

Topology and Turbulence

R. M. Kiehn

Physics Department, University of Houston, TX, 77204

Abstract: Over a given regular domain of independent variables $\{x,y,z,t\}$, every covariant vector field of flow can be constructed in terms of at most four functions and their first derivatives. These functions may be used to construct a differential 1-form of Action. The irreducible number of such functions required to describe the topological properties of the 1-form of Action is defined as the Pfaff dimension of the domain. These functions, their first derivatives, and certain algebraic intersections of these sets, may be used to construct a Cartan topology, and thereby define the four basic topological equivalence classes of Pfaff dimension one, two, three and four on space time. Those vector field solutions to the Navier-Stokes equations which are uniquely integrable in the sense of Frobenius emulate potential flows or streamline processes; they generate an Action 1-form of Pfaff dimension 1 and 2, respectively. Chaotic solutions to the Navier-Stokes equations must be associated with Action domains of Pfaff dimension 3; but such flows are deterministic and reversible in space time. Only domains of Pfaff dimension 4 admit of evolutionary flow processes that can break time reversal and parity symmetry, and therefore may be used to describe the irreversible processes inherent in turbulent flows. When the Pfaff dimension is 2 or less, the Cartan Topology induced by the vector field is connected. When the Pfaff dimension is greater than 2, the induced Cartan topology is disconnected. The topological implication is that the creation of turbulence (a state of Pfaff dimension 4 and a disconnected Cartan topology) from a streamline flow (a state of Pfaff dimension 2 and a connected topology) can take place only by discontinuous processes which induce shocks and tangential discontinuities. On the otherhand, the decay of turbulence can be described by continuous, but irreversible, processes. Numerical procedures that force continuity of slope and value cannot in principle describe the creation of turbulence, but such techniques of forced continuity can be used to describe the decay of turbulence.

INTRODUCTION

The turbulence problem of hydrodynamics is complicated by the fact that there does not exist a precise definition of the turbulent state. The visual complexity of the turbulent state leads to the assumption that the phenomenon is in some way random and statistical. It is however a matter of experience that a real viscous liquid which is isolated from its surroundings after being put into a turbulent state will decay into a streamline state or a state of rest.

Although the turbulent state is intuitively recognizable, only a small number of properties necessary for the turbulent state receive the support of a majority of researchers:

1. A turbulent flow is three dimensional
2. A turbulent flow is time dependent.
3. A turbulent flow is dissipative.
4. A turbulent flow may be intermittent.
5. A turbulent flow is irreversible.

Still, a precise mathematical definition of the turbulent state has not been established.

It sometimes is argued that a turbulent flow is random, a quality that has been reinforced by successes of certain statistical theories in describing average properties of the turbulent state. Recent advances in non-linear dynamics indicate a sensitivity to initial conditions can lead to deterministic "chaos", a quality which visually has some of the intuitive features often associated with turbulence. Moreover, the theory of non-linear dynamics has led to several new suggestions that describe the route to turbulence.

However, there are fundamental differences between irreversibility, chaos, and randomness that suggest that the turbulent state is not simply a chaotic regime. Similar statements can be made about the transition to turbulence and the formation of "coherent structures" in turbulent flows. Although the eye easily perceives the antithesis to turbulence as being a steady streamline flow, the transition from the pure streamline state to the turbulent state is complicated by the fact that there may be intermediate chaotic states in between an initial state of rest (or steady integrable streamline flow) and a final state of turbulent flow. The objective of this article is to focus attention on and apply topological methods, rather than the customary geometric methods, to the problem of defining the transition to turbulence and the turbulent state itself.

In hydrodynamics, it is generally accepted that the non-turbulent state seems to be described adequately in terms of solutions to the Navier-Stokes equations. The description of the turbulent state is not so clear, and to this author the reason may be due to a lack of definition of what is the turbulent state. As Ian Stewart [1988] states, "the Leray Theory of turbulence...asserts that when turbulence occurs, the Navier-Stokes equations break down. ... turbulence is a fundamentally different problem from smooth flow." In other words, one option would be that the turbulent state is not among the solutions to the Navier-Stokes equations. Counter to this option, and more in concurrence with the spirit of this article, would be the inclusion of discontinuous solutions to the Navier-Stokes equations into the class of evolutionary flows under consideration.

The Kolmogorov theory of turbulence [Kolmogorov, 1941] is a statistical option motivated by the assumption that the turbulent state consists of "vortices" of all "scales" with random intensities, but otherwise it is a theory which is not based upon the Navier-Stokes equations explicitly. The wavelet theory of Zimin [1991] is a method that does use a specific decomposition of the solutions to the Navier-Stokes equations, and a transformation to a set of collective variables, which mimic the Kolmogorov motivation of vortices of all scales. However from a topological view, scales can not have any intrinsic relevance.

The Hopf-Landau theory [Landau 1959] claims that the transition to turbulence is a quasi-periodic phenomenon in which infinitely many periods are sequentially generated in order to give the appearance of randomness. In contrast, the Ruelle-Takens theory [Berge, 1984] describes the transition to turbulence not in terms of an infinite cascade, but in terms of a few transitions leading to a chaotic state defined by a strange attractor. In other words, it would appear that the presence of a strange attractor defines the turbulent state, but again there is a degree of vagueness in that the "strange attractor" is ill-defined. A basic question arises: "Is chaos the same as turbulence?" [Kiehn, 1990b].

In this article, topological methods will be used to formulate a definition of the turbulent state. The method, being topological and not geometrical, will involve concepts of scale independence, a concept in

spirit recently utilized by Frisch [1991] in an approach based on multi-fractals. However, at the time of writing of the Frisch article a specific connection between fractals and solutions to the Navier-Stokes equations was unknown. Only recently has the suggestion been made that those special characteristic sets of points upon which the solutions to the Navier-Stokes equations can be discontinuous, and which at the same time are minimal surfaces, may be the generators of fractal sets [Kiehn, 1992].

A guiding feature of this article will be the idea that the abstract Cartan topology of interest is refined by the constraints imposed by the Navier-Stokes equations on the equations of topological evolution. Questions of smoothness and continuity are always related to a specific choice of a topology, and the topology chosen in this article is the topology generated by a constrained Cartan exterior differential system [Bryant, 1991]. The vector field solutions to the constrained topology (the Navier-Stokes equations) fall into four equivalence classes that characterize certain topological properties of the evolutionary system. Most of the mathematical details are left to the Appendix, with the primary discussion presented in a qualitative manner. The principle key feature of the constrained topology is that two of the equivalence classes imply that the Cartan topology induced by the vector flow field is disconnected, while the other two equivalence classes imply that the induced topology is connected.

