

```

> restart: with(liesymm):with(linalg):with(plots):with(diffforms):
> setup(x,y,z,u):
> deform(x=0,y=0,z=0,t=0,u=0,a=const,b=const,c=const,k=const,mu=const,m=const,n=const,p=const);
Warning, new definition for close
Warning, new definition for norm
Warning, new definition for trace
Warning, new definition for `&^`
Warning, new definition for d
Warning, new definition for mixpar
Warning, new definition for wdegree

```

The 3 Sphere in A4

The Instanton Map vs. the Hopf Map

R.M. Kiehn

Vierbein.mws

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Lecture Notes <http://www22.pair.com/csdc/pdf/defects2.pdf>

This work is a start at trying to understand the features of subspaces for which some or all of the Vierbein structures do not satisfy the equations of integrability. This means that they are not generated by a simply transitive group.

A frame field will be constructed on A4 {x,y,z,u}, such that the vierbein has three non-integrable 1-forms and one integrable 1-form. This will be called the Instanton map following Chandia and Zanelli. The properties of this frame field will be compared to the Hopf map on A4 where the vierbein has three perfect differentials and one non-integrable 1-form. The Torsion 2-forms and the Curvature 2-forms induced on the 3-D subspace {x,y,z} will be compared. (Note the Torsion 2-forms and the Curvature 2-forms on A4 are ZERO by definition.)

```

> dR:=[d(x),d(y),d(z),d(u)];

```

$$dR := [d(x), d(y), d(z), d(u)]$$

```

> r2:=subs((x^p+y^p+z^p+u^p)^(n/p));

```

$$r2 := (x^p + y^p + z^p + u^p)^{\left(\frac{n}{p}\right)}$$

r2 will be defined as the scaling factor. For p=2, n=1, the scaling factor is the (Euclidean) radius of the 3 sphere.

Next a Frame Field will be defined from a set of Vierbeins. The Vierbeins are a set of independent 1-forms on the domain of (x,y,z,u), which are assumed to be generated - to within a factor - by the inverse map of the Frame Field. Hence given the Vierbeins, the Frme Field is determined to within an Expansion, or renormalization, factor E(x,y,z,u)

The definitions for the vierebiens in this first example are taken out of the blue to agree with the presentation in Chandia and Zanelli. They called the Vierbein set the Instanton. The determinant of the Frame field is presumed to be not zero. For the CZ instanton, the value is the reciprocal of the 4th power of the euclidean radius of the 3 sphere, and corresponds to the choice p=2,n=0

Other scalings can be assumed, by choosing different values for p and n below. A unimodular frame is

determined from the choice $p=2, n=1$. The choice causes the CZ Vierbeins to be scaled (divided) by the length of the radius of the euclidean 3D sphere.

The example assumes the CZ scaling:

```
> p:=2;n:=0;
>
>
p:=2
n:=0
> MAT:=array([[u/r2,-z/r2,y/r2,-x/r2],[-z/r2,-u/r2,x/r2,y/r2],[y/r2,-x/r2,-u/r2,z/r2],[x/r2,y/r2,z/r2,u/r2]]):
FF:=(inverse(MAT));DETF:=factor(det(FF));FFINV:=simplify(evalm(MAT));
```

The FRAME MATRIX

$$FF := \begin{bmatrix} \frac{u}{x^2+y^2+z^2+u^2} & -\frac{z}{x^2+y^2+z^2+u^2} & \frac{y}{x^2+y^2+z^2+u^2} & \frac{x}{x^2+y^2+z^2+u^2} \\ -\frac{z}{x^2+y^2+z^2+u^2} & -\frac{u}{x^2+y^2+z^2+u^2} & -\frac{x}{x^2+y^2+z^2+u^2} & \frac{y}{x^2+y^2+z^2+u^2} \\ \frac{y}{x^2+y^2+z^2+u^2} & \frac{x}{x^2+y^2+z^2+u^2} & -\frac{u}{x^2+y^2+z^2+u^2} & \frac{z}{x^2+y^2+z^2+u^2} \\ -\frac{x}{x^2+y^2+z^2+u^2} & \frac{y}{x^2+y^2+z^2+u^2} & \frac{z}{x^2+y^2+z^2+u^2} & \frac{u}{x^2+y^2+z^2+u^2} \end{bmatrix}$$

$$DETF := \frac{1}{(x^2+y^2+z^2+u^2)^2}$$

$$FFINV := \begin{bmatrix} u & -z & y & -x \\ -z & -u & x & y \\ y & -x & -u & z \\ x & y & z & u \end{bmatrix}$$

The Vierbein

```
> Vierbein:=innerprod(FFINV,dR):
> sigma1:=Vierbein[1];sigma2:=Vierbein[2];sigma3:=Vierbein[3];omega:=Vierbein[4];
sigma1:=u d(x)-z d(y)+y d(z)-x d(u)
sigma2:=-z d(x)-u d(y)+x d(z)+y d(u)
sigma3:=y d(x)-x d(y)-u d(z)+z d(u)
omega:=x d(x)+y d(y)+z d(z)+u d(u)
```

The Topological Torsion of each of the Vierbeins will now be checked.

```
> TTORSION1:=factor(wcollect((sigma1)&^d(sigma1)));TTORSION2:=factor(wcollect((sigma2)&^d(sigma2)));TTORSION3:=factor(wcollect((sigma3)&^d(sigma3)));TTORSION_omega:=(wcollect(factor(omega&^d(omega)))));
```

TTORSION1 :=

$$-2 u \wedge^2(d(x), d(z), d(y)) - 2 z \wedge^2(d(y), d(u), d(x)) + 2 y \wedge^2(d(z), d(u), d(x)) + 2 x \wedge^2(d(u), d(z), d(y))$$

TTORSION2 :=

$$2 z \wedge^2(d(x), d(u), d(y)) + 2 u \wedge^2(d(y), d(z), d(x)) - 2 x \wedge^2(d(z), d(u), d(y)) - 2 y \wedge^2(d(u), d(z), d(x))$$

TTORSION3 :=

$$-2 y \wedge^2(d(x), d(u), d(z)) + 2 x \wedge^2(d(y), d(u), d(z)) - 2 u \wedge^2(d(z), d(y), d(x)) + 2 z \wedge^2(d(u), d(y), d(x))$$

$$TTORSION_omega := 0$$

The three vierbeins sigma(1,2,3) have nonzero parity, and non-zero topological torsion.
 The last vierbein is integrable (any n any p).

The determinant never vanishes except at the origin of R4, even p. IT is equal to 1 for the unimodular choice of scaling, p=2,n=1

Next create the induced metric: LUN is the Minkowski metric.

```
> LUN:=array([[1,0,0,0],[0,1,0,0],[0,0,1,0],[0,0,0,-1]]);
```

$$LUN := \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

```
> GUN:=simplify(subs(innerprod(transpose(FF),FF)));
```

$$GUN := \begin{bmatrix} \frac{1}{x^2+y^2+z^2+u^2} & 0 & 0 & 0 \\ 0 & \frac{1}{x^2+y^2+z^2+u^2} & 0 & 0 \\ 0 & 0 & \frac{1}{x^2+y^2+z^2+u^2} & 0 \\ 0 & 0 & 0 & \frac{1}{x^2+y^2+z^2+u^2} \end{bmatrix}$$

```
> LGUN:=simplify(subs(innerprod(transpose(FF),LUN,FF)));
```

$$LGUN := \begin{bmatrix} -\frac{-u^2-z^2-y^2+x^2}{(x^2+y^2+z^2+u^2)^2} & 2\frac{yx}{(x^2+y^2+z^2+u^2)^2} & 2\frac{xz}{(x^2+y^2+z^2+u^2)^2} & 2\frac{xu}{(x^2+y^2+z^2+u^2)^2} \\ 2\frac{yx}{(x^2+y^2+z^2+u^2)^2} & \frac{z^2+u^2+x^2-y^2}{(x^2+y^2+z^2+u^2)^2} & -2\frac{yz}{(x^2+y^2+z^2+u^2)^2} & -2\frac{yu}{(x^2+y^2+z^2+u^2)^2} \\ 2\frac{xz}{(x^2+y^2+z^2+u^2)^2} & -2\frac{yz}{(x^2+y^2+z^2+u^2)^2} & \frac{y^2+x^2+u^2-z^2}{(x^2+y^2+z^2+u^2)^2} & -2\frac{uz}{(x^2+y^2+z^2+u^2)^2} \\ 2\frac{xu}{(x^2+y^2+z^2+u^2)^2} & -2\frac{yu}{(x^2+y^2+z^2+u^2)^2} & -2\frac{uz}{(x^2+y^2+z^2+u^2)^2} & \frac{x^2+y^2+z^2-u^2}{(x^2+y^2+z^2+u^2)^2} \end{bmatrix}$$

The Instanton FRAME is NOT an element of the Lorentz Group.

