Topological Defects, Coherent Structures and Turbulence in Terms of Cartan's Theory of Differential Topology

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Abstract

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1. Introduction

Many of the interesting aspects of hydrodynamics can be described in terms of those features of a flow that are deformation invariants. These invariants are topological properties, and are to be distinguished from geometric properties of size and shape, which are invariants of rigid body, not fluid, motions of translations and rotations. Of interest to this article is the creation, the persistent lifetime, and the destruction of such topological properties by the dynamical evolution of a fluid flow. The dynamics of such topological evolution is not within the scope of a classical mechanics based upon geometric invariants. In this article, a qualitative discussion of Cartan's methods of differential topology is presented, demonstrating how the techniques may be used to develop a theory of topological evolution, and to describe the creation of dislocation and disclination defects, coherent structures and turbulence in deformable media. As an example of a hydrodynamic-topological defect consider the Falaco Solitons in Figure 1.

The Falaco Soliton is observed by means of the unique optics of Snell refraction from a surface of negative Gauss curvature (See Figure 2). The surface is almost a minimal surface, and the projection to the floor of the pool is almost conformal, preserving the circular appearance of the black disc, independent from the angle of solar incidence.

A remarkable feature of the Falaco Soliton [14] is that it consists of a pair of two dimensional defects, in a surface of discontinuity, which apparently are connected by means of a topological singular thread (See Figure 3). If dye drops are injected into the water, the dye particles will execute a *torsional* wave motion that oscillates up and down until the dye maps out the thread singularity (a circular arc) that connects the two vertices of the Falaco Soliton and acts as a guiding center for the torsion waves. If the thread is severed, the endcap singularities disappear almost immediately, and not diffusively. The long lifetime of the Falaco Soliton is due to this global stabilization of the connecting string singularity.

2. CARTAN'S DIFFERENTIAL TOPOLOGY.

Recall that a topological property is an invariant of a deformation, and that a topology, like a geometry, can be described in terms of its invariant properties [22]. However, on an arbitrary algebraic variety it is possible to define many inequivalent topologies. For example, the simplest way of defining a non-trivial topology on a euclidean domain is to specify a set of constraints. The differential form, xdx+ydy+zdz = 0, leads to the functional constraint (x^2 +y^2 +z^2 - b^2 = 0). For a particular choice of b, these constraints limit the euclidean 3-space to the topology of a compact spherical surface. Additional constraints may be used to refine the topology.

Cartan's theory of exterior differential systems [3] utilizes a specific set, Σ, of functional and differential constraints, called differential forms [8], to define a topological structure on a domain. For hydrodynamics, the fundamental Cartan constraints involve the specification of a 1-form of action, A, (which will be detailed in that which follows) and a density distribution, ρ. The Cartan 1-form, A, is related to the Lagrangian action in classical field theory, but the classical kinematic constraints, $d\mathbf{x} - \mathbf{v}dt = 0$, which in the Cartan theory are interpreted as topological constraints, are not subsumed, apriori. Otherwise, the Cartan extremal analysis is equivalent to the classical Lagrange-Euler analysis of mechanics and is related to the calculus of variations. The advantages of the Cartan methods over the classic methods reside with the fact that the Cartan differential forms carry both topological content as well as geometrical information, and the dynamics of such systems of differential forms are such that they include both reversible processes (conservative - with invariant topology) as well as irreversible processes (dissipative - with changing topology).

The very existence of a Cartan exterior differential system may be used to construct a well defined topology, with a limit point operator (the exterior derivative), an operation related to a logical intersection operator (the exterior product), and enough topological structure to determine if arbitrary evolutionary processes are continuous, reversible, or both. If topology is to be preserved by the evolutionary process, then the process (technically defined as a homeomorphism) must be both continuous and reversible. Reversible means that an inverse mapping both exists and is continuous. The Cartan topological structure permits such a determination of continuity to be made. Continuity is a concept which is defined in terms of a specified topology. A transformation is continuous if the limit points of the initial state, relative the topology of the initial state, permute into the closure of the images of these points in the final state, where the closure is defined in terms of the topology of the final state [9]. The topology of the final state can be different from the topology of the initial state.

For simplicity in this article, the set of Cartan forms, Σ, will be limited to the single 1-form of action, A. The exterior derivative of A produces a 2-form of closure points, $F = dA$, whose components are given by the expression, $F = F_{\mu\nu} dx^{\mu} \hat{} dx^{\nu}$. The combined set $\{A, F\}$ forms the closure of the set A. Next, construct all possible intersections of the closure, forming the set $\{A, F = dA, H = A\hat{}dA = A\hat{}F, K = dH = dA\hat{}dA = F\hat{}F\}$, which is defined as the Pfaff sequence on the domain $\{x, y, z, t\}$. The elements of the Pfaff sequence carry topological information, and are defined as:

TOPOLOGICAL ACTION $A = A_{\mu} dx^{\mu}$

TOPOLOGICAL VORTICITY $F = dA = F_{\mu\nu} dx^{\mu} \hat{} dx^{\nu}$

TOPOLOGICAL TORSION $H = A\hat{}dA = H_{\mu\nu\rho} dx^{\mu} \hat{} dx^{\nu} \hat{} dx^{\rho}$

TOPOLOGICAL PARITY $K = dA\hat{}dA = K_{\mu\nu\rho\sigma} dx^{\mu} \hat{} dx^{\nu} \hat{} dx^{\rho} \hat{} dx^{\sigma}$

The largest non-null element of the Pfaff sequence determines the class, or Pfaff dimension, of the domain [16].

The domains of support for each element of the Pfaff sequence may be considered as "points" that could be used to construct a topology. The union of the odd elements of the Pfaff sequence and their closures may be used to define the elements of the Cartan topological base: $\{A, A \cup F, H, H \cup K\}$ [2]. It is important to note that the complete Cartan topology built on this topological base consists of the union of the disjoint open subsets, $A \cup F$ and $H \cup K$; hence, the Cartan topology so defined is not connected, if H and K are not zero.

3. SIMPLE FACTS ABOUT THE CARTAN DISCONNECTED TOPOLOGY.

Of fundamental importance to the Cartan analysis is the idea that the 3-form, $H = A\hat{}dA$, is not zero over a chaotic or turbulent domain, but $H = 0$ over the domain of a globally laminar streamline flow. Assume without proof that the statement is valid. Then note from the arguments presented above, the Cartan topology for the non-chaotic, non-turbulent state is connected, where the Cartan topology for the chaotic or turbulent state is disconnected. Coherent structures in the turbulent state can be interpreted as realizations of the disconnected components of the Cartan topology.

To prove the statement, recall that a chaotic vector field may be used to define a set of deterministic solution trajectories, but with extreme sensitivity to initial conditions. Points that start out on "nearby" trajectories do not remain nearby. However, the chaotic system is deterministic, so that each trajectory may be retraced; that is, the chaotic system is reversible. The "chaotic" visualization occurs as the disconnected domains of trajectories fold and intertwine with one another. The difference between a chaotic system and the turbulent system is that the turbulent trajectories are not continuously reversible.

If a vector field is completely integrable in the sense of Frobenius, then there exists a global map to a space of two dimensions, and the associated system of ordinary differential equations is never chaotic [24]. The Frobenius criteria is equivalent to the topological constraint, $H = A\hat{}dA = 0$, which implies that the Pfaff dimension of such systems is less than 3. The concept of "globally nearby" means that there exists a unique function whose zero set defines a surface of synchronizeable points along the flow trajectories. The flow lines intersect this "timelike" surface transversely. For a chaotic system, there does not exist a globally unique transversal surface that can be used to synchronize points that evolve along the trajectories. The points along a chaotic system of trajectories cannot be smoothly connected everywhere; the topology is disconnected, and $H \neq$

0 everywhere. Although the flow lines form disconnected sets when $H \neq 0$, the flow lines may or may not intersect one another in the domain of space-time. If the flow lines never intersect the Pfaff dimension of the domain is 3 and then there is no ambiguity in retracing a given trajectory. Although the flow may be considered to be chaotic with extreme sensitivity to nearby, but disconnected, trajectories, each trajectory is reversible. On the other hand, if the trajectories intersect in the space time domain, then there is an ambiguity in retracing a given trajectory, and the system is not reversible. Such a concept of irreversibility is assumed to describe a necessary condition for the turbulent state. A result of this article is to demonstrate that irreversibility implies that the vector field must break time reversal symmetry [18] and the Pfaff dimension of such domains must be 4. As the turbulent state is irreversible, it must be on a domain of Pfaff dimension 4.

A laminar globally synchronizeable flow is an example of the topological constraint, $H = A\hat{}dA = 0$. It follows from the definition that the Cartan topology is connected in the globally laminar case, and is disconnected in the turbulent case. Not only is it impossible to proceed from a laminar state (connected topology) to a turbulent state (disconnected topology) by means of a continuous reversible transformation (the topology must change), it is also impossible to create the turbulent state from the laminar state by means of a continuous but irreversible transformation. The turbulent state (disconnected topology) can be created from the laminar state (connected topology) only by a discontinuous transformation (which may or may not be reversible). On the otherhand, it is remarkable that the decay of the turbulent state (disconnected topology) to the laminar state (connected topology) can be described by a continuous transformation, but such continuous transformations cannot be reversible. The transitions between disconnected and connected topologies have a built-in "arrow of time", which is not intrinsic to the classical evolutionary theories of mechanics.

Topological invariance is often subsumed in the usual treatments of classical mechanics; in fact, the concept of topological invariance is a natural consequence of rigid body dynamics. However, fluid dynamics is the epitome of a deformable dynamics, and topological invariance is not necessarily a property of fluid flow. As described above, topological evolution is a necessary consequence if coherent structures and turbulence are to be *created* in a hydrodynamical system, starting from a flow in the *laminar* state. Coherent structures are evidence of topologically disconnected domains. The creation of turbulence may be compared to the creation of mixed phase, disconnected, domains that occur during the process of a phase transition in thermodynamics.

It may be shown [2] that, relative to the Cartan topology constructed on differential forms composed of C2 functions, all evolutionary processes that can be described in terms of C2 vector fields are continuous. This theorem does not mean that these flow processes are reversible, for if the topology changes during the continuous process (say by the condensation of topological defects or holes) then that process must be irreversible. Another qualitative result of the Cartan method is that when any numeric or analytic procedure is used to force C2 differentiability (match of slope and value) on a solution, then that procedure can not be used to describe the transition to, or the "creation" of, turbulence from a laminar state. However, such continuous methods are applicable to the study of the "decay" of turbulence. Of practical importance is the idea that the no-slip boundary condition of conventional hydrodynamics may be inappropriate for the analysis of the "creation" of the turbulent state.

There are two extreme cases to consider: the global domain consists of a non-laminar fabric with islands of laminar flow, or the global domain consists of a laminar fabric (which is free of topological torsion) containing disconnected islands of non-null topological torsion. In the latter more readily understood case, the islands may have opposite signs, a result that can be related to the stability of the island domains. In certain instances the stable domains will become connected, while in the opposite circumstances the unstable domains will be connected throughout the domain. Numerical simulations indicate that "Lagrangian turbulence" or chaos is to be associated with the unstable domains of topological torsion. A transition takes place when the unstable islands become connected [7][15]

4. SHOCK WAVES AND TURBULENCE.

Another favorable feature of the Cartan methods is that differential forms can support non-C2 discontinuous structures, or shocks, on constrained subsets, where these subspaces are transversal to the discontinuities. For example, displacements confined to point sets along directions defined by those eigen vectors of an anti-symmetric matrix, $F_{\mu\nu}$, which have null eigen values will produce a null set for the product, $F_{\mu\sigma}v^{\sigma}$, even though the values of $F_{\mu\nu}$ are discontinuous across the selected transversal domain.

The Cartan methods thereby can be used to describe the evolution and propagation of point set domains which can support multivalued functions; i.e., discontinuities. The propagating singular point sets are defined as shocks [18]. The shocks may be points, jets (lines), surfaces (areas) or 3-forms of torsion on space-time varieties. As described above, the creation of turbulence requires the existence of propagating discontinuities, or shocks.

The point sets upon which multivalued functions or forms can exist are to be recognized as (topological and physical) defects in the otherwise homogeneous connected environment. From the arguments given above it becomes apparent that in order to create the turbulent state, the creation processes must involve propagating discontinuities or shock waves. On the other hand the decay of the turbulent state can be done continuously, but irreversibly.

For hydrodynamic purposes the vector fields of interest are solutions to the set of partial differential equations known as the Navier-Stokes equations. These equations of evolution may be deduced from a refinement of the Cartan topology, but the solutions are not necessarily generators of a single parameter pseudo group of transformations. This observation indicates that there may exist deviations or fluctuations from the kinematic constraints of a single parameter group, $d\mathbf{x}-\mathbf{v}dt = 0$. The Cartan 1-form of action, which is used both to define the Cartan topology and to deduce the partial differential equations of evolution, is closely related to the variational integrand proposed by Finsler [6]. This observation permits a correspondence to be made between the topological approach and certain non-Riemannian geometries, especially those with torsion.

5. THE CARTAN 1-FORM OF ACTION.

The format of the Cartan 1-form, A, will be that of the Cartan-Hilbert invariant integrand,

$$A = L(\mathbf{x}, t; \mathbf{v})dt + \mathbf{p}\cdot(d\mathbf{x} - \mathbf{v}dt) = L(\mathbf{x}, t; \mathbf{v})dt + \mathbf{p}\cdot\Delta\mathbf{x}. \qquad (5.1)$$

The Cartan technique prolongs the original space-time, $\{\mathbf{x}, t\}$ to a 10 dimension space of functions, $\{\mathbf{x}, t; \mathbf{v}, \mathbf{p}\}$. On this space, $\{\mathbf{x}, t; \mathbf{v}, \mathbf{p}\}$, it is useful to define a vector field, V , with 10 components, given by the arbitrary functions, $\{\mathbf{v}, 1; \mathbf{a}, \mathbf{F}\}$. Similar to the example of the first section, differential and functional constraints will be imposed on this 10 dimensional space thereby defining a topology. The basic idea is to compute the effect of topological constraints on the 10 dimensional space, and then by functional substitution pull back the results to 4 dimensional space-time.

Note that the Cartan-Hilbert 1-form of action involves a classic Lagrange function, $L(\mathbf{x}, t; \mathbf{v})$, and a linear combination of non-zero position "fluctuation or deformation" 1-forms, defined as

$$\Delta \mathbf{x} = (d\mathbf{x} - \mathbf{v}dt) \neq 0. \tag{5.2}$$

The 1-forms, $\Delta \mathbf{x}$, are defined as "position fluctuation-deformation" 1-forms for they represent deviations from the pure kinematic point of view associated with a rigid body dynamics or the evolution of a point particles in terms of a single parameter group of continuous transformations. If the evolution is described by a single parameter group, then the fluctuatuations can be interpreted as deviations in the initial conditions, or deviations in the origin. Although not always true, the deviations are often small corrections to the kinematic constraints of $\Delta \mathbf{x} = \mathbf{0}$, hence the terminology, "fluctuations". The details are presented in [19], but the basic idea is that the statement, $(d\mathbf{x} - \mathbf{v}dt) = 0$, must be interpreted as a topological constraint, just as the statement $xdx + ydy + zdz = 0$ of section 2 is to be interpreted as the topological constraint on Euclidean 3-space that produces the topology of a spherical surface.

In classical hydrodynamics, the non-zero fluctuations are usually constrained by the condition that their associated 3-form is closed in the Cartan sense. That is, it is often assumed that there exists a function, $\rho(x, y, z, t)$, such that the non-zero 3-form,

$$\Omega = \rho(dx - v^x dt)\hat{}(dy - v^y dt)\hat{}(dz - v^z dt) \tag{5.3}$$

has a vanishing exterior derivative:

$$d\Omega = \{div(\rho\mathbf{v}) + \partial\rho/\partial t\}dx\hat{}dy\hat{}dz\hat{}dt = 0. \tag{5.4}$$

This topological constraint is usually called the "equation of continuity" for deformable media. It may be shown that this topological constraint makes Ω an absolute invariant of the evolutionary process. If the flow lines are retraceable, implying that the Jacobian determinant of the assumed map is of rank 3, then the topological constraint on the fluctuations may be interpreted as the "conservation of mass". However, it is not apparent that nature always insists on the assumed topological constraint among the fluctuation-deformation 1-forms. Such a constraint is a matter for test, especially in the case of a turbulent, irreversible, evolutionary process.

The array, $\mathbf{p}$, of coefficients of the fluctuation-deformation 1-forms in **??** may be described as set of Lagrange multipliers. It can be demonstrated that this covariant field, $\mathbf{p}$, dual to the contravariant velocity field, $\mathbf{v}$, plays the role of the canonical momentum, when the system is subjected to additional, but classic, topological constraints that are equivalent to the constraint of zero temperature.

Consider first a Cartan 1-form of action where the fluctuations vanish over a domain. Then the 2-form of limit points is given by the expression, $F = dA = dL\hat{}dt$. It follows that $H = A\hat{}dA = 0$, and $K = 0$. The Pfaff dimension of such systems is 2 at most. Such systems can have vorticity but are without helicity, or Topological Torsion. Examples of systems that do support Topological Torsion are presented in reference [17].

A direct computation of the Topological Torsion, H, on the 10 dimensional space yields,

$$H = A\hat{}dA = Ldt\hat{}(\Delta\mathbf{p}\hat{}\Delta\mathbf{x}) + \{\partial L/\partial\mathbf{v} - \mathbf{p})\cdot\Delta\mathbf{v})\hat{}(\mathbf{p}\cdot\Delta\mathbf{x})\hat{}dt, \tag{5.5}$$

which may be evaluated, in principle, on 4 dimensional space time by functional substitution. The variables,

$$\Delta\mathbf{x} = (d\mathbf{x} - \mathbf{v}dt) \neq 0, \tag{5.6}$$

$$\Delta\mathbf{v} = (d\mathbf{v} - \mathbf{a}dt) \neq 0, \tag{5.7}$$

$$\Delta p = (d\mathbf{p} - (\partial L/\partial\mathbf{x})dt), \tag{5.8}$$

represent the fluctuations in position, velocity and momentum, respectively. A similar direct computation can be performed to yield the Topological Parity 4-form, $K = dH$. Details appear in [19].

6. EQUATIONS OF TOPOLOGICAL EVOLUTION.

In the Cartan method, the evolutionary process relative to a vector field, V , is described by the action of the Lie derivative [25] on the p-forms of interest. The action of the Lie derivative on 1-forms is equivalent to the "convective" derivative in Cartesian hydrodynamics, but the Lie derivative is defined without the geometric constraints of a metric or a connection. If any p-form is invariant with respect to the evolutionary process, V , then its Lie derivative vanishes:

$$L(V)A = 0. \tag{6.1}$$

If all p-forms that make up the topological base of the Cartan topology are invariant, then the topology is invariant, and the process described by V must be a homeomorphism. Such processes are both continuous and reversible, and are to be ignored in this article.

A more general set of evolutionary processes that can admit topological evolution is described by the constraint that forces limit sets, $dA = F$, to be evolutionary invariants, but permits the 1-form, A, to be an evolutionary variable. Such processes are described by the statement:

$$L(V)dA = 0,\ L(V)A = Q. \tag{6.2}$$

It may be demonstrated that such concepts are equivalent to Helmholtz theorem of the conservation of vorticity [11][26]. Direct computation on C2 functions indicates that the statement is equivalent to the topological constraint that the 1-form, $W = i(V)dA$, be closed:

$$dW = d(i(V)dA) = 0. \tag{6.3}$$

This constraint is satisfied by the more stringent sufficient condition: $i(V)dA = 0$.

For a given 1-form of Action, A, Cartan has demonstrated that the first variation of the Action integral is equivalent to the search for those vector fields, V , that satisfy the sufficient condition, $i(V)dA = 0$. Evaluation of the topological constraint yields a necessary set of partial differential equations that represent extremal evolution. [5] has shown that this projective extremal condition is necessary and sufficient for the dynamical system, V , relative to the action, A, to be Hamiltonian [10]. Such systems are not dissipative.

However, on the fluctuation space of 10 dimensions, $\{\mathbf{x}, t; \mathbf{v}, \mathbf{p}\}$, contraction of dA with the vector field, $V = \{\mathbf{v}, 1; \mathbf{a}, \mathbf{F}\}$ does not produce the Cartan-Hamilton constraint without further conditions. In fact, direct computation yields the expression,

$$i(V)dA = -\mathbf{f}\cdot\Delta\mathbf{x} - \mathbf{k}/S\cdot\Delta\mathbf{v}, \tag{6.4}$$

where $\mathbf{f} = \mathbf{F} - \partial L/\partial\mathbf{x}$ is recognized as the dissipative components of the force, $\mathbf{F}$. The RHS of 6.5 depends explicitly upon the fluctuations in position and velocity, and is explicitly independent from fluctuations in momentum. The two covariant vector fields, $f = i(V)\Delta\mathbf{p}$, and $\mathbf{k}/S$, represent the irreversible dissipative mechanisms of friction and radiation in the system fluctuation dynamics. If it is assumed that $\mathbf{k}\cdot\Delta\mathbf{v} = 0$, $\mathbf{f} = \upsilon\, curl\, curl\, \mathbf{v}$, $\mathbf{f}\cdot\Delta\mathbf{x} = 0$, then the equations of motion 6.5 become [19]

$$W = i(V)dA = \upsilon curl\, curl\, \mathbf{v}\cdot\Delta\mathbf{x} \tag{6.5}$$

For $L = \mathbf{v}\cdot\mathbf{v}-H$, and with H defined as $\mathbf{v}\cdot\mathbf{v}/2+\int dP/\rho+\lambda div\,\mathbf{v}$, substitution into 6.5 yields exactly the Navier-Stokes system of partial differential equations of evolution,

$$\partial\mathbf{v}/\partial t + grad(\mathbf{v}\circ\mathbf{v}/2) - \mathbf{v}\times curl\,\mathbf{v} = -gradP/\rho + \lambda div\,\mathbf{v} - \upsilon curl\, curl\,\mathbf{v}. \tag{6.6}$$

It is thereby demonstrated that the Navier-Stokes equations correspond to a refinement of the Cartan topology. Similar substitutions into equation 5.5 yield an engineering expression for the Torsion axial current,

$$\begin{aligned} \mathbf{T} &= \{h\mathbf{v} - L\, curl\,\mathbf{v}\} - \boldsymbol{\upsilon}\times curl\, curl\,\mathbf{v}, && (6.7)\\ with\ h &= \mathbf{v}\cdot curl\,\mathbf{v} && (6.8) \end{aligned}$$

which persists even for Euler flows, where $\upsilon = 0$. The measurement of the components of the Torsion vector have been completely ignored by experimentalists in hydrodynamics. Evaluation of the Topological Parity, $K = dH$, yields the equation.

$$div\ \mathbf{T} + \partial h/\partial t = -2\upsilon\ curl\,\mathbf{v}\circ curl\, curl\,\mathbf{v}, \tag{6.9}$$

and the helicity-torsion current conservation law, if the anomaly, $-2\upsilon\ curl\,\mathbf{v}\circ curl\, curl\,\mathbf{v}$, on the RHS vanishes. It is to be observed that when $K = 0$, the Euler index is zero, and the integral of H over a boundary of support vanishes by Stokes theorem. This idea is the generalization of the conservation of the integral of helicity density in an Eulerian flow. Note the result is true for a viscous fluid, subject to the constraint of zero Euler index. However, if the topological parity 4-form, K, does not vanish, then the lines of the torsion current, $\mathbf{T}$, do not satisfy a conservation law, and they can start or stop within the fluid interior in the sense of a defect. The parity 4-form, K, is the source for the production or destruction of topological defects in the evolutionary process. It is to be observed that the Topological Parity pseudo-scalar, K, is always zero for non-viscous Eulerian flows, and can be zero for viscous Navier-Stokes flows if the vorticity vector, $\boldsymbol{\omega} = curl\,\mathbf{v}$, satisfies the Frobenius integrability condition, $\boldsymbol{\omega}\cdot curl\,\boldsymbol{\omega} = \mathbf{0}$. If the flow is to be irreversible, the flowlines must have at least one point of intersection in space-time, and therefore the Euler index of the domain of support cannot be zero. It follows that K for the domain of support cannot be zero, and therefore the Pfaff dimension of the turbulent state must be 4. The production of defects is to be associated with a non-zero value of K, for the turbulent state, and a Pfaff dimension of the domain equal to 4.

7. ACKNOWLEDGEMENTS.

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8. Figures

Figure 1. Falaco Solitons, a few minutes after creation. The kinetic energy and the angular momentum of a pair of vortex structures created in the free surface of water quickly decay into dimpled, locally unstable, singular surfaces that have an extraordinary lifetime of more than 15 minutes in a still pool. The singular surfaces are connected by means of a stabilizing singular thread, which if severed will cause the endcaps to disappear in a non-diffusive manner. See Figures 2 and 3.

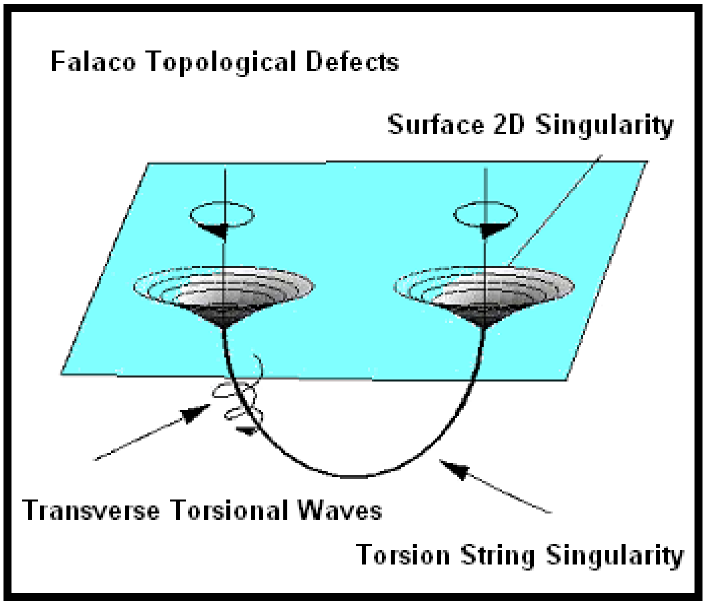

Figure 2. If dye drops are injected into the water, the dye particles will execute a transverse harmonic wave motion that oscillates up and down until the dye maps out the thread singularity (circular arc) that connects the two vertices of the Falaco Soliton. If the thread is severed, the endcaps disappear almost immediately, and not diffusively. The long lifetime of the Falaco Soliton is due to this global stabilization of the connecting string singularity.

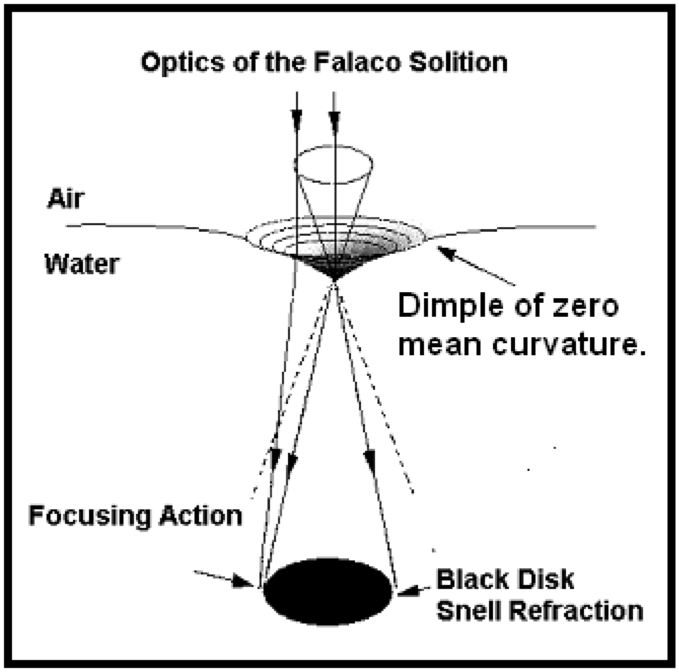

Figure 3. Explanation of the Snell refraction from a surface of discontinuity that has zero Mean curvature and in Euclieean space a negative Gaussian curvature. The surface is almost a minimal surface, and the projection to the floor of the pool is almost conformal, preserving the circular appearance of the black disc, independent from the angle of solar incidence.