According to the theory presented herein, the key feature of the turbulent state is that its representation as a vector field can only be associated with a *disconnected* Cartan topology, while the globally streamline integrable state is associated with a *connected* Cartan topology. It is a mathematical fact that a transition from a connected topology to disconnected topology can take place only by a discontinuous transformation. This discontinuous transformation is typically realized as a cutting or tearing operation of separation. However, a transition from a disconnected topology to a connected topology can take place by a continuous, but not reversible, transformation of pasting. The fundamental conclusion is that the creation of the turbulent state is intrinsically different from the decay of the turbulent state, but both processes involve topological change.

THE CARTAN TOPOLOGY

To come to grips with the topological issues of hydrodynamics it is necessary to be able to define a useful topology in a way naturally suited to the problem at hand. On the set $\{x,y,z,t\}$ it is possible to define many topologies. Indeed, for many evolutionary systems, particularly those which are dissipative and irreversible, the topology of the initial state need not be the same as the topology of the final state. Hence, not only is it necessary to construct a topology for a hydrodynamic system at some time, t , it is also necessary that the topology so constructed be a dynamical system in itself. The transition to turbulence will involve topological change.

As mentioned above, topological change can be induced either by a process that is discontinuous, or by a process that is continuous, but not reversible [Kiehn 1991a]. In this article attention is focused mainly on those processes that are continuous but irreversible. Recall that continuity is a topological property (not a geometrical property) that is defined in terms of the limit points of the initial and final state topologies. The methods used to define the topology used herein are based on those techniques that E. Cartan developed for his studies of exterior differential systems [Bishop, 1968]. For the hydrodynamic application it will be assumed that the evolutionary system can be defined in terms of a 1-form of Action built on a single covariant vector field. Typical of Lagrangian field theory, the equations of evolution are determined from a system of Pfaffian expressions constructed from the extremals to the 1-form of Action when subjected to constraints. The resultant Pfaffian system corresponds to a system of partial differential equations of evolution. The extremal process is equivalent to the first variation in the theory of the Calculus of Variations. The Action 1-form, A , may be composed from the functions that make up the evolutionary flow field itself, as well as other functions on the set $\{x,y,z,t\}$. A unreduced typical format for the Action 1-form on the set $\{x,y,z,t\}$ would be given by the expression,

$$A = A_x dx + A_y dy + A_z dz + A_t dt,$$

where the functions A_k are constructed from the components of the flow field, and other functions. Cartan's idea was to examine the irreducible representations of this 1-form of Action. For example, in certain domains, the action may be represented by the differentials of a single function, $A = d\phi$. In other

cases, the representation might require two functions, $A = \alpha d\beta$, and so on. Given an arbitrary action, A , how do you determine its irreducible representation? Is a given Action reducible? The answer to the last question is given by the Pfaff dimension or class of the 1-form A . The Pfaff dimension is computed by one differential and several algebraic processes producing a sequence of higher and higher order differential forms. The construction of these forms is detailed in the next section, but the remark to be made here is that these objects may be used to construct a topological basis, and thereby, a topology.

In short, a point set equivalent to the Cartan topology can be defined in terms of those functions that form the components of the covariant vector field use to define the Action 1-form, the first partial derivatives of these functions, and their algebraic intersections. The details of this coarse Cartan topology over a space, $\{x,y,z,t\}$, are given in Appendix A, with the refinement that constrains the coarse topology to yield those evolutionary fields that are solutions to the Navier-Stokes equations.

PSFAFF DIMENSION

The topological property of dimension is the key feature that distinguishes precisely four Cartan equivalence classes of covariant vector fields on the set $\{x,y,z,t\}$. The idea of Pfaff dimension (called the class of the Pfaffian system in the older literature [Forsyth, 1953]) is related to the irreducible number of functions which are required to describe an arbitrary Pfaffian form, in this case the 1-form of Action. An Action 1-form that can be generated globally in terms of a single scalar field, and its differentials, is of Pfaff dimension 1. Over space time, this single function or parameter is often called the "phase" or "potential" function, $\phi(x,y,z,t)$. When the Action is of Pfaff dimension 1, then it may be expressed in the reduced form, $A = d\phi$.

For the purposes described in this article, where the Action is defined in terms of the flow field itself, the constraint of Pfaff dimension 1 implies that the vector field representing the evolutionary flow is a "potential" flow. Such gradient vector flow fields represent a submersion from four dimensional space-time to a parameter space of one dimension, the potential function itself.

Actions constructed for flows that admit vorticity (but are completely integrable in the sense that through every point there exists a unique parameter function whose gradient determines the line of action of the flow) can be represented by a submersive map to a parameter space of two dimensions. The reduced format for the 1-form of Action is $A = \psi(x,y,z,t)d\phi$. Such integrable flows are defined as globally laminar flows (in the sense that there exists a globally synchronizable set of unique initial conditions, or parameters). Such flows are to be distinguished from flows that may have, for example, re-entrant domains, and are locally layered, but for which it is impossible to define a global *connected* set of initial conditions. Globally laminar flows and potential flows are of Pfaff dimension 2 or less, and are associated with a Cartan point set topology which is *connected*. This concept is to be interpreted as implying that there exists an $N-1$ dimensional set which intersects the flow lines in a unique set of points. (As Arnold says, the field is of co-dimension 1.) For three dimensional space, $N=3$, this set is a surface. For four-dimensional space, $N=4$, this set is a volume.

In a domain where the Cartan point set topology is connected (the 1-form of Action is of Pfaff dimension 2 or less) it is possible to define a single connected *parameter* of evolution which plays the role of "phase". In three dimensions, the parametric value is called "time". This idea of a uniquely defined global parameter $N-1$ surface is the heart of the Caratheodory theory of equilibrium thermodynamics. For such systems, there are infinitely close neighboring points which are *not* reachable by *closed equilibrium* processes. The equilibrium process is defined to be a process whose trajectory is confined to the $N-1$ hypersurface. The two irreducible functions of the integrable Pfaff representation are called temperature, T , and entropy, S , in the Caratheodory setting. The parameter space is connected, but a constant value of the parameter space does not intersect all points of the domain. The evolutionary vector fields associated with completely integrable Pfaffian systems are never chaotic [Schuster, 1984].

For a 1-form of Action that is associated with a disconnected point set topology, such a globally unique parameterization as described above is impossible. Those 1-forms of Action, if not of Pfaff dimension 2 (or less) globally, do not satisfy the Frobenius complete integrability conditions [Flanders,1963] The evolutionary vector fields associated with non-integrable 1-forms can be chaotic. If the 1-form of Action is of Pfaff dimension 3, then it has an irreducible representation as $A = d\beta + \psi d\phi$.