IT is an element of the orthogonal Group.

Note that the induced metric is conformal (for any n any p) which implies that the Frame matrix elements are orthogonal to one another. The conformality factor is a constant on a 3 sphere of fixed radius. The unimodular scaling results in a euclidean metric for the 3D sphere.

The forms sigma (1,2,3) and omega are the 1-forms (dx,dy,dz,du) relative to the basis frame FF. These 1-forms define the "vierbein" in Chandia and Zanelli to within a factor.

Note that for p = 2, the 1-form small omega is exact and equal to a differential of a power of the scaling coefficient, r2. Note further that on the 3 sphere, the 1-form small omega is ZERO.

Hence on the 3 Sphere of constant radius there is NO particle AFFINE TORSION (p=2)!

The Helicity of the 1-form small omega is always zero. The Helicity of the other vierbeins is never zero. However, it is remarkable that the 3-forms of $\sigma^a \wedge d\sigma^a$ are IDENTICAL for each of the three 1-forms. The 3-form is proportional to the contraction of the radius vector in 4D with the 4D volume element.

For the chosen (Instanton) Frame field, the normal field is integrable, and the tangent fields are NOT integrable in the sense of Frobenius.

In the Hopf map for the 3 sphere, the tangent fields are integrable, and the normal field is not integrable.

(See the second example below:

From the given Frame functions, use the standard methods to compute the Cartan Matrix of connection 1-forms.

```
> dFF:=simplify(array([[d(FF[1,1]),d(FF[1,2]),d(FF[1,3]),d(FF[1,4])],[d(FF[2,1]),d(FF[2,2]),d(FF[2,3]),d(FF[2,4])],[d(FF[3,1]),d(FF[3,2]),d(FF[3,3]),d(FF[3,4])],[d(FF[4,1]),d(FF[4,2]),d(FF[4,3]),d(FF[4,4])]]));
> cartan:=(evalm(FFINV*dFF));
```

The diagonal elements are zero on the constrained 3 sphere, but vary for the expanding sphere. The unimodular constraint is non-expanding. The unimodular constrain forces the Frame matrix to be an element of the orthonormal group. The Cartan matrix for the orthonormal Frame is anti-symmetric matrix of 1-forms. There is NO symmetric component.

The Shipov connection Delta is related to the Cartan connection by means of a similarity transformation. It is also completely anti-symmetric for the orthonormal group. When normality is not imposed there exists a diagonal component.

```
> SHIPOVDELTA:=innerprod(FFINV,cartan,FF);
> SHIPTORS:=evalm(SHIPOVDELTA-transpose(SHIPOVDELTA));
```

SHIPTORS :=

$$\begin{aligned} & [0, -2(-y d(x) u^2 - d(z) u^3 + d(y) x^3 - d(x) y^3 + 4 x y z d(z) + 4 x y u d(u) - u z^2 d(z) - 3 z d(u) x^2 + z d(u) u^2 \\ & - u d(z) y^2 + 3 u d(z) x^2 - y x^2 d(x) - 3 x d(y) z^2 - 3 x d(y) u^2 + x d(y) y^2 - y d(x) z^2 + d(u) z^3 + z d(u) y^2) / \\ & (x^2 + y^2 + z^2 + u^2)^2, -2(-z d(x) y^2 + x z^2 d(z) - 3 x d(z) u^2 - z d(x) u^2 - z d(x) x^2 - y u^2 d(u) + u d(y) z^2 \\ & - 3 u d(y) x^2 + u d(y) y^2 - 3 x d(z) y^2 + 3 y d(u) x^2 - y d(u) z^2 - d(x) z^3 + d(y) u^3 + d(z) x^3 - d(u) y^3 + 4 z x y d(y) \\ & + 4 z x u d(u)) / (x^2 + y^2 + z^2 + u^2)^2, -2(-3 x d(u) y^2 + d(z) y^3 + d(u) x^3 - d(x) u^3 - d(y) z^3 + 4 x u y d(y) \\ & + 4 x u z d(z) - z y^2 d(y) - 3 y d(z) x^2 + y d(z) u^2 + y d(z) z^2 - u x^2 d(x) - 3 x d(u) z^2 + x d(u) u^2 - u d(x) z^2 \\ & - u d(x) y^2 - z d(y) u^2 + 3 z d(y) x^2) / (x^2 + y^2 + z^2 + u^2)^2] \\ & [2(-y d(x) u^2 - d(z) u^3 + d(y) x^3 - d(x) y^3 + 4 x y z d(z) + 4 x y u d(u) - u z^2 d(z) - 3 z d(u) x^2 + z d(u) u^2 \\ & - u d(z) y^2 + 3 u d(z) x^2 - y x^2 d(x) - 3 x d(y) z^2 - 3 x d(y) u^2 + x d(y) y^2 - y d(x) z^2 + d(u) z^3 + z d(u) y^2) / \\ & (x^2 + y^2 + z^2 + u^2)^2, 0, -2(-3 x d(u) y^2 + d(z) y^3 + d(u) x^3 - d(x) u^3 - d(y) z^3 + 4 x u y d(y) + 4 x u z d(z) \\ & - z y^2 d(y) - 3 y d(z) x^2 + y d(z) u^2 + y d(z) z^2 - u x^2 d(x) - 3 x d(u) z^2 + x d(u) u^2 - u d(x) z^2 - u d(x) y^2 \end{aligned}$$

$$\begin{aligned}
& -z d(y) u^2 + 3 z d(y) x^2) / (x^2 + y^2 + z^2 + u^2)^2, 2 (-z d(x) y^2 + x z^2 d(z) - 3 x d(z) u^2 - z d(x) u^2 - z d(x) x^2 \\
& - y u^2 d(u) + u d(y) z^2 - 3 u d(y) x^2 + u d(y) y^2 - 3 x d(z) y^2 + 3 y d(u) x^2 - y d(u) z^2 - d(x) z^3 + d(y) u^3 + d(z) x^3 \\
& - d(u) y^3 + 4 z x y d(y) + 4 z x u d(u)) / (x^2 + y^2 + z^2 + u^2)^2] \\
& [2 (-z d(x) y^2 + x z^2 d(z) - 3 x d(z) u^2 - z d(x) u^2 - z d(x) x^2 - y u^2 d(u) + u d(y) z^2 - 3 u d(y) x^2 + u d(y) y^2 \\
& - 3 x d(z) y^2 + 3 y d(u) x^2 - y d(u) z^2 - d(x) z^3 + d(y) u^3 + d(z) x^3 - d(u) y^3 + 4 z x y d(y) + 4 z x u d(u)) / \\
& (x^2 + y^2 + z^2 + u^2)^2, 2 (-3 x d(u) y^2 + d(z) y^3 + d(u) x^3 - d(x) u^3 - d(y) z^3 + 4 x u y d(y) + 4 x u z d(z) - z y^2 d(y) \\
& - 3 y d(z) x^2 + y d(z) u^2 + y d(z) z^2 - u x^2 d(x) - 3 x d(u) z^2 + x d(u) u^2 - u d(x) z^2 - u d(x) y^2 - z d(y) u^2 \\
& + 3 z d(y) x^2) / (x^2 + y^2 + z^2 + u^2)^2, 0, -2 (-y d(x) u^2 - d(z) u^3 + d(y) x^3 - d(x) y^3 + 4 x y z d(z) + 4 x y u d(u) \\
& - u z^2 d(z) - 3 z d(u) x^2 + z d(u) u^2 - u d(z) y^2 + 3 u d(z) x^2 - y x^2 d(x) - 3 x d(y) z^2 - 3 x d(y) u^2 + x d(y) y^2 \\
& - y d(x) z^2 + d(u) z^3 + z d(u) y^2) / (x^2 + y^2 + z^2 + u^2)^2] \\
& [2 (-3 x d(u) y^2 + d(z) y^3 + d(u) x^3 - d(x) u^3 - d(y) z^3 + 4 x u y d(y) + 4 x u z d(z) - z y^2 d(y) - 3 y d(z) x^2 \\
& + y d(z) u^2 + y d(z) z^2 - u x^2 d(x) - 3 x d(u) z^2 + x d(u) u^2 - u d(x) z^2 - u d(x) y^2 - z d(y) u^2 + 3 z d(y) x^2) / \\
& (x^2 + y^2 + z^2 + u^2)^2, -2 (-z d(x) y^2 + x z^2 d(z) - 3 x d(z) u^2 - z d(x) u^2 - z d(x) x^2 - y u^2 d(u) + u d(y) z^2 \\
& - 3 u d(y) x^2 + u d(y) y^2 - 3 x d(z) y^2 + 3 y d(u) x^2 - y d(u) z^2 - d(x) z^3 + d(y) u^3 + d(z) x^3 - d(u) y^3 + 4 z x y d(y) \\
& + 4 z x u d(u)) / (x^2 + y^2 + z^2 + u^2)^2, 2 (-y d(x) u^2 - d(z) u^3 + d(y) x^3 - d(x) y^3 + 4 x y z d(z) + 4 x y u d(u) \\
& - u z^2 d(z) - 3 z d(u) x^2 + z d(u) u^2 - u d(z) y^2 + 3 u d(z) x^2 - y x^2 d(x) - 3 x d(y) z^2 - 3 x d(y) u^2 + x d(y) y^2 \\
& - y d(x) z^2 + d(u) z^3 + z d(u) y^2) / (x^2 + y^2 + z^2 + u^2)^2, 0]
\end{aligned}$$

