Cartan's topological structure

R. M. Kiehn and Phil Baldwin
Physics Department, University of Houston
http://www.cartan.pair.com

Abstract:A system of di®erential forms will establish a topology and a topological structure on a domain of independent variables such that is possible to determine which maps or processes acting on the system are continuous. Perhaps the most simple topology is that generated by the existence of a single 1-form of Action, its Pfa® sequence of exterior di®erentials, and their intersections. In such a topology the exterior derivative becomes a limit point generator in the sense of Kuratowski. The utilization of such techniques in physical systems is examined. A key feature of the Cartan topology is determined by the Pfa® dimension (representing the minimum number of functions to describe the 1-form generator). In particular, when the Pfa® dimension is 3 or more the Cartan topology becomes a disconnected topology, with the existence of topological torsion and topological parity. Most classical physical applications are constrained to cases where the Pfa® dimension is 2 or less, for such is the domain of unique integrability. The more interesting domain of non-unique solutions requires the existence of topological torsion, and can lead to an understanding of irreversible processes without the use of statistics.

1   Introduction

In the period from 1899 to 1926, Eli Cartan developed his theory of exterior di®erential systems [1,2], which included the ideas of spinor systems [3] and the di®erential geometry of projective spaces and spaces with torsion [4]. The theory was appreciated by only a few contemporary researchers, and made little impact on the main stream of physics until about the 1960's. Even specialists in di®erential geometry (with a few notable exceptions [5]) made

1

little use of Cartan's methods until the 1950's. Even today, many physical scientists and engineers feel that Cartan's theory of exterior differential forms is just another formalism of fancy.

However, Cartan's theory of exterior differential systems has several advantages over the methods of tensor analysis that were developed during the same period of time. The principle fact is that differential forms are well behaved with respect to functional substitution of C1 differentiable maps. Such maps need not be invertible even locally, yet differential forms are always deterministic in a retrodictive sense [6], by means of functional substitution. Such determinism is not to be associated with contravariant tensor fields, if the map is not a diffeomorphism.

Although the word "topology" had not become popular when Cartan developed his ideas (topological ideas were described as part of the theory of analysis situs), there is no doubt that Cartan's intuition was directed towards a topological development. For example, Cartan did not define what were the open sets of his topology, nor did he use in his early works the words "limit points or accumulation points" explicitly, but he did describe the union of a differential form and its exterior derivative as the "closure" of the form.

In a simplistic comparison it might be said that tensor methods are restricted to geometric applications, while Cartan's methods can be applied directly to topological concepts as well as geometrical concepts. Cartan's theory of exterior differential systems is a topological theory not necessarily limited by geometrical constraints and the class of diffeomorphic transformations that serve as the foundations of tensor calculus. A major objective of this article is to show how limit points, intersections, closed sets, continuity, connectedness and other elementary concepts of modern topology are inherent in Cartan's theory of exterior differential systems. These ideas do not depend upon the geometrical ideas of size and shape (hence Cartan's theory, as are all topological theories, is renormalizeable). In fact the most useful of Cartan's ideas do not depend explicitly upon the geometric ideas of a metric, nor upon the choice of a connection as in fiber bundle theories.

In this article the Cartan topology will be constructed explicitly for an arbitrary exterior differential system, §: All elements of the topology will be evaluated, and the limit points, the boundary sets and the closure of every subset will be computed abstractly. An earlier intuitive result [7] utilized the notion that Cartan's concept of the exterior product may be used as an intersection operator, and his concept of the exterior derivative may used as a limit point operator acting on differential forms. These ideas will be given

formal substance in this article. A major result of this article, with important physical consequences in describing evolutionary processes, is the demonstration that the Cartan topology is not necessarily a connected topology, unless the property of topological torsion vanishes.

1.1 The exterior product

Cartan's theory of exterior di®erential systems has its foundations in the Grassmann algebra, where the two combinatorial processes used to de¯ne the algebra are: vector space addition, and what is now called the exterior product [8]. The exterior product acts on pairs of algebraic elements called exterior p-forms. In this article, the exterior or wedge product operator is symbolized by the ^ symbol for ease of typing; the exterior product of A and B is then given by the expression A^B. The p-forms may have a di®erential basis, symbolized by the set dx, combined in the manner of a vector space with functional coe±cients. A di®erential 1-form is then given by the expression,

$$A = A_1 dx^1 : \qquad (1)$$

For 1-forms the multiplication rules are:

$$A\text{\textasciicircum}A = 0; \; A\text{\textasciicircum}B = \text{¡} \; B\text{\textasciicircum}A: \qquad (2)$$

At some regular point, fxg, the 1-form will admit N-1 vector ¯elds, V, to be constructed such that

$$i(V)A = A_1 V^1 = 0: \qquad (3)$$

Such vector ¯elds are de¯ned to be "associated" vectors of the 1-form, A. If these vector ¯elds have a vanishing Lie bracket, then they span a neighborhood of the point in terms of a simple (hyper) surface. The adjoint vector to this N-1 system acts as the "normal" ¯eld to the surface spanned by the N-1 vectors. The coe±cients A_1 form this "normal" ¯eld.

Now consider two such surface systems represented by the 1-forms A and B. Do the two (curved) surfaces intersect? The points in common to the

3

two surfaces are given by the non-null set formed by the exterior product of A with B. If A^B vanishes, then the two surfaces have no points in common. Consider the simple case where the 1-forms A and B have coe±cients which form the components of a gradient ¯eld (a Gauss Weingarten surface normal),

$$A = A_1 dx^1 = r \acute{A} ¢ dr \qquad B = B_1 dx^1 = r \tilde{A} ¢ dr \qquad (4)$$

Then (in 3 dimensions) A^B = (A £ B)$_z$dx^dy + cyclic permutations), a result that demonstrates that the Gibbs cross product, $A £ B = r \acute{A} £ r \tilde{A}$ in euclidean three dimensions is related to the "line" of points which are in common to both surfaces. This result pictorially cements the notion that the exterior product (acting on 1-forms) is an operator related to the concept of intersection. If the two surfaces do not intersect, and the exterior product vanishes, then the functions Á and Â are not functionally independent. These concept extends to p-forms of higher rank.

In three dimensions, the Gibbs cross product is considered to be a "vector" for it has the same number of components as the gradient. Yet it has di®erent behavior under transformations of the basis, and is sometimes called a "pseudovector" or an axial vector. In the exterior calculus, the exterior product of the two 1-forms, with components proportional to covariant tensor of rank 1, creates a 2-form with covariant components of rank 2. Only in constrained geometries, such as euclidean three space, do 2-forms have any resemblence to the Gibbs cross product (a rule which fails in dimension n >3). The pseudo-vector is an object that behaves like a contravariant tensor density of rank 1. Such objects are usually de¯ned as "currents". In general, there are two species of di®erential forms (that are often dual to one another and are well behaved with respect to functional substitution and the pullback operation: p-forms and N-p form densities or currents. One species pulls back (meaning that the form is well de¯ned with respect to functional substitution) with respect to the Jacobian transpose, while the other pulls back with respect to the Jacobian adjoint. Of course for orthogonal systems, these concepts are degenerate, for the inverse and the adjoint and the transpose of the Jacobian matrix are the same. Recall that at a point it is always possible to de¯ne a vector basis in terms of an orthogonal system (use the Gram-Schmidt process), but it may not be possible to extend the orthogonality concept smoothly (without singularities) from one neighborhood to another neighborhood. If the neighborhoods can be connected by

a singly parameterized vector field, then these concepts are at the basis of the Frenet-Serret moving frame analysis. Cartan extended these ideas to domains that are not so simply connected, and developed the notion of the moving basis Frame, which he called the Repere Mobile.

1.2 The exterior derivative

The second new operator found in Cartan's theory of exterior differential systems is the exterior derivative. The exterior derivative, like the exterior product, also has topological connotations when applied to differential forms, but the results are sometimes surprising and unfamiliar. For example consider the exterior derivative of the N-1 form density, D, in three dimensions, given by the expression,

$$dD = d(D^x dy{\wedge}dz - D^y dz{\wedge}dx + D^z dx{\wedge}dy) \qquad (5)$$
$$= \mathrm{div}_3(D) dx{\wedge}dy{\wedge}dz = \rho(x;y;z) dx{\wedge}dy{\wedge}dz$$

where ρ has been defined as the resultant of the action of the exterior derivative, $\mathrm{div}_3(D)$. The usual interpretation of Gauss' law is that the field lines of the vector D terminate (or have a limit or accumulation point) on the charges, Q. Gauss' law generates both the intuitive idea that sources are related to limit points, and the novel concept that the exterior derivative is a limit point operator creating these limits points when the operation is applied to a differential form. However, as demonstrated below, the concept that the exterior derivative is a limit point operator relative to the Cartan topology is a general idea, and is not restricted to Gauss' law.