The three independent functions form a non-zero three form, $H = A^{\wedge}dA = d\beta^{\wedge}d\psi^{\wedge}d\phi$, and represents a covariant current of rank 3, with a dual representation as a contravariant tensor density (the torsion current). The torsion current has zero divergence on the domain of space $\{x,y,z,t\}$ which is of Pfaff dimension 3 relative to the Action A. This result implies that the “lines” so generated by the solenoidal torsion current in $\{x,y,z,t\}$ can never stop or start within the domain interior. The lines representing the topological torsion 4-current either close on themselves, or start and stop on points of the boundary of the domain. The torsion lines never stop in the interior of the domain where the Pfaff dimension is 3. Such a torsion current does not exist in domains of Pfaff dimension 2 or less. Explicit formulas for the torsion current will be given below..

For a four dimensional domain, the 1-form of Action may be of Pfaff dimension 4, and the irreducible representation of the Action is given by the expression, $A = \alpha d\beta + \psi d\phi$. Each of these functions is independent, so the topological torsion current is of the form,

$$A^{\wedge}dA = \alpha d\beta^{\wedge}d\psi^{\wedge}d\phi + \psi d\phi^{\wedge}d\alpha^{\wedge}d\beta .$$

However, the divergence of this associated vector field is not zero! The lines of the torsion current can start or stop in the *interior* of the domain when the Pfaff dimension is 4.

In four dimensions, a solenoidal vector field, if homogeneous of degree 0, forms a minimal surface in space time. In fact, if a four dimensional vector field can be represented by a complex holomorphic curve, then the field is not only solenoidal, but also harmonic, and is *always* associated with a minimal surface. It will be shown below, that in the hydrodynamics governed by the Navier-Stokes equations, harmonic vector field solutions are not dissipative, no matter what the value of the viscosity coefficient. For dissipative irreversible systems, attention is therefore focused on systems of Pfaff dimension 4, for which the torsion current is not solenoidal.

The theory presented in this article insists that irreversible turbulence must be time dependent and irreducibly three dimensional. The idea of “two dimensional” turbulence, for time dependent continuous flows, is inconsistent, for such flows have a maximum Pfaff dimension of 3. Flows of Pfaff dimension 3 can be chaotic, but they are deterministically reversible, hence not turbulent. In agreement with the arguments expressed by Kida [1989], the turbulent state is more than just chaos. A turbulent domain must be of Pfaff dimension 4, for in space-time domains of Pfaff dimension 3, it is always possible to construct flow lines that never intersect. Hence such flow lines are always re-traceable, without ambiguity, and such flows are not irreversible. In order to break time-reversal symmetry, and hence to be irreversible, the flow lines must intersect in space-time such that they cannot be retraced without ambiguity. Such a result requires that the Euler characteristic of the four dimensional domain must be non-zero, for then it is impossible to construct a vector field without intersections. The Euler characteristic of space-time is only non-zero on domains of Pfaff dimension 4. Hence, the Pfaff dimension of turbulent domains on $\{x,y,z,t\}$ must be 4, while the chaotic domains need be only of Pfaff dimension 3 [Kiehn, 1991b].

TOPOLOGICAL CONNECTEDNESS VS. GEOMETRIC SCALES

In early studies of the turbulent state, from both the statistical point of view and the point of view of the Navier Stokes equations, the geometric concepts of large and small spatial scales, or short or long temporal scales, have permeated the discussions. From a topological point of view, length scales and time scales have no meaning. If things are too small, a topologist stretches them out, and conversely. If turbulence is a topological concept, then the ideas should be independent from scales. It is interesting to note that the original Kolmogorov statistical analysis of the turbulent state is now interpreted in terms of the multi-fractal concept of scale invariance [Frisch, 1991].

A key feature of the disconnected Cartan topology is that the domain supports non-null Topological Torsion, and is of Pfaff dimension 3 for chaotic flows, and of Pfaff dimension 4 for turbulent flows. Suppose the initial state is a turbulent state in which there exist disconnected striated or tubular domains that are of Pfaff dimension 4 and are embedded in domains of Pfaff dimension 2 or less. Then if the hydrodynamic system is left to decay, these striated domains can decay by continuous collapse into striations or filaments of measure zero. The Topological Torsion of the striated domains cannot be zero.

The size of the striated domains is not of issue, but the existence of such domains with non-zero measure is of interest, for if these domains do not exist, the flow is not chaotic and not turbulent.

The geometric idea of small domains versus large domains of space and/or time is transformed to a topological idea of connected domains versus disconnected domains. Points in disconnected components are not reachable [Hermann, 1968] in the sense of Caratheodory, hence are separated by "large scales", while points in the same component are reachable, and hence are separated by "small scales", compared to points in disconnected components. It is the view of this article that the geometric concept of scales is not germane to the problem of turbulence, but instead the basic issue is one of connectedness or disconnectedness.

TOPOLOGICAL TORSION

The difference between chaotic flows and turbulent flows is that chaotic flows preserve time reversal symmetry and turbulent flows do not. Chaotic flows can be reversible, while turbulent flows are not. Both chaotic and turbulent flows support a non-zero value of Topological Torsion tensor. As constructed in the Appendix for the Navier-Stokes system, the Topological Torsion 3-form on space-time, H , has 4 components, $\{\mathbf{T}, h\}$ that transform as the components of a third rank completely anti-symmetric covariant tensor field, H_{ijk} . If the vector field used to construct the Topological Torsion tensor is completely integrable in the sense of Frobenius, then all components, $\{\mathbf{T}, h\}$, vanish, and the Pfaff dimension of the domain is 2 or less. For the Navier-Stokes fluid, the torsion current is given by the engineering expression given in the appendix as equation (15). As a third rank tensor field, the Topological Torsion tensor is intrinsically covariant with respect to all coordinate transformations, including the galilean translation.

For engineers, the closest analog to the Topological Torsion tensor is the charge-current, 4-vector density, \mathbf{J} , of electromagnetism. The fundamental difference is that where the electromagnetic 4-current always satisfies the conservation law, $\text{div } \mathbf{j} + \partial \rho / \partial t = 0$, the Topological Torsion 4-current does not, unless the vector field use to construct the Topological Torsion tensor is an element of an equivalence class with Pfaff dimension less than 4. In other words, for the turbulent state, the Topological Torsion tensor does not satisfy a local conservation law, where for the chaotic state it does:

$$\text{div } \mathbf{T} + \partial h / \partial t = 0 \dots \text{Pfaff Dimension 3,}$$

(a necessary condition for chaos),

$$\text{div } \mathbf{T} + \partial h / \partial t \neq 0 \dots \text{Pfaff Dimension 4,}$$

(a necessary condition for turbulence).

