Applied Topology part 1: Cartan's Topological Structure

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Abstract

A system of differential forms will establish a topology and a topological structure on a domain of independent variables such that is possible to determine which evolutionary processes acting on the system are continuous. Of particular interest, to those many physical systems that can be encoded by means of a single 1-form of Action, is the Cartan topology generated by the single 1-form, and its Pfaff sequence of exterior differentials and their intersections. In the Cartan topology, the exterior differential becomes a limit point generator in the sense of Kuratowski, and the number of non-zero terms in the Pfaff sequence determines the Pfaff topological dimension (representing the minimum number of functions required to describe the 1-form generator). In particular, when the Pfaff dimension is 3 or more the Cartan topology becomes a disconnected topology, with non-zero topological torsion, which when combined with the constraint of continuous evolution establishes a logical arrow of time. These concepts do not depend upon the constraints of metric or connection. Classical dogma has constrained most deterministic physical theories to cases where the Pfaff dimension is 2 or less, for such is the domain of unique integrability. The more interesting domain of non-unique solutions and Pfaff dimension ≥ 4 leads to an understanding of thermodynamically irreversible processes without the use of statistics.

1. INTRODUCTION

In this presentation, a topological perspective will be used to extract those properties of physical systems and their evolution that are independent from the geometrical constraints of connections and/or metrics. It is subsumed that the presence of a physical system establishes a *topological structure* on a (possibly geometric) base space of independent variables. This concept is different from, but similar to, the geometric perspective of general relativity, whereby the presence of a physical system is presumed to establish a *metric* on a base space of independent variables. Note that a given base of independent variables may support many different topological structures; hence a given base may support many different physical systems. A major success of theory is that continuous non-homeomorphic processes of topological evolution establish a logical basis for the arrow of time [?]. This idea can be exploited to explain the concept of thermodynamic irreversibility without the use of statistics.

The fundamental axioms utilized in this article are:

Axiom 1. *The topological structure of Physical Systems on a domain of independent base variables can be encoded in terms of exterior differential forms (symbolically represented by A).*

Axiom 2. *Physical Processes can be defined in terms of contravariant vector direction fields, which may or may not be generators of 1-parameter groups, and in particular need not be homeomorphisms (symbolically represented by V).*

Axiom 3. *Equations of Continuous Evolution describing both reversible and irreversible Processes acting on Physical Systems are encoded by Cartan's magic formula :*

$$L_{(\mathbf{V})}A = i(\mathbf{V})dA + d(i(\mathbf{V})A) \tag{1.1}$$

In the period from 1899 to 1926, Eli Cartan developed his theory of exterior differential systems [1,2], which included the ideas of spinor systems [3] and the differential geometry of projective spaces and spaces with torsion [4]. The theory was appreciated by only a few contemporary researchers, and made little impact on the main stream of the physical sciences until about the 1960's. Even specialists in differential geometry (with a few notable exceptions [5]) made little use of Cartan's methods until the 1950's. Even today, many physical scientists and engineers have the impression that Cartan's theory of exterior differential forms is just another formalism of fancy.

However, Cartan's theory of exterior differential systems has several advantages over the methods of tensor analysis that were developed during the same period of time. The principle fact is that differential forms are well behaved with respect to functional substitution of C1 differentiable maps. Such maps need not be invertible even locally, yet differential forms are always deterministic in a retrodictive sense [6], by means of functional substitution. Such determinism is not to be associated with contravariant tensor fields, if the map is not a diffeomorphism. Cartan's theory of exterior differential systems contains topological information, and admits non-diffeomorphic maps which can describe topological evolution.

Although the word "topology" had not become popular when Cartan developed his ideas (topological ideas were described as part of the theory of analysis situs), there is no doubt that Cartan's intuition was directed towards a topological development. For example, Cartan did not define what were the open sets of his topology, nor did he use, in his early works, the words "limit points or accumulation points" explicitly, but he did describe the union of a differential form and its exterior differential as the "closure" of the form. With this concept, Cartan effectively used the idea that the closure of a subset is the union of the subset with its topological limit points. What was never stated (until 1990) is the idea that the exterior differential is indeed a limit point generator relative to a Cartan topology. The union of the identity operator and the exterior differential satisfy the axioms of a Kuratowski closure operator [7], which can be used to define a topology. The other operator of the Cartan calculus, the exterior product, also has topological connotations when it is interpreted as an intersection operator.

In a perhaps over simplistic comparison, it might be said that ubiquitous tensor methods are restricted to geometric applications, while Cartan's methods can be applied directly to topological concepts as well as geometrical concepts. Cartan's theory of exterior differential systems is a topological theory not necessarily limited by geometrical constraints and the class of diffeomorphic transformations that serve as the foundations of tensor calculus. A major objective of this article is to show how limit points, intersections, closed sets, continuity, connectedness and other elementary concepts of modern topology are inherent in Cartan's theory of exterior differential systems. These ideas do not depend upon the geometrical ideas of size and shape. Hence Cartan's theory, as are all topological theories, is renormalizeable (perhaps a better choice of words is that the topological components of the theory are independent from scale). In fact the most useful of Cartan's ideas do not depend explicitly upon the geometric ideas of a metric, nor upon the choice of a differential connection between basis frames, as in fiber bundle theories. The theme of this article is to explore the physical usefulness of those topological features of Cartan's methods which are independent from the constraints and refinements imposed by a connection and/or a metric.

In this article the Cartan topology will be constructed explicitly for an arbitrary exterior differential system, Σ. For a particular simple, but useful, system consisting of a single 1-form of Action, all elements of the Cartan topology will be evaluated, and the limit points, the boundary sets and the closure of every subset will be computed abstractly. Earlier intuitive results [7], which utilized the notion that Cartan's concept of the exterior product may be used as an intersection operator, and his concept of the exterior differential may used as a limit point operator acting on differential forms, will be given formal substance in this article. A major result of this article, with important physical consequences in describing topological evolutionary processes, is the demonstration that the Cartan topology is not necessarily a connected topology, unless the property of topological torsion vanishes, and that thermodynamic irreversibility is a consequence of 4 dimensions or more.

1.1. A Point Set Topology Example

As an example of a topological ideas, consider the set of 4 elements or points,

$$X : \{a, b, c, d\}. \tag{1.2}$$

and all possible subsets:

$$\emptyset, \tag{1.3}$$
$$\{a\}, \{b\}, \{c\}, \{d\}, \tag{1.4}$$
$$\{a, b\}, \{a, c\}, \{a, d\}, \{b, c\}, \{b, d\}, \{c, d\}, \tag{1.5}$$
$$\{a, b, c\}, \{a, c, d\}, \{b, c, d\}, \{a, b, d\}, \tag{1.6}$$
$$\{a, b, c, d\} = X \tag{1.7}$$

Select the following subset elements as a topological basis,

$$\text{basis selection} \quad \{a\}, \{a, b\}, \{c\}, \{c, d\}, \tag{1.8}$$

and then compose a topology $T4$ of open sets from all possible unions of the selected basis elements:

$$T4\{open\} : \emptyset, \{a\}, \{c\}, \{a,b\}, \{c,d\}, \{a,c\}, \{a,b,c\}, \{a,c,d\}, \{a,b,c,d\} \quad (1.9)$$

The closed sets are the compliments of the open sets:

$$T4\{closed\} : \{a,b,c,d\}, \{b,c,d\}, \{a,b,d\}, \{c,d\}, \{a,b\}, \{b,d\}, \{d\}, \{b\}, \emptyset \quad (1.10)$$

It is an easy exercise to demonstrate that the collections above indeed satisfy the axioms of a topology. (This is not the only topology that can be constructed over 4 elements).

This simple example of a point set topology permits explicit construction of all the topological concepts, which include limit sets, interiors, boundaries, and closures, for the all of subsets of X, relative to the topology, $T4$. The standard definitions are:

1. A limit point of a subset A is a point p such that all open sets that contain p also contain a point of A not equal to p.

2. The closure of a subset A is the union of the subset and its limit points, and is the smallest closed set that contains A.

3. The interior of a subset is the largest open set contained by the subset.

4. The exterior of a subset is the interior of its compliment.

5. A boundary of a subset is the set of points not contained in the interior or exterior.

6. The closure of a subset is also equal to the union of its interior and its boundary.

The results of applying these definitions to the $T4$ topology of 4 points are:

Table 1. **A T4 Topology of 4 points**

$X = \{a, b, c, d\}$

Basis subsets $\{a\}, \{a, b\}, \{c\}, \{c, d\}$

$T4\{open\} : \emptyset, \{a\}, \{c\}, \{a, b\}, \{c, d\}, \{a, c\}, \{a, b, c\}, \{a, c, d\}, X$

$T4\{closed\} : X, \{b, c, d\}, \{a, b, d\}, \{c, d\}, \{a, b\}, \{b, d\}, \{d\}, \{b\}, \emptyset$

Subset	**Limit Pts**	**Interior**	**Boundary**	**Closure**
$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
$\{a\}$	$\{b\}$	$\{a\}$	$\{b\}$	$\{a, b\}$
$\{b\}$	$\emptyset$	$\emptyset$	$\{b\}$	$\{b\}$
$\{c\}$	$\{d\}$	$\{c\}$	$\{d\}$	$\{c, d\}$
$\{d\}$	$\emptyset$	$\emptyset$	$\{d\}$	$\{d\}$
$\{a, b\}$	$\{b\}$	$\{a, b\}$	$\emptyset$	$\{a, b\}$
$\{a, c\}$	$\{b\}, \{d\}$	$\{a, c\}$	$\{b, d\}$	X
$\{a, d\}$	$\{b\}$	$\{a\}$	$\{b, d\}$	$\{a, b, d\}$
$\{b, c\}$	$\{d\}$	$\{c\}$	$\{b, d\}$	$\{b, c, d\}$
$\{b, d\}$	$\emptyset$	$\emptyset$	$\{b, d\}$	$\{b, d\}$
$\{c, d\}$	$\{d\}$	$\{c, d\}$	$\emptyset$	$\{c, d\}$
$\{a, b, c\}$	$\{b\}, \{d\}$	$\{a, b, d\}$	$\{d\}$	X
$\{b, c, d\}$	$\{d\}$	$\{c, d\}$	$\{b\}$	$\{b, c, d\}$
$\{a, c, d\}$	$\{b\}, \{d\}$	$\{a, c, d\}$	$\{b\}$	X
$\{a, b, d\}$	$\{b\}$	$\{a, b\}$	$\{d\}$	$\{b, c, d\}$
$\{a, b, c, d\}$	$\{b\}, \{d\}$	X	$\emptyset$	X

(1.11)

This $T4$ topology is quite interesting for many demonstrable reasons. First note that the all of the singletons of the topology are not closed. This implies that the topology is NOT a metric topology, NOT a Hausdorf topology, and even does NOT satisfy the separation axioms to be a T_1 topology. Note that all closed sets contain all of their limit points. Some open sets can contain limit points, but some open sets do not contain their limit points. Some subsets have boundaries that are composed of their limit points. Some subsets have limit points which are not boundary points. Certain subsets have a boundary, but do not have limit points, and in other cases there are subsets that have limit points, but do not have a boundary. There are certain subsets with a boundary, but without an interior. There are certain subsets with an interior, but without a boundary. These situations, though topologically correct, are not always intuitive to those accustomed to metric based topological concepts, which impose a number of additional constraints on the sets of interest. Yet all of these topological ideas, including the non-intuitive ones, are easy to grasp from the simple example of the $T4$ point set topology.

One other very important observation is that there are subsets of the $T4$ topology, $\{a, b\}$ and $\{c, d\}$, (other than $\emptyset$ and X) which are both open and closed. The union of these two subsets $\{a, b\}$ and $\{c, d\}$ is X. Topologies with this property are said to be disconnected topologies. What is important is that it is possible to construct a continuous map from a disconnected topology to a connected topology, but it is impossible to construct a continuous map from a connected topology to a disconnected topology. If the mapping process is interpreted as an evolutionary process, these facts establish a logical or topological basis for the arrow of time [?]. This idea will be exploited to explain the concept of thermodynamic irreversibility without the use of statistics.

What is even more remarkable is that properties of the $T4$ topology can be replicated in terms of the Pfaff sequence of exterior differential sets,

$$\text{Pfaff Sequence} : \{A, dA, A\hat{}dA, dA\hat{}dA...\}, \tag{1.12}$$

generated from any given 1-form of Action, A, on a N dimensional variety. The Pfaff sequence is readily computed, and will contain $M \leq N$ elements, where M is defined as the Pfaff topological dimension (or class) of the given 1-form, A. The realization of a $T4$ topology in terms of exterior differential forms is herein defined as the "Cartan topology", and is detailed in the next section. The Cartan topology has far reaching consequences in applications to physical problems.

1.2. Algebraic and Differential Closure

The concept of closure is one of the most important ideas in Cartan's theory. His methods center on two procedures of closure, one algebraic, and one differential. Both processes are closed in the sense that when they operate on a subset of a set of exterior differential forms, the objects created are also subsets of the set of exterior differential forms. There are no surprises. Cartan utilized the exterior algebra over a variety of dimension N to construct a vector space of exterior differential forms of dimension 2^N. The N subspaces of this (Grassmann) space are vector spaces of dimension equal to N things taken p at a time. The elements of the subspaces are called p-forms. In 4 dimensions, the subspace sets are 1 dimensional, N=4 dimensional, N(N+1)/2=6 dimensional, N=4 dimensional, and 1 dimensional. The elements of the subspaces are often called scalars (0-forms), vectors (1-forms), tensors (2-forms), pseudovectors (3-forms), pseudo-scalars (4-forms) in relativistic physical theories. The Exterior (Grassmann) algebra has a finite 2^N basis (equal to 16 elements in a space of 4 independent variables). The concept of closure means that the operations on elements of the 2^N dimensional space yield results that are contained within the 2^N dimensional space. When the operations are applied to elements of a subspace, the results usually are not contained in the same subspace, but they are contained within the 2^N dimensional vector space of p forms.