For example, extending this idea to four dimensions for the N-2 form density, G, of Maxwell excitations (D;H),

$$G = - D^x dy{\wedge}dz + D^y dz{\wedge}dx - D^z dx{\wedge}dy + H^x dx{\wedge}dt + H^y dy{\wedge}dt + H^z dz{\wedge}dt; \qquad (6)$$

the exterior derivative dG of G yields a three form, J, defined as the electromagnetic current 3-form,

$$J = J^x dy{\wedge}dz{\wedge}dt - J^y dx{\wedge}dz{\wedge}dt + J^z dx{\wedge}dy{\wedge}dt - \rho dx{\wedge}dy{\wedge}dt \qquad (7)$$

where in 3-vector language,

$$\text{curl } H - \partial D/\partial t = 0 \qquad \text{div } D = \tfrac{1}{2}: \qquad (8)$$

The charge current density act as the "limit points" of the Maxwell field excitations. Note that $dJ = 0$ for C2 functions by Poincare's lemma.

However, consider the N-1 current, C (not necessarily equal to J as defined above) in four dimensions

$$C = \{fV^x dy \wedge dz \wedge dt - V^y dx \wedge dz \wedge dt + V^z dx \wedge dy \wedge dt - 1 dx \wedge dy \wedge dt\} \qquad (9)$$

and its exterior derivative as given by the expression,

$$dC = \{div_3(\tfrac{1}{2}V) + \partial\tfrac{1}{2}/\partial t\}dx \wedge dy \wedge dz \wedge dt: = R dx \wedge dy \wedge dz \wedge dt = R - 4\_vol \qquad (10)$$

When thre 4-form R vanishes, the resultant expression is physically interpreted as the "equation of continuity" or as a "conservation law". Over a closed boundary, that which goes in is equal to that which goes out (when $dC = 0$). Note that the concept of the conservation law is a topological constraint: the "limit points" of the "current 3-form" in four dimensions must vanish if the conservation law is to be true. If the RHS of the above expression is not zero, then the current 3-form is said to have an "anomaly", or a source (or sink) . The anomaly acts as the source of the otherwise conserved quantity. The limit points, R, of the 3-form, C, are generated by its exterior derivative, $dC = \{div_3(\tfrac{1}{2}V) + \partial\tfrac{1}{2}/\partial t\} - 4:$ When the RHS is zero, the current "lines" do not stop or start within the domain. (It is possible for them to be closed on themselves in certain topologies).

As another example, consider the 1-form of vector and scalar potentials given by the expression,

$$A = A_x dx + A_y dy + A_z dz - \phi dt = A \cdot dr - \phi dt: \qquad (11)$$

The exterior derivative of the 1-form A generates the 2-form $F = dA$ of electromagnetic intensities, $(E; B)$:

$$F = dA = B_z dx \wedge dy + B_x dy \wedge dz + B_y dz \wedge dx + E_x dx \wedge dt + E_y dy \wedge dt + E_z dz \wedge dt; \qquad (12)$$

where

$$B = \text{curl}\,A; \quad \text{and} \quad E = -\nabla \phi - \partial A/\partial t. \quad (13)$$

The exterior derivative of F vanishes if the potential functions are C2 differentiable:

$$ddA = dF = 0 \quad ⅜ \quad \text{curl}\,E + \partial B/\partial t = 0 \quad \text{and} \quad \text{div}_3 B = 0. \quad (14)$$

Note that these derivations of Maxwell's equations are based on topological statements about limit points, and do not depend upon geometrical considerations of metric or connections.

This now almost classic generation of the Maxwell field equations [9] has another less familiar interpretation: The E and B field intensities are the topological limit "points" of the 1-form of potentials, {A; φ}, relative to the Cartan topology! The limit points of the 2-form of field intensities, F, are the null set. For C2 vector fields, the Cartan topology admits flux quanta, charge quanta, and spin quanta, but excludes magnetic monopoles [10]. When the differential system of interest is built upon the forms A, F and G, it is possible to show that superconductivity is to be associated with the constraints on the limit point sets of A; A^F; and A^G [11]. That is, superconductivity has its origins in topological, not geometrical, concepts. This remarkable idea that the exterior derivative is a limit point operator is given formal substance in the section 4.

2   The Cartan Point Set Topology.

Cartan built his theory around an exterior differential system, §, which consists of a collection of 0- forms, 1-forms, 2-forms, etc. [12]. He defined the closure of this collection as the union of the original collection with those forms which are obtained by forming the exterior derivatives of every p-form in the initial collection. In general, the collection of exterior derivatives will be denoted by d§, and the closure of § by the symbol, §ᶜ, where

$$§ᶜ = § \cup d§ \quad (15)$$

7

For notational simplicity in this article the systems of p-forms will be assumed to consist of the single 1-form, A. Then the exterior derivative of A is the 2-form $F = dA$, and the closure of A is the union of A and F : $A^c = A \cup F$. The other logical operation is the concept of intersection, so that from the exterior derivative it is possible to construct the set $A^\wedge F$ defined collectively as H : $H = A^\wedge F$: The exterior derivative of H produces the set defined as $K = dH$, and the closure of H is the union of H and K : $H^c = H \cup K$.

This ladder process of constructing exterior derivatives, and exterior products, may be continued until a null set is produced, or the largest p-form so constructed is equal to the dimension of the space under consideration. The set so generated is defined as a Pfaff sequence. The largest rank of the sequence determines the Pfaff dimension of the domain (or class of the form, [13]), which is a topological invariant. The idea is that the 1-form A (in general the exterior differential system, $\S$) generates on space-time four equivalence classes of points that act as domains of support for the elements of the Pfaff sequence, $A; F; H; K$. The union of all such points will be denoted by $X = A \cup F \cup H \cup K$. The fundamental equivalence classes are given specific names:

$$\text{Topological ACTION} : A = A_1 dx^1 \tag{16}$$

$$\text{Topological VORTICITY} : F = dA = F_{10} dx^1 {}^\wedge dx^0 \tag{17}$$

$$\text{Topological TORSION} : H = A^\wedge dA = H_{10\frac{3}{4}} dx^1 {}^\wedge dx^0 {}^\wedge dx^{\frac{3}{4}} \tag{18}$$

$$\text{Topological PARITY} : K = dA^\wedge dA = K_{10\frac{3}{4}i} dx^1 {}^\wedge dx^0 {}^\wedge dx^{\frac{3}{4}} {}^\wedge dx^i: \tag{19}$$

The Cartan topology is constructed from a basis of open sets, which are defined as follows: first consider the domain of support of A. Define this "point" by the symbol A. A is the first open set of the Cartan topology. Next construct the exterior derivative, $F = dA$, and determine its domain of support. Next, form the closure of A by constructing the union of these two

8

domains of support, $A^c = A \cup F$. $A \cup F$ forms the second open set of the Cartan topology. .

Next construct the intersection $H = A \wedge F$, and determine its domain of support. Define this "point" by the symbol H, which forms the third open set of the Cartan topology. Now follow the procedure established in the preceding paragraph. Construct the closure of H as the union of the domains of support of H and $K = dH$. The construction forms the fourth open set of the Cartan topology. In four dimensions, the process stops, but for $N > 4$, the process may be continued.

Now consider the basis collection of open sets that consists of the subsets,

$$B = \{A,\ A^c,\ H,\ H^c\} = \{A,\ A \cup F,\ H,\ H \cup K\} \tag{20}$$

The collection of all possible unions of these base elements, and the null set, $\emptyset$, generate the Cartan topology of open sets:

$$T(\text{open}) = \{X,\ \emptyset,\ A,\ H, A \cup F,\ H \cup K,\ A \cup H,\ A \cup H \cup K,\ A \cup F \cup H\}: \tag{21}$$

These nine subsets form the open sets of the Cartan topology constructed from the domains of support of the Pfaff sequence constructed from a single 1-form, A. The compliments of the open sets are the closed sets of the Cartan topology.

$$T(\text{closed}) = \{\emptyset,\ X,\ F \cup H \cup K,\ A \cup F \cup K, A \cup F,\ H \cup K,\ F \cup K,\ F,\ K\}: \tag{22}$$

From the set of 4 "points" $\{A, F, H, K\}$ that make up the Pfaff sequence it is possible to construct 16 subset collections by the process of union. It is possible to compute the limit points for every subset relative to the Cartan topology. The classical definition of a limit point is that a point p is a limit point of the subset Y relative to the topology T if and only if for every open set which contains p there exists another point of Y other than p [14]. The results of this definition are presented in Table I