The lines of torsion current, given by solutions to the system of first order differential equations,

$$dx/T_x = dy/T_y = dz/T_z, \tag{1}$$

can start or stop internally if the Pfaff dimension is 4, but only on boundary points or limit points of the domain, if the Pfaff dimension is 3. If the vector field is of Pfaff dimension less than 4, then the integral over a boundary of the Topological Torsion tensor is an evolutionary invariant, but some care must be taken to insure that the integration domain is a boundary, and not just a closed cycle. This result, which corresponds to a global helicity conservation theorem, is independent of any statement about viscosity. However, the invariance of the Topological Torsion integral over a boundary for a Navier-Stokes fluid implies that integral of the 4-form of Topological Parity must vanish, which in turn implies that the Euler index of the Cartan topology for this situation is zero. The compliment of this idea leads to a variable Topological Torsion integral, and the requirement that the Euler index is not zero for the irreversible decay of the turbulent state. For if the Euler index is not zero, then every vector field has at least one flow line with a singular point. If an evolutionary parameter carries the process through the singular point, a reversal of the process parameter will retrace the path only back to the singular point uniquely. Subsequently, the return path then becomes ambiguous, and the evolution is not reversible. Recall that a necessary condition for a vector field to exist on a manifold without self intersection singularities is that the Euler index of the

manifold must vanish. Hence a necessary condition for turbulence is that the Cartan topology must be of Pfaff dimension 4, and the topological torsion is not solenoidal.

It is remarkable to this author that experimentalists and theorists (including the present author) have been so brain-washed by the dogma of unique predictability in the physical sciences that they have completely ignored the measurement and implications of the Topological Torsion tensor. Although the solutions to the system of equations given by (1) is a problem that has found use in the older literature of differential equations (where it is known as the "subsidiary" system) [Forsyth 1959 p.95], its utilization in applied dynamical systems, especially hydrodynamics, is extremely limited. Of course, for uniquely integrable systems, the equations of topological torsion are evanescent, and not useful. The very existence of the Topological Torsion tensor is an indicator of when unique predictability is impossible [Kiehn, 1976], and attention should be paid to the system given by (1).

TORSION WAVES

One of the predictions of the Cartan topological approach is the fact that for systems of Pfaff dimension 4 it is possible to excite torsion waves. Torsion waves are essentially transverse waves but with enough longitudinal component to give them a helical or spiral signature. In the electromagnetic case, where such waves have been measured, they are represented by four component *quaternionic* solutions to Maxwell's equations [Schultz, 1979; Kiehn, 1991]. Such electromagnetic waves represent different states of left or right polarization (parity) traveling in opposite directions. The wave speeds in different directions can be distinct. In fluids, transverse torsion waves can be made visible by first constructing a Falaco soliton state [Kiehn 1991c] and then dropping dye near the rotating surface defect. The dye drop will execute transverse polarized helical motions about a guiding filamentary vortex that connects the pair of contra-rotating surface defects. There is some evidence that torsion waves can appear as traveling waves on Rayleigh cells [Croquette, 1989].

THE PRODUCTION VS. THE DECAY OF TURBULENCE

The Cartan topological theory predicts that the transition to turbulence from a globally laminar state involves a transition from a connected Cartan topology to a disconnected Cartan topology. From this fact it may be proved that such transitions can NOT be continuous, but they may be reversible! However, the theory also predicts that the decay of turbulence can be described by a continuous transformation, but the transformation can NOT be reversible.

The Cartan topology when combined with the Lie derivative may be used to define partial differential equations of evolution [Kiehn, 1990], which include the Navier-Stokes equations as a subset of a more refined topology. However, if the Cartan topology is constructed from p-forms and vector fields that are restricted to be C2 differentiable, then it may be shown that all such solutions to the Navier-Stokes equations are continuous relative to the Cartan topology. The *creation* of the turbulent state must involve discontinuous solutions to the Navier-Stokes equations, which are generated only by shocks or tangential discontinuities, and therefore are not describable by C2 fields. On the other hand, the *decay* of turbulence can be described by C2 differentiable, hence continuous, solutions which are not homeomorphisms, and are therefore not reversible. In this article, the decay of turbulence by C0 and C1 functions is left open.

Domains of finite Topological Torsion are topologically disconnected from domains that have zero Topological Torsion. The anomaly that permits the local creation or destruction of Topological Torsion is exactly the 4-form of Topological Parity (see Appendix). If the Topological Parity is zero, then the Topological Torsion obeys a pointwise conservation law. For a barotropic Navier-Stokes fluid, the anomaly, or source term for the Torsion current can be evaluated explicitly, and appears as the right hand side in the following equation:

$$\text{div } \mathbf{T} + \partial h / \partial t = - 2\nu \text{curl} \mathbf{v} \bullet \text{curl curl } \mathbf{v}. \quad (2)$$

It is remarkable that for flows of any viscosity, the Topological Torsion tensor satisfies a pointwise conservation law, and the integral over a bounded domain is a flow invariant, if the vorticity vector field satisfies the Frobenius integrability conditions, $\text{curl} \mathbf{v} \bullet \text{curl curl } \mathbf{v} = 0$. It would seem natural that the decay of turbulence would be attracted to such interesting limiting configurations of topological coherence

in a viscous fluid. These limit sets can be related to minimal surfaces of tangential discontinuities which can act as fractal boundaries of chaotic domains [Kiehn 1992b, 1992c, 1993].

SUMMARY

The topological perspective of Cartan indicates that:

1. A necessary condition for the turbulent state is that the flow field must generate a domain of support which is of Pfaff dimension 4, and is to be distinguished from the chaotic state, which is necessarily of Pfaff dimension 3. Moreover, the irreversible property of turbulence decay implies that the Euler index of the induced Cartan topology must be non-zero.

2. The geometric concept of length scales and time scales should be replaced by the topological concept of connectedness vs. disconnectedness.

3. The transition to turbulence can take place only by discontinuous solutions to the Navier-Stokes equations.

4. The decay of turbulence can be described by continuous but irreversible solutions to the Navier-Stokes equations. The decay of domains for which there exists finite Topological Torsion is dependent upon a finite viscosity, and the lack of integrability for the lines of vorticity. One model for the decay of turbulence might be described as the collapse of tubular domains of torsion current, becoming ever finer filamentary domains without helicity until ultimately they have a measure zero.

5. A coherent structure in a turbulent flow may be defined as a connected deformable component of a disconnected Cartan topology. For the Cartan topology, these coherent structures are of two species. One component will have null Topological Torsion and the other component will have non-null Topological Torsion. Bounded domains for which the integral of Topological Torsion is a flow invariant can form in a viscous turbulent fluid. In particular, domains for which $\text{curl } \mathbf{v} \bullet (\text{curl } \text{curl } \mathbf{v}) = 0$, but $\mathbf{v} \bullet \text{curl } \mathbf{v} \neq 0$, can have an evolutionary persistence in a viscous fluid. The existence of helicity density is a sufficient but not necessary signature that the Cartan topology is disconnected

ACKNOWLEDGMENTS

This work was supported in part by the Energy Laboratory at the University of Houston. The author owes the late A. Hildebrand, the director of the Energy Lab, a debt of gratitude for first, his skepticisms, then his conversion, and finally his insistence that the Cartan ideas of differential forms would have practical importance in physical systems.

