The Topological Hall Effect

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1. The Topological Plasma

For electromagnetic systems, a particular interesting choice of specialized processes are those that leave the closed integrals (around cycles $z2$) of the 2 form of field excitations, G, a deformation (relative) integral invariant. Such processes βV which preserve the net *number* of charges, globally, are defined as elements of the category of plasma processes:

$$L_{\beta V}(\iint_{z2} G) = \iint_{z2} i(\beta V)dG + \iint_{z2} d(i(\beta V)G) = \iint_{z2} i(\beta V)J \Rightarrow 0. \tag{1.1}$$

The criteria for relative integral invariance with an arbitrary deformation parameter, β, implies that $i(\beta V)J = \beta\{(V^4\mathbf{J} - \rho\mathbf{V})^x dy\hat{}dz - ... + (\mathbf{J}\times\mathbf{V})^x dx\hat{}dt..\} \Rightarrow 0$. This constraint has expression in engineering language as,

$$\text{Plasma Processes:}\quad \mathbf{J} = \rho\mathbf{V}, \qquad \mathbf{J}\times\mathbf{V} = 0. \tag{1.2}$$

A plasma process need not conserve energy. Again, the 3-forms, J, $A\hat{}G$ and $A\hat{}F$ are of particular interested for their tangent manifolds define "lines" in the 4-dimensional variety of space and time. Relative to plasma processes, the topological evolution associated with such lines, and their entanglements, is of utility in understanding solar corona and plasma instability.

The invariance principle that defines a plasma process on G should be compared to the Helmholtz process on F:

$$L_{\beta\mathbf{V}}(\underset{z2}{\iint} F) = \underset{z2}{\iint} i(\beta V)dF \Rightarrow 0. \tag{1.3}$$

The closed integral of electromagnetic flux is an intrinsic topological (deformation) invariant of an electromagnetic system, for the 2-form F is exact by construction (the postulate of potentials). In a plasma, for which the evolutionary processes are constrained such that $\mathbf{J} = \rho\mathbf{V}$, both the closed integrals of F and G are deformation invariants. In the sense, the plasma is a topological refinement of the complete Maxwell system.

The "perfect" plasma is defined as a process that is both a plasma process and a Hamiltonian extremal process. It follows that the virtual work 1-form must vanish. The plasma process will have the form $J = \rho[\mathbf{V}, 1]$ such that the Hamiltonian extremal criteria yields the "Work Free" equation:

$$W = i(J)dA = \rho(\mathbf{E} + \mathbf{V} \times \mathbf{B}) \circ d\mathbf{r} + \rho(\mathbf{V} \circ \mathbf{E}\ )dt \Rightarrow 0 \tag{1.4}$$

The "perfect or ideal" plasma is therefor a "Force Free" plasma when the Lorentz force is zero. If the plasma is a "force free" plasma, then if follows that

$$\text{Force Free plasma process:}\ \ \mathbf{V} \circ \mathbf{E} = \mathbf{0}, \quad \mathbf{E} \circ \mathbf{B} = \mathbf{0}, \quad \mathbf{J} \circ \mathbf{E} = 0. \tag{1.5}$$

Other plasma dynamics [**?**] belong to categories that depend upon the Pfaff dimension of the Work 1-form.

2. Extremal vs. Bernouilli plasmas

The development that follows is guided by Cartan's pioneering work, in which he examined those specialized processes for mechanical systems that leave the closed integrals of the 1-form of Action, A, a deformation (relative integral) invariant. Cartan proved that such processes always have a Hamiltonian representation. There are two classes of such Hamiltonian processes, the Extremal class and the Bernoulli class:

$$\text{Hamiltonian Processes :} \quad L_{(\beta\mathbf{V})} \int_{z1} A = \int_{z1} i(\beta V)dA + di(\beta V)A \Rightarrow 0. \tag{2.1}$$

$$\mathit{Extremal:} \qquad i(\beta\mathbf{V})dA = 0, \tag{2.2}$$

$$Bernoulli: \quad i(\beta \mathbf{V})dA = d\Theta. \tag{2.3}$$

The closed integration chain $z1$ is not necessarily a boundary.

An electromagnetic system has not only the primitive 1-form, A, but also the N-2 form, G, which can undergo evolutionary processes. For electromagnetic systems, a set of equations similar to those that define Hamiltonian processes can be used to define specialized processes that leave the closed integrals of the N-2=2 form, G, of field excitations, a deformation (relative integral) invariant. These special processes will be defined as Plasma processes. Such process do not create free charge, but they can cause a change in the number of charge pairs of opposite sign. The equations that must be satisfied are of the form:

$$\text{Plasma Processes} \tag{2.4}$$

$$L_{(\beta \mathbf{V})} \int_{z2} G = \int_{z2} i(\beta V)dG + di(\beta V)G \Rightarrow 0. \tag{2.5}$$

$$Extremal : \quad i(\beta V)dG = 0, \tag{2.6}$$

$$Bernoulli : \quad i(\beta V)dG = d\omega. \tag{2.7}$$

In the Extremal case,

$$i(\beta V)dG = \beta\{(\mathbf{J} - \rho \mathbf{V})^x dy\hat{}dz - ... + (\mathbf{J} \times \mathbf{V})^x dx\hat{}dt... \Rightarrow 0, \tag{2.8}$$

implies that the extremal Plasma process obeys the classic expression: $\mathbf{J}_E = \rho \mathbf{V}$.

2.1. The Topological Hall effect

In the Bernoulli case of a Plasma process the integrand must be proportional to an exact 2-form, $d\omega$. There is one obvious candidate, the 2-form, F :

$$i(\beta V)dG = \beta\{(\mathbf{J} - \rho \mathbf{V})^x dy\hat{}dz - ... + (\mathbf{J} \times \mathbf{V})^x dx\hat{}dt... = \sigma_{Hall} F. \tag{2.9}$$

The conductivity coefficient σ_{Hall} in the expression must be a domain constant. Comparing the components of the equation of constraint yields the properties of a Bernoulli Plasma process:

$$\text{Bernoulli Plasma process,} \quad \mathbf{J}_B = \rho \mathbf{V} + (\sigma_{Hall}/\beta)\mathbf{B}, \tag{2.10}$$

$$\text{and,} \quad (\mathbf{J}_B \times \mathbf{V}) = (\sigma_{Hall}/\beta)\mathbf{E}. \tag{2.11}$$

$$(\sigma_{_{Hall}}/\beta)(\mathbf{J}_B \circ \mathbf{E}) \Rightarrow 0 \tag{2.12}$$

$$(\sigma_{_{Hall}}/\beta)(\mathbf{V} \circ \mathbf{E}) \Rightarrow 0 \tag{2.13}$$

$$(\sigma_{_{Hall}}/\beta)(\mathbf{E} \circ \mathbf{B}) \Rightarrow 0. \tag{2.14}$$

Thus the Bernoulli plasma process leads to a current $\mathbf{J}_B$ which is orthogonal to the $\mathbf{E}$ field and whose magnitude is proportional to the $\mathbf{B}$ field. To quote Landau and Lifshitz [1] "As we see, it (the Hall effect) gives rise to a current perpendicular to the electric field, whose magnitude is proportional to the magnetic field." The conclusion is that the Bernoulli Plasma process generates a Hall effect, and requires that the second Poincare coefficient must vanish. It follows that the Topological Hall effect exists in non-equilibrium systems where the 1-form A cannot be of Pfaff dimension 4. Bernoulli plasma processes are not dissipative in the sense that such that $(\mathbf{J}_B \circ \mathbf{E}) = 0$.

The appearance of a magnetic conductance, $\sigma_{_{Hall}}$, is novel to the topological format of electromagnetism as presented herein, and is deduced from the sole assumption that the Plasma current defines a process direction field that preserves the closed integrals of the 2-form, G. Plasma processes do not change the net charge within the closed integration domain. That is, charges can be produced only in equal and opposite pairs by a "Plasma process".

The conclusion is that the Hall effect is a topological property of electromagnetism, and can appear at all scales, from the microworld to the macroworld to the cosmological world. In domains where the non-equilibrium 3-forms of Topological Torsion and Topological Spin are closed, the Topological Hall effect can have an impedance multiplied by a rational fraction [2]. That is, the rational fraction Hall impedance is a topological result, independent from quantum theory.

References

[1] Landau, L. D. and Lifshitz, E. M., "Electrodynamics of Continuous Media" (Pergamon Press, London 1960) p. 97.

[2] Kiehn, R. M. (1991) " Are there three kinds of superconductivity", Int. Journ. of Modern Physics, vol5 #10 p. 1779