> SHIPOVDELTA[1,1];SHIPOVDELTA[2,2];SHIPOVDELTA[3,3];SHIPOVDELTA[4,4];SHIPOVDELTA[4,1]+SHIPOVDELTA[1,4];SHIPOVDELTA[3,1]+SHIPOVDELTA[1,3];

$$\begin{aligned}
& - \frac{x d(x) + y d(y) + z d(z) + u d(u)}{x^2 + y^2 + z^2 + u^2} \\
& - \frac{x d(x) + y d(y) + z d(z) + u d(u)}{x^2 + y^2 + z^2 + u^2} \\
& - \frac{x d(x) + y d(y) + z d(z) + u d(u)}{x^2 + y^2 + z^2 + u^2} \\
& - \frac{x d(x) + y d(y) + z d(z) + u d(u)}{x^2 + y^2 + z^2 + u^2} \\
& 0 \\
& 0
\end{aligned}$$

The Shipov Delta for this Frame field has a SYMMETRIC DIAGONAL part, dependent upon the differential of the radius of the 3 sphere squared. IF there is no "expansion" and the radius of the 3sphere is a constant, then in Shipov's terms, the entire connection is a pure "torsion" field, and is related directly to the anti-symmetric portions of the Cartan matrix.

> Gamma11:=wcollect(factor(wcollect(simplify(cartan[1,1]))));

$$\Gamma_{11} := - \frac{x d(x)}{x^2 + y^2 + z^2 + u^2} - \frac{y d(y)}{x^2 + y^2 + z^2 + u^2} - \frac{z d(z)}{x^2 + y^2 + z^2 + u^2} - \frac{u d(u)}{x^2 + y^2 + z^2 + u^2}$$

> Gamma12:=factor(wcollect(cartan[1,2]));

$$\Gamma_{12} := - \frac{-y d(x) + x d(y) + u d(z) - z d(u)}{x^2 + y^2 + z^2 + u^2}$$

> Gamma13:=factor(wcollect(cartan[1,3]));

$$\Gamma_{13} := - \frac{-z d(x) - u d(y) + x d(z) + y d(u)}{x^2 + y^2 + z^2 + u^2}$$

```

> Gamma21:=factor(wcollect(cartan[2,1]));
      Γ21 :=  $\frac{-y d(x) + x d(y) + u d(z) - z d(u)}{x^2 + y^2 + z^2 + u^2}$ 
> Gamma22:=factor(wcollect(cartan[2,2]));
      Γ22 :=  $-\frac{x d(x) + y d(y) + z d(z) + u d(u)}{x^2 + y^2 + z^2 + u^2}$ 
> Gamma23:=factor(wcollect(cartan[2,3]));
      Γ23 :=  $-\frac{-u d(x) + z d(y) - y d(z) + x d(u)}{x^2 + y^2 + z^2 + u^2}$ 
> Gamma31:=factor(wcollect(cartan[3,1]));
      Γ31 :=  $\frac{-z d(x) - u d(y) + x d(z) + y d(u)}{x^2 + y^2 + z^2 + u^2}$ 
> Gamma32:=factor(wcollect(cartan[3,2]));
      Γ32 :=  $\frac{-u d(x) + z d(y) - y d(z) + x d(u)}{x^2 + y^2 + z^2 + u^2}$ 
> Gamma33:=factor(wcollect(cartan[3,3]));
      Γ33 :=  $-\frac{x d(x) + y d(y) + z d(z) + u d(u)}{x^2 + y^2 + z^2 + u^2}$ 
[ The "Space-S" components are:
> h1:=factor(wcollect(cartan[4,1]));
>
      h1 :=  $\frac{-u d(x) + z d(y) - y d(z) + x d(u)}{x^2 + y^2 + z^2 + u^2}$ 
> gamma1:=factor(wcollect(factor(wcollect(cartan[1,4]))));
      γ1 :=  $-\frac{-u d(x) + z d(y) - y d(z) + x d(u)}{x^2 + y^2 + z^2 + u^2}$ 
> h2:=factor(wcollect(cartan[4,2]));
      h2 :=  $-\frac{-z d(x) - u d(y) + x d(z) + y d(u)}{x^2 + y^2 + z^2 + u^2}$ 
> gamma2:=factor(wcollect(cartan[2,4]));
      γ2 :=  $\frac{-z d(x) - u d(y) + x d(z) + y d(u)}{x^2 + y^2 + z^2 + u^2}$ 
> h3:=factor(wcollect(cartan[4,3]));
      h3 :=  $\frac{-y d(x) + x d(y) + u d(z) - z d(u)}{x^2 + y^2 + z^2 + u^2}$ 
> gamma3:=factor(wcollect(cartan[3,4]));
      γ3 :=  $-\frac{-y d(x) + x d(y) + u d(z) - z d(u)}{x^2 + y^2 + z^2 + u^2}$ 
> Omega:=wcollect(factor(subs(p=p,simplify(wcollect(cartan[4,4])))));TTORSION_Omega:=Omega&^d(Omega);
>
      Ω :=  $-\frac{x d(x)}{x^2 + y^2 + z^2 + u^2} - \frac{y d(y)}{x^2 + y^2 + z^2 + u^2} - \frac{z d(z)}{x^2 + y^2 + z^2 + u^2} - \frac{u d(u)}{x^2 + y^2 + z^2 + u^2}$ 
      TTORSION_Omega := 0
>

```

Note that the Big Omega term is a perfect differential, and is zero if n=1,p=2.

That is - off the sphere $r^2 = \text{constant}$ the Omega term does not vanish, unless $n=1, p=2$. Hence if the radius of the two sphere is "expanding", then r^2 is not constant and one has a dilatation. (The source of dilatons?)

There are in general two sets of torsion two forms.

The Particle Affine torsion 2-forms, big Sigma, which depend upon the product of little omega and the connection components, little gamma.

The second set or Wave Affine torsion 2-forms, Big Phi, depends upon big Omega and the connection components, little gamma.

The two species of Torsion 2-forms are related as they have the same common factor (the vector of 1-forms, small gamma) . However, in the example (with the unimodular scaling) there is a difference, for in the neighborhood of the 3D sphere of constant radius little omega is proportional to the gradient of the sphere implicit function, and is not zero identically. However, Big Omega vanishes identically for the unimodular scaling.