APPENDIX A : THE CARTAN TOPOLOGY

Starting in 1899, Cartan [1899,1937,1945] developed his theory of exterior differential systems built on the Grassmann algebraic concept of exterior multiplication, and the novel calculus concept of exterior differentiation. These operations are applied to sets called exterior p-forms, which are often described as the objects that form an integrand under the integral sign. The Cartan concepts may still seem unconventional to the engineer, and only during the past few years have they slowly crept into the mainstream of physics. There are several texts at an introductory level that the uninitiated will find useful [Flanders, 1963; Bamberg, 1989; Bishop, 1968; Lovelock, 1989; Nash, 1989; Greub, 1973]. A reading of Cartan's many works in the original French will yield a wealth of ideas that have yet to be exploited in the physical sciences. It is not the purpose of this article to provide such a tutorial of Cartan's methods, but suffice it to say the "raison d'être" for these, perhaps unfamiliar, but simple and useful methods is that they permit topological properties of physical systems and processes to be sifted out from the chaff of geometric ideas that, at present, seem to dominate the engineering and physical sciences.

For hydrodynamics, the combination of the exterior derivative and the interior product to form the Lie derivative acting on p-forms should be interpreted as the fundamental way of expressing an evolution operator with properties that are independent of geometric concepts such as metric and connection [Kiehn, 1975a]. Cartan's Lie derivative yields the equivalent of a convective derivative that may be used to demonstrate that the laws of hydrodynamic evolution are topological laws, not geometric laws. This philosophy is similar to that championed by Van Dantzig [1934] with regard to the topological content of Maxwell's equations of electrodynamics.

Cartan built his theory around an exterior differential system, Σ , which consists of a collection of 0-forms, 1-forms, 2-forms, etc. [Slebodzinsky, 1970]. He defined the closure of this collection as the union of the original collection with those forms which are obtained by forming the exterior derivatives of every p-form in the initial collection. In general, the collection of exterior derivatives will be denoted by $d\Sigma$, and the closure of Σ by the symbol, Σ^c , where

$$\Sigma^c = \Sigma \cup d\Sigma.$$

Cartan's interest in this closure was that he was able to prove that the system of 1-forms adjoint to the closure were completely integrable. The result allowed him to devise schemes for prolonging a non-integrable system until it became integrable.

For notational simplicity in this article the systems of p-forms will be assumed to consist of the single 1-form, A. Then the exterior derivative of A is the 2-form $F = dA$, and the closure of A is the union of A and F: $A^c = A \cup F$. The other logical operation is the concept of intersection, so that from the exterior product it is possible to construct the set $A \wedge F$ defined collectively as H: $H = A \wedge F$. The exterior derivative of H produces the set defined as $K = dH$, and the closure of H is the union of H and K: $H^c = H \cup K$.

This ladder process of constructing exterior derivatives, and exterior products, may be continued until a null set is produced, or the largest p-form so constructed is equal to the dimension of the space under consideration. The collection of sets so generated is defined as a Pfaff sequence. The largest rank of the sequence determines the Pfaff dimension of the domain (or class of the form, [Schouten, 1949], which is a topological invariant.

The idea for evolutionary systems is that the 1-form A (in general the exterior differential system, Σ) generates on space-time, $\{x,y,z,t\}$, four equivalence classes of points that act as domains of support for the elements of the Pfaff sequence, $\{A, F, H, K\}$. The union of all such points will be denoted by $X = A \cup F \cup H \cup K$. The fundamental equivalence classes are given specific names:

TOPOLOGICAL ACTION

$$A = A_\mu dx^\mu$$

TOPOLOGICAL VORTICITY

$$F = dA = F_{\mu\nu} dx^\mu dx^\nu$$

TOPOLOGICAL TORSION

$$H = A \wedge dA = H_{\mu\nu\rho} dx^\mu dx^\nu dx^\rho$$

TOPOLOGICAL PARITY

$$K = dA \wedge dA = K_{\mu\nu\rho\sigma} dx^\mu dx^\nu dx^\rho dx^\sigma.$$

The Cartan topology is constructed from a basis of open sets, which are defined as follows: First consider the domain of support of A. Define this "point set" by the symbol A. A is the first open set of the Cartan topology. Next construct the exterior derivative, $F = dA$, and determine its domain of support. Next, form the closure of A by constructing the union of these two domains of support, $A \cup F = A^c$. $A \cup F$ forms the second open set of the Cartan topology.

Next construct the intersection $H = A \wedge F$, and determine its domain of support. Define this "point set" by the symbol H. H forms the third open set of the Cartan topology. Now follow the procedure established in the preceding paragraph. Construct the closure of H as the union of the domains of support of H and $K = dH$. The construction forms the fourth open set of the Cartan topology. In 4 dimensions, the process stops, but for $N > 4$, the process may be continued.

Now consider the basis collection of open sets that consists of the subsets,

$$B = \{A, A^c, H, H^c\} = \{A, A \cup F, H, H \cup K\}$$

The collection of all possible unions of these base elements, and the null set, $\{0\}$ generate the Cartan topology of open sets:

$$T_{\text{open}} = \{X, 0, A, H, A^c, H^c, A \cup H, A \cup H^c, A^c \cup H\}.$$

These nine subsets form the open point sets of the Cartan topology constructed from the domains of support of the Pfaff sequence $\{A, F, H, K\}$.

Subsets (σ)	Limit Pts ($d\sigma$)	Interior	Boundary ($\partial\sigma$)	Closure ($\sigma \cup d\sigma$)
0	0	0	0	0
A	F	A	F	$A \cup F$
F	0	0	F	F
H	K	H	K	$H \cup K$
K	0	0	K	K
$A \cup F$	F	$A \cup F$	0	$A \cup F$
$A \cup H$	F, K	$A \cup H$	$F \cup K$	X
$A \cup K$	F	A	$F \cup K$	$A \cup F \cup K$
$F \cup H$	K	H	$F \cup K$	$F \cup H \cup K$
$F \cup K$	0	0	$F \cup K$	$F \cup K$
$H \cup K$	K	$H \cup K$	0	$H \cup K$
$A \cup F \cup H$	F, K	$A \cup F \cup H$	K	X
$F \cup H \cup K$	K	$H \cup K$	F	$F \cup H \cup K$
$A \cup H \cup K$	F, K	$A \cup H \cup K$	F	X
$A \cup F \cup K$	F	$A \cup F$	K	$A \cup F \cup K$
X	F, K	X	0	X

Table 1. The Cartan Topology

It is possible to compute the limit points for every subset relative to the Cartan topology. The classical definition of a topological limit point is that a point p is a limit point of the subset Y relative to the topology T if and only if for every open set which contains p there exists another point of Y other than p [Lipschutz, 1965]. The results of this definition are presented in Table I [Baldwin, 1991]. Note that the Cartan exterior derivative is a limit point operator relative to the Cartan topology.

It is apparent that the Cartan topology as given in Table 1 is composed of the union of two subsets which are both open and closed ($X = A^c \cup H^c$), a result that implies that the Cartan topology is not connected, unless the Topological Torsion, H , and hence its closure, vanishes. This extraordinary result has a number of physical consequences, some of which are described in [Kiehn, 1975, 1975a].

To prove that a turbulent flow must be a consequence of a Cartan topology that is not connected, consider the following argument: First consider a fluid at rest and from a global set of unique, synchronous, initial conditions generate a vector field of flow. Such flows must satisfy the Frobenius complete integrability theorem, which requires that $A \wedge dA = 0$. The Torsion current is zero for such systems. The Cartan topology for such systems ($H = A \wedge dA = 0$) is connected, and the Pfaff dimension of the domain is 2 or less. Such domains do not support Topological Torsion. Such globally laminar flows are to be distinguished from flows that reside on surfaces, but do not admit a unique set of connected synchronizable initial conditions. Next consider turbulent flows which, as the antithesis of laminar flows, can not be integrable in the sense of Frobenius; such turbulent domains support Topological Torsion ($H = A \wedge dA \neq 0$), and therefore a disconnected Cartan topology. The connected components of the disconnected Cartan topology can be defined as the coherent structures of the turbulent flow. The transition from an initial laminar state ($H = 0$) to the turbulent state ($H \neq 0$) is a transition from a connected topology to a disconnected topology. Therefore the transition to turbulence can NOT be continuous. However, the decay of turbulence can be described by a continuous transformation from a disconnected topology to a connected topology. Condensation is continuous, gasification is not.

A topological structure is defined to be enough information to decide whether a transformation is continuous or not [Gellert, 1977]. A transformation is continuous if and only if the image of the closure of every subset is included in the closure of the image. This means that the concept of closure and the concept of transformation must commute for continuous processes. Suppose the forward image of a set A is Q and the forward image of the set $F (= dA)$ is Z . Then if the closure, $A^c = A \cup F$ is included in the closure of $Q^c = Q \cup dQ$, for all subsets, the transformation is defined to be continuous. The idea of continuity becomes equivalent to the concept that the forward image Z of the limit points, dA , is an element of the closure of Q [Hocking, 1961]. (This idea will be tested below, to demonstrate that for C^2 fields, the evolutionary process generated by the action of the Lie derivative is continuous with respect to the Cartan topology.)

The Cartan theory of exterior differential systems can now be interpreted as a topological structure, for every subset of the topology given in Table 1. can be tested to see if the process of closure commutes with the process of transformation. The evolutionary transformation of interest to this article is generated by the action of the Lie derivative with respect to a vector field, \mathbf{V} , acting on differential forms, Σ [Bishop, 1968]. The Lie derivative has a number of interesting and useful properties.

1. The Lie derivative does not depend upon a metric or a connection.
2. The Lie derivative has a simple action on differential forms, ω , producing a resultant form that is decomposed into a transversal and an exact part:

$$L(\mathbf{V}) \omega = i(\mathbf{V})d\omega + di(\mathbf{V})\omega . \quad (A1)$$

3. The Lie derivative may be used to describe deformations and topological evolution. By suitable choice of the 1-form of action it is possible to show that the action of the Lie derivative on the 1-form of action can generate the Navier Stokes partial differential equations [Kiehn, 1974, 1975a, 1991a].

4. With respect to vector fields and forms constructed over C^2 functions, the Lie derivative commutes with the closure operator [Baldwin, 1991]. Hence, the Lie derivative (restricted to C^2 functions) generates transformations on differential forms which are continuous with respect to the Cartan topology. The analysis above indicates that C^2 differentiable solutions to the Navier-Stokes equations can not be used to describe the transition to or the creation of turbulence. The C^2 solutions can, however, describe the irreversible decay of turbulence to the globally laminar state.

For a given Lagrange Action, A , Cartan has demonstrated that the first variation of the Action integral is equivalent to the search for those vector fields, \mathbf{V} , that for any renormalization factor, ρ , satisfy the equations

$$i(\rho\mathbf{V})dA = 0. \quad (A2)$$

Such vector fields are called extremal vector fields [Klein, 1962, Kiehn, 1975a]. The Cartan theorem [Cartan, 1958] states that the conditions (A2) are both necessary and sufficient that there exists a Hamiltonian representation for \mathbf{V} . The resultant equations are a set of partial differential equations that represent extremal evolution. The renormalization condition is common place in the projective geometry of lines, and does not require the Riemannian or euclidean concept of an inner product or a metric [Meserve, 1983]. Hamiltonian systems are not considered to be dissipative.

The constraints given by (A2) are extremely strong. A more relaxed system of constraints would require only that $d(i(\rho\mathbf{V})dA) = 0$. For C^2 vector fields, both constraints imply that such vector fields are not only continuous, but they are uniformly continuous.

A more general expression for the RHS of the Cartan condition (A2) is given by the transversal condition,

$$i(\mathbf{V})dA = -\mathbf{f} \bullet (dx - \mathbf{v} dt) - d(kT). \quad (A3)$$

This extension of the Cartan Hamiltonian constraint is transversal for the first term is orthogonal to the vector field, (\mathbf{V}) . The Lagrange multipliers, \mathbf{f} , are arbitrary for such a transversal constraint, but if chosen to be of the form, $\mathbf{f} = \mathbf{v} \text{ curl curl } \mathbf{v}$, then the constrained Cartan topology (A3) will generate the Navier-Stokes equations as a subset of those evolutionary processes that generate the Cartan topology.

As an example of this Cartan technique, substitute the 1-form of action given by the expression,

$$A = \sum_{i=1}^3 v_i dx^i - Hdt \quad (\text{A4})$$

where the "Hamiltonian" function, H , is defined as,

$$H = \mathbf{v} \bullet \mathbf{v} / 2 + \int dP / \mathbf{r} - \mathbf{l} \text{ div } \mathbf{v} + kT, \quad (\text{A5})$$

into the constraint equation given by (A3). Carry out the indicated operations of exterior differentiation and exterior multiplication to yield a system of necessary partial differential equations yields of the form,

$$\partial \mathbf{v} / \partial t + \text{grad}(\mathbf{v} \bullet \mathbf{v} / 2) - \mathbf{v} \times \text{curl } \mathbf{v} = -\text{grad } P / \rho + \lambda \text{grad div } \mathbf{v} - \mathbf{v} \text{ curl curl } \mathbf{v}. \quad (\text{A6})$$

Equations (A6) are exactly the Navier-Stokes partial differential equations for the evolution of a compressible viscous irreversible flowing fluid.

By direct computation, the 2-form $F = dA$ has components,

$$F = dA = \omega_z dx^{\wedge} dy + \omega_x dy^{\wedge} dz + \omega_y dz^{\wedge} dx + \mathbf{a}_x dx^{\wedge} dt + \mathbf{a}_y dy^{\wedge} dt + \mathbf{a}_z dz^{\wedge} dt, \quad (\text{A7})$$

where by definition

$$\omega = \text{curl } \mathbf{v}, \quad \mathbf{a} = -\partial \mathbf{v} / \partial t - \text{grad } H \quad (\text{A8})$$

The 3-form of Helicity or Topological Torsion, H , is constructed from the exterior product of A and dA as,

$$H = A^{\wedge} dA = H_{ijk} dx^i \wedge dx^j \wedge dx^k = -\mathbf{T}_x dy^{\wedge} dz^{\wedge} dt - \mathbf{T}_y dz^{\wedge} dx^{\wedge} dt - \mathbf{T}_z dx^{\wedge} dy^{\wedge} dt + h dx^{\wedge} dy^{\wedge} dz, \quad (\text{A9})$$

where \mathbf{T} is the fluidic Torsion axial vector current, and h is the torsion (helicity) density:

$$\mathbf{T} = \mathbf{a} \times \mathbf{v} + H \omega, \quad h = \mathbf{v} \bullet \omega \quad (\text{A10})$$

The Torsion current, \mathbf{T} , consists of two parts. The first term represents the shear of translational accelerations, and the second part represents the shear of rotational accelerations. The topological torsion tensor, H_{ijk} , is a third rank completely anti-symmetric covariant tensor field, with four components on the variety $\{x, y, z, t\}$.

The Topological Parity becomes

$$K = dH = dA^{\wedge} dA = -2(\mathbf{a} \bullet \omega) dx^{\wedge} dy^{\wedge} dz^{\wedge} dt. \quad (\text{A11})$$

The identity given by eq. (A11) is in the form of a divergence when expressed on $\{x, y, z, t\}$,

$$\text{div } \mathbf{T} + \partial h / \partial t = -2(\mathbf{a} \bullet \omega), \quad (\text{A12})$$

and yields the helicity-torsion current conservation law if the anomaly, $-2(\mathbf{a} \bullet \omega)$, on the RHS vanishes. It is to be observed that when $K = 0$, the integral of K vanishes, which implies that the Euler index, χ , is zero. It

follows that the integral of H over a boundary of support vanishes by Stokes theorem. This idea is the generalization of the conservation of the integral of helicity density in an Eulerian flow. Note the result is independent from viscosity, subject to the constraint of zero Euler index, $\chi = 0$

The Navier-Stokes constraint given by (A6) may be used to express the acceleration term, \mathbf{a} , kinematically; i.e.,

$$\mathbf{a} = -\partial\mathbf{v}/\partial t - \text{grad } H = -(\mathbf{v} \times \boldsymbol{\omega}) + \nu\{\text{curl } \boldsymbol{\omega}\} \quad (\text{A13})$$

By substituting (A13) into (A10), the Torsion current becomes expressible in terms of the helicity density, h , the Lagrangian function,

$$L = \mathbf{v} \bullet \mathbf{v} - H, \quad (\text{A14})$$

and the viscosity as:

$$\begin{aligned} \mathbf{T} &= \{ h \mathbf{v} - L \boldsymbol{\omega} \} - \nu\{\mathbf{v} \times \text{curl } \boldsymbol{\omega}\} \\ &= \{ h \mathbf{v} - [\mathbf{v} \bullet \mathbf{v}/2 - \int dP/\rho + \lambda \text{div } \mathbf{v}] \boldsymbol{\omega} \} - \nu\{\mathbf{v} \times \text{curl } \boldsymbol{\omega}\}, \end{aligned} \quad (\text{A15})$$

Note that the torsion axial vector current persists even for Euler flows, where ν and λ vanish. The measurement of the components of the Torsion current have been completely ignored by experimentalists in hydrodynamics.

By a similar substitution of (A13) into (A11), the Topological Parity pseudo-scalar for the Navier-Stokes fluid becomes expressible in terms of engineering quantities as,

$$K = 2\nu\{ \boldsymbol{\omega} \bullet \text{curl } \boldsymbol{\omega} \} dx^{\wedge} dy^{\wedge} dz^{\wedge} dt. \quad (\text{A16})$$

The Euler index for the Navier-Stokes fluid is proportional to the integral of the Topological Parity 4-form, which is the "top Pfaffian" in Chern's analysis [Chern, 1944, 1988]. When

$$\boldsymbol{\omega} \bullet \text{curl } \boldsymbol{\omega} = 0, \quad (\text{A17})$$

the Euler index of the induced Cartan topology must vanish. This result is to be compared to the classic hydrodynamic principle of minimum rate of energy dissipation [Lamb, 1932]. For a barotropic Navier-Stokes fluid of Pfaff dimension 4, the viscosity cannot be zero, and the lines of vorticity must be non-integrable in the sense of Frobenius.

REFERENCES

- Baldwin, P. and Kiehn, R. M., 1991, "Cartan's Topological Structure", talk present at the Santa Barbara summer school (1991)
- Bamberg, P. and Sternberg, S., 1990, *A Course in Mathematics for Students of Physics*. Vol I and II, (Cambridge University Press, Cambridge)
- Berge, B., Pomeau, Y. and Vidal, C., 1984, *Order within Chaos*, (Wiley, NY) p. 161.
- Bishop, R. L. and Goldberg, S. I., 1968, *Tensor Analysis on Manifolds*, (Dover, N. Y.) p. 199.
- Bryant, R.L., Chern, S.S., Gardner, R.B., Goldschmidt, H.L., and Griggriths, P. A. 1991, *Exterior Differential Systems*, (Springer Verlag)
- Cartan, E., 1899, "Sur certaine expressions differentielles et le systeme de Pfaff" *Ann Ec. Norm.* 16, p. 329.
- Cartan, E., 1937, *La Theorie des Spaces a Connexion Projective*, Hermann, Paris, (1937)

- Cartan, E., 1945, "Systems Differentials Exterieurs et leurs Applications Geometriques", Actualites sci. et industrielles , p .944.
- Cartan, E., *Lecons sur les invariants integraux*, Hermann, Paris (1958).
- Chern, S. S., 1944, "A Simple intrinsic proof of the Gauss -Bonnet formula for Closed Riemannian Manifolds", *Annals of Math.* **45**, p. 747- 752.
- Chern, S.S. ,1988, "Historical Remarks on Gauss-Bonnet", MSRI 04808-88, Berkeley, CA.
- Croquette, V. and Williams, H., 1989, "Nonlinear of oscillatory instability on finite convective rolls", in *Advances in Fluid Turbulence*, edited by G. Doolen, R. Ecke, D. Holm and V. Steinberg, (North Holland, Amsterdam) p. 300.
- von Dantzig, D., 1934, "Electromagnetism Independent of metrical geometry", *Proc. Kon. Ned. Akad. v. Wet.* **37**, pp. 521-531, 644-652, 825-836 .
- Flanders, H., 1963, *Differential Forms*, (Academic Press, New York (1963), p. 92 .
- Forsyth, A. R. (1959), *Theory of Differential Equations* (Dover, N.Y.) p.96
- Frisch, U., 1991, "From global (Kolmogorov 1941) scaling to local (multifractal) scaling in fully developed turbulence" to appear in the *Proc. Roy. Soc. A*
- Gellert, W., 1977, et.al. Editors, *The VNR Concise Encyclopedia of Mathematics* (Van Nostrand, New York), p.686.
- Greub, W., Halperin, S. and Vanstone, R., 1973, *Connections, Curvature and Cohomology*, (Academic Press, New York).
- Hermann, R., 1968, *Differential Geometry and the Calculus of Variations*, (Academic Press, New York), p. 254.
- Hocking, J. G.,1961, *Topology* , (Addison Wesley, N.Y.), p.2.
- Kida, S., Yamada, M., and Ohkitani, K., 1989, " A route to chaos and turbulence" in *Advances in Fluid Turbulence*, edited by G. Doolen, R. Ecke, D. Holm and V. Steinberg, (North Holland, Amsterdam) p. 116.
- Kiehn, R. M., 1974, "Modifications of Hamilton's Principle to include Dissipation", *J. Math Phys*
- Kiehn, R. M., 1975, "Submersive equivalence classes for metric fields", *Lett. al Nuovo Cimento* **14**, p. 308.
- Kiehn, R. M., 1975,a "A Geometric model for supersymmetry", *Lett. al Nuovo Cimento* **12**, p. 300.
- Kiehn, R. M. , 1975a, "Intrinsic hydrodynamics with applications to space-time fluids", *Int. J. of Eng. Sci.* **13**, p. 941.
- Kiehn, R. M., 1976, "Retrodicitive Determinism", *Int. J. of Eng. Sci.* **14**, p. 749
- Kiehn, R. M., 1990a "Topological Torsion, Pfaff Dimension and Coherent Structures", in: H. K. Moffatt and T. S. Tsinober eds, *Topological Fluid Mechanics*, (Cambridge University Press), 449-458 .
- Kiehn, R. M., 1990b, "Irreversible Topological Evolution in Fluid Mechanics" in *Some Unanswered Questions in Fluid Mechanics ASME- Vol. 89-WA/FE-5*, Trefethen, L. M. and Panton, R. L. Eds.
- Kiehn, R.M., Kiehn, G. P., and Roberds, R. B., 1991, "Parity and Time-reversal symmetry Breaking, Singular Solutions and Fresnel Surfaces", *Phys Rev A*, **43** , p. 5665
- Kiehn, R. M. , 1991a, "Continuous Topological Evolution" submitted to *Phys. Rev. B*
- Kiehn, R. M.,1991b "Topological Parity and the turbulent state of a Navier Stokes Fluid", submitted to *Fluid Dynamics Research*
- Kiehn, R. M., 1991c, "Compact Dissipative Flow Structures with Topological Coherence Embedded in Eulerian Environments", in: *Non-linear Dynamics of Structures*, edited by R.Z. Sagdeev, U. Frisch, F. Hussain, S. S. Moiseev and N. S. Erokhin, (World Scientific Press, Singapore) p.139-164.
- Kiehn, R. M., 1992a "Topological Defects, Coherent Structures and Turbulence in Terms of Cartan's Theory of Differential Topology" in "Developments in Theoretical and Applied Mathematics" Proceedings of the SECTAM XVI conference, B. N. Antar, R. Engels, A.A. Prinaris and T. H. Moulden, Editors, The University of Tennessee Space Institute, Tullahoma, TN 37388 USA.
- Kiehn, R. M., 1992b "Instabilty Patterns, Wakes and Topological Limit Sets" Poitier IUTAM conference 1992. To be published.
- Kiehn, R. M., 1992c "Hydrodynamic Wakes and Minimal Surfaces with Fractal Boundaries" Barcelona ERCOFTAC meeting. To be published.
- Klein, J., 1962, *Ann. Inst. AIFUB Fourier*, Grenoble 12, p. 1 .
- Kolmogorov, A. V. ,1941, *Dolk. Akad. Nauk SSSR*, 31, p. 99 .

- Lamb, H., 1937, *Hydrodynamics*, (Cambridge University Press) p. 617-619.
- Landau, L. and Lifshitz, M, (1959) , *Fluid Mechanics*, (Pergamon Press, London) p.106.
- Lipschutz, S.1965, *General Topology*, (Schaum, New York) p.88.
- Lovelock, D. and Rund, H., 1989, *Tensors, Differential Forms and Variational Principles*, (Dover, New York).
- Meserve, B. E., 1983, *Projective Geometry*, (Dover, New York).
- Nash, C. and Sen, S., 1989, *Topology and Geometry for Physicists*, (Academic Press, San Diego).
- Schouten, J. A. and Van der Kulk, W.,1949, *Pfaff's Problem and its Generalizations*, (Oxford Clarendon Press)
- Schultz, A., Kiehn, R. M., Post, E. J., and Roberds, R. B., 1979, "Lifting of the four fold EM degeneracy and PT asymmetry", *Phys Lett* **74A**, p.384.
- Schuster, H. G.,1984, *Deterministic Chaos*, Physics Verlag (1984)
- Slebodzinsky, W., 1970, *Exterior Forms and their Applications*, (PWN Warsaw).
- Stewart, I., 1988, *The Problems of Mathematics*, (Oxford, NY) p 177.
- Zimin, V., 1991, "Hierarchical Models of Turbulence" in: *Non-linear Dynamics of Structures*, edited by R.Z. Sagdeev, U. Frisch, F. Hussain, S. S. Moiseev and N. S. Erokhin, (World Scientific Press, Singapore) p.273-288.