9

The Cartan Topology

$$A = A_k dx^k$$
$$F = dA; \quad H = A{\wedge}F; \quad K = F{\wedge}F$$
Basis $\{A, A^c, H, H^c\} = \{A; A \cup F; H; H \cup K\}$
$T(\text{open}) = \{X; \varnothing; A; H; A \cup F; H \cup K; A \cup H; A \cup H \cup K; A \cup F \cup H\}$
$T(\text{closed}) = \{\varnothing; X; F \cup H \cup K; A \cup F \cup K; H \cup K; A \cup F; F \cup K; F; K\}$

Subset	Limit Pts	Interior	Boundary	Closure
¾	d¾	:	@¾	¾ ∪ d¾
∅	∅	∅	∅	∅
A	F	A	F	A ∪ F
F	∅	∅	F	F
H	K	H	K	H ∪ K
K	∅	∅	K	K
A ∪ F	F	A ∪ F	∅	A ∪ F
A ∪ H	F; K	A ∪ H	F ∪ K	X
A ∪ K	F	A	F ∪ K	A ∪ F ∪ K
F ∪ H	K	H	F ∪ K	F ∪ H ∪ K
F ∪ K	∅	∅	F ∪ K	F ∪ K
H ∪ K	K	H ∪ K	∅	H ∪ K
A ∪ F ∪ H	F; K	A ∪ F ∪ K	K	X
F ∪ H ∪ K	K	H ∪ K	F	F ∪ H ∪ K
A ∪ H ∪ K	F; K	A ∪ H ∪ K	F	X
A ∪ F ∪ K	F	A ∪ F	K	A ∪ F ∪ K
X	F; K	X	∅	X

By examining the set of limit points so constructed for every subset of the Cartan system, and presuming that the functions that make up the forms are C2 differentiable (such that the Poincare lemma is true, $dd\omega = 0$; any ω), it is easy to show that for all subsets of the Cartan topology every limit set is composed of the exterior derivative of the subset, thereby proving the claim that the exterior derivative is a limit point operator relative to the Cartan topology. For example, the open subset, A ∪ H, has the limit points that consist of F and K: The limit set consists of F ∪ K which can be derived directly by taking the exterior derivatives of the elements that make up A ∪ H; that is, (F ∪ A = d(A ∪ H) = (dA ∪ dH). Note that this open set, A ∪ H, does not contain its limit points. Similarly for the closed set, A ∪ F, the limit points

are given by F which may be deduced by direct application of the exterior derivative to (A [F) : (F) = d(A [F) = (dA [dF) = (F [;) = (F).

3   Topological Torsion and Connected vs Non-connected Cartan topologies.

The Cartan topology as given in Table 1 is composed of the union of two sub-sets which are both open and closed (X = A^c [H^c), a result that implies that the Cartan topology is not necessarily connected. An exception exists if the topological torsion, H, and hence its closure, vanishes, for then the Cartan topology is connected. This extraordinary result has broad physical consequences. The connected Cartan topology based on a vanishing topological torsion is at the basis of most physical theories of equilibrium. In mathematics, the connected Cartan topology corresponds to the Frobenius integrability condition for Pfa±an forms. In thermodynamics, the connected Cartan topology is associated with the Caratheodory concept of inaccessible thermodynamic states [15], and the existence of an equilibrium thermodynamic surface. If the 1-form, Q, of heat generates a Cartan topology of null topological torsion, H = Q^dQ = ; ; then the Cartan topology built on Q is connected. Such systems are "isolated" in a topological sense, and the heat 1-form has a representation in terms of two and only two functions, conventionally written as: Q = TdS. Note again that a fundamental physical concept, in this case the idea of equilibrium, is a topological concept independent from geometrical properties of size and shape. Processes that generate the 1-form Q such that Q^dQ = ; are thermodynamically reversible.   If Q^dQ 6 ; ; the process that generates Q is thermodynamically irreversible.

When the Cartan topology is connected, it might be said that all forces are extendible over the whole of the set, and that these forces are of "long range". Conversely when the Cartan topology is disconnected, the "forces" cannot be extended inde¯nitely over the whole   domain of independent variables, but perhaps only over a single component.   In this sense, such forces are said to be of short range, as they are con¯ned. Note that this notion of short or long range forces does not depend upon geometrical size or scale. The physical idea of short or long range forces is a topological idea of connectivity, and not a geometrical concept of how far.

In an earlier article,   these ideas were formulated intuitively in order

to give an explanation of the "four forces" of physics. The earlier work was based upon differential geometry, before the construction of the Cartan topology based upon differential topology as presented herein. [16] The features of the Pfaff sequence were used to establish equivalence classes for known example metric field solutions, g_{ij}; to the Einstein field equations. The ideas originally presented upon experience with systems in differential geometry can now be given credence based upon the construction differential topology of the disconnected Cartan topology, which will divide the classes into the long range connected category and the short range disconnected category. The 1-form used to build the Cartan topology was constructed from the space-time interactions, $A = g_{j4}dx^j$: Long range parity preserving forces due to gravity (Pfaff dimension 1) and electromagnetism (Pfaff dimension 2) are to be associated with a Cartan Topology that is connected $(H = A \wedge F = A \wedge dA = 0)$: Both the strong force (Pfaff dimension 3) and the weak force (Pfaff dimension 4) are "short" range $(H \neq 0)$ and are to be associated with a disconnected Cartan topology. The strong force is parity preserving $(K = 0)$ and the weak force is not $(K \neq 0)$: The fact that the Cartan topology is not necessarily connect is the topological (not metrical) basis that may be used to distinguish between short and long range forces.

In much of our physical experience with nature it appears that the disconnected domains of Pfaff dimension 3 or more are often isolated as nuclei, while the surrounding connected domains of Pfaff dimension 2 or less appears as fields of charged or non-charged molecules and atoms. However, part of the thrust of this article is to demonstrate that such disconnected topological phenomena are not confined to microscopic systems, but also appear in a such mundane phenomena as the flow of a turbulent fluid. Physical examples of the existence of topological torsion (and hence a non-connected Cartan topology) are given by the experimental appearance of what appear to be coherent structures in a turbulent fluid flow.

To prove that a turbulent flow must be a consequence of a Cartan topology that is not connected, consider the following argument: First consider a fluid at rest and from a global set of unique, synchronous, initial conditions generate a vector field of flow. Such flows must satisfy the Frobenius complete integrability theorem, which requires that $A \wedge dA = 0$. The Cartan topology for such systems is connected, and the Pfaff dimension of the domain is 2 or less. Such domains do not support topological torsion (the Helicity vanishes). Such globally laminar flows are to be distinguished from flows that reside on surfaces, but do not admit a unique set of connected sychronizeable initial

12

conditions. Next consider turbulent °ows which, as the anti-thesis of laminar °ows, can not be integrable in the sense of Frobenius; such turbulent domains support topological torsion (A^dA ⊕ 0), and therefore a disconnected Cartan topology. The connected components of the disconnected Cartan topology can be de¯ned as the (topologically) coherent structures of the turbulent °ow.

Note that a domain can support a homogeneous topology of one component and then undergo continuous topological evolution to a domain with some interior holes. The domain is simply connected in the initial state, and multiply connected in the ¯nal state, but still connected. However, consider the dual point of view where the originally connected domain consists of a homogeneous space that becomes separated into multiple components. The evolution to a topological space of multiple components is not continuous. It follows that the case of a transition from an initial laminar state (H = 0) to the turbulent state (H ⊕ 0) is a transition from a connected topology to a disconnected topology. Therefore the transition to turbulence is NOT continuous. However, note that the decay of turbulence can be described by a continuous transformation from a disconnected topology to a connected topology. Condensation is continuous, gasi¯cation is not. It is demonstrated below that relative to the Cartan topology all C2 di®erentiable, V, acting on C2 p-forms by means of the Lie derivative are continuous. The conclusion is reached that the transition to turbulence must involve transformations that are not C2, hence can occur only in the presence of shocks or tangential discontinuities.

4   The Cartan Topological Structure

A topological structure is de¯ned to be enough information to decide whether a transformation is continuous or not [18]. The classical de¯nition of continuity depends upon the idea that every open set in the range must have an inverse image in the domain. This means that topologies must be de¯ned on both the initial and ¯nal state, and that somehow an inverse image must be de¯ned. Note that the open sets of the ¯nal state may be di®erent from the open sets of the initial state, because the topologies of the two states can be di®erent.