The exterior product (with symbol ˆ) takes elements of the 2^N base space and multiplies them together in a manner such that the result is contained as an element of the 2^N base space. This process of exterior multiplication is closed, for the action of the process on any subset of the 2^N base space produces another subset of the 2^N base space. However, the exterior product takes a p-form times a q-form into a p+q form. The elements of the product can be from different or from the same vector subspaces, but the resultant is always a linear combination of the subspaces of the Exterior algebra.

Similarly the concept of exterior differentiation (with symbol d) is defined such that the operation produces a p+1 form from a p-form. This process of exterior differentiation is "closed", for the action of the process on any subset of the 2^N base space produces another subset of the 2^N base space. A differential ideal is defined as the union of a collection of given p-forms and their exterior derivatives.

An "interior" product with respect to a direction field $\mathbf{V}$ (with symbol $i(\mathbf{V})$ and of dimension N) can be defined on the Grassmann algebra of exterior differential forms. The interior product takes a p-form to a p-1 form, and in this sense is another operation which is closed within the Grassman algebra. The resultant product is still an element of the 2^N base space. Where the exterior differential raises the rank of a p-form to a p+1 form, the inner product lowers the rank of a p-form to a p-1form. (There are other useful operators that lower the rank of the exterior differential p-form, and involve integration.)

By composition of the exterior derivative and the inner product operators, the Lie differential operator (with symbol $L_{(\mathbf{V})} = i(\mathbf{V})d + di(\mathbf{V})$) can be constructed, such that when the Lie differential operates on an exterior p-form, the resultant object is another p-form. For a 1-form of Action, A, the process reads:

$$L_{(\mathbf{V})}A = i(\mathbf{V})dA + d(i(\mathbf{V})A) = Q. \tag{1.13}$$

The resultant is not only closed relative to the Grassmann algebra, it also remains within the same Grassman vector subspace. The Lie differential does not depend upon a metric nor upon a connection. When the Lie differential acting on a p-form vanishes, the p-form is said to be an invariant of the process, $\mathbf{V}$. When the Lie derivative of a p-form does not vanish, the topological features of the resultant p-form permit the processes, $\mathbf{V}$, that produce such a result, to be put into equivalence classes, depending on the Pfaff dimension of the resultant form. For example, if in the formula given above for a 1-form, A, yields a result Q such that $dQ = 0$, then the process $\mathbf{V}$ belongs to the class of process known as Hamiltonian processes in mechanics, and to the Helmholtz class of processes that conserve vorticity in Hydrodynamics. Of particular interest to this article are processes where Q is of Pfaff dimension greater than 2. The Pfaff sequence constructed from Q contains three or more elements. Such processes are thermodynamically irreversible.

The Lie differential will be used extensively in physical applications of Cartan's theory, especially to the study of processes that involve topological evolution. The perhaps more familiar covariant derivative, highly constrained by connection or metric assumptions, is a special case of the Lie differential. The use of the covariant derivative leads to useful, but limited, physical theories for which the description of topological evolution is awkward, if not impossible.

1.3. The Exterior Product and Set Intersection

Cartan's theory of exterior differential systems has its foundations in the Grassmann algebra, where the two combinatorial processes are defined to produce algebraic and differential closure. The algebra is based upon the concepts of vector space addition, and an algebraic closure multiplication process now called the exterior product [8]. The Cartan calculus is defined in terms of the another closure operator now called the exterior differential[1]. In that which follows the operators of the exterior product and exterior differential will be applied to objects defined as exterior differential p-forms.

An exterior differential p-form is a function of independent variables, x^v, and their differentials, dx^μ. An exterior differential 1-form, A, is given by the expression,

$$A = A_\mu(x^v)dx^\mu. \tag{1.14}$$

[1]Cartan originally defined the calculus operation as the exterior derivative. Then in the later years defined calculus operation as the exterior differential.

The Cartan operations of exterior product (symbol ˆ) and exterior differential (symbol d), when operating on 1-forms, A and B, obey the rules

$$A\hat{}A = 0, \tag{1.15}$$

$$A\hat{}B = -B\hat{}A. \tag{1.16}$$

and

$$dA = d(A_\mu\, dy^\mu) = (dA_\mu)\hat{}dy^\mu + A_\mu\, d(dy^\mu) \tag{1.17}$$

$$= (dA_\mu)\hat{}dy^\mu + 0 \tag{1.18}$$

$$d(A\hat{}B) = dA\hat{}B - A\hat{}dB. \tag{1.19}$$

The non-zero product, $A\hat{}B$, defines an exterior differential 2-form; the product of three 1-forms defines a 3-form; etc.. For more detail consult Flanders [?] or Liebermann [?].

In simple cases, a 1-form can be constructed from the differential of an ordinary function. In such cases, the coefficients of the 1-form are proportional to the gradient of the function.

$$A = A_\mu dx^\mu = \nabla\phi \cdot d\mathbf{r} = (\partial\, \boldsymbol{\phi}/\partial x^\mu)dx^\mu \tag{1.20}$$

In surface theory, the gradient is classically interpreted as vector direction field orthogonal to the implicit surface, $\phi(x^\mu) = 0$. Consider the simple case where the 1-forms A and B each have coefficients which form the components of (different) gradient fields,

$$A = A_\mu dx^\mu = \nabla\phi \cdot d\mathbf{r} \qquad B = B_\mu dx^\mu = \nabla\psi \cdot d\mathbf{r}. \tag{1.21}$$

Do the two implicit (curved) surfaces intersect? The answer is yes if the two surfaces have points in common. The classic analysis in 3D says there is a curve of points in common defined by a non-zero value of the Gibbs cross product of the two gradient fields:

$$\text{Intersection of two implicit surfaces: } \mathbf{J} = \nabla\phi \times \nabla\psi \neq 0. \tag{1.22}$$

Note that (in 3 dimensions) the exterior product of the two 1-forms has coefficients exactly equal to the Gibbs cross product:

$$A\hat{}B = \mathbf{J}_z dx\hat{}dy\ + \mathbf{J}_x dy\hat{}dz + \mathbf{J}_y dz\hat{}dx. \tag{1.23}$$

This result pictorially cements the notion that the exterior product (acting on 1-forms) is an operator related to the concept of intersection. If the two surfaces do not intersect, the exterior product vanishes, and then the direction fields of the gradients of ϕ and χ are proportional to one another. The two functions, ϕ and χ, are not functionally independent if the exterior product vanishes.

These concepts extend to 1-forms which are not representable by gradient fields, and to p-forms of higher rank. If the exterior product of two p-forms is not zero, then the p-forms have non-zero intersections. The coefficient functions are functionally independent.

An exterior differential 1-form A is deterministic, as a predictive (or retrodictive) invariant, with respect to all tensor diffeomorphisms. The coefficient functions, $A_\mu(x^v)$, are presumed to behave as a co-variant vector, and the differentials, dx^μ, behave as a contravariant vector, with respect to tensor diffeomorphisms. (Exterior differential form densities will be discussed later.) However, the exterior differential 1- form, and hence all p-forms, are also well behaved with respect to a larger class of transformations, which contain the tensor diffeomorphisms as special cases. The exterior differential 1- form is deterministic in a retrodictive sense (but not in a predictive sense) with respect to C1 mappings that do not have a local or a global inverse. These C1 mappings do not preserve all topological features, where diffeomorphisms of tensor theory, are special cases of homeomorphisms, which do preserve all topological properties. These extraordinary features demonstrate that Cartan's theory is not just another formalism of fancy, and goes well beyond the theory of tensor analysis. In fact, these features of exterior differential forms can be exploited to develop something that has slipped through the net of tensor analysis: a non-statistical theory of thermodynamic irreversibility.

1.4. The Exterior Differential and Limit Points

The second closure operator found in Cartan's theory of exterior differential systems is the exterior differential. The exterior differential, like the exterior product, also has topological connotations when applied to differential forms, but the results are sometimes surprising and unfamiliar. Where the exterior product is related to the topological concept of set intersection, the exterior differential is related to the topological idea of limit points. It will be demonstrated that:

Theorem 1.1. *With respect to the Cartan topology, the exterior differential is a limit point generator.*

The exterior differential is a differential operator which takes the p-forms into p+1 forms. Hence, like the exterior product, the exterior differential generates a vector in a different vector subspace of the exterior algebra.

$$d(\omega^p) \Rightarrow \omega^{p+1}. \tag{1.24}$$

The exterior differential of a function (0-form) is equivalent to the total differential of a scalar function, and yields a 1-form with coefficients proportional to the gradient field. The exterior differential of a 1-form is defined as

$$\begin{aligned} d\omega^1 &= d(A_b\, dy^b) = (dA_b)\hat{}dy^b + A_b\, d(dy^b) \\ &= (\partial A_b/\partial y^e\, dy^e)\hat{}dy^b + 0 \\ &= (\partial A_b/\partial y^e - \partial A_e/\partial y^b)\, dy^e\, \hat{}\, dy^b \\ &= F_{[eb]}\, dy^{[eb]} = F_{[H]} dy^{[H]}. \end{aligned} \tag{1.25}$$

It has been assumed that $dd(\omega^p) = 0$. The collective index notation $[H] = [eb]$ permits the formula defining exterior differentiation to be generalized:

$$d\omega^p = d(A_H\, dy^H) = (dA_H)\hat{}dy^H \tag{1.26}$$
$$= (\{\partial A_H/\partial y^e\}\, dy^e)\hat{}dy^H \tag{1.27}$$

Other properties of the exterior differential will be exemplified by the rules for distributing the operator over a product of 1-forms, A and B,

$$d(A\hat{}B) = dA\hat{}B - A\hat{}dB. \tag{1.28}$$

It can be shown that the operator $KCl = I \cup d$, where I is the identity and d is the exterior differential, acting on a system of differential forms satisfies the "Kuratowski closure" axioms [?], and therefor can be used to define a topology. Starting from a single 1-form, A, on a 4 dimensional space, it is possible to generate the Pfaff Sequence

$$Pfaff\ Sequence : \{A, dA, A\hat{}dA, dA\hat{}dA\} \tag{1.29}$$
$$= \{A, F, H, K\}. \tag{1.30}$$

The subsets of the Cartan topological space consist of all possible unions of the subsets that make up the Pfaff sequence. The Cartan topology will be constructed from a topological basis which consists of the odd elements of the Pfaff sequence, and their closures:

$$\textit{the Cartan topological base}: \{A, KCl(A), A\hat{}dA, KCl(A\hat{}dA)\}. \tag{1.31}$$

When applied to the Pfaff sequence generated by a single 1-form of Action, A, on a space of 4 dimensions, the base elements correspond to the set

$$\textit{the Cartan topological base} \quad : \quad \{A, A \cup F, H, H \cup K\} \tag{1.32}$$

$$\text{compare to the point set example above} \quad \approx \quad \{a, b, c, d\}\ . \tag{1.33}$$

When it is realized that the exterior differential acts a limit point generator, it becomes apparent why Cartan referred to the union of Σ and $d\Sigma$ as the closure of Σ,

$$\text{Closure} = (KCl) \circ \Sigma = (I \cup d) \circ \Sigma = \Sigma + d\Sigma = \text{subset} + \text{limit points}. \tag{1.34}$$

In the next section, the topological features of the Cartan topology, based on the Cartan topological base, will be worked out in detail. It will turn out that the Cartan topology can be put into correspondence with the T4 topology of 4 points displayed in a previous section. It will be evident, indeed, that the exterior differential is a limit point generator for any subset relative to the Cartan topology. This is a remarkable result, for as will be demonstrated below, all C2 vector fields acting through the concept of the Lie differential on a set of differential forms, with C2 coefficients, generate continuous transformations with respect to the Cartan topology. Moreover, the Cartan topology is disconnected if $A\hat{}dA \neq 0$. As the conditions for unique integrability of the 1-form A are given by the Frobenius theorem, which requires $A\hat{}dA = 0$, it should be expected that one of the features of the disconnected Cartan topology is that if solutions exist, they are not unique.

2. The Cartan "Point Set" Topology.

Cartan built his theory around an exterior differential system, Σ, which consists of a collection of 0- forms, 1-forms, 2-forms, etc. [12]. He defined the closure of this collection as the union of the original collection with those forms which are obtained by forming the exterior differentials of every p-form in the initial collection. In general, the collection of exterior differentials will be denoted by $d\Sigma$, and the closure of Σ by the symbol, $KCl(\Sigma)$, where

$$\text{Kuratowski Closure operator:} \quad KCl(\Sigma) = \Sigma \cup d\Sigma \tag{2.1}$$

For notational simplicity in this article the systems of p-forms will be assumed to consist of the single 1-form, A. Then the exterior differential of A is the 2-form $F = dA$, and the closure of A is the union of A and $F : KCl(A) = A \cup F$. The other logical operation is the concept of intersection, so that from the exterior differential it is possible to construct the set $A\hat{}F$ defined collectively as $H : H = A\hat{}F$. The exterior differential of H produces the set defined as $K = dH$, and the closure of H is the union of H and $K : KCl(H) = H \cup K$.