It follows that the Particle Affine Torsion 2-forms are not Zero, but the Wave Affine Torsion 2-forms are identically zero.

AFFINE TORSION 2-forms Big Sigma

```
> `omega` := omega ; d(omega) ;
```

$$\omega := x d(x) + y d(y) + z d(z) + u d(u)$$

CARTAN TORSION AND CURVATURE

Next compute the induced Cartan Torsion 2-forms and the Cartan Curvature 2-forms on the 2D subspace

```
> Sigma1 := wcollect(omega^gamma1) ; Sigma2 := wcollect(omega^gamma2) ; Sigma3 := wcollect(omega^gamma3) ;
```

$$\begin{aligned} \Sigma 1 &:= -\frac{(-y^2 - z^2)(d(y) \wedge d(z))}{x^2 + y^2 + z^2 + u^2} - \frac{(x^2 + u^2)(d(x) \wedge d(u))}{x^2 + y^2 + z^2 + u^2} - \frac{(-yx + uz)(d(x) \wedge d(z))}{x^2 + y^2 + z^2 + u^2} \\ &\quad - \frac{(yx - uz)(d(y) \wedge d(u))}{x^2 + y^2 + z^2 + u^2} - \frac{(xz + yu)(d(z) \wedge d(u))}{x^2 + y^2 + z^2 + u^2} - \frac{(xz + yu)(d(x) \wedge d(y))}{x^2 + y^2 + z^2 + u^2} \\ \Sigma 2 &:= \frac{(yx + uz)(d(y) \wedge d(z))}{x^2 + y^2 + z^2 + u^2} + \frac{(yx + uz)(d(x) \wedge d(u))}{x^2 + y^2 + z^2 + u^2} + \frac{(x^2 + z^2)(d(x) \wedge d(z))}{x^2 + y^2 + z^2 + u^2} \\ &\quad + \frac{(y^2 + u^2)(d(y) \wedge d(u))}{x^2 + y^2 + z^2 + u^2} + \frac{(-xu + yz)(d(z) \wedge d(u))}{x^2 + y^2 + z^2 + u^2} + \frac{(-xu + yz)(d(x) \wedge d(y))}{x^2 + y^2 + z^2 + u^2} \\ \Sigma 3 &:= -\frac{(yu - xz)(d(y) \wedge d(z))}{x^2 + y^2 + z^2 + u^2} - \frac{(yu - xz)(d(x) \wedge d(u))}{x^2 + y^2 + z^2 + u^2} - \frac{(xu + yz)(d(x) \wedge d(z))}{x^2 + y^2 + z^2 + u^2} \\ &\quad - \frac{(-yz - xu)(d(y) \wedge d(u))}{x^2 + y^2 + z^2 + u^2} - \frac{(-z^2 - u^2)(d(z) \wedge d(u))}{x^2 + y^2 + z^2 + u^2} - \frac{(y^2 + x^2)(d(x) \wedge d(y))}{x^2 + y^2 + z^2 + u^2} \end{aligned}$$

```
> Phi1 := wcollect(simplify(Omega^h1)) ; Phi2 := wcollect(simplify(Omega^h2)) ; Phi3 := wcollect(simplify(Omega^h3)) ;
```

$$\begin{aligned} \Phi_1 := & -\frac{(-y^2 - z^2)(d(y) \wedge d(z))}{(x^2 + y^2 + z^2 + u^2)^2} - \frac{(x^2 + u^2)(d(x) \wedge d(u))}{(x^2 + y^2 + z^2 + u^2)^2} - \frac{(-yx + uz)(d(x) \wedge d(z))}{(x^2 + y^2 + z^2 + u^2)^2} \\ & - \frac{(yx - uz)(d(y) \wedge d(u))}{(x^2 + y^2 + z^2 + u^2)^2} - \frac{(xz + yu)(d(z) \wedge d(u))}{(x^2 + y^2 + z^2 + u^2)^2} - \frac{(xz + yu)(d(x) \wedge d(y))}{(x^2 + y^2 + z^2 + u^2)^2} \\ \Phi_2 := & -\frac{(-yx - uz)(d(y) \wedge d(z))}{(x^2 + y^2 + z^2 + u^2)^2} - \frac{(-yx - uz)(d(x) \wedge d(u))}{(x^2 + y^2 + z^2 + u^2)^2} - \frac{(-x^2 - z^2)(d(x) \wedge d(z))}{(x^2 + y^2 + z^2 + u^2)^2} \\ & - \frac{(-y^2 - u^2)(d(y) \wedge d(u))}{(x^2 + y^2 + z^2 + u^2)^2} - \frac{(xu - yz)(d(z) \wedge d(u))}{(x^2 + y^2 + z^2 + u^2)^2} - \frac{(xu - yz)(d(x) \wedge d(y))}{(x^2 + y^2 + z^2 + u^2)^2} \\ \Phi_3 := & -\frac{(yu - xz)(d(y) \wedge d(z))}{(x^2 + y^2 + z^2 + u^2)^2} - \frac{(yu - xz)(d(x) \wedge d(u))}{(x^2 + y^2 + z^2 + u^2)^2} - \frac{(xu + yz)(d(x) \wedge d(z))}{(x^2 + y^2 + z^2 + u^2)^2} \\ & - \frac{(-yz - xu)(d(y) \wedge d(u))}{(x^2 + y^2 + z^2 + u^2)^2} - \frac{(-z^2 - u^2)(d(z) \wedge d(u))}{(x^2 + y^2 + z^2 + u^2)^2} - \frac{(y^2 + x^2)(d(x) \wedge d(y))}{(x^2 + y^2 + z^2 + u^2)^2} \end{aligned}$$

As the frame group is orthogonal, the Torsion 2-forms are proportional to each other.
However if normality is assumed, then the Wave Affine torsion forms vanish

```
>
> CURV2FORMS := evalm(array([[wcollect(h1&^gamma1), wcollect(h1&^gamma2), wcollect(h1&
^gamma3)], [wcollect(h2&^gamma1), wcollect(h2&^gamma2), wcollect(h2&^gamma3)], [wcol
lect(h3&^gamma1), wcollect(h3&^gamma2), wcollect(h3&^gamma3)]]));
```

CURV2FORMS :=

$$\begin{aligned} & \left[0, \frac{(xz - yu)(d(y) \wedge d(z))}{(x^2 + y^2 + z^2 + u^2)^2} + \frac{(xz - yu)(d(x) \wedge d(u))}{(x^2 + y^2 + z^2 + u^2)^2} + \frac{(-yz - xu)(d(x) \wedge d(z))}{(x^2 + y^2 + z^2 + u^2)^2} \right. \\ & + \frac{(xu + yz)(d(y) \wedge d(u))}{(x^2 + y^2 + z^2 + u^2)^2} + \frac{(-x^2 - y^2)(d(z) \wedge d(u))}{(x^2 + y^2 + z^2 + u^2)^2} + \frac{(u^2 + z^2)(d(x) \wedge d(y))}{(x^2 + y^2 + z^2 + u^2)^2}, \\ & - \frac{(yx + uz)(d(y) \wedge d(z))}{(x^2 + y^2 + z^2 + u^2)^2} - \frac{(yx + uz)(d(x) \wedge d(u))}{(x^2 + y^2 + z^2 + u^2)^2} - \frac{(-y^2 - u^2)(d(x) \wedge d(z))}{(x^2 + y^2 + z^2 + u^2)^2} \\ & \left. - \frac{(-x^2 - z^2)(d(y) \wedge d(u))}{(x^2 + y^2 + z^2 + u^2)^2} - \frac{(-xu + yz)(d(z) \wedge d(u))}{(x^2 + y^2 + z^2 + u^2)^2} - \frac{(-xu + yz)(d(x) \wedge d(y))}{(x^2 + y^2 + z^2 + u^2)^2} \right] \\ & \left[\frac{(yu - xz)(d(y) \wedge d(z))}{(x^2 + y^2 + z^2 + u^2)^2} + \frac{(yu - xz)(d(x) \wedge d(u))}{(x^2 + y^2 + z^2 + u^2)^2} + \frac{(xu + yz)(d(x) \wedge d(z))}{(x^2 + y^2 + z^2 + u^2)^2} \right. \\ & + \frac{(-yz - xu)(d(y) \wedge d(u))}{(x^2 + y^2 + z^2 + u^2)^2} + \frac{(y^2 + x^2)(d(z) \wedge d(u))}{(x^2 + y^2 + z^2 + u^2)^2} + \frac{(-z^2 - u^2)(d(x) \wedge d(y))}{(x^2 + y^2 + z^2 + u^2)^2}, 0, \\ & \frac{(-u^2 - x^2)(d(y) \wedge d(z))}{(x^2 + y^2 + z^2 + u^2)^2} + \frac{(y^2 + z^2)(d(x) \wedge d(u))}{(x^2 + y^2 + z^2 + u^2)^2} + \frac{(yx - uz)(d(x) \wedge d(z))}{(x^2 + y^2 + z^2 + u^2)^2} + \frac{(-yx + uz)(d(y) \wedge d(u))}{(x^2 + y^2 + z^2 + u^2)^2} \\ & \left. + \frac{(-xz - yu)(d(z) \wedge d(u))}{(x^2 + y^2 + z^2 + u^2)^2} + \frac{(-xz - yu)(d(x) \wedge d(y))}{(x^2 + y^2 + z^2 + u^2)^2} \right] \\ & \left[-\frac{(-yx - uz)(d(y) \wedge d(z))}{(x^2 + y^2 + z^2 + u^2)^2} - \frac{(-yx - uz)(d(x) \wedge d(u))}{(x^2 + y^2 + z^2 + u^2)^2} - \frac{(y^2 + u^2)(d(x) \wedge d(z))}{(x^2 + y^2 + z^2 + u^2)^2} \right. \\ & \left. - \frac{(x^2 + z^2)(d(y) \wedge d(u))}{(x^2 + y^2 + z^2 + u^2)^2} - \frac{(xu - yz)(d(z) \wedge d(u))}{(x^2 + y^2 + z^2 + u^2)^2} - \frac{(xu - yz)(d(x) \wedge d(y))}{(x^2 + y^2 + z^2 + u^2)^2}, \frac{(x^2 + u^2)(d(y) \wedge d(z))}{(x^2 + y^2 + z^2 + u^2)^2} \right] \end{aligned}$$

$$\left[\begin{aligned} & + \frac{(-y^2 - z^2) (d(x) \&^{\wedge} d(u))}{(x^2 + y^2 + z^2 + u^2)^2} + \frac{(-y x + u z) (d(x) \&^{\wedge} d(z))}{(x^2 + y^2 + z^2 + u^2)^2} + \frac{(y x - u z) (d(y) \&^{\wedge} d(u))}{(x^2 + y^2 + z^2 + u^2)^2} \\ & + \frac{(x z + y u) (d(z) \&^{\wedge} d(u))}{(x^2 + y^2 + z^2 + u^2)^2} + \frac{(x z + y u) (d(x) \&^{\wedge} d(y))}{(x^2 + y^2 + z^2 + u^2)^2}, 0 \end{aligned} \right]$$

>

THE HOPF MAP in A4

In this example, the Hopf map will be used to fabricate the Frame field. The Hopf map is a map from $\{x,y,z,u\}$ to $\{X,Y,Z\}$ such that the square of the radius of the Euclidean Sphere in 3D is equal to the 4th power of the radius of the Euclidean Sphere in 4D. Hence, 3 of the "Vierbiens" are perfect differentials to within a factor.

The Hopf map has both a right and a left handed version, depending upon a choice of a variable herein called rot. rot = plus 1 or rot = minus 1. The fact leads to differences in the computable results.

```
> restart: with(liesymm):with(linalg):with(plots):with(diffforms):
> setup(x,y,z,u):
> deform(x=0,y=0,z=0,t=0,u=0,a=const,b=const,c=const,k=const,mu=const,m=const,n=const,p=const,rot=const);
Warning, new definition for close
Warning, new definition for norm
Warning, new definition for trace
Warning, new definition for `&^`
Warning, new definition for d
Warning, new definition for mixpar
Warning, new definition for wdegree
> #rot:=1;
> sigma1:=-d(rot*x*z+u*y);
                                σ1 := -rot z d(x) - rot x d(z) - y d(u) - u d(y)
> sigma2:=d(-rot*x*u+z*y);
                                σ2 := -rot u d(x) - rot x d(u) + y d(z) + z d(y)
> sigma3:=d((-x^2+y^2+rot*(u^2+z^2))/2);
                                σ3 := -x d(x) + y d(y) + rot u d(u) + rot z d(z)
> omega:=innerprod([-y],x,[-rot*u],rot*z],[d(x),d(y),d(z),d(u)]);
                                ω := -y d(x) + x d(y) - rot u d(z) + rot z d(u)
> FFINVV:=array([-rot*z,u,rot*x,y],[-rot*u,z,y,-rot*x],[-x,y,rot*z,rot*u],[-y,+x,-rot*u,+rot*z]);
DETINVV:=simplify(subs(rot^2=1,det(FFINVV)));
DETRH:=subs(rot=1,DETINVV);
DETLH:=subs(rot=-1,DETINVV);
                                FFINVV :=
                                [ -rot z  -u  -rot x  -y
                                  -rot u  z   y   -rot x
                                  -x   y   rot z   rot u
                                  -y   x   -rot u  rot z ]
DETINVV :=
-rot^3 z^4 - 2 rot^3 z^2 u^2 + z^2 y^2 - rot^3 z^2 x^2 - rot^3 u^4 + u^2 y^2 - rot^3 u^2 x^2 - x^2 u^2 - x^2 z^2 - x^4 - y^2 rot u^2 - y^2 rot z^2 + y^4
DETRH := -z^4 - 2 z^2 u^2 - 2 x^2 z^2 - u^4 - 2 x^2 u^2 - x^4 + y^4
DETLH := z^4 + 2 z^2 u^2 + 2 z^2 y^2 + u^4 + 2 u^2 y^2 - x^4 + y^4
> FF:=(inverse(FFINVV/(DETINVV)^(0/4))):DETF:=factor(det(FF));FFINV:=inverse(FF):
DETRH:=factor(subs(rot=1,DETF));DETLH:=factor(subs(rot=-1,DETF));
```

$$DETF := -\frac{1}{(u^2 \text{rot} + z^2 \text{rot} + x^2 - y^2)(\text{rot}^2 z^2 + \text{rot}^2 u^2 + \text{rot}^2 x^2 + y^2)}$$

$$DETRH := -\frac{1}{(u^2 + z^2 + x^2 - y^2)(z^2 + u^2 + x^2 + y^2)}$$

$$DETLH := \frac{1}{(u^2 + z^2 - x^2 + y^2)(z^2 + u^2 + x^2 + y^2)}$$

```
> Vierbein:=(innerprod(FFINV,[d(x),d(y),d(z),d(u)])):`sigma1`:=factor(Vierbein[1])
;`sigma2`:=factor(Vierbein[2]);`sigma3`:=factor(Vierbein[3]);`omega`:=factor(Vie
rbein[4]);
```

$$\sigma_1 := -\text{rot } z \, d(x) - \text{rot } x \, d(z) - y \, d(u) - u \, d(y)$$

$$\sigma_2 := -\text{rot } u \, d(x) - \text{rot } x \, d(u) + y \, d(z) + z \, d(y)$$

$$\sigma_3 := -x \, d(x) + y \, d(y) + \text{rot } u \, d(u) + \text{rot } z \, d(z)$$

$$\omega := -y \, d(x) + x \, d(y) - \text{rot } u \, d(z) + \text{rot } z \, d(u)$$

Indeed the Viebeins are to within a factor as stated initially.

```
> TTORSION1:=factor(wcollect((sigma1)&^d(sigma1)));TTORSION2:=factor(wcollect((sig
ma2)&^d(sigma2)));TTORSION3:=factor(wcollect((sigma3)&^d(sigma3)));TTORSION_omeg
a:=(wcollect(factor(omega&^d(omega)))):PARITY_omega:=(wcollect(factor(d(omega)&^
d(omega)))):TTORSION_omegaLH =
subs(rot=-1,TTORSION_omega);PARITY_omegaLH:=subs(rot=-1,PARITY_omega);PARITY_ome
gaRH:=subs(rot=1,PARITY_omega);qqqRH:=(factor(getcoeff(PARITY_omegaRH)^2))^(1/2)
;qqqLH:=(factor(getcoeff(PARITY_omegaLH)^2))^(1/2);RATIO:=(qqqRH/qqqLH)^2;MULT:=
getcoeff(TTORSION_omega&^d(u)):SQMULTRH=factor(subs(rot=1,MULT/u)^2);SQMULTLH=fa
ctor(subs(rot=-1,MULT/u)^2);
```

$$TTORSION1 := 0$$

$$TTORSION2 := 0$$

$$TTORSION3 := 0$$

$TTORSION_omegaLH =$

$$-2 \&^{(d(x), d(u), d(z))} y + 2 \&^{(d(y), d(u), d(z))} x - 2 u \&^{(d(z), d(y), d(x))} + 2 \&^{(d(u), d(y), d(x))} z$$

$$PARITY_omegaLH := -8 \&^{(d(u), d(z), d(y), d(x))}$$

$$PARITY_omegaRH := 8 \&^{(d(u), d(z), d(y), d(x))}$$

$$qqqRH := \sqrt{64}$$

$$qqqLH := \sqrt{64}$$

$$RATIO := 1$$

$$SQMULTRH = 4$$

$$SQMULTLH = 4$$

The first three vierbeins sigma are integrable.

The last vierbein is NOT integrable, and its Topological Parity depends upon the choice of a RH or LH map!!!!

THE LH AND RH RESULTS ARE NOT REAL OVER THE SAME DOMAINS!

```
> LUN:=array([[1,0,0,0],[0,1,0,0],[0,0,1,0],[0,0,0,-1]]);
```

$$LUN := \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

Next create the induced metric:

```
> GUN:=simplify((innerprod(transpose(FF),LUN,FF))):
> GUN11RH:=(factor(subs(rot=1,GUN[1,1]))^2)^(1/2);GUN11LH:=(factor(subs(rot=-1,GUN
[1,1]))^2)^(1/2);GUN12RH:=(factor(subs(rot=1,GUN[1,4]))^2)^(1/2);GUN12LH:=(facto
r(subs(rot=-1,GUN[1,4]))^2)^(1/2);DETGUN:=factor(subs(rot=-1,det(GUN))):
```

$$GUNI1RH := \sqrt{\frac{(6x^2u^2y^2 - 6x^2z^2y^2 + 3x^2u^4 + 3x^2y^4 - y^6 + 3x^4u^2 - 3x^4y^2 + z^6 + 6u^2z^2x^2 + 3u^2z^4 - 3y^2z^4 + y^2u^4 + 3u^4z^2 + 3z^2y^4 - u^2y^4 + 3x^2z^4 + 3x^4z^2 + x^6 + 6u^2y^2z^2 + u^6)^2}{((u^2 + z^2 + x^2 - y^2)^4 (z^2 + u^2 + x^2 + y^2)^4)}}$$

$$GUNI1LH := \sqrt{\frac{(2x^2u^2y^2 + 6x^2z^2y^2 - x^2u^4 + 3x^2y^4 - y^6 - x^4u^2 - 3x^4y^2 + z^6 - 6u^2z^2x^2 + 3u^2z^4 + y^2z^4 + y^2u^4 + 3u^4z^2 - 8x^3yz - z^2y^4 - u^2y^4 + 3x^2z^4 + 3x^4z^2 + x^6 - 8xz^3xu - 8xz^3uy + 8x^3uyz + 2u^2y^2z^2 + u^6)^2}{(z^2 + u^2 + x^2 + y^2)^4 (u^2 + z^2 - x^2 + y^2)^4}}$$

$$GUN12RH := \sqrt{4} \sqrt{\frac{y^2(-z^5 - y^4z + 2y^2z^3 + 2y^2x^2z - x^4z - 2z^3x^2 + zu^4 + 4yx^3u + 4uyz^2x)^2}{(u^2 + z^2 + x^2 - y^2)^4 (z^2 + u^2 + x^2 + y^2)^4}}$$

$$GUN12LH := \sqrt{4} \sqrt{\frac{(yz^5 + y^5z + 2y^3z^3 + 2yz^3u^2 - 2y^2xu^3 + 2y^3zu^2 + 2uxz^4 + 3yx^4z + yz^4u^4 + 2xu^3z^2 - 2x^3uz^2 + 2y^2x^3u - 2y^4xu - 4x^2u^2zy)^2}{((z^2 + u^2 + x^2 + y^2)^4 (u^2 + z^2 - x^2 + y^2)^4)}}$$

$$DETGUN := -\frac{1}{(z^2 + u^2 + x^2 + y^2)^2 (u^2 + z^2 - x^2 + y^2)^2}$$

The Hopf Frame is not an element of the Lorentz group.

Nor is it a member of the orthogonal group.

>

The induced UNIMODULAR metrics are not the same for the LH or the RH Hopf maps!!!
They also have different domains of reality.

In the Hopf map for the 3 sphere, the tangent fields are exact, and the normal field is not integrable.
This is different form the Instanton map above.

From the given Frame functions, use the standard methods to compute the Cartan Matrix of connection 1-forms.

```
> dFF:=simplify(array([[d(FF[1,1]),d(FF[1,2]),d(FF[1,3]),d(FF[1,4])],[d(FF[2,1]),d
(FF[2,2]),d(FF[2,3]),d(FF[2,4])],[d(FF[3,1]),d(FF[3,2]),d(FF[3,3]),d(FF[3,4])],[
d(FF[4,1]),d(FF[4,2]),d(FF[4,3]),d(FF[4,4])]])):
```

```
> cartan:=(evalm(FFINV*dFF)):
```

Evaluate each component of the connection coefficients on transverse space of E1,E2,E3.

The diagonal elements are zero on the constrained 3 sphere, but vary for the expanding sphere

```
> Gamma11:=wcollect(factor(wcollect(simplify(cartan[1,1])))):
```

```
> Gamma12:=factor(wcollect(cartan[1,2])):
```

```
> Gamma13:=factor(wcollect(cartan[1,3])):
```

```
> Gamma21:=factor(wcollect(cartan[2,1])):
```

```
> Gamma22:=factor(wcollect(cartan[2,2])):
```

```
> Gamma23:=factor(wcollect(cartan[2,3])):
```

```
> Gamma31:=factor(wcollect(cartan[3,1])):
```

```
> Gamma32:=factor(wcollect(cartan[3,2])):
```

```
> Gamma33:=factor(wcollect(cartan[3,3])):
```


> **gamma3:=wcollect(simplify(cartan[3,4]));**

$$\begin{aligned} \gamma_3 := & \left((-y \operatorname{rot} u^2 + y \operatorname{rot}^2 x^2 - y \operatorname{rot} z^2 + y^3) d(x) \right) / \left(\operatorname{rot}^3 z^4 + 2 \operatorname{rot}^3 z^2 u^2 - \operatorname{rot}^2 z^2 y^2 + \operatorname{rot}^3 z^2 x^2 + \operatorname{rot}^3 u^4 - \operatorname{rot}^2 u^2 y^2 \right. \\ & \left. + \operatorname{rot}^3 u^2 x^2 + \operatorname{rot}^2 x^2 u^2 + \operatorname{rot}^2 x^2 z^2 + \operatorname{rot}^2 x^4 + x^2 y^2 + y^2 \operatorname{rot} u^2 + y^2 \operatorname{rot} z^2 - y^2 \operatorname{rot}^2 x^2 - y^4 \right) + \left(\right. \\ & \left. (-\operatorname{rot}^3 z^2 x - \operatorname{rot}^3 u^2 x - \operatorname{rot}^2 x^3 - x y^2) d(y) \right) / \left(\operatorname{rot}^3 z^4 + 2 \operatorname{rot}^3 z^2 u^2 - \operatorname{rot}^2 z^2 y^2 + \operatorname{rot}^3 z^2 x^2 + \operatorname{rot}^3 u^4 - \operatorname{rot}^2 u^2 y^2 \right. \\ & \left. + \operatorname{rot}^3 u^2 x^2 + \operatorname{rot}^2 x^2 u^2 + \operatorname{rot}^2 x^2 z^2 + \operatorname{rot}^2 x^4 + x^2 y^2 + y^2 \operatorname{rot} u^2 + y^2 \operatorname{rot} z^2 - y^2 \operatorname{rot}^2 x^2 - y^4 \right) + \left(\right. \\ & \left. (\operatorname{rot}^3 z^2 u + x y \operatorname{rot}^3 z + \operatorname{rot}^3 u^3 - \operatorname{rot}^2 u y^2 + \operatorname{rot}^2 x^2 u + x y \operatorname{rot} z) d(z) \right) / \left(\operatorname{rot}^3 z^4 + 2 \operatorname{rot}^3 z^2 u^2 - \operatorname{rot}^2 z^2 y^2 + \operatorname{rot}^3 z^2 x^2 \right. \\ & \left. + \operatorname{rot}^3 u^4 - \operatorname{rot}^2 u^2 y^2 + \operatorname{rot}^3 u^2 x^2 + \operatorname{rot}^2 x^2 u^2 + \operatorname{rot}^2 x^2 z^2 + \operatorname{rot}^2 x^4 + x^2 y^2 + y^2 \operatorname{rot} u^2 + y^2 \operatorname{rot} z^2 - y^2 \operatorname{rot}^2 x^2 - y^4 \right) + \left(\right. \\ & \left. (-\operatorname{rot}^3 z u^2 + u \operatorname{rot}^3 x y + x y \operatorname{rot} u - \operatorname{rot}^3 z^3 + \operatorname{rot}^2 z y^2 - \operatorname{rot}^2 x^2 z) d(u) \right) / \left(\operatorname{rot}^3 z^4 + 2 \operatorname{rot}^3 z^2 u^2 - \operatorname{rot}^2 z^2 y^2 + \operatorname{rot}^3 z^2 x^2 \right. \\ & \left. + \operatorname{rot}^3 u^4 - \operatorname{rot}^2 u^2 y^2 + \operatorname{rot}^3 u^2 x^2 + \operatorname{rot}^2 x^2 u^2 + \operatorname{rot}^2 x^2 z^2 + \operatorname{rot}^2 x^4 + x^2 y^2 + y^2 \operatorname{rot} u^2 + y^2 \operatorname{rot} z^2 - y^2 \operatorname{rot}^2 x^2 - y^4 \right) \end{aligned}$$

> **Omega:=wcollect(factor(subs(simplify(wcollect(cartan[4,4]))));**

$$\begin{aligned} \Omega := & - \frac{(\operatorname{rot}^3 z^2 x + \operatorname{rot}^3 u^2 x + \operatorname{rot}^2 x^3 + x y^2) d(x)}{(u^2 \operatorname{rot} + z^2 \operatorname{rot} + x^2 - y^2) (\operatorname{rot}^2 z^2 + \operatorname{rot}^2 u^2 + \operatorname{rot}^2 x^2 + y^2)} \\ & - \frac{(y \operatorname{rot} u^2 + y \operatorname{rot} z^2 - y \operatorname{rot}^2 x^2 - y^3) d(y)}{(u^2 \operatorname{rot} + z^2 \operatorname{rot} + x^2 - y^2) (\operatorname{rot}^2 z^2 + \operatorname{rot}^2 u^2 + \operatorname{rot}^2 x^2 + y^2)} \\ & - \frac{(\operatorname{rot}^3 z u^2 - u \operatorname{rot}^3 x y - x y \operatorname{rot} u + \operatorname{rot}^3 z^3 - \operatorname{rot}^2 z y^2 + \operatorname{rot}^2 x^2 z) d(z)}{(u^2 \operatorname{rot} + z^2 \operatorname{rot} + x^2 - y^2) (\operatorname{rot}^2 z^2 + \operatorname{rot}^2 u^2 + \operatorname{rot}^2 x^2 + y^2)} \\ & - \frac{(\operatorname{rot}^3 z^2 u + x y \operatorname{rot}^3 z + \operatorname{rot}^3 u^3 - \operatorname{rot}^2 u y^2 + \operatorname{rot}^2 x^2 u + x y \operatorname{rot} z) d(u)}{(u^2 \operatorname{rot} + z^2 \operatorname{rot} + x^2 - y^2) (\operatorname{rot}^2 z^2 + \operatorname{rot}^2 u^2 + \operatorname{rot}^2 x^2 + y^2)} \end{aligned}$$

> **TTORSION_Omega:=wcollect(simplify((Omega)&^d(Omega))):TTORSIONRH:=subs(rot=1,TTORSION_Omega);TTORSIONLH:=subs(rot=-1,TTORSION_Omega);**

>

TTORSIONRH :=

$$\begin{aligned} & \frac{(4 x y^4 + 4 x z^2 u^2 + 2 x z^2 y^2 + 2 x u^2 y^2 + 2 x^3 u^2 + 4 x^3 y^2 + 2 x z^4 + 2 x u^4 + 2 x^3 z^2) \&^{\wedge}(d(y), d(u), d(z))}{(u^2 + z^2 + x^2 - y^2)^2 (z^2 + u^2 + x^2 + y^2)^2} + \left(\right. \\ & \left. (2 x^2 u^3 + 8 y x z^3 + 8 y x^3 z - 2 y^2 u^3 - 2 y^2 u z^2 + 2 x^2 u z^2 + 8 x z y u^2 + 2 x^4 u + 2 u y^4 - 4 u y^2 x^2) \right. \\ & \left. \&^{\wedge}(d(x), d(z), d(y)) \right) / \left((u^2 + z^2 + x^2 - y^2)^2 (z^2 + u^2 + x^2 + y^2)^2 \right) \\ & + \frac{(-2 y z^2 x^2 + 4 y z^2 u^2 - 2 y u^2 x^2 - 4 y x^4 + 2 y z^4 - 2 y^3 z^2 - 2 y^3 u^2 + 2 y u^4 - 4 y^3 x^2) \&^{\wedge}(d(x), d(u), d(z))}{(u^2 + z^2 + x^2 - y^2)^2 (z^2 + u^2 + x^2 + y^2)^2} + \left(\right. \\ & \left. (8 y x u^3 + 2 z y^2 u^2 - 2 x^2 z u^2 + 4 y^2 x^2 z - 2 x^4 z - 2 z^3 x^2 + 8 y x^3 u + 8 u y z^2 x - 2 y^4 z + 2 y^2 z^3) \right. \\ & \left. \&^{\wedge}(d(x), d(u), d(y)) \right) / \left((u^2 + z^2 + x^2 - y^2)^2 (z^2 + u^2 + x^2 + y^2)^2 \right) \end{aligned}$$

TTORSIONLH :=

$$\begin{aligned} & - \frac{(4 x y^4 - 4 x z^2 u^2 + 2 x z^2 y^2 + 2 x u^2 y^2 + 2 x^3 u^2 + 4 x^3 y^2 - 2 x z^4 - 2 x u^4 + 2 x^3 z^2) \&^{\wedge}(d(y), d(u), d(z))}{(-u^2 - z^2 + x^2 - y^2)^2 (z^2 + u^2 + x^2 + y^2)^2} - \left(\right. \\ & \left. (-2 x^2 u^3 + 8 y x z^3 + 2 y^2 u^3 + 2 y^2 u z^2 - 2 x^2 u z^2 + 8 x z y u^2 + 2 x^4 u + 8 x z y^3 + 2 u y^4 - 4 u y^2 x^2) \right. \\ & \left. \&^{\wedge}(d(x), d(z), d(y)) \right) / \left((-u^2 - z^2 + x^2 - y^2)^2 (z^2 + u^2 + x^2 + y^2)^2 \right) \\ & - \frac{(-2 y z^2 x^2 - 4 y z^2 u^2 - 2 y u^2 x^2 - 4 y x^4 - 2 y z^4 - 2 y^3 z^2 - 2 y^3 u^2 - 2 y u^4 - 4 y^3 x^2) \&^{\wedge}(d(x), d(u), d(z))}{(-u^2 - z^2 + x^2 - y^2)^2 (z^2 + u^2 + x^2 + y^2)^2} - \left(\right. \\ & \left. (8 y x u^3 - 2 z y^2 u^2 + 2 x^2 z u^2 + 4 y^2 x^2 z - 2 x^4 z + 2 z^3 x^2 + 8 u y z^2 x - 2 y^4 z - 2 y^2 z^3 + 8 y^3 x u) \right. \\ & \left. \&^{\wedge}(d(x), d(u), d(y)) \right) / \left((-u^2 - z^2 + x^2 - y^2)^2 (z^2 + u^2 + x^2 + y^2)^2 \right) \end{aligned}$$

> **TPARITY_Omega:=wcollect(simplify(d(Omega)&^d(Omega)));**

$$\text{TPARITY_Omega} := 0$$

[>

Note that the Big Omega term is NOT a perfect differential, and is zero if $n=1, p=2$. That is - off the sphere $r^2 = \text{constant}$ the Omega term does not vanish, unless $n=1, p=2$. Hence if the radius of the two sphere is expanding, then r^2 is not constant and one has a dilatation. (The source of dilatons?)

The Big Omega is not a perfect differential in the Hopf case, and its topological torsion is sensitive to the chiral factor rot . The topological torsion of Big Omega vanishes.

There are in general two sets of torsion two forms.

The Particle Affine torsion 2-forms, big Sigma, which depend upon the product of little omega and the connection components, little gamma.

The second set or Wave Affine torsion 2-forms, Big Phi, depends upon big Omega and the connection components, little gamma.

For the ORTHOGONAL FRAME FIELD the two sets of torsion 2-forms are the same to within a factor. The result because small omega and Big omega are proportional.

See <http://www.uh.edu/~rkiehn/pdf/defects2.pdf>

AFFINE TORSION 2-forms Big Sigma

> ``omega`:=omega;d(omega);`

$$\omega := -y \, d(x) + x \, d(y) - \text{rot} \, u \, d(z) + \text{rot} \, z \, d(u) \\ - 2 \, \text{rot} \, (d(u) \wedge d(z)) - 2 \, (d(y) \wedge d(x))$$

CARTAN TORSION AND CURVATURE

Next compute the induced Cartan Torsion 2-forms and the Cartan Curvature 2-forms on the 2D subspace

> `Sigma1:=wcollect(omega^gamma1);Sigma2:=wcollect(omega^gamma2);Sigma3:=wcollect(omega^gamma3);`

$\Sigma_1 := - \text{rot}$

$$\begin{aligned} & (-y^2 \text{rot} \, u^2 - y^2 \text{rot} \, z^2 + y^2 \text{rot}^2 \, x^2 + y^4 + \text{rot}^3 \, z^2 \, u^2 + u \, \text{rot}^3 \, z \, x \, y + \text{rot}^3 \, u^4 - \text{rot}^2 \, u^2 \, y^2 + \text{rot}^2 \, x^2 \, u^2 + u \, x \, y \, \text{rot} \, z) \\ & (d(x) \wedge d(z)) / (\text{rot}^3 \, z^4 + 2 \, \text{rot}^3 \, z^2 \, u^2 - \text{rot}^2 \, z^2 \, y^2 + \text{rot}^3 \, z^2 \, x^2 + \text{rot}^3 \, u^4 - \text{rot}^2 \, u^2 \, y^2 + \text{rot}^3 \, u^2 \, x^2 + \text{rot}^2 \, x^2 \, u^2 + \text{rot}^2 \, x^2 \, z^2 \\ & + \text{rot}^2 \, x^4 + x^2 \, y^2 + y^2 \, \text{rot} \, u^2 + y^2 \, \text{rot} \, z^2 - y^2 \, \text{rot}^2 \, x^2 - y^4) + \text{rot} \\ & (-y \, x \, u^2 \, \text{rot} - y \, x \, z^2 \, \text{rot} + y \, x^3 \, \text{rot}^2 + y^3 \, x + u \, \text{rot}^2 \, z^3 - z \, y^2 \, \text{rot} \, u + z \, \text{rot}^2 \, u^3 - y \, \text{rot}^2 \, x \, u^2 - x \, u^2 \, y + z \, \text{rot} \, x^2 \, u) \\ & (d(y) \wedge d(z)) / (\text{rot}^3 \, z^4 + 2 \, \text{rot}^3 \, z^2 \, u^2 - \text{rot}^2 \, z^2 \, y^2 + \text{rot}^3 \, z^2 \, x^2 + \text{rot}^3 \, u^4 - \text{rot}^2 \, u^2 \, y^2 + \text{rot}^3 \, u^2 \, x^2 + \text{rot}^2 \, x^2 \, u^2 + \text{rot}^2 \, x^2 \, z^2 \\ & + \text{rot}^2 \, x^4 + x^2 \, y^2 + y^2 \, \text{rot} \, u^2 + y^2 \, \text{rot} \, z^2 - y^2 \, \text{rot}^2 \, x^2 - y^4) + (\\ & (-y \, x \, \text{rot}^3 \, z^2 - y \, x \, \text{rot}^3 \, u^2 - y \, x^3 \, \text{rot}^2 - y^3 \, x + z^3 \, \text{rot}^4 \, u + z^2 \, \text{rot}^4 \, y \, x + z \, \text{rot}^4 \, u^3 - z \, \text{rot}^3 \, y^2 \, u + z \, \text{rot}^3 \, x^2 \, u + y \, \text{rot}^2 \, x \, z^2) \\ & (d(x) \wedge d(u))) / (\text{rot}^3 \, z^4 + 2 \, \text{rot}^3 \, z^2 \, u^2 - \text{rot}^2 \, z^2 \, y^2 + \text{rot}^3 \, z^2 \, x^2 + \text{rot}^3 \, u^4 - \text{rot}^2 \, u^2 \, y^2 + \text{rot}^3 \, u^2 \, x^2 + \text{rot}^2 \, x^2 \, u^2 + \text{rot}^2 \, x^2 \, z^2 \\ & + \text{rot}^2 \, x^4 + x^2 \, y^2 + y^2 \, \text{rot} \, u^2 + y^2 \, \text{rot} \, z^2 - y^2 \, \text{rot}^2 \, x^2 - y^4) - (\\ & (-\text{rot}^3 \, z^2 \, x^2 - \text{rot}^3 \, u^2 \, x^2 - \text{rot}^2 \, x^4 - x^2 \, y^2 + \text{rot}^3 \, z^4 - \text{rot}^2 \, z^2 \, y^2 + \text{rot}^3 \, z^2 \, u^2 - u \, \text{rot}^3 \, z \, x \, y - u \, x \, y \, \text{rot} \, z + \text{rot}^2 \, x^2 \, z^2) \\ & (d(y) \wedge d(u))) / (\text{rot}^3 \, z^4 + 2 \, \text{rot}^3 \, z^2 \, u^2 - \text{rot}^2 \, z^2 \, y^2 + \text{rot}^3 \, z^2 \, x^2 + \text{rot}^3 \, u^4 - \text{rot}^2 \, u^2 \, y^2 + \text{rot}^3 \, u^2 \, x^2 + \text{rot}^2 \, x^2 \, u^2 + \text{rot}^2 \, x^2 \, z^2 \\ & + \text{rot}^2 \, x^4 + x^2 \, y^2 + y^2 \, \text{rot} \, u^2 + y^2 \, \text{rot} \, z^2 - y^2 \, \text{rot}^2 \, x^2 - y^4) - \end{aligned}$$

[All of the 2-forms of the x,y,z subspace are dependent upon the chirality factor, rot.
The long printouts have been eliminated for the non-zero curvature 2-forms.

COMMENT:

[The result that Big Omega is not integrable is a feature of the Hopf map not to be found in the instanton
map. Hence, Wave Affine Torsion 2-forms are to be expected, and cannot be mapped away.

[>

[> **SHIPOVDELTA:=innerprod(FFINV, cartan, FF) :**

[The Shipov Torsion field is not ZERO, and depends on the chirality rot:

[The long formulas have not been printed

[>

[> **SHIPTORS:=evalm(SHIPOVDELTA-transpose(SHIPOVDELTA)) :**

[>

[>

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