There is another de¯nition of continuity that is more useful for it depends only on the transformation and not its inverse explicitly. A transformation is

continuous if and only if the image of the closure of every subset is included in the closure of the image. This means that the concept of closure and the concept of transformation must commute for continuous processes. Suppose the forward image of a 1-form A is Q; and the forward image of the set F = dA is Z. Then if the closure, A^c = A $\cup$ F is included in the closure of Q^c = Q $\cup$ dQ, for all sub-sets, the transformation is defined to be continuous. The idea of continuity becomes equivalent to the concept that the forward image Z of the limit points, dA, is an element of the closure of Q [18]:

> A function f that produces an image f[A] = Q is continuous iff for every subset A of the Cartan topology, Z = f[dA] $\subseteq$ Q^c = (Q $\cup$ dQ).

The Cartan theory of exterior differential systems can now be interpreted as a topological structure, for every subset of the topology can be tested to see if the process of closure commutes with the process of transformation. For the Cartan topology, this emphasis on limit points rather than on open sets is a more convenient method for determining continuity. A simple evolutionary process, X $\to$ Y, is defined by a map ϕ. The map, ϕ, may be viewed as a propagator that takes the initial state, X, into the final state, Y. For more general physical situations the evolutionary processes are generated by vector fields of flow, V. The trajectories defined by the vector fields may be viewed as propagators that carry domains into ranges in the manner of a convective fluid flow. The evolutionary propagator of interest to this article is the Lie derivative with respect to a vector field, V, acting on differential forms, ω [19].

The Lie derivative has a number of interesting and useful properties.

1. The Lie derivative does not depend upon a metric or a connection.

2. The Lie derivative has a simple action on differential forms producing a resultant form that is decomposed into a transversal and an exact part:

$$L_{(V)}\omega = i(V)d\omega + di(V)\omega .\qquad(23)$$

This formula is known as "Cartan's magic formula". For those vector fields V which are "associated" with the form ω; such that $i(V)\omega = 0$; the Lie

14

derivative becomes equivalent to the covariant derivative of tensor analysis. Otherwise the two derivative concepts are distinct.

3. The Lie derivative may be used to describe deformations and topological evolution. Note that the action of the Lie derivative on a 0-form (scalar function) is the same as the directional or convective derivative of ordinary calculus,

$$L_{(V)}\phi = i(V)d\phi + di(V)\phi = i(V)d\phi + 0 = V \cdot \mathrm{grad}\,\phi: \qquad (24)$$

It may be demonstrated that the action of the Lie derivative on a 1-form will generate equations of motion of the hydrodynamic type.

4. With respect to vector ﬁelds and forms constructed over C2 functions, the Lie derivative commutes with the closure operator. Hence, the Lie derivative (restricted to C2 functions) generates transformations on diﬀerential forms which are continuous with respect to the Cartan topology. To prove this claim:

First construct the closure, $\{\sigma \ [\ d\sigma\}$. Next propagate σ and $d\sigma$ by means of the Lie derivative to produce the decremental forms, say Q and Z,

$$L_{(V)}\sigma = Q \quad \text{and} \quad L_{(V)}d\sigma = Z: \qquad (25)$$

Now compute the contributions to the closure of the ﬁnal state as given by $\{Q\ [\ dQ\}$. If $Z = dQ$, then the closure of the initial state is propagated into the closure of the ﬁnal state, and the evolutionary process deﬁned by V is continuous. However,

$$dQ = dL_{(V)}\sigma = di(V)d\sigma + dd(i(V)\sigma) = di(V)d\sigma \qquad (26)$$

as $dd(i(V)\sigma) = 0$ for C2 functions. But,

$$Z = L_{(V)}d\sigma = d(i(V)d\sigma) + i(V)dd\sigma = di(V)d\sigma \qquad (27)$$

as $i(V)dd\sigma = 0$ for C2 p-forms. It follows that $Z = dQ$, and therefore V generates a continuous evolutionary process relative to the Cartan topology. QED

Certain special cases arise for those subsets of vector ﬁelds that satisfy the equations, $d(i(V)\S) = 0$. In these cases, only the functions that make up the p-form, $\S$, need be C2 diﬀerentiable, and the vector ﬁeld need only be C1. Such processes will be of interest to symplectic processes, with Bernoulli-Casimir invariants.

By suitable choice of the 1-form of action it is possible to show that the action of the Lie derivative on the 1-form of action can generate the Navier Stokes partial diﬀerential equations [20]. The analysis above indicates that C2 diﬀerentiable solutions to the Navier-Stokes equations can not be used to describe the transition to turbulence. The C2 solutions can, however, describe the irreversible decay of turbulence to the globally laminar state.

5 APPLICATIONS

5.1 Frozen - in Fields, the Master Equation

A starting point for many discussions of the magnetic dynamo and allied problems in hydrodynamics starts with what has been called the "master equation" [21],

$$\mathrm{Curl}(V \times B) = \partial B / \partial t.\tag{28}$$

Using the Cartan methods it may be shown that this equation is equivalent to the constraint of "uniform" continuity relative to the Cartan topology. Moreover, it is easy to show these constraints generate symplectic processes which include Hamiltonian evolutionary systems, such as Euler ﬂows, as well as a number of other evolutionary processes which are continuous, but not homeomorphic. In addition a criteria can be formulated to develop an extension of the "helicity" conservation law to a more general setting.

The proof of these results produces a nice exercise in use of the Cartan theory. Consider a 1-form A that satisﬁes the exterior diﬀerential system

$$F - dA = 0,\tag{29}$$

where A is a 1-form of Action, with twice diﬀerentiable coeﬃcients (potentials proportional to momenta) which induce a 2-form, F, of electromagnetic

16

intensities (E and B, related to forces). The exterior differential system is a topological constraint that in effect defines field intensities in terms of the potentials. On a four dimensional space-time of independent variables, (x, y, z, t) the 1-form of Action (representing the postulate of potentials) can be written in the form

$$A = \sum_{k=1}^{3} A_k(x, y, z, t)dx^k - \acute{A}(x, y, z, t)dt = \mathbf{A} \cdot d\mathbf{r} - \acute{A}dt. \qquad (30)$$

Subject to the constraint of the exterior differential system, the 2-form of field intensities, F, becomes:

$$F = dA = \{\partial A_k / \partial x^j - \partial A_j / \partial x^k\}dx^j \wedge dx^k = F_{jk}dx^j \wedge dx^k \qquad (31)$$
$$= B_z dx \wedge dy + B_x dy \wedge dz + B_y dz \wedge dx + E_x dx \wedge dt + E_y dy \wedge dt + E_z dz \wedge dt.$$

where in usual engineering notation,

$$E = -\partial \mathbf{A}/\partial t - \text{grad}\,\acute{A}; \qquad B = \text{curl } \mathbf{A} \approx \partial A_k / \partial x^j - \partial A_j / \partial x^k. \qquad (32)$$

The closure of the exterior differential system, $dF = 0$, vanishes for C2 differentiable p-forms, to yield

$$dF = ddA = \{\text{curl } E + \partial B/\partial t\}_x dy \wedge dz \wedge dt - \cdots + \cdots - \text{div } \mathbf{B}\,dx \wedge dy \wedge dz\} \Rightarrow 0. \qquad (33)$$

Equating to zero all four coefficients leads to the Maxwell-Faraday equations,

$$\{\text{curl } E + \partial B/\partial t = 0; \qquad \text{div } B = 0\}. \qquad (34)$$

The component functions (E and B) of the 2-form, F, transform as covariant tensor of rank 2. The topological constraint that F is exact, implies that the domain of support for the field intensities cannot be compact without boundary, unless the Euler characteristic vanishes. These facts distinguish classical electromagnetism from Yang-Mills field theories. Moreover, the fact that F is subsumed to be exact and C1 differentiable excludes the concept

of magnetic monopoles from classical electromagnetic theory on topological grounds.

Now search for all vector fields that leave the 2-form F an absolute invariant of the flow; that is, search for all vectors that satisfy Cartan's magic formula

$$L_{(V)}F = i(V)dF + di(V)F = 0 + di(V)F = 0. \qquad (35)$$

For C2 functions, the term involving dF vanishes, leaving the expression,

$$L_{(V)}F = di(V)F \qquad (36)$$
$$= d\{(E + V \times B) \cdot dr - (E \cdot V)dt\} \qquad (37)$$
$$= \{curl(E + V \times B)\}_z dy^\wedge dz::: \qquad (38)$$
$$+ \{\partial(E + V \times B)/\partial t + grad(E \cdot V)\} \cdot dr^\wedge dt \qquad (39)$$
$$= 0. \qquad (40)$$

Setting the first three factors to zero yields

$$curl(E + V \times B) = 0 \qquad (41)$$

But for C2 functions, $curlE = -\partial B/\partial t$, and when this expression is substituted into the above equation, the "master equation given by the first equation results. Now recall that dF generates the limit points of A, and if $F = dA$ is a flow invariant, then all limit points are flow invariants relative to the Cartan topology. This result implies that the vector fields, V, that satisfy the constraints of the "master equation" are uniformly continuous evolutionary processes, the limit points, $F = dA$; of the 1-form A are flow invariants, and the lines of vorticity are "frozen-in" the flow. Non-uniform continuity would imply that the limit points are not invariants of the process, but that the closure of the limit points of the target range include the vanishes limit points of the initial domain. Such processes would correspond to a folding of the "lines" of vorticity, which preserve the limit points, but not their sequential order.