This ladder process of constructing exterior differentials, and exterior products, may be continued until a null set is produced, or the largest p-form so constructed is equal to the dimension of the space under consideration. The set so generated is defined as a Pfaff sequence. The largest rank of the sequence determines the Pfaff dimension of the domain (or class of the form, [13]), which is a topological invariant. The idea is that the 1-form A (in general the exterior differential system, Σ) generates on space-time four equivalence classes of points that act as domains of support for the elements of the Pfaff sequence, A, F, H, K . The union of all such points will be denoted by $X = A \cup F \cup H \cup K$. The fundamental equivalence classes are given specific names:

$$\text{Topological ACTION:} \quad A = A_\mu dx^\mu \tag{2.2}$$

$$\text{Topological VORTICITY:} \quad F = dA = F_{\mu\nu} dx^\mu \hat{}dx^\nu \tag{2.3}$$

$$\text{Topological TORSION:} \quad H = A\hat{}dA = H_{\mu\nu\sigma} dx^\mu \hat{}dx^\nu \hat{}dx^\sigma \tag{2.4}$$

$$\text{Topological PARITY:} \quad K = dA\hat{}dA = K_{\mu\nu\sigma\tau} dx^\mu \hat{}dx^\nu \hat{}dx^\sigma \hat{}dx^\tau . \tag{2.5}$$

The Cartan topology is constructed from a basis of open sets, which are defined as follows: first consider the domain of support of A. Define this "point" by the symbol A. A is the first open set of the Cartan topology. Next construct the exterior differential, $F = dA$, and determine its domain of support. Next, form the closure of A by constructing the union of these two domains of support, $KCl(A) = A \cup F$. $A \cup F$ forms the second open set of the Cartan topology.

Next construct the intersection $H = A\hat{\ }F$, and determine its domain of support. Define this "point" by the symbol H, which forms the third open set of the Cartan topology. Now follow the procedure established in the preceding paragraph. Construct the closure of H as the union of the domains of support of H and $K = dH$. The construction forms the fourth open set of the Cartan topology. In four dimensions, the process stops, but for N > 4, the process may be continued.

Now consider the basis collection of open sets that consists of the subsets,

$$B = \{A,\ KCl(A),\ H,\ KCl(H)\} = \{A,\ A \cup F,\ H,\ H \cup K\} \tag{2.6}$$

The collection of all possible unions of these base elements, and the null set, $\emptyset$, generate the Cartan topology of open sets:

$$T(open) = \{X,\ \emptyset,\ A,\ H, A \cup F,\ H \cup K,\ A \cup H,\ A \cup H \cup K,\ A \cup F \cup H\}. \tag{2.7}$$

These nine subsets form the open sets of the Cartan topology constructed from the domains of support of the Pfaff sequence constructed from a single 1-form, A, in 4 dimensions. The compliments of the open sets are the closed sets of the Cartan topology.

$$T(closed) = \{\emptyset,\ X,\ F \cup H \cup K,\ A \cup F \cup K, A \cup F,\ \ H \cup K, F \cup K, F,\ K\}. \tag{2.8}$$

From the set of 4 "points" $\{A, F, H, K\}$ that make up the Pfaff sequence it is possible to construct 16 subset collections by the process of union. It is possible to compute the limit points for every subset relative to the Cartan topology. The classical definition of a limit point is that a point p is a limit point of the subset Y relative to the topology T if and only if for every open set which contains p there exists another point of Y other than p [14]. The results of this and other standard definitions are presented in Table 2, and are to be compared to Table 1.

Table 2. The Cartan T4 Topology

A 1-form in 4D: $A = A_k(x)dx^k$

$X = \{A, \quad F = dA, \quad H = A\hat{}F, \quad K = F\hat{}F\}$

Basis subsets {A, $KCl(A)$, H, $KCl(H)\} = \{A, A \cup F, H, H \cup K\}$

$T(open) = \{X, \emptyset, A, H, A \cup F, H \cup K, A \cup H, A \cup H \cup K, A \cup F \cup H\}$

$T(closed) = \{\emptyset, X, F \cup H \cup K, A \cup F \cup K, H \cup K, A \cup F, F \cup K, F, K\}$

Subset	**Limit Pts**	**Interior**	**Boundary**	**Closure**
σ	$d\sigma$	.	$\partial\sigma$	$\sigma \cup d\sigma$
$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
A	F	A	F	$A \cup F$
F	$\emptyset$	$\emptyset$	F	F
H	K	H	K	$H \cup K$
K	$\emptyset$	$\emptyset$	K	K
$A \cup F$	F	$A \cup F$	$\emptyset$	$A \cup F$
$A \cup H$	F, K	$A \cup H$	$F \cup K$	X
$A \cup K$	F	A	$F \cup K$	$A \cup F \cup K$
$F \cup H$	K	H	$F \cup K$	$F \cup H \cup K$
$F \cup K$	$\emptyset$	$\emptyset$	$F \cup K$	$F \cup K$
$H \cup K$	K	$H \cup K$	$\emptyset$	$H \cup K$
$A \cup F \cup H$	F, K	$A \cup F \cup K$	K	X
$F \cup H \cup K$	K	$H \cup K$	F	$F \cup H \cup K$
$A \cup H \cup K$	F, K	$A \cup H \cup K$	F	X
$A \cup F \cup K$	F	$A \cup F$	K	$A \cup F \cup K$
X	F, K	X	$\emptyset$	X

(2.9)

By examining the set of limit points so constructed for every subset of the Cartan system, and presuming that the functions that make up the forms are C2 differentiable (such that the Poincare lemma is true, $dd\omega = 0, any\ \omega$), it is easy to show that for all subsets of the Cartan topology every limit set is composed of the exterior differential of the subset, thereby proving the claim that the exterior differential is a limit point operator relative to the Cartan topology. For example, the open subset, $A \cup H$, has the limit points that consist of F and K. The limit set consists of $F \cup K$ which can be derived directly by taking the exterior differentials of the elements that make up $A \cup H$; that is, $(F \cup A = d(A \cup H) = (dA \cup dH)$. Note that this open set, $A \cup H$, does not contain its limit points. Similarly for the closed set, $A \cup F$, the limit points are given by F which may be deduced by direct application of the exterior differential to $(A \cup F) : (F) = d(A \cup F) = (dA \cup dF) = (F \cup \emptyset) = (F)$.

3. Topological Torsion and Connected vs. Non-connected Cartan topologies.

Topological torsion of a 1-form is defined as the exterior product of the 1-form and its exterior derivative. Topological torsion is different from, but can be related to, the Frenet torsion of a space curve and the affine torsion of a connection. If non-zero, Topological torsion has important topological properties. The Cartan topology as given in Table 2 is composed of the union of two sub-sets which are both open and closed,

$$(X = KCl(A) \cup KCl(H) = \{A \cup F\} \cup \{H \cup K\}), \tag{3.1}$$

a result that implies that the Cartan topology is not necessarily a connected topology. An exception exists if the topological torsion, H, and hence its closure, vanishes, for then the Cartan topology is connected. This extraordinary result has broad physical consequences. The connected Cartan topology based on a vanishing topological torsion is at the basis of most physical theories of equilibrium. In mathematics, the connected Cartan topology corresponds to the Frobenius integrability condition for Pfaffian forms. In thermodynamics, the connected Cartan topology is associated with the Caratheodory concept of inaccessible thermodynamic states [15], and the existence of an equilibrium thermodynamic surface. If the non-exact 1-form, Q, of heat generates a Cartan topology of null topological torsion, $H = Q\hat{}dQ = \emptyset$, then the Cartan topology built on Q is connected. Such systems are "isolated" in a topological sense, and the heat 1-form has a representation in terms of two and only two functions, conventionally written as: $Q = TdS$. Note again that a fundamental physical concept, in this case the idea of equilibrium, is a topological concept independent from geometrical properties of size and shape. Processes that generate the 1-form Q such that $Q\hat{}dQ = \emptyset$ are thermodynamically reversible. If $Q\hat{}dQ \neq \emptyset$, the process that generates Q is thermodynamically irreversible.

When the Cartan topology is connected, it might be said that all forces are extendible over the whole of the set, and that these forces are of "long range". Conversely when the Cartan topology is disconnected, the "forces" cannot be extended indefinitely over the whole domain of independent variables, but perhaps only over a single component. The components are not arc connected. In this sense, such forces are said to be of short range, as they are confined to a specific component. Note that this notion of short or long range forces does not depend upon geometrical size or scale. The physical idea of short or long range forces is a topological idea of connectivity, and not a geometrical concept of how far.

In an earlier article, these ideas were formulated intuitively in order to give an explanation of the "four forces" of physics. The earlier work was based upon experience with differential geometry [16]. The features of the Pfaff sequence were used to establish equivalence classes for 1-forms constructed from known example metric field solutions, $g_{\mu v}$, to the Einstein field equations. The original ideas, based upon experience with systems in differential geometry, can now be given credence based upon differential topology. The construction of a 1-form, $A = g_{\mu 4}dx^{\mu}$, whose coefficients are the space time components of a metric tensor, will divide the topology into equivalence classes depending upon the number of non-zero elements of its Pfaff sequence. This number has been defined above as the Pfaff topological dimension. Long range parity preserving forces due to gravity (Pfaff dimension 1) and electromagnetism (Pfaff dimension 2) are to be associated with a Cartan Topology that is connected ($H = A\hat{}F = A\hat{}dA = 0$). Both the strong force (Pfaff dimension 3) and the weak force (Pfaff dimension 4) are "short" range ($H \neq 0$) and are to be associated with a disconnected Cartan topology. The strong force is parity preserving ($K = 0$) and the weak force is not ($K \neq 0$). The fact that the Cartan topology is not necessarily connected is the topological (not metrical) basis that may be used to distinguish between short and long range forces.

In much of our physical experience with nature, it appears that the disconnected domains of Pfaff dimension 3 or more are often isolated as nuclei, while the surrounding connected domains of Pfaff dimension 2 or less appears as fields of charged or non-charged molecules and atoms. However, part of the thrust of this article is to demonstrate that such disconnected topological phenomena are not confined to microscopic systems, but also appear in a such mundane phenomena as the flow of a turbulent fluid. Physical examples of the existence of topological torsion (and hence a non-connected Cartan topology) are given by the experimental appearance of what appear to be coherent structures in a turbulent fluid flow.

To prove that a turbulent flow must be a consequence of a Cartan topology that is not connected, consider the following argument: First consider a fluid at rest and from a global set of unique, synchronous, initial conditions generate a vector field of flow. Such flows must satisfy the Frobenius complete integrability theorem, which requires that $A\hat{}dA = 0$. The Cartan topology for such systems is connected, and the Pfaff dimension of the domain is 2 or less. Such domains do not support topological torsion (the Helicity vanishes). Such globally laminar flows are to be distinguished from flows that reside on surfaces, but do not admit a unique set of connected sychronizeable initial conditions. Next consider turbulent flows which, as the anti-thesis of laminar flows, can not be integrable in the sense of Frobenius; such turbulent domains support topological torsion ($A\hat{}dA \neq 0$), and therefore a disconnected Cartan topology. The connected components of the disconnected Cartan topology can be defined as the (topologically) coherent structures of the turbulent flow.

Note that a domain can support a homogeneous topology of one component and then undergo continuous topological evolution to a domain with some interior holes. The domain is simply connected in the initial state, and multiply connected in the final state, but still connected. However, consider the dual point of view where the originally connected domain consists of a homogeneous space that becomes separated into multiple components. The evolution to a topological space of multiple components is not continuous. It follows that the case of a transition from an initial laminar state ($H = 0$) to the turbulent state ($H \neq 0$) is a transition from a connected topology to a disconnected topology. Therefore the transition to turbulence is NOT continuous. However, note that the decay of turbulence can be described by a continuous transformation from a disconnected topology to a connected topology. Condensation is continuous, gasification is not. It is demonstrated below that relative to the Cartan topology all C2 differentiable, $\mathbf{V}$, acting on C2 p-forms by means of the Lie differential are continuous. The conclusion is reached that the transition to turbulence must involve transformations that are not C2, hence can occur only in the presence of shocks or tangential discontinuities.

4. The Cartan Topological Structure

A topological structure is defined to be enough information to decide whether a transformation is continuous or not [18]. The classical definition of continuity depends upon the idea that every open set in the range must have an inverse image in the domain. This means that topologies must be defined on both the initial and final state, and that somehow an inverse image must be defined. Note that the open sets of the final state may be different from the open sets of the initial state, because the topologies of the two states can be different.

There is another definition of continuity that is more useful for it depends only on the transformation, and not its inverse, explicitly. A transformation is continuous if and only if the image of the closure of every subset is included in the closure of the image. This means that the concept of closure and the concept of transformation must commute for continuous processes. Suppose the forward image of a 1-form A is Q, and the forward image of the set $F = dA$ is Z. Then if the closure, $KCl(A) = A \cup F$ is included in the closure of $KCl(Q) = Q \cup dQ$, for all sub-sets, the transformation is defined to be continuous. The idea of continuity becomes equivalent to the concept that the forward image Z of the limit points, dA, is an element of the closure of Q [18]:

> A function that produces an image $f[A] = Q$ is continuous iff for every subset A of the Cartan topology, $Z = f[dA] \subset KCl(Q) = (Q \cup dQ)$.

The Cartan theory of exterior differential systems can now be interpreted as a topological structure, for every subset of the topology can be tested to see if the process of closure commutes with the process of transformation. For the Cartan topology, this emphasis on limit points rather than on open sets is a more convenient method for determining continuity. A simple evolutionary process, $X \Rightarrow Y$, is defined by a map Φ. The map, Φ, may be viewed as a propagator that takes the initial state, X, into the final state, Y. For more general physical situations the evolutionary processes are generated by vector fields of flow, $\mathbf{V}$. The trajectories defined by the vector fields may be viewed as propagators that carry domains into ranges in the manner of a convective fluid flow. The evolutionary propagator of interest to this article is the Lie differential with respect to a vector field, $\mathbf{V}$, acting on differential forms, Σ [19].

The Lie differential has a number of interesting and useful properties.

1. The Lie differential does not depend upon a metric or a connection.

2. The Lie differential has a simple action on differential forms producing a resultant form that is decomposed into a transversal and an exact part:

$$L_{(\mathbf{V})}\omega = i(V)d\omega + di(V)\omega. \tag{4.1}$$

This formula is known as "Cartan's magic formula". For those vector fields V which are "associated" with the form ω, such that $i(V)\omega = 0$, the Lie differential becomes equivalent to the covariant differential of tensor analysis. Otherwise the two differential concepts are distinct.

3. The Lie differential may be used to describe deformations and topological evolution. Note that the action of the Lie differential on a 0-form (scalar function) is the same as the directional or convective differential of ordinary calculus,

$$L_{(\mathbf{V})}\Phi = i(V)d\Phi + di(V)\Phi = i(V)d\Phi + 0 = \mathbf{V} \cdot grad\Phi. \tag{4.2}$$

It may be demonstrated that the action of the Lie differential on a 1-form will generate equations of motion of the hydrodynamic type. In fact Arnold calls the Lie differential the "convective" or "Fisherman's" differential.

4. With respect to vector fields and forms constructed over C2 functions, the Lie differential commutes with the closure operator. Hence, the Lie differential (restricted to C2 functions) generates transformations on differential forms which are continuous with respect to the Cartan topology. To prove this claim:

First construct the closure, $\{\Sigma \cup d\Sigma\}$. Next propagate Σ and $d\Sigma$ by means of the Lie differential to produce the decremental forms, say Q and Z,

$$L_{(\mathbf{V})}\Sigma = Q \quad and \quad L_{(\mathbf{V})}d\Sigma = Z. \tag{4.3}$$

Now compute the contributions to the closure of the final state as given by $\{Q \cup dQ\}$. If $Z = dQ$, then the closure of the initial state is propagated into the closure of the final state, and the evolutionary process defined by $\mathbf{V}$ is continuous. However,

$$dQ = dL_{(\mathbf{V})}\Sigma = di(V)d\Sigma + dd(i(V)\Sigma) = di(V)d\Sigma \tag{4.4}$$

as $dd(i(V)\Sigma) = 0$ for C2 functions. But,

$$Z = L_{(\mathbf{V})}d\Sigma = d(i(V)d\Sigma) + i(V)dd\Sigma = di(V)d\Sigma \tag{4.5}$$

as $i(V)dd\Sigma = 0$ for C2 p-forms. It follows that $Z = dQ$, and therefore $\mathbf{V}$ generates a continuous evolutionary process relative to the Cartan topology. QED It is to be noticed that this concept of a topological structure is developed in terms of the action of the Lie differential acting on a 1-form. The method does not depend upon metric or connection.

Certain special cases arise for those subsets of vector fields that satisfy the equations, $d(i(\mathbf{V})\Sigma) = 0$. In these cases, only the functions that make up the p-form, Σ, need be C2 differentiable, and the vector field need only be C1.

Such processes will be of interest to symplectic processes, with Bernoulli-Casimir invariants.

By suitable choice of the 1-form of action it is possible to show that the action of the Lie differential on the 1-form of action can generate the Navier Stokes partial differential equations [20]. The analysis above indicates that C2 differentiable solutions to the Navier-Stokes equations can not be used to describe the transition to turbulence. The C2 solutions can, however, describe the irreversible decay of turbulence to the globally laminar state.

5. APPLICATIONS

5.1. Maxwell Equations

Maxwell's PDE's are topological statements deduced from an exterior differential system. The two postulates are

$$\text{The Postulate of Potentials:} \quad F - dA = 0. \tag{5.1}$$

$$\text{The Postulate of conserved Charge Current densities:} \quad \overline{J} - d\overline{G} = 0. \tag{5.2}$$

No constraints of geometrical connection or metric are required. Such geometric constraints refine the Maxwell topology, and are useful for understanding constitutive equations that distinguish, for example, birefringent media from optically active media. The Maxwell-Faraday PDE's are not restricted to spaces of topological dimension N = 4. For an exterior differential system $F - dA = 0$ on a space of any dimension N>3, the closure conditions, $ddA = dF = 0$, always yield the same identical Maxwell-Faraday PDE's for the first 4 variables. Additional PDE's are also generated for N>4, but the system of PDE's created forms a nested set, with the Maxwell-Faraday equations as topological kernel, of invariant format for any dimension N. A remarkable result is that Faraday induction is a topological idea, and does not depend upon metric or connection. The concept of Faraday induction applies to any system that satisfies the Postulate of Potentials.

As demonstrated below, the Postulate of Potentials establishes the field intensities, **E** and **B**, (think forces),and the Postulate of Conserved Charge current densities establishes the field excitations, **D** and **H**, (think sources). The topological perspective subsumes that the two species are independent ideas. The experimental justification of such ideas can be demonstrated with a simple parallel plate capacitor experiment. First connect the plates to a battery of constant potential and let it remain connected. Insert a slab of plastic dieletric halfway between the plates. Release the plastic slab. Does the slab remain motionless, or is the motion such that the slab is expelled or attracted? For a second experiment, attach the plates of the capacitor to a battery and then disconnect the battery after charging the capacitor. Now insert the plastic slab halfway, and release it. Does the slab remain motionless, or is the motion such that the slab is expelled or attracted? In the first case, the **E** field remains constant (the potential does not change), and motion of the dielectric slab causes the **D** field to change (the battery adjusts the charge distribution). In the second experiment, the charge distribution is constant, so that the **D** field remains constant, but the **E** field changes. Consider the simple constitutive constraint, $\mathbf{D} = \varepsilon\mathbf{E}$. In the first experiment, insertion would cause the average ε to increase, hence even though **E** remains constant, the **D** field would increase. However, the total energy density $\mathbf{D} \circ \mathbf{E}$ would decrease if the slab was expelled, and that is what happens. In the second experiment, motion of the slab would cause the **E** field to change, as the **D** field remains constant, and the minimum energy density occurs when the slab is fully inserted.

Current electromagnetic dogma presents the idea that from a given charge current density distribution, $[\mathbf{J}, \rho]$,it is possible to deduce the **E** and **B** fields. However, the Postulate of conserved Charge-Current densities indicates that it is **D** and **H** that are the related quantities, not **E** and **B**. The Postulate of Potentials indicates that the field intensities **E** and **B** are deduced from the potentials $[\mathbf{A}, \phi]$. It takes some constitutive constraint to convert **D** and **H** into **E** and **B**, or $[\mathbf{J}, \rho]$ into $[\mathbf{A}, \phi]$. Both types of constraints appear in the literature in great detail and variety. Such assumptions obscure the topological basis and differences between exterior differential forms and exterior differential form densities.

The postulate of potentials indicates that the domain of support for the 2-form F is not compact without boundary[2]. The postulate also demonstrates that magnetic monopoles are not compatible with the assumption of C2 differentiability. Such a statement does not apply to the density N-2 form $\overline{G}$, which can have closed and non-closed components. The closed but not exact components of $\overline{G}$ lead to the quantization of charge as a topological result. As $\overline{G}$ is a density, it also follows that quantized charge is a pseudo-scalar [?Post]. The historical assumptions of charge as a scalar are not compatible with the topological format. Experiments

[2] There are two exceptions: the Klein bottle and the torus.

with piezo electric crystals indicate that volume deformations can cause electrical phenomena. If $\overline{G}$ was not a density, there would be no Piezo electricity.

5.1.1. The D H field excitations: differential N-2 form densities.

For example consider the exterior differential of the N-1 form density[3], $\overline{D}$, in three dimensions, given by the expression,

$$\begin{aligned} d\overline{D} &= d(\overline{D}^x dy\hat{}dz - \overline{D}^y dz\hat{}dx + \overline{D}^z dx\hat{}dy) \\ &= div_3(\overline{D})dx\hat{}dy\hat{}dz \Rightarrow \rho(x,y,z)dx\hat{}dy\hat{}dz \end{aligned} \tag{5.3}$$

where ρ has been defined as the resultant of the action of the exterior differential, $div_3(\overline{\mathbf{D}})$. The usual interpretation of Gauss' law is that the field lines of the vector (density) $\overline{\mathbf{D}}$ terminate (or have a limit or accumulation point) on the charges, Q. Gauss' law generates both the intuitive idea that sources are related to limit points, and demonstrates the novel concept that the exterior differential is a limit point operator. The exterior differential creates limits points when the operation is applied to a differential form. However, as demonstrated above, the concept that the exterior differential is a limit point operator relative to the Cartan topology is a general idea, and is not restricted to Gauss' law.

Maxwell's PDE's are topological statements deduced from an exterior differential system. The two postulates are

$$\text{The Postulate of Potentials:} \quad F - dA = 0. \tag{5.4}$$

$$\text{The Postulate of conserved Charge current densities:} \quad \overline{J} - d\overline{G} = 0. \tag{5.5}$$

Extending this idea to four dimensions for the N-2 form density, G, of Maxwell excitations $(\mathbf{D}, \mathbf{H})$,

[3]There are two species of differential forms considered in this article. The first specie transforms as a scalar with respect to diffeomorphisms. The second specie transforms as a scalar density, and is proportional to the determinant of the diffeomorphism. The coefficients pull back with respect to the transpose of a differential Jacobian mapping, whether it is a diffeomorphism or not. The second species, the differential form densities, pull back with respect to the adjoint of a differential Jacobian mapping.

$$G = -D^x dy\hat{}dz + D^y dz\hat{}dx - D^z dx\hat{}dy + H^x dx\hat{}dt + H^y dy\hat{}dt + H^z dz\hat{}dt, \quad (5.6)$$

the exterior differential dG of G yields a three form, J, defined as the electromagnetic current 3-form,

$$J = J^x dy\hat{}dz\hat{}dt - J^y dx\hat{}dz\hat{}dt + J^z dx\hat{}dy\hat{}dt - \rho dx\hat{}dy\hat{}dt \quad (5.7)$$

where in 3-vector language,

$$curl\ \mathbf{H} - \partial\mathbf{D}/\partial t = 0 \qquad div\,\mathbf{D} = \rho. \quad (5.8)$$

The charge current density act as the "limit points" of the Maxwell field excitations. Note that $dJ = 0$ for C2 functions by Poincare's lemma.

However, consider the N-1 current, C (not necessarily equal to J as defined above) in four dimensions

$$C = \rho\{V^x dy\hat{}dz\hat{}dt - V^y dx\hat{}dz\hat{}dt + V^z dx\hat{}dy\hat{}dt - 1dx\hat{}dy\hat{}dt\} \quad (5.9)$$

and its exterior differential as given by the expression,

$$dC = \{div_3(\rho\mathbf{V}) + \partial\rho/\partial t\}dx\hat{}dy\hat{}dz\hat{}dt. = Rdx\hat{}dy\hat{}dz\hat{}dt = R\,\Omega_{4\_vol} \quad (5.10)$$

When the 4-form R vanishes, the resultant expression is physically interpreted as the "equation of continuity" or as a "conservation law". Over a closed boundary, that which goes in is equal to that which goes out (when $dC = 0$). Note that the concept of the conservation law is a topological constraint: the "limit points" of the "current 3-form" in four dimensions must vanish if the conservation law is to be true. If the RHS of the above expression is not zero, then the current 3-form is said to have an "anomaly", or a source (or sink) . The anomaly acts as the source of the otherwise conserved quantity. The limit points, R, of the 3-form, C, are generated by its exterior differential, $dC = \{div_3(\rho\mathbf{V}) + \partial\rho/\partial t\}\Omega_4$. When the RHS is zero, the current "lines" do not stop or start within the domain. (It is possible for them to be closed on themselves in certain topologies).

5.1.2. The E B Field Intensities: differential 2-forms

On a four dimensional space-time of independent variables, (x, y, z, t) the 1-form of Action (constrained by the postulate of potentials, $F - dA = 0$) can be written in the form

$$A = \Sigma_{k=1}^{3} A_k(x, y, z, t)dx^k - \phi(x, y, z, t)dt = \mathbf{A}\circ d\mathbf{r} - \phi dt. \tag{5.11}$$

Subject to the constraint of the exterior differential system, the 2-form of field intensities, F, becomes:

$$\begin{aligned} F = dA = \{\partial A_k/\partial x^j - \partial A_j/\partial x^k\}dx^j\hat{}dx^k = F_{jk}dx^j\hat{}dx^k \qquad (5.12) \\ = \mathbf{B}_z dx\hat{}dy + \mathbf{B}_x dy\hat{}dz + \mathbf{B}_y dz\hat{}dx + \mathbf{E}_x dx\hat{}dt + \mathbf{E}_y dy\hat{}dt + \mathbf{E}_z dz\hat{}dt. \end{aligned}$$

where in usual engineering notation,

$$\mathbf{E} = -\partial \mathbf{A}/\partial t - grad\phi, \qquad \mathbf{B} = curl\ \mathbf{A} \equiv \partial A_k/\partial x^j - \partial A_j/\partial x^k. \tag{5.13}$$

The closure of the exterior differential system, $dF = 0$, vanishes for C2 differentiable p-forms, to yield

$$dF = ddA = \{curl\ \mathbf{E} + \partial \mathbf{B}/\partial t\}_x dy\hat{}dz\hat{}dt - .. + .. - div\,\mathbf{B}dx\hat{}dy\hat{}dz\} \Rightarrow 0. \tag{5.14}$$

Equating to zero all four coefficients leads to the Maxwell-Faraday partial derivative equations,

$$\{curl\ \mathbf{E} + \partial \mathbf{B}/\partial t = 0, \qquad div\,\mathbf{B} = 0\}. \tag{5.15}$$

This topological development of the Maxwell-Faraday equations has made no use of a connection nor of a metric.

The component functions ($\mathbf{E}$ and $\mathbf{B}$) of the 2-form, F, transform as covariant tensor of rank 2. The topological constraint that F is exact, implies that the domain of support for the field intensities cannot be compact without boundary, unless the Euler characteristic vanishes. These facts distinguish classical electromagnetism from Yang-Mills field theories. Moreover, the fact that F is subsumed to be exact and C1 differentiable excludes the concept of magnetic monopoles from classical electromagnetic theory on topological grounds.

This now almost classic generation of the Maxwell field equations [9] has another less familiar interpretation: The $\mathbf{E}$ and $\mathbf{B}$ field intensities are the topological limit "points" of the 1-form of potentials, $\{\mathbf{A}, \phi\}$, relative to the Cartan topology! The limit points of the 2-form of field intensities, F, are the null set. For C2 vector fields, the Cartan topology admits flux quanta, charge quanta, and spin quanta, but excludes magnetic monopoles [10]. When the differential system of interest is built upon the forms A, F and G, it is possible to show that superconductivity is to be associated with the constraints on the limit point sets of $A,\ A\hat{}F,\ and\ A\hat{}G$ [11]. That is, superconductivity has its origins in topological, not geometrical, concepts. This remarkable idea that the exterior differential is a limit point operator is based upon Kuratowski's closure operator is equivalent to the union of the identity and the exterior differential.

5.2. Frozen - in Fields, the Master Equation

A starting point for many discussions of the magnetic dynamo and allied problems in hydrodynamics starts with what has been called the "master equation" [21],

$$Curl(\mathbf{V} \times \mathbf{B}) = \partial \mathbf{B}/\partial t. \tag{5.16}$$

Using the Cartan methods it may be shown that this equation is equivalent to the constraint of "uniform" continuity relative to the Cartan topology. Moreover, it is easy to show these constraints generate symplectic processes which include Hamiltonian evolutionary systems, such as Euler flows, as well as a number of other evolutionary processes which are continuous, but not homeomorphic. In addition a criteria can be formulated to develop an extension of the "helicity" conservation law to a more general setting.

The proof of these results produces a nice exercise in use of the Cartan theory. Consider a 1-form A that satisfies the exterior differential system

$$F - dA = 0, \tag{5.17}$$

where A is a 1-form of Action, with twice differentiable coefficients (potentials proportional to momenta) which induce a 2-form, F, of electromagnetic intensities ($\mathbf{E}$ and $\mathbf{B}$, related to forces). The exterior differential system is a topological constraint that in effect defines field intensities in terms of the potentials.

Now search for all vector fields that leave the 2-form F an absolute invariant of the flow; that is, search for all vectors that satisfy Cartan's magic formula

$$L_{(\mathbf{V})}F = i(V)dF + di(V)F = 0 + di(V)F = 0. \tag{5.18}$$

For C2 functions, the term involving dF vanishes, leaving the expression,

$$\begin{aligned} L_{(\mathbf{V})}F &= di(V)F && (5.19)\\ &= d\{(\mathbf{E} + \mathbf{V} \times \mathbf{B}) \cdot\ d\mathbf{r} - (\mathbf{E} \cdot \mathbf{V})dt\} && (5.20)\\ &= \{curl(\mathbf{E} + \mathbf{V} \times \mathbf{B})\}_z dy\hat{}dz... && (5.21)\\ &+ \{\partial(\mathbf{E} + \mathbf{V} \times \mathbf{B})/\partial t + grad(\mathbf{E} \cdot \mathbf{V})\} \cdot\ d\mathbf{r}\hat{}dt && (5.22)\\ &= 0. && (5.23)\end{aligned}$$

Setting the first three factors to zero yields

$$curl(\mathbf{E} + \mathbf{V} \times \mathbf{B}) = 0 \tag{5.24}$$

From the Maxwell Faraday equations for C2 functions, $curl\mathbf{E} = -\partial\mathbf{B}/\partial t$, and when this expression is substituted into the above equation, the "master" equation given above is the result. Now recall that dF generates the limit points of A, and if $F = dA$ is a flow invariant, then all limit points are flow invariants relative to the Cartan topology. This result implies that the vector fields, $\mathbf{V}$, that satisfy the constraints of the "master equation" are uniformly continuous evolutionary processes, as the limit points, $F = dA$, of the 1-form A are flow invariants, and the lines of vorticity are "frozen-in" the flow. Non-uniform continuity would imply that the limit points are not invariants of the process, but that the closure of the limit points of the target range includes the limit points of the initial domain. Such processes would correspond to a folding of the "lines" of vorticity, which preserve the limit points, but not their sequential order. A second criteria for limit point invariance is given by the equation,

$$\{\partial(\mathbf{E} + \mathbf{V} \times \mathbf{B})/\partial t + grad(\mathbf{E} \cdot \mathbf{V})\} = 0. \tag{5.25}$$

The formula indicates that limit point invariance can occur in the presence of input-output power, $\mathbf{E} \cdot \mathbf{V} \neq 0$.

The criteria for frozen-in fields is established as a constraint of uniform continuity on the admissable vector fields,

$$\text{Uniform Continuity:} \quad di(V)dA = di(V)F = 0. \tag{5.26}$$

The solution vector fields, V, subject to this constraint can be put into three global categories:

1. Extremal (Hamiltonian) $i(V)F = 0.$
2. Bernoulli-Casimir (Hamiltonian) $i(V)F = d\Theta.$
3. Symplectic $i(V)F = d\Phi + \gamma_{harmonic}$

The first category can exist only on domains of support of F which are of odd Pfaff dimension, but then the solution vector is unique to within a factor. In the other categories, the solution vector need not be unique. Vector fields that satisfy the equation for uniform continuity are said to be symplectic relative to the 1-form, A. Vector fields that belong to categories 1 and 2 have a Hamiltonian representation. Vector fields that belong to category 1, are said to be "extremal" relative to the 1-form, A.

When the concepts are applied to the integrals of the 2-form F, then the criteria for invariance of the flux integral depends on the topology of the integration domain. If the integration area of the 2-form is a boundary or a cycle of a 3 dimensional domain, the flux integral over the closed boundary or cycle is always a flow invariant. If the integration area is bounded, then by Stokes theorem the flux integral depends only on the boundary conditions: F or $i(V)F$ must vanish on the boundary, or when integrated over the boundary.

5.3. Euler flows and Hamiltonian systems.

In 1922 Cartan established the idea that the necessary and sufficient conditions for a system to admit a unique Hamiltonian representation for its evolution, $\mathbf{V}$, on a space of 2n+1 dimensions is given by the category 1 constraint,

$$\text{Extremal process } V: \quad W = i(V)dA = i(V)F = 0. \tag{5.27}$$

In the language of mechanics on state space $\{p, q, t\}$, the 1-form of Action is given by an expression of the form,

$$A = p_\mu dq^\mu - H(p, q, t)dt. \tag{5.28}$$

For a given 1-form, those vector direction fields, $V = [\dot{p_\mu}, \dot{q}^\mu, 1]$,that satisfy this constraint are said to be extremal vector fields. The word extremal comes from the theory of the calculus of variations. By direct computation,

$$dA = dp_\mu \hat{} dq^\mu - dH(p,q,t)\hat{} dt, \tag{5.29}$$

$$i(V)dA = \dot{p_\mu} dq^\mu - \dot{q}^\mu dp_\mu + dH(p,q,t) - \{i(V)dH\}dt, \tag{5.30}$$

$$= \dot{p_\mu}\{dq^\mu - (\partial H/\partial p_\mu)dt\} - \dot{q}^\mu\{dp_\mu + (\partial H/\partial q^\mu)dt\} \tag{5.31}$$

$$-(\partial H/\partial p_\mu \dot{p_\mu} + \partial H/\partial q^\mu \dot{q}^\mu)dt \tag{5.32}$$

It follows that the extremal condition $i(V)dA = i(V)F = 0$, if the extremal vector exists, is satisfied by

$$\text{Hamiltonian equations} \quad \dot{p}_\mu = -\partial H/\partial q^\mu, \quad \dot{q}^\mu = +\partial H/\partial p_\mu. \tag{5.33}$$

When the space is of odd Pfaff topological dimension, then the extremal field is the unique null eigen vector of the antisymmetric matrix of functions that make up the 2-form, $F = dA$. For an even dimensional space of maximum topological dimension, the anti-symmetric matrix, F, has no null eigen vectors - the extremal field does not exist as there are no null eigenvalues. However, for category 2 situations, a function can be found, $H' = H + \theta$, such that the Hamiltonian equations are valid using the variable, H' instead of H.

It is apparent that this extremal condition is more stringent than that given in the preceding section for uniform continuity, $di(V)F = 0$. Such extremal vector fields are independent of parameterization. That is, for extremal processes, $i(\rho V)dA = 0$ if $i(V)dA = 0$, for any function, ρ. Extremal vector fields do not exist on domains where the Pfaff dimension of the Cartan 1-form is even. In classical mechanics, the 1-form W is defined as the 1-form of Virtual Work, and the Cartan constraint is typical of problems in the variational calculus where it is presumed that the Virtual Work vanishes.

As an example,consider a 1-form of Action defined as

$$A = \mathbf{v}\ d\mathbf{r} - (\mathbf{v} \cdot \mathbf{v}/2 + \Psi)dt, \tag{5.34}$$

where $d\Psi = dP/\rho$. Application of the extremal constraint yields the resulting necessary system of partial differential equations is given by known as the Euler equations of hydrodynamics.

$$\partial \mathbf{v}/\partial t + grad(\mathbf{v} \cdot \mathbf{v}/2) - \mathbf{v} \times \mathbf{w} = -gradP/\rho, \tag{5.35}$$

It also follows that the Master equation is valid, with the only difference being that $curl\mathbf{v}$ is defined as $\boldsymbol{\omega}$, the vorticity of the hydrodynamic flow. The master equation becomes,

$$curl(\mathbf{v} \times \boldsymbol{\omega}) = \partial\boldsymbol{\omega}/\partial t, \tag{5.36}$$

and this equation is to be recognized as Helmoltz' equation for the conservation of vorticity. In the hydrodynamic sense, conservation of vorticity implies uniform continuity. In other words, the Eulerian flow is not only Hamiltonian, it is also uniformly continuous, and satisfies the master equation and the conservation of vorticity constraints. In addition, it may be demonstrated that such systems are at most of Pfaff dimension 3, and admit a relative integral invariant which generalizes the hydrodynamic concept of invariant helicity. In the electromagnetic topology, the Hamiltonian constraint is equivalent to the statement that the Lorentz force vanishes, a condition that has been used to define the "ideal" plasma or "force-free" plasma state.

5.4. Conservation of Topological Torsion

A slightly more general class of evolutionary processes (flows) is given by the constraints which are gauge equivalent to the Hamiltonian extremal case; a search is made for those flows that satisfy the (non-extremal, but Hamiltonian) constraint:

$$i(\rho V)dA = i(\rho V)F = d\Theta. \tag{5.37}$$

Such flows admit two topological invariants of the relative integral invariant form. The first integral invariant is 1-dimensional:

$$L_{(\rho\mathbf{V})} \oint_{1d\_closed} A = \oint_{1d\_closed} i(\rho V)dA + di(\rho V)A = \tag{5.38}$$

$$\oint_{1d\_closed} d\Theta + di(\rho V)A = \oint_{1d\_closed} d\{\Theta - i(\rho V)A\} \Rightarrow 0, \tag{5.39}$$

expressing the relative integral invariance of circulation (Kelvin's theorem). The second integral invariant is 3-dimensional:

$$L_{(\rho\mathbf{V})} \oint_{3d\_closed} A\hat{}dA = \oint_{3d\_closed} d\{\Theta - i(\rho V)A\}\hat{}dA \Rightarrow 0, \tag{5.40}$$

a result expressing the generalization of the law which in hydrodynamics is called the conservation of Helicity. The integrations are over closed 1 and 3 dimensional domains. These closed integration domains can be either cycles or boundaries. For exampled the 1-dimensional closed curve in the punctured disc that encircles the central hole is a cycle but not a boundary. As the integrands are exact differentials, the closed integrals vanish.

Note that on the domain $\{x, y, z, t\}$, the 3-form of topological torsion, $A\hat{}dA$, has the general representation with coefficients, $Z_{\mu\nu\sigma}$, that transform as a covariant tensor field of third rank. On a 4 dimensional space, the components of $A\hat{}dA$ are proportional to a contravariant tensor density of rank 1, whose four components have a vector part defined as, $\mathbf{T}$, the torsion (pseudo) current, and a (pseudo) density part, h. The 3-form $A\hat{}dA$ is not an impair form (density). In electromagnetic engineering language, the general formula for the torsion 3-form has a component expression given by:

$$T = [\mathbf{T}, h] = [\mathbf{E} \times \mathbf{A} + \phi\mathbf{B}, \mathbf{A} \cdot \mathbf{B}]. \tag{5.41}$$

For the constraints of an Eulerian flow, the 4 components of the Torsion three form reduce to

$$T = [\mathbf{T}, h] = [(\mathbf{v} \cdot \boldsymbol{\omega})\mathbf{v} - (\mathbf{v} \cdot \ \mathbf{v}/2 + \Psi)\boldsymbol{\omega}, \mathbf{v} \cdot \boldsymbol{\omega}]. \tag{5.42}$$

Recall that the closed integration domain used to evaluate the relative integral invariant is *not* necessarily restricted to a spatial volume integral with a boundary upon which the normal component of $\mathbf{v}$ vanishes. Also note that the helicity density of hydrodynamic fame is the fourth component, $h = \mathbf{v} \cdot \boldsymbol{\omega}$, of a contravariant vector density, equivalent to a covariant tensor of third rank. Care must be used in its transformation with respect to diffeomorphisms, such as the Galilean transformation. Furthermore, for the constraints of an Eulerian flow (an extremal field) described above, the topological parity 4-form vanishes globally, such that there exists a pointwise conservation law of the 3-form, equivalent to the expression,

$$div_3\mathbf{T} + \partial h/\partial t = 0. \tag{5.43}$$

5.5. Topological Invariants and Period Integrals

Besides the invariant structures considered above, the Cartan methods may be used to generate other sets of topological invariants. Realize that over a domain of

Pfaff dimension n less than or equal to N, the Cartan criteria admits a submersive map to be made from N to a space of minimal dimension n. The map may be viewed as a vector field of functional components,

$$[V^x(x,y,z..), V^y(x,y,z..), V^z(x,y,z..), ...],$$

of dimension n, and will have a representation in the projective geometry of n+1 homogeneous coordinates. The n+1 component will be generated by a function λ, related to the Holder norm,

$$\rho = 1/\lambda = 1/\{a(V^x)^p + b(V^y)^p + c(V^z)^p + ...\}^{n/p}. \tag{5.44}$$

For any vector field, construct the n dimensional volume element,

$$Vol = \rho(V)\ dV^x\hat{}dV^y\hat{}dV^z... \tag{5.45}$$

and the n-1 form density (current) J as:

$$J = i(V^x, V^y, V^z, ...)Vol = \\ \rho\{V^x\ dV^y\hat{}dV^z...\ - V^y\ dV^x\hat{}dV^z... + V^z\ dV^x\hat{}dV^y... - ...\} \quad . \tag{5.46}$$

It is remarkable that the current J so defined has a vanishing exterior differential, independent of the value of p for a given n, and for all values of the constants, plus or minus a,b,c...). All such currents define a "conservation law". As the map defining the components of the vector field in terms of the base {x,y,z..} is presumed to be differentiable, then the n-1 form, J, has a well defined pull back on the base space (almost every where), and its exterior differential on the base space also vanishes everywhere mod the defects. That is, the form J is locally exact.

In the expression for λ, the factors $\{a, b, c, d...\}$ are arbitrary constants of either sign. The most familiar format is when $p = 2$, and then the function λ has a null set which is a conic. For positive isotropic signature, the only defect is the origin in the space defined by the functions, V. The construction produces the algebraic dual or adjoint vector field from the functional components of the original vector field with integrating factors $\rho = 1/\lambda$ that create conservation laws for physical systems. The integrals of these closed currents when integrated over closed N-1 dimensional chains form deformation invariants, with respect to any evolutionary process that can be described by a vector field, for

$$L_{(\rho\mathbf{V})}\oint_{n-1} J = \oint_{n-1} i(\rho\mathbf{V})dJ + \oint_{n-1} d(i(\rho\mathbf{V}))J) = 0 + 0 = 0 \tag{5.47}$$

These integral objects appear as "topological coherent" structures, which may have defects or anomalous sources, when the integrating factor $1/\lambda$ is not defined.

The compliment to the zero sets of the function λ determine the domain of support associated with the specified vector field. The closed n-1 form, J, that satisfies the conservation law, $dJ = 0$, has integrals over closed domains that have rational fraction ratios. As this n-1 current is closed globally, it may be deduced on a connected local domain from a n-2 form, G. In every case J has a well defined pull-back to the base variety, x,y,z,t. Note that the n functions $[V^x(x, y, z..), V^y(x, y, z..), V^z(x, y, z..), ...]$ represent the minimum number of Clebsch variables that are equivalent to the original action, A, over the domain of support. As each of these integrals is intrinsically closed, the Lie differential with respect to any C2 vector field, $\mathbf{V}$, is a perfect differential, such that (when integrated over closed domains that are p-1 boundaries) the evolutionary variation of these closed integrals vanishes. These n-1 integrals are relative integral invariants for any C2 evolutionary processes, or flows. The values of the integrals are zero if the closed integration domains are boundaries, or completely enclose a simply connected region. If the closed integration domains encircle the zeros of the function λ, then the values of the integrals are proportional to the integers; i.e., their ratios are rational. Note that each signature must be investigated. For the elliptic (positive definite) signature, the singular points are the stagnation points, and the domain of support excludes those singularities. For the hyperbolic signatures, the domain of support excludes the hyperbolic singularities of lower dimension, such as the light cone. Further note that a given vector field may not generate real domains of support for all possible signatures of the quadratic form, λ.

5.6. The Flux or Circulation Integral 1-form

For the Cartan topology constructed from a fundamental 1-form of Action and a fundamental N-1 form of Current, several period integrals of closed forms integrated over closed chains appear in a natural manner. In particular on an N=4 dimensional domain, the four period integrals of most interest are the period integrals of flux (circulation), charge, spin and torsion [9]. The fundamental period integral over a closed 1-form will be defined as the "Circulation" or "flux" integral. When the Pfaff dimension is 2, there exists a submersive map to two dimensions, and the vector fields on this domain will have two irreducible components, say $[\Phi(x,y,z,t),\Psi(x,y,z,t)]$. Following the procedure of the preceding section, construct the 2-dimensional volume element defined as $\Omega=\rho d\Phi\hat{}d\Psi$, and the $n-1=2-1=1$ form $A=(\Phi d\Psi-\Psi d\Phi)/\{\pm a\Phi^p\pm b\Psi^p\}^{2/p}$. The exterior differential of such a 1-form is exactly zero for all point sets that exclude the null set of the denominator. The classic choice is for $p=2$,and $a=1$, $b=1$, (+,+) signature. The closed integrals of these closed 1-forms then can be expressed as

$$\text{Circulation }\Gamma=\oint_1 A=\oint_1(\Phi d\Psi-\Psi d\Phi)/\{\Phi^2+\Psi^2\} \tag{5.48}$$

By substituting the functional forms in terms of (x,y,z,t) the circulation integral can be written in terms of functions on (x,y,z,t) and their differentials, $\{dx,\ dy,\ dz,\ dt...\}$

As an example, suppose that the domain is three dimensional, N=3. Then the zero sets of $\Phi(x,y,z)=0$ and $\Psi(x,y,z)=0$, represent two 2 dimensional surfaces which may or may not have one or more lines of intersection. If the surfaces intersect, then

$$Intersection=d\Phi\hat{}d\Psi\neq0. \tag{5.49}$$

If the closed integration paths cannot be contracted to a point, because they encircle these lines of intersection, the values of the integrals have rational ratios depending on how many lines are encircled and how many times the integration path encircles a line. The lines of intersection must have zero divergence (and therefore must stop or start on boundary points, or are closed on themselves). Otherwise the integration chains can be deformed and then contracted to a point. The classic example is given by the 1-form, $A=(ydx-xdy)/(+x^2+y^2)$ in three dimensions. For integration contours that encircle the z axis, the value of $\Gamma=\oint_1 A=2\pi$. In hydrodynamics, this vector field is called a potential "vortex", even though the vorticity $\boldsymbol{\omega}=curl\mathbf{v}=0$. Stokes theorem does not apply as the closed integration chain is a cycle that is not a boundary.

An interesting application of the circulation integral is given when there exists a map to the complex domain. Then $\Psi\Rightarrow\Phi^*$ and the circulation integral has the form of the integral of the probability current in standard quantum mechanics.

$$Period=(1/2i)\oint_1(\Phi d\Phi^*-\Phi^*d\Phi)/\{\Phi\cdot\Phi^*\}. \tag{5.50}$$

5.7. The Gauss Linking or Charge Integral 2-form

Many different options exist for construction of these invariant topological structures from closed p-forms. The idea is to find a formulation for a closed form on a domain, and then to specify a closed and compatible integration chain. The integration chain need not be a boundary, but only a closed cycle. For example, from the components of the specified vector, A_μ , the Jacobian matrix, $[\partial A_\mu/\partial x^\nu]$ can be constructed. The rows or columns of the matrix of cofactors of the Jacobian (the adjoint matrix) forms a set of vector fields that have zero divergence [21], and therefore these vectors could be used to construct relative integral invariants. In every case there exists an algebraic construction which produces a vector that is divergence free and whose line of action is uniquely related to original vector that was used to construct the Cartan topology. That vector may be constructed by multiplying the original vector A_μ by the matrix of cofactors and then dividing by the function λ defined above. The construction replicates the previous procedure. As an application for n = 3, p=2, consider the vector that represents the difference between two space curves, $\mathbf{z} = \mathbf{R}_2 - \mathbf{R}_1$. Then compute the two form $G(z)$ from the "volume" element $\Omega = dz^1\hat{}dz^2\hat{}dz^3/\lambda$, to give

$$G_{n=3} = \{z^1dz^2\hat{}dz^3 - z^2dz^3\hat{}dz^1 + z^3dz^1\hat{}dz^2\}/\lambda \tag{5.51}$$

where

$$\lambda = (\pm(z^1)^2 \pm (z^2)^2 \pm (z^3)^2)^{3/2}. \tag{5.52}$$

Next assert that the displacements of interest are constrained by two parametric curves given by

$$d\mathbf{R}_1 = \mathbf{V}_1 dt \quad \textit{and} \quad d\mathbf{R}_2 = \mathbf{V}_2 dt', \tag{5.53}$$

where the parameters dt and $dt\prime$ are not functionally related (which would imply that $dt\hat{}dt\prime = 0$).

It is important to realize that kinematic constraints are topological constraints that refine the Cartan topology, a topology based solely upon the specified 1-form of action, A. From a physical point of view, these constraints can be interpreted as constraints of null fluctuations and in certain circumstances can be associated physically with the limit of zero temperature. To demonstrate the utility of such constraints, substitute these differential expressions into the expression for the 2-form G of "current" in N=3 dimensions, and carry out the exterior products, using $dt\hat{}dt' \neq 0$, *but* $dt\hat{}dt = 0$ and $dt'\hat{}dt' = 0$. The result is the vector triple product representation for the Gauss integral,

$$Q = \oint_2 G = \oint_2 \{\mathbf{z} \circ \mathbf{V}_1 \times \mathbf{V}_2\} dt\hat{}dt' / (\mathbf{R}_1 \circ \mathbf{R}_1 - 2\mathbf{R}_1 \circ \mathbf{R}_{21} + \mathbf{R}_2 \circ \mathbf{R}_2)^{3/2}. \quad (5.54)$$

The integration domain is the closed "2-dimensional area" formed by the displacements along the non-intersecting curves defined by the two distinct parameters, dt, and dt'. This double integral is to be recognized as the Gauss linking integral of Knot Theory [7]. (Without the kinematic substitutions, it may also be interpreted as the charge integral of electromagnetic theory.) When integrations are computed along closed curves whose tangent vectors are $\mathbf{V}_1$ and $\mathbf{V}_2$, then the integer values of the closed integral may be interpreted as how many times the two curves are linked. Note that the same integer result is obtained when the vector $\mathbf{z}$ is interpreted as the sum of the two vectors, $\mathbf{z} = \mathbf{R}_2 + \mathbf{R}_1$, although the values of the integrals have different scales.

The constraint that $dt\hat{}dt' \neq 0$ implies that the "motion" along the curve generated by $\mathbf{R}_1$ is independent of the "motion" along the curve generated by $\mathbf{R}_2$. If the curve generated by $\mathbf{R}_1$ is a conic in the xy plane and the curve generated by $\mathbf{R}_2$ is a conic in the xz plane, then the surface swept out by the vector $\mathbf{z}$ is a Dupin cyclide. Such surfaces have application to the propagation of waves in electromagnetic systems.

From another point of view, consider the ruled surface [22] defined by the vector field of two parameters,

$$\mathbf{z}(\mu, t) = \mathbf{R}(t) \pm \mu\mathbf{V}(t). \quad (5.55)$$

Vector fields of this type are primitive types of "strings" for fixed values of the parameter, t, and string parameter, μ. Direct substitution of the physical constraints, $d\mathbf{R} - \mathbf{V}dt = 0$, and $d(\mathbf{V}) - \mathbf{A}dt = 0$ leads to the topological Gauss integral,

$$Q = \oint_2 G = \oint_2 \{\mathbf{R} \circ \mu\mathbf{V} \times \mathbf{A}\}/\lambda =$$
$$\oint_2 \{\mathbf{A} \circ \mathbf{R} \times \mu\mathbf{V}\} dt\hat{}d\mu / (\mathbf{R} \circ \mathbf{R} \pm 2\mu\mathbf{R} \circ \mathbf{V} + \mu\mathbf{V} \circ \mu\mathbf{V})^{3/2}. \quad (5.56)$$

It is apparent that the interaction of the "angular" momentum, $\mathbf{L} = \mathbf{R} \times \mu\mathbf{V}$, and the acceleration, $\mathbf{A}$, produces a topological invariant whose values are "quantized" (in the sense that the ratios of the integrals are rational). Note that for the classical central field problem where the force (acceleration) and the angular momentum are orthogonal, the orbits are in a plane and the Gauss–linking number is zero. Further note that the triple vector product of the integrand is proportional to the Frenet torsion of the orbit. An orbit that is planar has Frenet torsion zero everywhere. The Gauss linking integral is a special case of the Gauss two dimensional period integral of electromagnetic theory when the integration domains can be factored into independent products, $dt\hat{}dt\prime \neq 0$.

5.8. Chaos and the Unknot

Much interest of late has been shown in knot theory and its application to an understanding of the trajectories of dynamical systems. The conjecture is that somehow an understanding of knot theory will give a better understanding of chaos. Counter intuitively is the idea that chaos is to be related to the unknot. Of particular interest will be those cases where lines of vorticity have an oscillatory Frenet torsion with a period equal to 2/3 of the fundamental period of closure. The topological Gauss integral will average to zero for such systems; but these systems can be created by continuous deformations of folding and twisting a closed loop of vorticity, producing a period 3 system which is known to be related to chaos [23]. In the undeformed circular state, tubular neighborhoods guided by the vortex lines can continuously evolve into domains without stagnation points or tangential singularities, or knots, or twists. However, when the closed vortex line is in the deformed period 3 configuration, tangential (hyperbolic) singularities are created by the flow lines of the velocity field, and the evolution becomes highly convoluted and chaotic. See Figure 1.

These topological features may be demonstrated visually by taking a long strip of paper and wrapping the strip three times around your fingers. Close the strip by going under one strand and over the next before pasting together. The strip is of obvious period three. Now slide the closed strip from the fingers and note that it can be deformed 9continuously into a cylindrical strip without twists or knots (Spin 0). If the same procedure is used, except that a double over or a double under crossing is used before pasting the strip ends together, the resulting closed loop will have a continuously irreducible 4π twist (Spin 2). Both the Spin 2 and the Spin 0 strips have a zero Euler characteristic. However, the Spin 2 strip can be continuously deformed into a Klein bottle, or a double lapped Mobius band, and is not homeomorphic to the spin zero strip [24].

If a model of the Spin0 and Spin 2 systems (deformed to their period 3 configurations) is made from a copper tube, and if flexible bands are created to link any pair of neighboring tubular strands, then it is readily observed that the paired domain twists and folds as it is propagated unidirectionally along the vortex lines. For the spin 2 system the flexible bands will return to their original state in 3 revolutions. However, the paired domain continues to twist and fold, becoming ever more complicated as it follows the evolution around the Spin 0 configuration. The folded spin 0 system has chaotic neighborhoods. This result indicates that the source of chaos in dynamical systems may be due to the unknot, and not the knot! The Cartan theory thereby predicts that the source of chaos in turbulent systems does not require a discontinuous cut and connect process, but may be induced by vortex lines that continuously evolve by twisting and folding into a closed, spin 0, period three configuration.

5.9. The Torsion 3-form and the Braid integral

For n = 4 the same procedures used above can be used to produce a period integral over a closed 3-dimensional domain. In fact, the same vector field that is used to define the Cartan 1-form of Action may be used to construct a dual N-1 form that is closed. The algorithm is to substitute for the functions of the vector field, V, the functions that make up the covariant 1-form of Action, A. This construction is equivalent to constructing the Jacobian matrix of the original vector field on the N-dimensional velocity space, computing its cofactor matrix, multiplying the original vector by the cofactor matrix, and then dividing by the quadratic form, λ. When these operations are completed, functional substitution will lead to an conserved axial vector current density on (x,y,z,t). Another form of the topological integral invariant is constructed in the following way. First, for the classic Cartan action, $A = P_k dx^k - Edt/c$, construct the N-volume, $\Omega = -dP_x\hat{}dP_y\hat{}dP_z\hat{}dE/c$. Next contract Ω with the vector, $(Px, Py, Pz, -E/c)$, and then divide by $\lambda = \{\pm P \circ P \pm (E/c)^2\}^2$. For sake of simplicity, assume that E/c is a constant such the $dE = 0$. Then the closed 3-form or current becomes equivalent to

$$J = (E/c)dP_x\hat{}dP_y\hat{}dP_z/\lambda \qquad with \qquad dJ = 0 \tag{5.57}$$

Now invoke the same Cartan trick of individual parametrization as uses above.

Consider a total momentum vector composed of three individual vector components, $\mathbf{P} = \mathbf{p_1}+\mathbf{p_2}+\mathbf{p_3}$. Assume that the Cartan topology is constrained in such a way that for each vector component a Newtonian kinematic law of parametrization is maintained such that

$$d\mathbf{p}_1 - \mathbf{f}_1 dt = 0, \quad d\mathbf{p}_2 - \mathbf{f}_2 dt' = 0, \quad d\mathbf{p}_3 - \mathbf{f}_3 dt'' = 0. \tag{5.58}$$

Also note that $dt\hat{}dt'\hat{}dt'' \neq 0$; that is, the parameters used in the Newtonian kinematic descriptions are not sychronizeable. If they were functionally related the value of J must be zero. Substitute these expressions into the equation for the closed current J and integrate over a closed 3 dimensional chain to yield a triple Braid integral,

$$\begin{aligned} Braid &= \oint_3 J = \oint_3 (E/c) dP_x\hat{}dP_y\hat{}dP_z/\lambda \\ &= \oint_3 (E/c)\{\mathbf{f}_1 \circ (\mathbf{f}_2 \times \mathbf{f}_3)\}\, dt\hat{}dt'\hat{}dt''/\{\pm\mathbf{P}\circ\mathbf{P} \pm (E/c)^2\}^2 \end{aligned} \tag{5.59}$$

The integrations are now over three closed curves whose tangents are the Newtonian forces, $\mathbf{f}$, on three "particles". Where in the two dimensional Gauss integral, of the previous section, the evaluation was along the closed curves of two particles that formed the ends of a string, in this case the integrations are along the closed trajectories of three "particles" which form the vertices of a triangle. In every case, the trajectories are the trajectories of a system of limit points.

The idea that three "lines" are used to form the integral (whose values form rational ratios) is the reason that this topological integral in the format given above is defined as the braid integral. Of course the three form of topological torsion is a variant of the braid integral, but applies to those topologies where the system is not reducible to three factors $dt, dt\prime$ and dt" (such systems are said to have torsion cycles). An example of a period 3 braid with Braid integral zero (chaotic) and Braid integral 2 (non-chaotic) is given in Figure 1

The equivalent to this Figure, and the fact that there are two distinct period 3 configurations, one chaotic and one non-chaotic, was brought to the attention of the present authors during a stimulating lecture given by J. Los at the August, 1991, Pedagogical Workshop on Topological Fluid Mechanics held at the Institute for Theoretical Physics, Santa Barbara UCSB.

It is to be noted that the 3-form of topological torsion is related to the braid integral, a three dimensional thing in four dimensions, and not the Gauss linkage integral, which is a two dimensional thing in three dimensions. The literature of helicity is sometimes confused on this point, and often attempts to relate the helicity integral to the linkage integral.

5.10. Navier Stokes flows and Pfaff Dimension 4

As a last example consider a system where the strong kinematic (topological) constraint $d\mathbf{x} - \mathbf{V}dt = 0$ is not imposed apriori. In other words, the admissable evolutionary processes, $\mathbf{V}$, may have anholonomic fluctuations about kinematic perfection.

$$\Delta\mathbf{x} = d\mathbf{x} - \mathbf{V}dt \neq 0 \tag{5.60}$$

The physical system will be built on the Cartan topology of the 1-form, A, given previously for the Euler flow. However, the Cartan topology will be constrained, not by the Hamiltonian conditions required to generate an extremal system (which is free of kinematic fluctuations), but by a more relaxed set of conditions that permit finite kinematic fluctuations, $d\mathbf{x} - \mathbf{V}dt \neq 0$. As it is known that $i(V)dA$ must be transversal to the vector field, V, it follows that a weaker topological constraint might exist in the form,

$$i(V)dA = f_k(dx^k - V^k dt) + d\theta, \tag{5.61}$$

where the functions θ are Bernoulli-Casimir first integrals in the sense that $i(V)d\theta = 0$.

When $f_k = 0$, these fluctuation constraints reduce to the more stringent Hamiltonian conditions for an extremal flow, or in the case where $d\theta \neq 0$, to the Bernoulli-Casimir symplectic conditions. If is assumed that

$$f_k = v(curlcurl\,\mathbf{V})_k, \tag{5.62}$$

it follows that the expression given above, $i(v)dA = f_k(dx^k - V^k dt)$, is exactly equivalent to the Navier-Stokes partial differential system [25] for an incompressible viscous flow on the variety x, y, z, t.

$$\{\partial\mathbf{V}/\partial t + grad(\mathbf{V} \circ \mathbf{V}/2) - \mathbf{V} \times curl\mathbf{V}\} = \{\nu\nabla^2\mathbf{V}\} - grad\,P/\rho \tag{5.63}$$

These relaxed topological constraints, which admit evolutionary fluctuations in the Cartan system, permit the Topological Parity 4-form to be computed for the Navier Stokes fluid; the result is:

$$K = F\hat{}F = -2\nu\,(curl\,\mathbf{V} \circ curlcurl\,\mathbf{V})dx\hat{}dy\hat{}dz\hat{}dt. \tag{5.64}$$

From this result it is apparent that the Pfaff dimension of the domain is 4, unless the viscosity is zero, or the vorticity field satisfies the conditions of Frobenius integrability. The Torsion current anomaly is equal to $-2\nu\,(curl\,\mathbf{V} \circ curlcurl\,\mathbf{V})$. The torsion lines can stop or stop within the domain producing defect structures that effect the cohomology of the Cartan topology.

An interesting result is the proof that the closed integral of topological Torsion-Helicity is a relative integral invariant for the viscous, compressible fluid, if the Cartan sequence has a Pfaff dimension equal to 3. Recall that the evolution of the 3-form $H = A\hat{}dA$ is given by the Lie differential expression,

$$L_{(\beta V)} \oint_3 H = \oint_3 \{i(\beta V)dH + d(i(\beta V)H\} = \oint_3 \{i(\beta V)dH\} + 0 \tag{5.65}$$

But if $curl\,\mathbf{V} \circ curlcurl\,\mathbf{V}$ vanishes (for any viscosity) then $dH = dA\hat{}dA = 0$, and the RHS of the above expression vanishes, for any reparameterization, β. Therefore, the closed integral of the Topological Torsion three form is a deformation invariant not only of Eulerian flows, but also of viscous flows for which the vorticity field is of Pfaff dimension 2 (the velocity field is Pfaff dimension 3). The folklore concept that viscosity destroys the helicity invariant is not necessarily true.

6. APPENDIX

6.1. Distributions and the Adjoint Field

Although the emphasis in this article is on concepts that are independent from the choice of metric or connection[4], it is useful to demonstrate how a 1-form of Action, A, may be used to generate a compatible Frame field $[F]$ and a Cartan connection $[C]$ on the variety. The symmetry features of $[F]$ lead to metric ideas, and certain anti-symmetry features of $[C]$ lead to the concept of Affine torsion

[4]The evolutionary processes of primary interest herein are those described by operating on differential forms with the Lie differential with respect to a direction field.

(which is not the same as Topological torsion, or Frenet torsion). The concept of a differential connection also leads to the famous geometrical structural equations of Cartan, which are different from the topological structure concept utilized in this article. The topological structure concepts uses herein are independent from the choice of connection or metric. The details of the refined topological features of subspaces based upon the constraints of a global Cartan connection or metric will be the topic of another article.

The construction of the Frame field $[F]$ can be done in such a manner that it admits differential closure of a vector basis over the domain of support (det$[F]$ $\neq 0$). That is, the differential of any basis vector (contra-variant columns of functions) of the matrix Frame field creates a displaced vector which can be linearly composed of the vectors of the basis frame, each multiplied by differential 1-forms. The N^2 differential 1-forms that make up the coefficients of the vector differentials can be used to define the Cartan right connection matrix, $[C]$. The differential closure condition can be expressed by the equation,

$$d[F] = [F]\hat{}[C]. \tag{6.1}$$

The differential closure process on a Frame of independent vectors which is based on a connection is not the same as the operation of forming the exterior differential of a p-form. The exterior differential of a p form takes a p form into a p+1 form, where the p-form is an element of one vector subspace, and the p+1 form is an element of a different vector subspace of the . Grassman algebra of dimension 2^N. The differential process constrained by a connection takes a vector of dimension N into a vector of dimension N. In other words the connection based closure process is a process where the initial and final state are within the same vector subspace.

It should be realized from the outset that Frame fields are not uniquely determined by a given 1-form of Action. When a Frame field exists, the differential connections $[C]$ which generate differential closure can be placed into equivalence classes, determined by the group properties of the matrices involved. The investigation of the properties of these various group equivalence classes has become known as the study of "gauge theories", and the method enjoys great popularity at present. However, the choice of a gauge group in physics is often just that, a choice made by guessing, followed by attempts to put the constrained results into correspondence with physical properties and measurements. In this article, the focus is on those topological features that can be put into correspondence with experiment, and yet are independent from a specific choice of connection and/or metric.

Even though a Frame field is not necessarily unique, and goes beyond the primitive topological concepts that do not depend upon metric or connection, an algorithm for producing a Frame field will be discussed in the next section. From a given 1-form, A, there are two important types of procedures that can be used to construct a useful Frame field. One procedure is differential, and is related to parametric surface theory. The second procedure is algebraic, and is more closely related to implicit surface theory. The algebraic procedure will be discussed first.

6.1.1. The implicit algebraic Frame field

To construct the Frame field from a given 1-form, note that at a regular point, $\{x\}$, of an N dimensional space, any given 1-form, A, will admit N-1 linearly independent vector direction fields, $\mathbf{V}(x)$. Each vector direction field has N component functions, V^μ, to be determined algebraically from the following formula:

$$\text{algebraic orthogonality} \quad A_\mu V^\mu = 0. \tag{6.2}$$

The collection of N-1 vectors orthogonal to the 1-form are called elements of a distribution direction field, for multiplication of each vector field $\mathbf{V}$ by any non-zero function $1/\lambda(x^\mu)$ is also a solution to the algebraic orthogonality equation:

$$A_\mu(V^\mu/\lambda(x^\mu)) = 0 \text{ if } A_\mu V^\mu = 0. \tag{6.3}$$

This independence from scale is typical of projective geometries.

One possible (algebraic) construction, using the given functional coefficients, A_μ, of the 1-form, A, yields a Frame matrix of the form:

$$[F] = \begin{bmatrix} A_n & 0 & ... & A_1/\lambda \\ 0 & A_n & ... & A_2/\lambda \\ ... & ... & ... & ... \\ -A_1 & -A_2 & ... & A_n/\lambda \end{bmatrix}. \tag{6.4}$$

The first N-1 columns satisfy the algebraic orthogonality constraint, which implies that the last column vector is proportional to the adjoint of the matrix $[F]$. The determinant of the Frame field is given by the expression:

$$\det [F] = (A_n)^{(n-2)}\{(A_1)^2 + (A_2)^2 + ... + (A_n)^2\}/\lambda^2 \tag{6.5}$$

If the rescaling factor, $1/\lambda$, is chosen such that the determinant is unity over the domain of support of A, then on that domain the Frame field is globally defined and always has an inverse. The N-1 vector (direction fields) which satisfy the orthogonality relations, each of N components, are defined to be a basis of the "associated or horizontal" vectors relative to the given 1-form, A. Note that in the construction above, the coefficient A_n appears to have a privileged position. However, in spaces of odd topological (Pfaff) dimension, a canonical (Darboux) format indicates that there is one coefficient (presumed to be A_n) that is equal to unity. The differential 1-form then has the canonical format, $A = p_\mu dq^\mu + 1ds$. For even topological dimensions, the canonical format is $A = p_\mu dq^\mu + Hdt = Ldt + p_\mu(dq^\mu - V^\mu dt)$, where H is an independent function. Note that the classical Hamiltonian constraint that $H \Rightarrow H(p, q, t)$ reduces the topological dimension 2n+2 to 2n+1.

The Cartan connection matrix for a Frame field constructed in an implicit algebraic manner can admit certain anti-symmetries of subspace that have been defined as Affine translational torsion. The parametric method described below, will not produce a connection with affine translational torsion of subspaces.

6.1.2. The parametric differential Frame field

If a parametric mapping of N functions in terms of N-1 parameters is given,

$$\xi^\alpha \Rightarrow x^k = X^k(\xi^\alpha) \quad (1 \leq k \leq N) \ (1 \leq \alpha \leq N-1) \tag{6.6}$$

is given, then the N-1 associated vectors can be defined differentially. That is, the partial derivatives of the N mapping functions with respect to the N-1 parameters can be used to form the first N-1 columns (associated vectors) of the matrix, $[M]$.

$$[M] = \begin{bmatrix} \mathbf{e}_\alpha^1 & \mathbf{e}_\beta^1 & ... & 0 \\ \mathbf{e}_\alpha^2 & \mathbf{e}_\beta^2 & ... & 0 \\ ... & ... & ... & ... \\ \mathbf{e}_\alpha^n & \mathbf{e}_\beta^n & ... & 0 \end{bmatrix} \tag{6.7}$$

$$\mathbf{e}_\alpha^k = \partial X^k(\xi^\beta)/\partial \xi^\alpha \tag{6.8}$$

The adjoint vector direction field to this N-1 system of associated vectors can be interpreted as a "normal or vertical" direction field via the algebraic orthogonality relations. Given the N-1 associated vectors, the adjoint vector, $\mathbf{n}$, can be constructed algebraically by adding a column of zeros to the N by N-1 matrix $[M]$ of contravariant associated vectors, $\mathbf{e}^k_\alpha$. (The component index k ranges from 1 to N and the index α ranges from 1 to N-1) .

$$[M] = \begin{bmatrix} \mathbf{e}^1_\alpha & \mathbf{e}^1_\beta & ... & 0 \\ \mathbf{e}^2_\alpha & \mathbf{e}^2_\beta & ... & 0 \\ ... & ... & ... & ... \\ \mathbf{e}^n_\alpha & \mathbf{e}^n_\beta & ... & 0 \end{bmatrix} \tag{6.9}$$

The determinant of $[M]$ is zero, but there always exists an adjoint matrix consisting of a column of N-1 by N-1 sub determinants.

$$\text{transpose of the Adjoint of } [M] = \begin{bmatrix} 0 & 0 & ... & \mathbf{n}^1 \\ 0 & 0 & ... & \mathbf{n}^2 \\ ... & ... & ... & ... \\ 0 & 0 & ... & \mathbf{n}^n \end{bmatrix} \tag{6.10}$$

The parametric method permits the creation of the (orthogonal) Adjoint 1-form given the N-1 distribution vectors, while the implicit method permits the creation of the N-1 orthogonal distribution vectors from a given 1-form, that is adjoint to the vectors of the distribution.

The adjoint direction field, $\mathbf{n}$, exists algebraically whether or not the distribution of N-1 vectors, $\mathbf{e}^k_\alpha$, span a simple hypersurface. By construction via the orthogonality constraint, the coefficients of the given 1-form A_μ are in effect proportional to the adjoint direction field. As discussed in the previous subsection, in the more simple situations the coefficients of a differential 1-form, A, can be viewed as a representation of the normal field to a hypersurface. In all cases the coefficients of a differential 1-form can be viewed as being an adjoint direction field.

Perhaps even more remarkably, it is possible to scale the adjoint direction field (hence the differential 1- form) by a function λ such that the determinant of the N x N matrix,

$$[F] = \begin{bmatrix} \mathbf{e}^1_\alpha & \mathbf{e}^1_\beta & ... & \mathbf{n}^1/\lambda \\ \mathbf{e}^2_\alpha & \mathbf{e}^2_\beta & ... & \mathbf{n}^2/\lambda \\ ... & ... & ... & ... \\ \mathbf{e}^n_\alpha & \mathbf{e}^n_\beta & ... & \mathbf{n}^n/\lambda \end{bmatrix} \tag{6.11}$$

is globally equal to a constant. The procedure thereby defines a Frame of N basis vectors everywhere over the N dimensional domain of support of the 1-form, A. It follows that exterior differentials of each of the basis vectors of the

Frame are linear combinations of the set of the basis vectors. That is, the exterior differential process acting on the basis vectors of the Frame is closed. The process of exterior differentiation acting on elements of the set creates objects that remain within the set. Although this parametric procedure is similar to the implicit method described previously, the parametric method never generates a Frame with a connection that supports translational Affine torsion of subspaces.

When acting on p-forms, the exterior derivative carries a p-form from one vector space into a p+1 form in a different vector space. The concept of a connection constrains the differential process to transport a initial vector of one vector space into a final vector in the same vector space. Both vectors have the same basis.

6.1.3. Projective Frames.

In each of the "adjoint" methods given above, the orthogonality conditions are in effect 2(N-1) constraints on the general N^2 variables of a Frame matrix. A determinantal constraint of the type det $[F] = 1$ adds one more constraint condition. Quadratic (metric) symmetry features implies that symmetric product of the Frame fields constructed by the adjoint procedure above yields a matrix with a fixed point.

$$[\widetilde{F}] \circ [F] = \begin{bmatrix} \mathbf{g}_\alpha^1 & \mathbf{g}_\beta^1 & ... & 0 \\ \mathbf{g}_\alpha^2 & \mathbf{g}_\beta^2 & ... & 0 \\ ... & ... & ... & ... \\ 0 & 0 & ... & \det[F]/A_n^{n-2} \end{bmatrix} \tag{6.12}$$

The coefficients of a projective frame would have only one constraint.

The utility of the "adjoint" procedure is that quadratic geometric metric properties of the tangent space can be decoupled from the geometric properties of the "adjoint" or "normal" space with an appropriate choice of $1/\lambda$.

6.1.4. Remarks

In three dimensions, the Gibbs cross product of engineering vector calculus is considered to be a "vector" for it has the same number of components as the gradient. Yet it has different behavior under transformations of the basis, and is therefor called a "pseudovector" or an axial vector. In the exterior calculus, the exterior product of the two 1-forms, with components proportional to covariant tensor of rank 1, creates a 2-form with covariant components of rank 2. Only in constrained geometries, such as euclidean three space, do 2-forms have any resemblance to the Gibbs cross product (a rule which fails in dimension n > 3). The pseudo-vector is an object that behaves like a contravariant tensor density of rank 1. Such objects are usually defined as "currents". In general, there are two species of differential forms (that are often dual to one another and are well behaved with respect to functional substitution and the pullback operation: p-forms and N-p form densities or currents. One species pulls back (meaning that the form is well defined with respect to functional substitution) with respect to the Jacobian transpose, while the other pulls back with respect to the Jacobian adjoint. Of course for orthogonal systems, these concepts are degenerate, for the inverse and the adjoint and the transpose of the Jacobian matrix are the same. Recall that at a point it is always possible to define a vector basis in terms of an orthogonal system (use the Gram-Schmidt process), but the possibility of extending, or mapping, the property of orthogonality smoothly and uniquely (without singularities) from one neighborhood to another neighborhood in a global sense requires that the mapping process be constrained to the be an element of the orthogonal group. Such constraints apply nicely to rigid body motion, but fail to describe the deformation of a solid. Hence the reader is advised that the automatic or indiscriminate use of orthonormal basis frames will not yield a complete understanding of nature.

If the neighborhoods can be connected by a singly parameterized vector field, then these concepts are at the basis of the Frenet-Serret moving frame analysis. Cartan extended these ideas to domains that are not so simply connected, and developed the notion of the moving basis Frame, which he called the Repere Mobile. In that which follows, it will be demonstrated how to construct these moving basis frames. There will be two distinct problems. The first problem will be how to construct a matrix frame of basis vectors at some point p of a space. Depending on the constraints inherent in their construction, the basis frames can be elements of an equivalence class. The equivalence class can be refined by imposition of other constraints. The second problem will be how to determine the origin, O, such that the point p can be defined. The intuitive idea is that the origin can be uniquely defined. However, it will soon be discovered that the origin need not be unique, and might even incorporate fluctuations.

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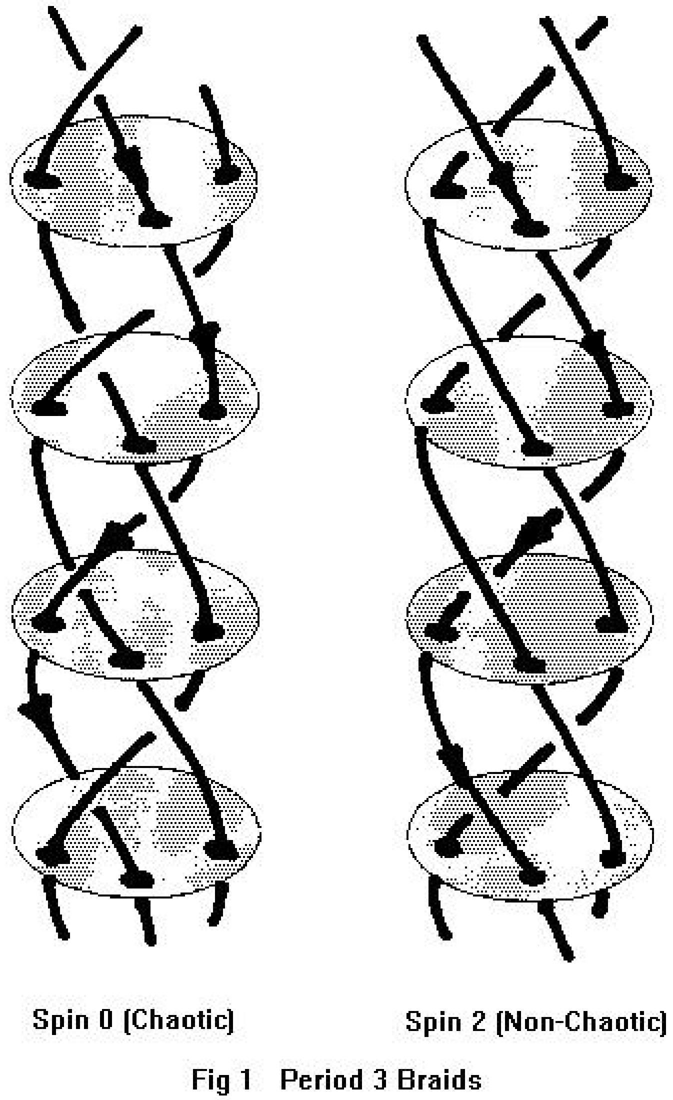

Figure 8.1: