says that all period integrals are integer multiples of some smallest value. The maximal cycle is a closed set that is not a boundary but can contain a system with internal defects, hence the name, the "container". As a simple example consider a disc with several internal holes; the maximal cycle is the cycle which would be the boundary if the disc had no holes. The global conservation laws stated above imply that radiation through the maximal cycle must be compensated by a change in the cohomology class, or the production of a defect of inhomogeneity in the interior. Radiation defects ("holes and torsion handles") are quantized, for it is impossible to create half a hole.

It would appear from the above argument that Planck's hypothesis of quantized radiation oscillators may be considered a consequence of theorem IV and Uniformly CONTINUOUS evolution as defined by equation (17).

8.4. The Navier-Stokes Fluid

Although the bulk of this article is limited to the study of uniformly continuous evolution ($dQ = 0$), some remarks should be made about continuous evolution of the Navier-Stokes category ($dQ \neq 0$). The kinematic topology is often too course for direct application to a typical physical system. Additional topological constraints must be applied. For a Navier-Stokes fluid, the additional topological constraints on the admissible flow fields, $V = \{\mathbf{v}, 1\}$ implies a specific format is required for the dissipative force, f. Let f take the form $\nu\ curl\boldsymbol{\omega}$ such that upon dividing through by μ, the equation for the Work 1-form becomes:

$$W = i(V)dA = -\sum_i \{(\nu\ curl\boldsymbol{\omega})_i(dx^i - V^i dt)\}. \tag{8.2}$$

Evaluating both sides explicity and comparing coefficients of the terms dx^i yields the Navier-Stokes partial differential equations,

$$\partial\mathbf{v}/\partial t + grad(\mathbf{v}\circ\mathbf{v}/2) - \mathbf{v}\times curl\mathbf{v} = -gradP/\rho + \nu\ curlcurl\mathbf{v} \tag{8.3}$$

This process is typical of the Cartan method, where by the coefficients of a system of differential forms are equivalent to a system of partial differential equations. For the kinematic Action, A, the equation above expressing and constraining the 1-form of Work is the differential form equivalent to the Navier-Stokes equations. The constraint limits the class of all V to those V that are solutions to the Navier-Stokes partial differential equations.

The constraint given by (39) may be used evaluate the behavior of the topological base with respect to the evolution described by V. For example, the evolution of the Action is given by the expression,

$$L(V)A = i(V)dA + d\{i(V)A\} = -\{(\nu\ curl\boldsymbol{\omega})\circ(d\mathbf{x}^i - \mathbf{v}^i dt)\} + d\{(\mathbf{v}\circ\mathbf{v}/2) + H\} \tag{8.4}$$

The evolution of the limit sets is given by

$$L(V)dA = -d\{(\nu\ curl\boldsymbol{\omega}) \circ (d\mathbf{x}^i - \mathbf{v}^i dt)\}. \tag{8.5}$$

If the flow V is uniformly continuous, then the RHS of 8.4 must vanish, making $F = dA$ a flow invariant. The Navier-Stokes equations have $C2$ solutions that belong to the Stokes category of closed flows. This result is an extension of the Helmholtz theorem on the conservation of vorticity. It would follow that the 4-form, $K = dA\hat{}dA$ is also a flow invariant, for uniformly continuous flows. A remarkable result is that even for dissipative Navier Stokes flows where $\nu\ curl\boldsymbol{\omega} \neq \mathbf{0}$, it is still possible that the RHS of 8.4 may vanish, and the flow is uniformly continuous. Examples of such harmonic solutions to the Navier Stokes equations were presented by this author at the Permb conference on Large Scale Structures [3]. One such harmonic closed form solution was shown to develop a tertiary Hopf bifurcation in terms of the parameter of mean flow. The surface of null helicity density, $h = \mathbf{v} \circ \boldsymbol{\omega} = 0$ went through a topological phase change as the bifurcation took place similar to that presented by soap films initially forming a single sheeted surface between two rings, and then with increased ring displacement, forming a double sheeted surface.

According to theorem II, the even dimensional topological properties $\{F, K\}$ are invariants of a uniformly continuous flow. If topology is to change in a uniformly continuous manner, the only possible candidates for topological evolution must be the 1-dimensional circulation, A, and the 3-dimensional torsion, H. For incompressible flows ($div\mathbf{v} = 0$) circulation defects must be associated with boundaries; however, if $K \neq 0$, then torsion defects can occur within the bulk media. It is the author's perception that the production of torsion defects is the key to the understanding of large scale structures in continuous media, and the transition to turbulence.

In general, as has been stated above, if the flow is continuous, then the limit sets $d\Sigma$ must remain within the closure of Σ. Abstractly this idea can be written as,

$$L(V)d\Sigma = d\Sigma + \Sigma\hat{}\Sigma. \tag{8.6}$$

Uniform continuity is the stronger constraint,

$$L(V)d\Sigma = 0. \tag{8.7}$$

For the Navier-Stokes flows, where the evolution is not necessarily uniformly continuous, the Navier-Stokes constraint may be used to express the acceleration term, a, dynamically; i.e.,

$$\mathbf{a} = -gradH - \partial v/\partial t = -\mathbf{v} \times curl\mathbf{v} + \nu\{curl\ curl\mathbf{v}\}. \tag{8.8}$$

By substituting this expression for **a** into equation 7.35 a simple engineering representation is obtained for the torsion vector current, T, of a Navier-Stokes fluid:

$$\mathbf{T} = \{h\,\mathbf{v} - L\,curl\,\mathbf{v}\} - \nu\{\mathbf{v} \times (curl\ curl\mathbf{v})\}(45) \tag{8.9}$$

Note that the torsion axial vector current persists even for Euler flows, where $\nu = 0$. When $h = 0$, the torsion axial vector is proportional to the vorticity of the flow. It is the opinion of this author that many of the visual phenomena of fluid dynamics which have been associated with "vortices" are actually representations of torsion defects. In fact, a closed form solution to the Navier-Stokes equations was presented at the Perm conference [3] which indicates that the experimental phenomena of "vortex" bursting can be emulated by the streamlines of a flow for which there is no parametric evolutionary change of vorticity, but for which there is a parametric evolution and topological phase change of the 3-form of topological torsion. As the critical value of flow is achieved, a re-entrant compact torsion bubble is produced in what was originally a unidirectional flow. The measurement of the components of the Torsion vector have been completely ignored by experimentalists (and theorists) in hydrodynamics (and other dynamical systems).

The measurement of the components of the Torsion vector have been completely ignored by experimentalists in hydrodynamics.

By a similar substitution using the value of a given by the constraint 8.4, the topological parity pseudo-scalar becomes expressible in terms of engineering quantities as,

$$K = dH = dA\hat{}dA = -2\nu(\boldsymbol{\omega} \circ curl\,\boldsymbol{\omega})dx\hat{}dy\hat{}dz\hat{}dt. \tag{8.10}$$

From this expression it is apparent that if the vorticity field is integrable in the sense of Frobenius, then viscosity does NOT contribute to the creation of torsion defects. As described below, the integral of K over $\{x, y, z, t\}$ gives the Euler index induced by the flow on the space time variety. If $K = 0$, the flow lines never intersect.

9. Pfaff's Problem, Characteristics, and the Torsion Current.

Closely related to the concept of topological torsion is the Pfaff problem that asks about the solubility of the system of differential equations defined by setting each element of the Cartan closure to zero. The problem is equivalent to finding characteristic vector fields which, if continuous, generate an evolutionary flow that preserves the Cartan topology. The key idea of Pfaff's problem is to find maps from spaces of q dimensions into the variety, X, such that when these maps and their differentials are substituted into the system of forms that make up the Cartan closure, then the new forms are equal to zero. In this sense, the pullback of the forms of the Cartan closure to the spaces of dimension q are zero. In the case of usual interest to physics, the maps are of a single parameter which almost always is associated with the concept of time. However, they may exist higher dimensional solutions of say two parameters or more.

The question arises as to the largest dimension of such a "solution" and is determined in terms of the "characters" and "genus" of the Pfaff system [22]. It is the objective of this section to demonstrate that the genus of the Pfaff system built from a single 1-form of action is 3 if the Torsion current, $\mathbf{T}$, vanishes, and can be 2 only if $\mathbf{T} \neq 0$. The genus is an arithmetic invariant and a topological property. A change of genus implies topological evolution. However for the special Pfaff system described, the characters are such that only 1-parameter solutions are possible, when $\mathbf{T} = 0$, and a unique 2 parameter solution is admissible only when $\mathbf{T} \neq 0$. In other words the Pfaff problem admits a "string" solution (a two parameter solution) only when the Torsion current is not zero.

Consider an electromagnetic format. For the electromagnetic case, the Cartan 1-form may be defined in terms of the vector and scalar potentials,

$$A = \mathbf{A} \bullet d\mathbf{r} - \varphi dt. \tag{9.1}$$

Using the classical notation of Sommerfeld, define the $\mathbf{E}$ and $\mathbf{B}$ field intensities as

$$\mathbf{B} = curl\mathbf{A}, \quad \mathbf{E} = -\partial\mathbf{A}/\partial t - grad\varphi. \tag{9.2}$$

Then the components of the Darboux-Cartan-Maxwell field, $F_{\mu\nu}$, may be written as an anti-symmetric matrix (or as a Sommerfeld six-vector) of components :

$$F_{12} = B_z,\ F_{13} = -B_y,\ F_{23} = B_x,\ F_{14} = E_x,\ F_{24} = E_y,\ F_{34} = E_z \tag{9.3}$$

such that the components of $dA = F = F_{\mu\nu}dx^\mu\hat{}dx^\nu$

The Topological torsion, H, becomes

$$H = A\hat{}dA = -i\{\mathbf{E}\times\mathbf{A}+\varphi\mathbf{B},\mathbf{A}\bullet\mathbf{B}\}dx\hat{}dy\hat{}dz\hat{}dt. \tag{9.4}$$

with the torsion current defined as,

$$\mathbf{T} = \mathbf{E}\times\mathbf{A}+\varphi\mathbf{B} \tag{9.5}$$

and the helicity density,

$$h = \mathbf{A}\bullet\mathbf{B}. \tag{9.6}$$

The Topological Parity 4-form becomes the global top Pfaffian on the 4 dimensional space-time variety, and is equal to

$$K = dA\hat{}dA = -2\mathbf{E}\bullet\mathbf{B}dx\hat{}dy\hat{}dz\hat{}dt. \tag{9.7}$$

Note that

$$div\mathbf{T} + \partial h/\partial t = -2\mathbf{E}\bullet\mathbf{B} \tag{9.8}$$

. The 3-form of axial current, H, is NOT conserved when $K \neq 0$. This result has been observed by Berger [23]. Following Chern, the Euler index on a compact manifold would be the integral

$$\chi = \int_{z4} 2\mathbf{E}\bullet\mathbf{B}dx\hat{}dy\hat{}dz\hat{}dt. \tag{9.9}$$

Now the Pfaff problem is determined by the equations

$$A = 0, \quad F = 0. \tag{9.10}$$

Following Slebodzinsky, as there is only one 1-form in the Pfaff system, the first character, $s0$, of the Pfaff system is equal to 1. Multiply F by φ, and use $A = 0$ to eliminate φdt in the equation $F = 0$. The result is given by the equation,

$$\{\mathbf{E}\times\mathbf{A}+\varphi\mathbf{B}\}_{\mu\nu}dx^\mu\hat{}dx^\nu = \{\mathbf{T}\}_{\mu\nu}dx^\mu\hat{}dx^\nu = 0, \tag{9.11}$$

which is an expression that does not contain dt. The polar system of these resultant equations determines the genus of the Pfaff system. In particular, if $\mathbf{T}$, the torsion current vanishes, then (9.11) vanishes, the second character, $s1$ is zero and the genus of the Pfaff system is 3. All higher characters vanish, so the Pfaff system is special. Only 1-parameter homeomorphic evolutionary solutions are possible for the Pfaff system in 4 dimensions, when $\mathbf{T} = 0$.

On the other hand, for any arbitrary vector field, $\mathbf{V}$, such that the two 1-forms $\{\mathbf{T} \times \mathbf{V}\}_\mu dx^\mu$ and A, are linearly independent, then the second character, $s1$, equals 1, and the genus is 2. There then exists a two parameter characteristic evolutionary system (a string). In other words, the presence of the torsion current is necessary for the existence of a two parameter solution to the Pfaff problem. There are no 3 parameter solutions to this Pfaff problem in 4-dimensions. This extraordinary connection between the concept of the Torsion current and the solubility of Pfaff's problem serves to further emphasize the content of the often neglected quantity of topological torsion.

9.1. The Euler index

The coefficients of the Action 1-form globally define a covariant vector field on the variety. This vector field need not be a section without singularities. As mentioned in section 13 Arnold has shown how the singular points (zeros) of the Action 1-form, A, can be used to define the Euler index of the topology induced on the variety. Another method for evaluating this key topological property has been devised by Chern [24]. Following Chern, the Euler index becomes the integral

$$\chi = \int_{z4} K = \int_{z4} 2\mathbf{E} \bullet \mathbf{B} dx\hat{}dy\hat{}dz\hat{}dt. \tag{9.12}$$

In Lagrangian field theories, a non-zero value for K implies that the second Chern class is not empty and signals the demise of time reversal and parity symmetry [25] (hence, the name Topological Parity 4-form). It should be remarked that K is the exterior derivative of the 3-form of topological torsion, H, and that this 3-form can be put into correspondence with the Chern-Simons 3-form of differential geometry. In effect the evolutionary law for the 3-form of Topological Torsion given by (10) is a Lagrangian field theory built on a Chern-Simons action. In this article, no constraint of self dualism is imposed, as is usually the case in current string theories.

When the electric field is orthogonal to the magnetic field, then the Euler index is zero. The idea that this Poincare invariant might have deeper meaning led Eddington [26] to state: "It is somewhat curious that the scalar-product of the electric and magnetic forces is of so little importance in classical theory, for ..(eq (53)) .. would seem to be the most fundamental invariant of the field. Apart from the fact that it vanishes for electromagnetic waves propagated in the absence of any bound electric field (i.e., remote from electrons), this invariant seems to have no significant properties. Perhaps it may turn out to have greater importance when the study of electron-structure is more advanced."

A non-zero value of the Topological Parity 4-form, K, implies that the divergence of $\mathbf{T}$ is not zero. Therefore, torsion lines can stop or start within the variety even though the evolution is C2 continuous. The torsion current is not necessarily conserved and 3-dimensional defects can be produced internally. String theorists describe this effect as an anomaly of the axial (Torsion) current. In the same sense that the closed but not exact 1-form leads to a complex representation involving ordered pair of variables, a closed but not exact 3-form leads to a quaternionic representation.

The concept of a domain of non-null Euler index ($K \neq 0$) now appears to be useful to the theory of magnetic reconnection in the electromagnetic case [27] and to vortex reconnection [28] in the hydrodynamic case. The correspondence between the bridging and rib structures produced in numerical simulations of turbulent fluid flows and the 4-string interaction of superstring theory is remarkable [29]. The concept ($K \neq 0$) appears to be applicable to the understanding of the stretching of lines and surfaces in turbulent flows where time-reversal symmetry is violated [30]. The appearance of large scale structures in certain flows has been associated with the lack of parity invariance [31]. The concepts of macroscopic violations of P and T symmetries appear to have application to the theory of the quantum Hall effect [32].

With regards to hydrodynamic systems, the evolution of a flow from a laminar flow to a turbulent flow involves topological evolution. For the Navier-Stokes system, the Euler index depends upon the viscosity and the lack of Frobenius integrability of the vorticity field (see equation 36). Such a term yields a local source for the creation of Torsion currents. The lack of reversibility of such flows, and the irreducible time dependent, 3 dimensional features of such flows, implies that K can not be zero for the turbulent state. It is conjectured that the Euler index of the flow (the integral of K over the domain) is not zero during the transition to turbulence. That is, K is not a last multiplier of the spatial volume element, $dx\hat{}dy\hat{}dz$ for the flow describing the continuous (relative to the Cartan C2 topology) transition to turbulence. If $dQ\hat{}F = 0$ then the function K defines an integrating actor in the sense of a mass density such that

$$div(K\mathbf{V}) + \partial K/\partial t = 0. \tag{9.13}$$

If K were a mass density, this equation is often called the "equation of continuity", but it is more accurately described as the "conservation of mass". Relative to the Cartan topology all C2 vector fields are continuous. The transition to the turbulent state, however, must be discontinuous, for the Cartan topology in the turbulent state is disconnected.

10. SUMMARY

To review, a topology has been constructed on a variety in terms of the elements of closure of a Cartan system of C2 differential forms and their intersections. The associated topological structure indicates that all processes generated by the Lie convective derivative (relative to a C2 vector field, **V**) are continuous relative to the Cartan topology. However, the processes so generated are not necessarily homeomorphisms for they need not be reversible; i.e., the topology of the initial state can evolve continuously into a different topology on the final state. The method for constructing the Cartan topology is the same on both the initial and the final state, but, for example, the "hole and handle" count on the initial state can be different from the "hole and handle" count in the final state.

In terms of a single 1-form of Action, A, a Cartan topological base was constructed in terms of a set of distinct elements, defined as a Pfaff sequence, and their closures. The fundamental laws of evolution of each of the elements of the topological base was formulated relative to an arbitrary vector field. It was determined that there are two categories of continuous flows, those which are "closed" and those which are "open". A special sub-category of closed flows describe a Hamiltonian evolution, an evolutionary process which preserves the number of "holes and handles".

Relative to the closed category of continuous processes, all even dimension elements of the Cartan topological base are evolutionary invariants. For closed flows, topological evolution takes place only in terms of the odd elements of the topological base. The first odd element of the topological base is the Action, and its law of evolution is equivalent to the evolution of energy. The next odd element (and the only other odd element on space-time) of the Cartan topological base is formulated as the novel 3-form of Topological Torsion. The evolution of this 3-form is studied, for although it does not necessarily satisfy a local conservation law, the anomalous source term, defined as topological parity, can be computed. It is a source of system evolutionary defects. However, it is still possible to establish a set of global conservation laws for the category of closed, (uniformly) continuous but irreversible evolutionary flows. Although the evolution of topooloigcal torsion may be described by a continuous process, the creation of topological torsion from a state without topological torsion is not described by a continuous process. As the Cartan topology is not connected, the creation of topological torsion must involve discontinuous processes or shocks.

The fundamental equation of topological evolution, $L_{(\rho\mathbf{V})}A = Q$, is equivalent to cohomological format of the first law of thermodynamics, $W + dU = Q$. The heat 1-form Q may be used to form a Pfaff sequence whose Pfaff dimension may be used to further classify evolutionary flows. For example, if the Pfaff dimension of Q is 2 or less, then Q can be written in the equilibrium format, $Q = TdS$. An example of an open system of flows (defined as $dQ \neq 0$) was presented in terms of the Navier-Stokes equations, for which the anomalous source term, can be computed. In effect it was demonstrated that C2 irreversible flows are among the solution set to the Navier-Stokes system. An abstract example was also given for an electromagnetic Action, in which the concept of time reversal and parity symmetry breaking was associated with a non-null Euler characteristic of the Cartan topology.

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