A second criteria for limit point invariance is given by the equation,

$$\{\partial(E + V \times B)/\partial t + grad(E \cdot V)\} = 0. \qquad (42)$$

18

The formula indicates that limit point invariance can occur in the presence of dissipation, $E \cdot V \neq 0$:

The criteria for frozen-in ̄elds is established as a constraint on the admissable vector ̄elds, $di(V)dA = di(V)F = 0$: The solution vector ̄elds, V; subject to this constraint can be put into three global categories:

1. Extremal (Hamiltonian) $i(V)F = 0$:
2. Bernoulli-Casimir (Hamiltonian) $i(V)F = d\pounds$:
3. Symplectic $i(V)F = d\copyright + {}^\circ_{harmonic}$

The ̄rst category can exist only on domains of support of F which are of odd Pfa® dimension, but then the solution vector is unique to within a factor. In the other categories, the solution vector need not be unique. Vector ̄elds that satisfy the equation for uniform continuity are said to be symplectic relative to the 1-form, A: Vector ̄elds that belong to categories 1 and 2 have a Hamiltonian representation. Vector ̄elds that belong to category 1, are said to be "extremal" relative to the 1-form, A:

5.2 Euler °ows and Hamiltonian systems.

In 1922 Cartan established the idea that the necessary and su±cient conditions for a system to admit a unique Hamiltonian representation for its evolution, V, is given by the category 1 constraint,

$$W = i(V)dA = i(V)F = 0: \qquad (43)$$

It is apparent that this extremal condition is more stringent than that given above for uniform continuity, $di(V)F = 0$. Such extremal vector ̄elds are independent of parameterization. That is, for extremal processes, $i(\not{h}V)dA = 0$ if $i(V)dA = 0$; for any function, $\not{h}$: Extremal vector ̄elds do not exist on domains where the Pfa® dimension of the Cartan 1-form is even. In classical mechanics, the 1-form W is de ̄ned as the 1-form of Virtual Work, and the Cartan constraint is typical of problems in the variational calculus where it is presumed that the Virtual Work vanishes.

As an example, consider a 1-form of Action de ̄ned as

$$A = v \, dr \, {}_i \, (v \cdot v_{=2} + {}^a)dt; \qquad (44)$$

19

where $d^a = dP/\rho$. Application of the extremal constraint yields the resulting necessary system of partial differential equations is given by known as the Euler equations of hydrodynamics.

$$\partial v/\partial t + grad(v \cdot v/2) - v \times w = - gradP/\rho, \qquad (45)$$

It also follows that the Master equation is valid, with the only difference being that curlv is defined as ω, the vorticity of the hydrodynamic flow. The master equation becomes,

$$curl(v \times \omega) = \partial \omega /\partial t, \qquad (46)$$

and is to be recognized as Helmoltz' equation for the conservation of vorticity. In the hydrodynamic sense, conservation of vorticity implies uniform continuity. In other words, the Eulerian flow is not only Hamiltonian, it is also uniformly continuous, and satisfies the master equation and the conservation of vorticity constraints. In addition, it may be demonstrated that such systems are at most of Pfaff dimension 3, and admit a relative integral invariant which generalizes the hydrodynamic concept of invariant helicity. In the electromagnetic topology, the Hamiltonian constraint is equivalent to the statement that the Lorentz force vanishes, a condition that has been used to define the "ideal" plasma or "force-free" plasma state.

5.3   Conservation of Topological Torsion

A slightly more general class of evolutionary processes (flows) is given by the constraints which are gauge equivalent to the Hamiltonian extremal case; a search is made for those flows that satisfy the (non-exrtremal, but Hamiltonian) constraint:

$$i(\rho V)dA = i(\rho V)F = dW: \qquad (47)$$

Such flows admit two topological invariants of the relative integral invariant form. The first integral invariant is 1-dimensional:

$$L_{(\frac{1}{2}V)} \oint_{1d\_closed} A = \oint_{1d\_closed} i(\frac{1}{2}V)dA + di(\frac{1}{2}V)A = \quad (48)$$

$$\oint_{1d\_closed} dW + di(\frac{1}{2}V)A = \oint_{1d\_closed} dfW \, ¡ \, i(\frac{1}{2}V)Ag) \, 0; \quad (49)$$

expressing the relative integral invariance of circulation (Kelvin's theorem). The second integral invariant is 3-dimensional:

$$L_{(\frac{1}{2}V)} \oint_{3d\_closed} A^\wedge dA = \oint_{3d\_closed} dfW \, ¡ \, i(\frac{1}{2}V)Ag^\wedge dA) \, 0; \quad (50)$$

a result expressing the generalization of the law which in hydrodynamics is called the conservation of Helicity. The integrations are over closed 1 and 3 dimensional domains. These closed integration domains can be either cycles or boundaries. For exampled the 1-dimensional closed curve in the punctured disc that encircles the central hole is a cycle but not a boundary. As the integrands are exact di®erentials, the closed integrals vanish.

Note that on the domain fx; y; z; tg, the 3-form of topological torsion, $A^\wedge dA$, has the general representation with coe±cients, $Z_{1o\frac{3}{4}}$, that transform as a covariant tensor ¯eld of third rank. On a 4 dimensional space, the components of $A^\wedge dA$ are proportional to a contravariant tensor density of rank 1, whose four components have a vector part de¯ned as, T, the torsion (pseudo) current, and a (pseudo) density part, h. The 3-form $A^\wedge dA$ is not an impair form (density). In electromagnetic engineering language, the general formula for the torsion 3-form has a component expression given by:

$$T = [T; h] = [E \, £ \, A + \acute{A}B; A \, ¢ \, B]: \quad (51)$$

For the constraints of an Eulerian °ow, the 4 components of the Torsion three form reduce to

$$T = [T; h] = [(v \, ¢ \, !)v \, ¡ \, (v \, ¢ \, v = 2 + {}^{a})! \, ; v \, ¢ \, !]: \quad (52)$$

Recall that the closed integration domain used to evaluate the relative integral invariant is not necessarily restricted to a spatial volume integral

with a boundary upon which the normal component of v vanishes. Also note that the helicity density of hydrodynamic fame is the fourth component, $h = v \cdot \omega$; of a contravariant vector density, equivalent to a covariant tensor of third rank. Care must be used in its transformation with respect to diffeomorphisms, such as the Galilean transformation. Furthermore, for the constraints of an Eulerian flow (an extremal field) described above, the topological parity 4-form vanishes globally, such that there exists a pointwise conservation law of the 3-form, equivalent to the expression,

$$\mathrm{div}_3 T + \partial h / \partial t = 0. \tag{53}$$

5.4 Topological Invariants and Period Integrals

Besides the invariant structures considered above, the Cartan methods may be used to generate other sets of topological invariants. Realize that over a domain of Pfaff dimension n less than or equal to N, the Cartan criteria admits a submersive map to be made from N to a space of minimal dimension n. The map may be viewed as a vector field of functional components,

$$[V^x(x, y, z..), V^y(x, y, z..), V^z(x, y, z..), ...];$$

of dimension n, and will have a representation in the projective geometry of $n+1$ homogeneous coordinates. The $n+1$ component will be generated by a function σ, related to the Holder norm,

$$\lambda = 1/\sigma = 1/\{a(V^x)^p + b(V^y)^p + c(V^z)^p + ...\}^{n=p}. \tag{54}$$

For any vector field, construct the n dimensional volume element,

$$Vol = \lambda(V)\, dV^x{\wedge}dV^y{\wedge}dV^z... \tag{55}$$

and the n-1 form density (current) J as:

$$J = i(V^x, V^y, V^z, ...)Vol =$$
$$\lambda\{V^x\, dV^y{\wedge}dV^z... - V^y\, dV^x{\wedge}dV^z... + V^z\, dV^x{\wedge}dV^y... - ...\} . \tag{56}$$

It is remarkable that the current J so de¯ned has a vanishing exterior deriva-
tive, independent of the value of p for a given n, and for all values of the
constants, plus or minus a,b,c...). All such currents de¯ne a "conservation
law". As the map de¯ning the components of the vector ¯eld in terms of the
base fx,y,z..g is presumed to be di®erentiable, then the n-1 form, J; has a
well de¯ned pull back on the base space (almost every where), and its exte-
rior derivative on the base space also vanishes everywhere mod the defects.
That is, the form J is locally exact.

In the expression for ¸; the factors fa; b; c; d:::g are arbitrary constants
of either sign.   The most familiar format is when p = 2; and then the
function ¸ has a null set which is a conic. For positive isotropic signature,
the only defect is the origin in the space de¯ned by the functions, V: The
construction produces the algebraic dual or adjoint vector ¯eld from the
functional components of the original vector ¯eld with integrating factors
½ = 1=¸ that create conservation laws for physical systems.   The integrals
of these closed currents when integrated over closed N-1 dimensional chains
form deformation invariants, with respect to any evolutionary process that
can be described by a vector ¯eld, for

$$
L_{(\frac{1}{2}V)} \oint_{n_i 1} J = \oint_{n_i 1} i(\frac{1}{2}V)dJ + \oint_{n_i 1} d(i(\frac{1}{2}V))J) = 0 + 0 = 0 \qquad (57)
$$

These integral objects appear as "topological coherent" structures, which
may have defects or anomalous sources, when the integrating factor 1=¸ is
not de¯ned.

The compliment to the zero sets of the function ¸ determine the domain
of support associated with the speci¯ed vector ¯eld. The closed n-1 form, J,
that satis¯es the conservation law, dJ = 0, has integrals over closed domains
that have rational fraction ratios.   As this n-1 current is closed globally, it
may be deduced on a connected local domain from a n-2 form, G. In every
case J has a well de¯ned pull-back to the base variety, x,y,z,t. Note that the n
functions $[V^x (x; y; z::); V^y(x; y; z::); V^z(x; y; z::); :::]$ represent the minimum
number of Clebsch variables that are equivalent to the original action, A, over
the domain of support. As each of these integrals is intrinsically closed, the
Lie derivative with respect to any C2 vector ¯eld, V, is a perfect di®erential,
such that (when integrated over closed domains that are p-1 boundaries) the
evolutionary variation of these closed integrals vanishes. These n-1 integrals
are relative integral invariants for any C2 evolutionary processes, or °ows.
The values of the integrals are zero if the closed integration domains are

23

boundaries, or completely enclose a simply connected region. If the closed integration domains encircle the zeros of the function σ, then the values of the integrals are proportional to the integers; i.e., their ratios are rational. Note that each signature must be investigated. For the elliptic (positive definite) signature, the singular points are the stagnation points, and the domain of support excludes those singularities. For the hyperbolic signatures, the domain of support excludes the hyperbolic singularities of lower dimension, such as the light cone. Further note that a given vector field may not generate real domains of support for all possible signatures of the quadratic form, σ.

5.5   The Flux or Circulation Integral 1-form

For the Cartan topology constructed from a fundamental 1-form of Action and a fundamental N-1 form of Current, several period integrals of closed forms integrated over closed chains appear in a natural manner.   In particular on an N=4 dimensional domain, the four period integrals of most interest are the period integrals of flux (circulation), charge, spin and torsion [9]. The fundamental period integral over a closed 1-form will be defined as the "Circulation" or "flux" integral.   When the Pfaff dimension is 2, there exists a submersive map to two dimensions, and the vector fields on this domain will have two irreducible components, say $[\odot(x;y;z;t); a(x;y;z;t)]$:   Following the procedure of the preceding section, construct the 2-dimensional volume element defined as $\sigma = \frac{1}{2}d\odot \wedge d a$; and the $n - 1 = 2 - 1 = 1$ form $A = (\odot d a - a d\odot) = f \S a \odot^p \S b^a {}^p g^{2-p}$: The exterior derivative of such a 1-form is exactly zero for all point sets that exclude the null set of the denominator. The classic choice is for $p = 2$; and $a = 1$; $b = 1$; $(+,+)$ signature. The closed integrals of these closed 1-forms then can be expressed as

$$\text{Circulation} \; \equiv = \oint_1 A = \oint_1 (\odot d a - a d\odot) = f\odot^2 + a {}^2 g \tag{58}$$

By substituting the functional forms in terms of (x,y,z,t) the circulation integral can be written in terms of functions on (x,y,z,t) and their differentials, $fdx; dy; dz; dt:::g$

As an example, suppose that the domain is three dimensional, N=3. Then the zero sets of $\odot(x;y;z) = 0$ and $a(x;y;z) = 0$; represent two 2 dimensional surfaces which may or may not have one or more lines of intersection.   If the surfaces intersect, then

$$\text{Intersection} = d\copyright^\wedge d^a \not= 0: \qquad\qquad (59)$$

If the closed integration paths cannot be contracted to a point, because they encircle these lines of intersection, the values of the integrals have rational ratios depending on how many lines are encircled and how many times the integration path encircles a line. The lines of intersection must have zero divergence (and therefore must stop or start on boundary points, or are closed on themselves). Otherwise the integration chains can be deformed and then contracted to a point. The classic example is given by the 1-form, $A = (ydx - xdy)/(+x^2 + y^2)$ in three dimensions. For integration contours that encircle the z axis, the value of $\int = \oint_1 A = 2\pi$: In hydrodynamics, this vector field is called a potential "vortex", even though the vorticity $! =$ curl $v = 0$: Stokes theorem does not apply as the closed integration chain is a cycle that is not a boundary.

An interesting application of the circulation integral is given when there exists a map to the complex domain. Then ^{a}) $\copyright^\pi$ and the circulation integral has the form of the integral of the probability current in standard quantum mechanics.

$$\text{Period} = \oint_1 (\copyright d\copyright^\pi - \copyright^\pi d\copyright) = f\copyright \, \complement \, \copyright^\pi g: \qquad\qquad (60)$$

5.6   The Gauss Linking or Charge Integral 2-form

Many different options exist for construction of these invariant topological structures from closed p-forms. The idea is to find a formulation for a closed form on a domain, and then to specify a closed and compatible integration chain. The integration chain need not be a boundary, but only a closed cycle. For example, from the components of the specified vector, A_1 , the Jacobian matrix, $[@A_1 = @x^\circ]$ can be constructed. The rows or columns of the matrix of cofactors of the Jacobian (the adjoint matrix) forms a set of vector fields that have zero divergence [21], and therefore these vectors could be used to construct relative integral invariants. In every case there exists an algebraic construction which produces a vector that is divergence free and whose line of action is uniquely related to original vector that was used to

construct the Cartan topology. That vector may be constructed by multiplying the original vector A_1 by the matrix of cofactors and then dividing by the function σ de¯ned above. The construction replicates the previous procedure. As an application for n = 3, p=2, consider the vector that represents the di®erence between two space curves, $z = R_2 ¡ R_1$. Then compute the two form G(z) from the "volume" element $\varpi = dz^1 \wedge dz^2 \wedge dz^3 = \sigma$; to give

$$G_{n=3} = \{z^1 dz^2 \wedge dz^3 ¡ z^2 dz^3 \wedge dz^1 + z^3 dz^1 \wedge dz^2\}/\sigma \tag{61}$$

where

$$\sigma = ((z^1)^2 + (z^2)^2 + (z^3)^2)^{3=2}: \tag{62}$$

Next assert that the displacements of interest are constrained by two parametric curves given by

$$dR_1 = V_1 dt \quad \text{and} \quad dR_2 = V_2 dt^0; \tag{63}$$

where the parameters dt and dt0 are not functionally related (which would imply that $dt \wedge dt0 = 0$).

It is important to realize that kinematic constraints are topological constraints that re¯ne the Cartan topology, a topology based solely upon the speci¯ed 1-form of action, A. From a physical point of view, these constraints can be interpreted as constraints of null °uctuations and in certain circumstances can be associated physically with the limit of zero temperature. To demonstrate the utility of such constraints, substitute these di®erential expressions into the expression for the 2-form G of "current" in N=3 dimensions, and carry out the exterior products, using $dt \wedge dt^0 \neq 0$; but $dt \wedge dt = 0$ and $dt0 \wedge dt0 = 0$: The result is the vector triple product representation for the Gauss integral,

$$Q = \oint_2 G = \oint_2 \{z \bullet V_1 \pounds V_2\} dt \wedge dt^0 = (R_1 \bullet R_1 ¡ 2R_1 \bullet R_{21} + R_2 \bullet R_2)^{3=2}: \tag{64}$$

The integration domain is the closed "2-dimensional area" formed by the displacements along the non-intersecting curves de¯ned by the two distinct

parameters, dt, and dt^0: This double integral is to be recognized as the Gauss linking integral of Knot Theory [7]. (Without the kinematic substitutions, it may also be interpreted as the charge integral of electromagnetic theory.) When integrations are computed along closed curves whose tangent vectors are V_1 and V_2, then the integer values of the closed integral may be interpreted as how many times the two curves are linked. Note that the same integer result is obtained when the vector z is interpreted as the sum of the two vectors, $z = R_2 + R_1$, although the values of the integrals have di®erent scales.

The constraint that $dt \wedge dt^0 \neq 0$ implies that the "motion" along the curve generated by R_1 is independent of the "motion" along the curve generated by R_2: If the curve generated by R_1 is a conic in the xy plane and the curve generated by R_2 is a conic in the xz plane, then the surface swept out by the vector z is a Dupin cyclide. Such surfaces have application to the propagation of waves in electromagnetic systems.

From another point of view, consider the ruled surface [22] de¯ned by the vector ¯eld of two parameters,

$$z(^1; t) = R(t) \S {}^1V(t): \qquad (65)$$

Vector ¯elds of this type are primitive types of "strings" for ¯xed values of the parameter, t, and string parameter, ^{1}: Direct substitution of the physical constraints, $dR \ _{i} \ Vdt = 0$, and $d(V) \ _{i} \ Adt = 0$ leads to the topological Gauss integral,

$$Q = \overset{\text{H}}{} G = \overset{\text{H}}{} fR \pm {}^1V \pounds Ag_{=\,} =$$
$$\overset{\text{H}}{}_2 fA \pm R \pounds {}^1Vgdt \wedge d^1 = (R \pm R \S 2^1R \pm V + {}^1V \pm {}^1V)^{3=2}: \qquad (66)$$

It is apparent that the interaction of the "angular" momentum, $L = R \pounds {}^1V$, and the acceleration, A, produces a topological invariant whose values are "quantized" (in the sense that the ratios of the integrals are rational). Note that for the classical central ¯eld problem where the force (acceleration) and the angular momentum are orthogonal, the orbits are in a plane and the Gauss{linking number is zero. Further note that the triple vector product of the integrand is proportional to the Frenet torsion of the

orbit. An orbit that is planar has Frenet torsion zero everywhere. The Gauss linking integral is a special case of the Gauss two dimensional period integral of electromagnetic theory when the integration domains can be factored into independent products, dt^dt0 ⊖ 0:

5.7   Chaos and the Unknot

Much interest of late has been shown in knot theory and its application to an understanding of the trajectories of dynamical systems. The conjecture is that somehow an understanding of knot theory will give a better understanding of chaos. Counter intuitively is the idea that chaos is to be related to the unknot. Of particular interest will be those cases where lines of vorticity have an oscillatory Frenet torsion with a period equal to 2/3 of the fundamental period of closure. The topological Gauss integral will average to zero for such systems; but these systems can be created by continuous deformations of folding and twisting a closed loop of vorticity, producing a period 3 system which is known to be related to chaos [23]. In the undeformed circular state, tubular neighborhoods guided by the vortex lines can continuously evolve into domains without stagnation points or tangential singularities, or knots, or twists. However, when the closed vortex line is in the deformed period 3 con¯guration, tangential (hyperbolic) singularities are created by the °ow lines of the velocity ¯eld, and the evolution becomes highly convoluted and chaotic. See Figure 1.

These topological features may be demonstrated visually by taking a long strip of paper and wrapping the strip three times around your ¯ngers. Close the strip by going under one strand and over the next before pasting together. The strip is of obvious period three. Now slide the closed strip from the ¯ngers and note that it can be deformed 9continuously into a cylindrical strip without twists or knots (Spin 0). If the same procedure is used, except that a double over or a double under crossing is used before pasting the strip ends together, the resulting closed loop will have a continuously irreducible 4¼ twist (Spin 2). Both the Spin 2 and the Spin 0 strips have a zero Euler characteristic. However, the Spin 2 strip can be continuously deformed into a Klein bottle, or a double lapped Mobius band, and is not homeomorphic to the spin zero strip [24].

If a model of the Spin0 and Spin 2 systems (deformed to their period 3 con¯gurations) is made from a copper tube, and if °exible bands are created

to link any pair of neighboring tubular strands, then it is readily observed that the paired domain twists and folds as it is propagated unidirectionally along the vortex lines. For the spin 2 system the °exible bands will return to their original state in 3 revolutions. However, the paired domain continues to twist and fold, becoming ever more complicated as it follows the evolution around the Spin 0 con⁻guration. The folded spin 0 system has chaotic neighborhoods. This result indicates that the source of chaos in dynamical systems may be due to the unknot, and not the knot! The Cartan theory thereby predicts that the source of chaos in turbulent systems does not require a discontinuous cut and connect process, but may be induced by vortex lines that continuously evolve by twisting and folding into a closed, spin 0, period three con⁻guration.

5.8   The Torsion 3-form and the Braid integral

For $n = 4$ the same procedures used above can be used to produce a period integral over a closed 3-dimensional domain. In fact, the same vector ⁻eld that is used to de⁻ne the Cartan 1-form of Action may be used to construct a dual N-1 form that is closed. The algorithm is to substitute for the functions $[V^x(x;y;z::); V^y(x;y;z::); V^z(x;y;z::); :::]$ the functions $[A_x; A_y; A_z :::]$, that make up the covariant 1-form of Action.. This construction is equivalent to constructing the Jacobian matrix of the original vector ⁻eld on the N-dimensional velocity space, computing its cofactor matrix, multiplying the original vector by the cofactor matrix, and then dividing by the quadratic form, ͺ. When these operations are completed, functional substitution will lead to an conserved axial vector current density on (x,y,z,t). Another form of the topological integral invariant is constructed in the following way. First, for the classic Cartan action, $A = P_k dx^k$ ¡ $E dt=c$, construct the N-volume, – = ¡ $dP_x^\wedge dP_y^\wedge dP_z^\wedge dE =c$: Next contract – with the vector, $(Px; Py; Pz; ¡ E=c)$, and then divide by ͺ = f§P ±P §$(E=c)^2 g^2$: For sake of simplicity, assume that E=c is a constant such the $dE = 0$. Then the closed 3-form or current becomes equivalent to

$$J = (E=c)dP_x^\wedge dP_y^\wedge dP_{z=ͺ} \qquad \text{with} \qquad dJ = 0 \qquad (67)$$

Now invoke the same Cartan trick of individual parametrization as uses above. Consider a total momentum vector composed of three individual

29

vector components, $P = p_1 + p_2 + p_3$: Assume that the Cartan topology is constrained in such a way that for each vector component a Newtonian kinematic law of parametrization is maintained such that

$$dp_{1\,i}\ f_1 dt = 0; \quad dp_{2\,i}\ f_2 dt^0 = 0; \quad dp_{3\,i}\ f_3 dt^{00} = 0: \tag{68}$$

Also note that $dt^\wedge dt^0 {}^\wedge dt^{00} \neq 0$; that is, the parameters used in the Newtonian kinematic descriptions are not sychronizeable. If they were functionally related the value of J must be zero. Substitute these expressions into the equation for the closed current J and integrate over a closed 3 dimensional chain to yield a triple Braid integral,

$$\text{Braid} = \oint J = \oint (E = c) dP_x{}^\wedge dP_y{}^\wedge dP_z = ,$$
$$= \oint_3 (E = c) f\, f_1 \pm (f_2 \,£\, f_3) g\, dt^\wedge dt^0 {}^\wedge dt^{00} = f \S P \pm P \S (E = c)^2 g^2 \tag{69}$$

The integrations are now over three closed curves whose tangents are the Newtonian forces, f, on three "particles". Where in the two dimensional Gauss integral, of the previous section, the evaluation was along the closed curves of two particles that formed the ends of a string, in this case the integrations are along the closed trajectories of three "particles" which form the vertices of a triangle. In every case, the trajectories are the trajectories of a system of limit points.

The idea that three "lines" are used to form the integral (whose values form rational ratios) is the reason that this topological integral in the format given above is de¯ned as the braid integral. Of course the three form of topological torsion is a variant of the braid integral, but applies to those topologies where the system is not reducible to three factors dt; dt0 and dt'' (such systems are said to have torsion cycles). An example of a period 3 braid with Braid integral zero (chaotic) and Braid integral 2 (non-chaotic) is given in Figure 1

The equivalent to this Figure, and the fact that there are two distinct period 3 con¯gurations, one chaotic and one non-chaotic, was brought to the attention of the present authors during a stimulating lecture given by J. Los at the August, 1991, Pedagogical Workshop on Topological Fluid Mechanics held at the Institute for Theoretical Physics, Santa Barbara UCSB.

It is to be noted that the 3-form of topological torsion is related to the braid integral, a three dimensional thing in four dimensions, and not the Gauss linkage integral, which is a two dimensional thing in three dimensions. The literature of helicity is sometimes confused on this point, and often attempts to relate the helicity integral to the linkage integral.

5.9   Navier Stokes flows and Pfaff Dimension 4

As a last example consider a system where the strong kinematic (topological) constraint $dx - Vdt = 0$ is not imposed apriori. In other words, the admissable evolutionary processes, V, may have anholonomic fluctuations about kinematic perfection.

$$\mathfrak{C}x = dx - Vdt \neq 0 \tag{70}$$

The physical system will be built on the Cartan topology of the 1-form, A, given previously for the Euler flow. However, the Cartan topology will be constrained, not by the Hamiltonian conditions required to generate an extremal system (which is free of kinematic fluctuations), but by a more relaxed set of conditions that permit finite kinematic fluctuations, $dx - Vdt \neq 0$. As it is known that $i(V)dA$ must be transversal to the vector field, V, it follows that a weaker topological constraint might exist in the form,

$$i(V)dA = f_k(dx^k - V^k dt) + d\mu; \tag{71}$$

where the functions μ are Bernoulli-Casimir first integrals in the sense that $i(V)d\mu = 0$:

When $f_k = 0$, these fluctuation constraints reduce to the more stringent Hamiltonian conditions for an extremal flow, or in the case where $d\mu \neq 0$; to the Bernoulli-Casimir symplectic conditions. If is assumed that

$$f_k = v(\text{curlcurl } V)_k; \tag{72}$$

it follows that the expression given above, $i(v)dA = f_k(dx^k - V^k dt)$; is exactly equivalent to the Navier-Stokes partial differential system [25] for an incompressible viscous flow on the variety $x; y; z; t$.

31

$$f@V=@t + \mathrm{grad}(V \cdot V=2) \; ¡ \; V \pounds \mathrm{curl} V g = f° r^2 V g ¡ \; \mathrm{grad} P =½ \qquad (73)$$

These relaxed topological constraints, which admit evolutionary °uctuations in the Cartan system, permit the Topological Parity 4-form to be computed for the Navier Stokes °uid; the result is:

$$K = F \wedge F = ¡ \; 2° \, (\mathrm{curl}\, V \cdot \mathrm{curlcurl}\, V) dx \wedge dy \wedge dz \wedge dt: \qquad (74)$$

From this result it is apparent that the Pfa® dimension of the domain is 4, unless the viscosity is zero, or the vorticity ¯eld satis¯es the conditions of Frobenius integrability. The Torsion current anomaly is equal to ¡ 2° (curl V · curlcurl V). The torsion lines can stop or stop within the domain producing defect structures that e®ect the cohomology of the Cartan topology.

An interesting result is the proof that the closed integral of topological Torsion-Helicity is a relative integral invariant for the viscous, compressible °uid, if the Cartan sequence has a Pfa® dimension equal to 3. Recall that the evolution of the 3-form $H = A \wedge dA$ is given by the Lie derivative expression,

$$L_{(¯V)} \oint_3^H H = \oint_3^H fi(¯V)dH + d(i(¯V)Hg = \oint_3^H fi(¯V)dHg + 0 \qquad (75)$$

But if curl V·curlcurl V vanishes (for any viscosity) then $dH = dA \wedge dA = 0$, and the RHS of the above expression vanishes, for any reparameterization, ¯. Therefore, the closed integral of the Topological Torsion three form is a deformation invariant not only of Eulerian °ows, but also of viscous °ows for which the vorticity ¯eld is of Pfa® dimension 2 (the velocity ¯eld is Pfa® dimension 3). The folklore concept that viscosity destroys the helicity invariant is not necessarily true.

6   Acknowledgments

7 REFERENCES

[1] E. Cartan "Sur certaines expressions d² erentielles et le systeme de Pfa®" Ann Ec. Norm. 16 329 (1899)

[2] E. Cartan, "Systems Di®erentials Exterieurs et leurs Applications Geometriques", Actualites sci. et industrielles 944 (1945)

[3] E. Cartan, "Lecons sur la theorie des spineurs" (Hermann, Paris 1938)

[4] E. Cartan, "La Theorie des Spaces a Connexion Projective", (Hermann, Paris, 1937)

[5] S.S.Chern, Annais of Math. 45, 747- 752 (1944).

[6] R. M. Kiehn, "Retrodictive Determinism" Int. Journ. Eng Sci (1976)

[7] R.M. Kiehn, 'Topological Torsion, Pfa® Dimension and Coherent Structures" in Topological Fluid Mechanics, H. K. Mo®att and A. Tsinober, editors, (Cambridge University Press, 1990), p. 225.

[8] H.Flanders, "Di®erential Forms", (Academic Press, N. Y. 1963).

[9] R.M. Kiehn and J. F. Pierce, The Physics of Fluids 9 1941 (1969)

[10] R. M. Kiehn, J. Math Phys. 9 1975

[11] R. M. Kiehn, "Are there three kinds of superconductivity" INt J. of Mod. Phys.10 1779 (1991)

[12] W. Slebodzinsky, "Exterior Forms and their Applications

[13] Van der Kulk and Schouten "Pfa®s Problem Oxford University Press.

[14] S. Lipschutz, "General Topology", (Schaums Publishing Co.,New York, 1965) p.97

[15] R. Hermann, "Di®erential Geometry and the Calculus of Variations", (Academic Press, New York, 1968).

[16] R. M. Kiehn, Lett al Nuovo Cimento 14, 308 (1975) Submersive Equivalence Classes for Metric Fields"

[17] W. Gellert, et.al. Editors 'The VNR Concise Encyclopedia of Mathematics", (Van Nostrand, New York 1977), p.686.

[18] J. G. Hocking, "Topology", (Addison Wesley, N. Y., 1961), p.2.

[19] R. L Bishop and S. I. Goldberg, "Tensor Analysis on Manifolds", (Dover, N. Y., 1968).

[20] R. M. Kiehn, 'Topological Parity and the Turbulent State" submitted to Jap. J. of Fluid Res.

[21] N. E. Kochin, I. A. Kibel, and N. V. Roze Theoretical Hydrodynamics" (Inter-science, New York 1965)

[22] H. W. Turnbull, "The Theory of determinants, matrices and invariants" (Dover, New York 1960)

[23] D. Struik, "Di®erential Geometry" , Addison Wesley, (Reading, Mass 1961)

[24] J. Yorke and T.U, Amercan Mathematical Monthly 82, 985 (1975).

[25] R. M. Kiehn, Lett al Nuovo Cimento 12, 300 (1975); Lett al Nuovo Cimento 22, 308 (1978)
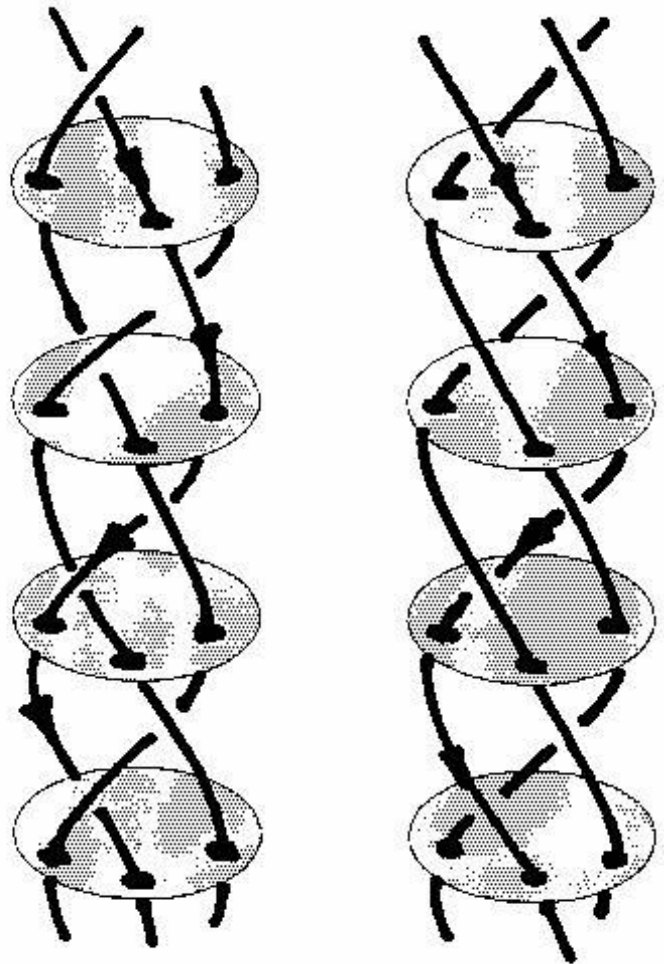
Spin 0 (Chaotic)   Spin 2 (Non-Chaotic)

Fig 1   Period 3 Braids

Figure 1: