The Photon Spin and other Topological Features of Classical Electromagnetism

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Abstract

Cartan's methods of exterior differential forms can be used to demonstrate that the laws of classical electromagnetism are topological constraints on a domain of independent variables and are independent from the geometric constraints of a metric or a connection. In Cartan's language of differential forms and exterior differential systems [1], the two topological constraints, $F - dA = 0$, and $J - dG = 0$ represent the PDE,s of Maxwell$'$s equations in a manner free from any choice of coordinates, metric or connection. The differential forms that make up the two postulates can be used to construct other topological statements that depend upon two independent 3-forms: the 3-form of topological torsion, $A\hat{}F$, and the 3-form of topological spin, $A\hat{}G$. The exterior derivative (divergence) of these two 3-forms creates the two familiar Poincare deformation invariants of an electromagnetic system, valid in the vacuum or plasma state. In domains where the Poincare invariants vanish, the closed integrals of $A\hat{}F$ and $A\hat{}G$ exhibit topological invariant or coherent properties similar to the "quantized" helicity and spin properties of a photon. The possible evolution of these and other topological properties can be studied with respect to equivalence classes of processes that can be defined in terms of singly parameterized vector fields. Non-zero values of the Poincare invariants are the source of topological change and non-equilibrium thermodynamics. Example solutions to the Maxwell exterior differential system exhibiting the properties of topological torsion and topological spin are given in terms of 1-forms of Pfaff dinension 4, deduced from coupled Hopf maps that

have spinor and minimal surface properties. A non-zero topological spin for the photon implies the existence of a longitudinal component to the electromagnetic field.

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1. The Topological Torsion and Topological Spin 3-forms

1.1. Topological Torsion and Topological Spin

The two fundamental postulates of an electromagnetic system, $F - dA = 0$, and $J - dG = 0$, require the existence of four fundamental exterior differential forms, $\{A, F, G, J\}$, which form a differential ideal on the variety, {x,y,z,t}. The postulates constrain the topology of the variety. For example, the first postulate implies that domain of support for the **E** and **B** fields cannot be compact without boundary (like the surface of a sphere). The elements of the ideal can be used to construct the complete Pfaff sequence of forms

$$Pfaff\ Sequence = \{A, F = dA, G, J = dG, A\hat{}F, A\hat{}G, A\hat{}J, F\hat{}F, G\hat{}G\}. \quad (1.1)$$

by the processes of exterior differentiation and exterior multiplication. A (Cartan) topology constructed on this system of forms has the useful feature that the exterior derivative may be interpreted as a limit point, or closure, operator in the sense of Kuratowski [2].

The complete Maxwell system of differential forms (which assumes the existence and physical importance of the potentials, A) also generates two other exterior differential systems.

$$d(A\hat{}G) - (F\hat{}G - A\hat{}J) = 0, \qquad d(A\hat{}F) - F\hat{}F = 0. \quad (1.2)$$

These equations introduce the (apparently novel to many researchers) 3-forms of Topological Spin Current density, $A\hat{}G$,[3] and Topological Torsion-Helicity, $A\hat{}F$ [4]. For an electromagnetic system, the Action 1-form, A, has the physical dimensions of the flux quantum, h/e. The 2-form, G, has the physical dimensions of charge, e. The 3-form, $A\hat{}G$, has the physical dimensions of angular momentum, h, and the 3-form $A\hat{}F$, has the physical dimensions of spin multiplied by the Hall impedance, $(h/e)^2 = h(h/e^2) = hZ_{hall}$. [5]

By direct evaluation of the exterior product on a domain of 4 independent variables, each 3-form ($A\hat{}G$, $A\hat{}F$, and the charge current J) will have 4 components that can be symbolized by the 4-vector arrays given in engineering format below. Each 3-form can be composed by the equivalent contraction process with the 4D volume element, $Vol4 = dx\hat{}dy\hat{}dz\hat{}dt$.

$$\begin{aligned} Topological\ Spin\ Current &: \quad A\hat{}G = i(\mathbf{S}_4)Vol4 && (1.3)\\ \mathbf{S}_4 &= [\mathbf{A}\times\mathbf{H}+\mathbf{D}\phi, \mathbf{A}\circ\mathbf{D}] \equiv [\mathbf{S},\sigma], && (1.4)\\ Topological\ Torsion\ vector &: \quad A\hat{}F = i(\mathbf{T}_4)Vol4 && (1.5)\\ \mathbf{T}_4 &= -[\mathbf{E}\times\mathbf{A}+\mathbf{B}\phi, \mathbf{A}\circ\mathbf{B}] \equiv [\mathbf{T},h], && (1.6) \end{aligned}$$

Charge-Current density: $J = dG = i(\mathbf{J}_4)Vol4 \qquad \mathbf{J}_4 = [\mathbf{J}, \rho], \qquad (1.7)$

Note that the ubiquitous helicity density, $\mathbf{A}\circ\mathbf{B}$, is merely the fourth component of $A\hat{}F$.

The vanishing of $A\hat{}G$ is an additional topological constraint on the domain that defines topologically transverse electric (TTE) waves: the vector potential, $\mathbf{A}$, is orthogonal to $\mathbf{D}$, in the sense that $\mathbf{A}\circ\mathbf{D} = 0$. The vanishing of $A\hat{}F$ is a additional topological constraint on the domain that defines topologically transverse magnetic (TTM) waves: the vector potential, $\mathbf{A}$, is orthogonal to $\mathbf{B}$,in the sense that $\mathbf{A}\circ\mathbf{B} = 0$. When both 3-forms vanish, the topological constraint on the domain defines topologically transverse (TTEM) waves. For classic real fields this double constraint would require that the vector potential, $\mathbf{A}$, is collinear with the field momentum, $\mathbf{D}\times\mathbf{B}$, and in the direction of the wave vector, $\mathbf{k}$. Such constraints permit the definition of singular solutions of propagating discontinuities, or electromagnetic "signals" [6], and in the general case lead to the result that signal propagation has 4 different phase speeds dependent upon both polarization and direction.

Note that if the 2-form F was not exact, such topological concepts of transversality would be without distinct meaning, for the 3-forms of Topological Spin and Topological Torsion depend explicitly upon the existence of the 1-form of Action. For future developments, also observe that the torsion vector $\mathbf{T}_4$ and the Spin vector $\mathbf{S}_4$ are associated vectors to the 1-form of Action, in the sense that

$$i(\mathbf{T}_4)A = 0 \quad and \quad i(\mathbf{S}_4)A = 0. \tag{1.8}$$

The two distinct concepts of Spin Current and the Torsion vector have had almost no utilization in applications of classical electromagnetic theory, for they are explicitly dependent upon the potentials, A. Examples, both novel and well-known, of vacuum and plasma solutions to the electromagnetic system which satisfy (and which do not satisfy) these topological constraints are given elsewhere [7].

1.2. The Poincare Invariants

The exterior derivatives of the 3-forms of Spin and Torsion produce two 4-forms, $F\hat{}G - A\hat{}J$ and $F\hat{}F$, whose closed integrals are deformation invariants for *any* continuous evolutionary process that can be defined in terms of a singly parameterized vector field. These topological objects are related to the conformal invariants of a Lorentz system as discovered by Poincare and Bateman [8]. Note that their topological properties are valid even in the plasma domain of dissipative charge currents and radiation, as well as in the vacuum. In the format of independent variables $\{x, y, z, t\}$, the exterior derivative corresponds to the 4-divergence of the 4-component Spin and Torsion vectors, $\mathbf{S}_4$ and $\mathbf{T}_4$.

$$\begin{aligned} Poincare\ 1 &= d(A\hat{}G) = F\hat{}G - A\hat{}J \qquad (1.9) \\ &= \{div_3(\mathbf{A}\times\mathbf{H}+\mathbf{D}\phi) + \partial(\mathbf{A}\circ\mathbf{D})/\partial t\}Vol4 \\ &= \{(\mathbf{B}\circ\mathbf{H}-\mathbf{D}\circ\mathbf{E}) - (\mathbf{A}\circ\mathbf{J}-\rho\phi)\}Vol4 \end{aligned}$$

$$\begin{aligned} Poincare\ 2\ \text{(or topological parity)} &= d(A\hat{}F) = F\hat{}F \qquad (1.10) \\ &= \{div_3(\mathbf{E}\times\mathbf{A}+\mathbf{B}\phi) + \partial(\mathbf{A}\circ\mathbf{B})/\partial t\}Vol4 \\ &= \{2\mathbf{E}\circ\mathbf{B}\}Vol4 \qquad (1.11) \end{aligned}$$

For the vacuum state, defined by $J = 0$, zero values of the Poincare invariants require that the magnetic energy density is equal to the electric energy density $(1/2\mathbf{B}\circ\mathbf{H} = 1/2\mathbf{D}\circ\mathbf{E})$, and, respectively, that the electric field is orthogonal to the magnetic field $(\mathbf{E}\circ\mathbf{B} = 0)$. Note that these constraints often are used as elementary textbook definitions of what is meant by electromagnetic waves. Consider the definitions:

Definition 1: Spin-chirality is defined as the closed integral of the 3-form $A\hat{}G$

$$Topo\log ical\ Spin\_chirality = \iiint_{closed} A\hat{}G \tag{1.12}$$

Definition 2: Torsion-helicity is defined as the closed integral of the 3-form $A\hat{}F$

$$Topo\log ical\ Torsion\_helicity = \iiint_{closed} A\hat{}F \tag{1.13}$$

By using Cartan's magic formula [9] it is possible to prove:

Theorem 1: If the First Poincare Invariant vanishes, topological Spin is an evolutionary deformation invariant with values whose ratios are rational.

Theorem 2. If the second Poincare Invariant vanishes, topological Torsion is an evolutionary deformation invariant with values whose ratios are rational.

The quantized (integer) ratios comes from the deRham cohomology theorems on closed integrals of closed p-forms [10]. All of the above development has been without the constraint of a metric and without the choice of a connection, in the spirit of Van Dantzig [11]. It is important to realize that these topological conservation laws are valid in a plasma as well as in the vacuum, subject to the conditions of zero values for the Poincare invariants. On the other hand, topological transitions between "quantized" states of Spin or Torsion require that the respective Poincare invariants are not zero. On space time domains where both 3-forms are closed, the ratio of the two integrals of topological torsion and topological spin have the physical dimensions of the fractional Hall impedance [12].

1.3. Why use differential forms?

The reasons for studying electromagnetism in the Cartan's language of exterior differential forms can be listed as follows

1. Differential Forms are independent of the choice of coordinate system (they are diffeomorphic invariants).

2. Constrained Differential Forms contain topological information as well as geometrical (tensor) information

3. The exterior derivative, d, generates Limit Points; exterior product, $\hat{}$, generates Intersections; both are topological properties.

4. $ddA = dF = 0$ generates the Maxwell-Faraday PDE's, $dG = J$ generates the Maxwell-Ampere PDE's, for any coordinate system.

5. The topological properties of dimension, orientability, intersections, connectedness (components, holes and handles) induced by the electromagnetic field have explicit realization in terms of exterior differential forms.

6. Closed integrals of closed exterior differential forms are evolutionary invariants.

7. When the 3-form of topological torsion $A\hat{}F \neq 0$ the electromagnetic system can exhibit enantiomorphic pairs.

1.4. The Pfaff dimension.

One of the basic objects in a topological study of physical systems is the 1 form of Action, which for electromagnetism is constructed from the vector and scalar potentials, $.A = A_k dx^k - \phi dt$. The limit sets (exterior derivatives) of the 1-form are the Field Intensities (**E** and **B**). The question can be asked: What is the minimum number of functions A_k, ϕ, x^k, t required to represent the exterior features of the 1-form of Action? The answer can be determined by computing the number of non-zero terms in the Pfaff sequence, $\{A, dA, A\hat{}dA, dA\hat{}dA, ...\}$.Most historical investigations are constrained to the sequence, $\{A, dA, 0, 0....\}$ so that the minimum Pfaff topological dimension required is two. That is, there exists a submersive map from the original space of 2k+2 variables to a space of dimension 2. Why is this the usual case studied? The answer is that when $A\hat{}dA =$

0 the system satisfies the Frobenius unique integrability theorem. This result enables predictive determinism. For a fluid $A\hat{}dA = 0$ defines a laminar flow, and $A\hat{}dA \neq 0$ must be a property of a turbulent flow, which is not laminar [13]. In another example, the work 1-form $W = i(V)dA$ for all Hamiltonian reversible evolutionary processes is of Pfaff dimension 1 [14]. Such is not the case for irrreversible processes where the work 1-form is of Pfaff dimension 4 [15]

In this article the non-zero 3-form of topological torsion, $A\hat{}F = A\hat{}dA \neq 0,$ is of special interest. When $A\hat{}dA \neq 0$ the Pfaff dimension of the Action, A, is therefore 3 or more, and global unique integrability of the Action is not possible. The implication is that there does not exist a single function on the domain whose gradient defines the covariant direction field of the form, or a unique subspace foliation that is global. Instead when $A\hat{}dA \neq 0$ there can be enantiomorphic pairs and Faraday Rotation. In addition, the 3-form of topological spin, $A\hat{}G$, which has the units of angular momentum, is important to electromagnetic systems that involve enantiomorphic pairs and Optical Activity. As mentioned above, the ratio of the closed integrals of the two 3-forms yields the Hall Impedance.

2. Examples in terms of the Hopf Map

2.1. Why use the Hopf Map?

1. The Hopf map is a map from 4 to 3 (real or complex) dimensions that has interesting and useful topological properties related to links and braids and other forms of entanglement.

2. The Hopf map is a projection which can be used to determine a global basis frame for the variety in terms of 3 exact 1-forms and 1 adjoint 1-form which is of Pfaff dimension 4. The Frame field so defined has non-zero affine torsion.

3. The Hopf adjoint field can be used to represent, within a factor, the 1-form of Action (potentials) for a certain class of electromagnetic fields that exhibit propagating non-zero topological torsion and non-zero topological spin.

4. The Hopf map yields two pairs of orthogonal 3 vectors, one which is left-handed and the other which is right handed. The 4 form of topological parity, $dA\hat{}dA$ constructed from the respective adjoint fields is either negative or positive.

5. The complex sum of two Hopf vectors generates a Cartan spinor.

2.2. The Hopf Map and its Adjoint field of Pfaff dimension 4.

Consider the map from R4(X,Y,Z,S) to R3(u,v,w) given by the formulas

$$\mathbf{H}1 = [u1, v1, w1] = [2(XZ + YS), 2(XS - YZ), (X^2 + Y^2) - (Z^2 + S^2)] \quad (2.1)$$

These formulas define the format of a Hopf map. The 3 component Hopf vector **H**1 is real and has the property that

$$\mathbf{H}1 \cdot \mathbf{H}1 = (u1)^2 + (v1)^2 + (w1)^2 = (X^2 + Y^2 + Z^2 + S^2)^2. \quad (2.2)$$

Hence a real (and imaginary) 4 dimensional sphere maps to a real 3 dimensional sphere. If the functions $[u1, v1, w1]$ are defined as $[x/ct, y/ct, z/ct]$, then the 4D sphere $(X^2 + Y^2 + Z^2 + S^2)^2 = 1$, implies that the Hopf map formulas are equivalent to the 4D light cone. The Hopf map can also be represented in terms of complex functions by a map from C2 to R3, as given by the formulas

$$\mathbf{H}1 = [u1, v1, w1] = [\alpha \cdot \beta^* + \beta \cdot \alpha^*, i(\alpha \cdot \beta^* - \beta \cdot \alpha^*), \alpha \cdot \alpha^* - \beta \cdot \beta^*. \quad (2.3)$$

The variables α and β also can be viewed as two distinct complex variables defining ordered pairs of the four variables $[X, Y, Z, S]$. For example, the classic format given above for **H**1 can be obtained from the expansion, $\alpha = X + iY$, $\beta = Z + iS$. Other selections for the ordered pairs of (X, Y, Z, S) (along with permutations of the 3 vector components) give distinctly different Hopf vectors. For example, the ordered pairs, $\alpha = X + iZ$, $\beta = Y + iS$, give

$$\mathbf{H}2 = [2(YX - SZ), X^2 + Z^2 - Y^2 - S^2, -2(ZY + SX)] \quad (2.4)$$

which is another Hopf vector, a map from R4 to R3, but with the property that **H**2 is orthogonal to **H**1 :

$$\mathbf{H}2 \cdot \mathbf{H}1 = 0. \quad (2.5)$$

Similarly, a third linearly independent orthogonal Hopf vector **H**3 can be found

$$\mathbf{H}3 = [X^2 + Y^2 - Z^2 - S^2, -2(YX + SZ), 2(-ZX + SY)] \quad (2.6)$$

such that

$$\mathbf{H}2 \cdot \mathbf{H}1 = \mathbf{H}3 \cdot \mathbf{H}2 = \mathbf{H}2 \cdot \mathbf{H}3 = 0. \tag{2.7}$$

$$\mathbf{H}1 \cdot \mathbf{H}1 = \mathbf{H}2 \cdot \mathbf{H}2 = \mathbf{H}3 \cdot \mathbf{H}3 = (X^2 + Y^2 + Z^2 + S^2)^2. \tag{2.8}$$

The three linearly independent Hopf vectors can be used as a basis of R3 excluding thos points where the quartic form vanishes. The mapping functions (u, v, w) of the Hopf vector can be differentiated with respect to (X, Y, Z, S) to produce a set of three exact 1- form whose coefficients form 3 independent 4 component vectors on R4. A 4th linearly independent vector can be created algebraically and forms the "adjoint" field for the given Hopf vector. This direction field can be used to construct a non-integrable 1-form, A, of Pfaff dimension 4. These three exact 1-forms and the non-integrable 1-form can be used as a basis frame for the space. The exterior derivatives of the basis frame produce the usual Cartan connection which is not affine-torsion free in its subspaces. By this mechanism the differential structure of R4 as induced by the Hopf map is determined.

For $\mathbf{H}1$, the 4 independent 1 forms are given by the expressions (where $\Lambda(X, Y, Z, S)$ is an arbitrary scaling function):

$$d(u1) = 2Zd(X) + 2Sd(Y) + 2Xd(Z) + 2Yd(S) \tag{2.9}$$

$$d(u2) = 2Sd(X) - 2Zd(Y) - 2Yd(Z) + 2Xd(S) \tag{2.10}$$

$$d(u3) = 2Xd(X) + 2Yd(Y) - 2Zd(Z) - Sd(S) \tag{2.11}$$

$$A = \{-Yd(X) + Xd(Y) - Sd(Z) + Zd(S)\}/\Lambda \tag{2.12}$$

A Frame Matrix can be generated by the coefficients of the 4 independent 1-forms, such that $Det\,F = (Z^2 + S^2 + Y^2 + X^2)^2/\Lambda$. It is some interest to examine the properties of the adjoint 1-form, A,defined hereafter as the Hopf 1-form. For $\Lambda = 1$, it follows that the Hopf 1-form is of Pfaff dimension 4. It is also of interest to consider factors Λ that are of the format of the Holder norm, where n and p are integers, and (a,b,k,m) are arbitrary constants.

$$\Lambda = (aX^p + bY^p + kZ^p + mS^p)^{n/p} \tag{2.13}$$

The exponents n and p determine the homogeneity of the resulting 1-form, which is given below an ambiguous format (the plus of minus sign)

$$A_{\pm} = A(\pm)/\Lambda = \{\pm(Yd(X) - Xd(Y)) - Sd(Z) + Zd(S)\}/\Lambda. \qquad (2.14)$$

For example, for $n = p = 2$, the scaling factor becomes related to the classic quadratic form. The scaled Hopf 1-form, A, is then homogeneous of degree zero. For arbitrary n and p, the 3-form of topological (Hopf) torsion becomes:

$$\text{Topological Torsion } = (A_{\pm})\hat{}d(A_{\pm}) = i(\mathbf{T}_4)d(X)\hat{}d(Y)\hat{}d(Z)\hat{}d(S), \qquad (2.15)$$

where the topological torsion 4 vector is equal to:.

$$\mathbf{T}_4 = \pm[X, Y, Z, S]/\Lambda. \qquad (2.16)$$

The Torsion vector, $\mathbf{T}_4$, for the Hopf map is proportional to the position vector from the four dimensional origin and represents an expansion or a contraction process. The factor Λ depends upon the integers n and p as well as the constants (a,b,k,m).

The Topological Parity 4-form, whose coefficient is the 4 divergence of the Torsion vector, $\mathbf{T}_4$, becomes

$$\text{Topological Parity} = d(A_{\pm})\hat{}(d(A_{\pm}) \qquad (2.17)$$
$$= -4(\pm\Lambda)^{(-2n/p)})(n-2)d(X)\hat{}d(Y)\hat{}d(Z)\hat{}d(S) \qquad (2.18)$$

It is most remarkable that for n=2, any p and any (a,b,k,m), the topological parity vanishes; the scaled Hopf 1-form is of Pfaff dimension 3, not 4. In such cases the ratios of the integrals of the topological torsion 3 form over various closed manifiolds are rational, and the closed integrals of the 3-form are topological deformation invariants. (coherent structures).

Also note that if the scaling factor is restricted to values such that $n = 4,\ p = 2,\ a = b = k = m = 1,$ then the Frame matrix is unimodular, and the scaled Hopf 1-form is homogeneous of degree -2, relative to the substitution $X \Rightarrow \gamma X$, *etc.* (A somewhat different definition of homogeneity relative to the volume element will be given below.) Emphasis is to be placed on those examples for which $n = 4,\ p = 2,\ a = b = k = 1,\ \ m = \pm 1.$

2.3. Spinors as linear combinations of Hopf Maps

The 3D isotropic (null) complex position vector, $[z1, z2, z3]$ can be decomposed into a real and an imaginary part, such that both parts have the same magnitude and are orthogonal. In short, the Cartan Spinor, [16] can be represented as the complex sum of two Hopf vectors. The spinors come in two triples of the form

$$|\boldsymbol{\sigma}_{12}\rangle = |\mathbf{H}1\rangle + i\ |\mathbf{H}2\rangle \qquad with \quad \langle\boldsymbol{\sigma}_{12}| \circ |\boldsymbol{\sigma}_{12}\rangle = 0 \tag{2.19}$$

$$|\boldsymbol{\sigma}_{23}\rangle = |\mathbf{H}2\rangle + i\ |\mathbf{H}3\rangle \qquad with \quad \langle\boldsymbol{\sigma}_{23}| \circ |\boldsymbol{\sigma}_{23}\rangle = 0 \tag{2.20}$$

$$|\boldsymbol{\sigma}_{31}\rangle = |\mathbf{H}3\rangle + i\ |\mathbf{H}1\rangle \qquad with \quad \langle\boldsymbol{\sigma}_{31}| \circ |\boldsymbol{\sigma}_{31}\rangle = 0 \tag{2.21}$$

These complex combinations of Hopf vectors can be used to generate solutions for which the topological torsion vanishes, and yet the topological spin is finite and quantized..

2.4. Electromagnetism of Index zero Hopf 1-forms

Guided by prior investigations, it is of interest to use the scaled Hopf 1-form as the generator of electromagnetic field intensities. The coefficients of the scaled Hopf 1-form can be put into correspondence with the classic vector and scalar potentials, $[\mathbf{A}, \phi]$ (using $S = CT$ where C is a constant). The Action for the first example is then of the format,

$$A_{\pm} = A_{0\pm}/\Lambda = \{\pm(+Yd(X) - Xd(Y)) - CTd(Z) + CZd(T)\}/\Lambda \tag{2.22}$$

When the number of minus signs in the quadratric form is zero (index 0), and the exponents are $n = 4, p = 2$, define

$$\lambda_0^2 = (X^2 + Y^2 + Z^2 + C^2T^2), \quad and \;\; \Lambda = \lambda_0^4 \tag{2.23}$$

The coefficients of the scaled Hopf 1-form can be put into correspondence with the classic vector and scalar potentials, $[\mathbf{A}, \phi]$, using S = CT where C is a constant. The Action for the first example is then of the format,

$$A_{\pm} = A_{0\pm}/\lambda_0^4 = \{\pm(+Yd(X) - Xd(Y)) - CTd(Z) + CZd(T)\}/\lambda_0^4 \tag{2.24}$$

For this choice it is remarkable that the derived 2-form has coefficients (**E** and **B**) that are proportional to the same Hopf Map. The adjoint 1-form generated from one Hopf map has a limit set which is another Hopf map. Using the minus ambiguity (parity) sign, leads to the classic result that $\mathbf{E}^2 = C^2\mathbf{B}^2$, but with the not-usual result that the **E** field is anti-parallel to the **B** field (If the positive ambiguity (parity) sign is used, the **E** and **B** fields are parallel.):

$$\begin{aligned}
F &= dA && (2.25)\\
\mathbf{B} &= curl\mathbf{A} && (2.26)\\
&= [2(CTY + XZ), -2(-YZ + CTX), (-X^2 - Y^2 + Z^2 + (CT)^2)](2/\lambda_0{}^6) &&\\
\mathbf{E} &= -grad\phi - \partial\mathbf{A}/\partial T && (2.27)\\
&= [-2(CTY + XZ), 2(-YZ + CTX), -(-X^2 - Y^2 + Z^2 + (CT)^2)](2C/\lambda_0{}^6) &&
\end{aligned}$$

It is natural to ask if these **E** and **B** fields admit a Lorentz symmetry constitutive constraint such that vacuum state is charge current free. Recall that a constitutive constraint is a relationship between a 2-form, F, and a 2-form density G, such that the coefficients of $G(\mathbf{D}, \mathbf{H})$ are related to the coefficients of $F(\mathbf{E}, \mathbf{B})$. A Lorenz vacuum condition implies that the fields are solutions of the vector wave equation. The question becomes, "If is presumed that $\mathbf{D} = \varepsilon\mathbf{E}$ and $\mathbf{B} = \mu\mathbf{H}$, do the Maxwell Ampere equations generate a zero 3 form of charge current? ". Direct computation of the index zero Hopf 1-form indicates that $dG = J \neq 0$, unless $\varepsilon\mu C^2 + 1 = 0$. Hence the scaled Hopf Action, where the scaling is of signature zero, does **not** describe a charge current free vacuum, for real positive values of ε, μ, and C. On the other hand, if it is presumed that the domain is such that say μ,or ε, is negative, then the Hopf Map, scaled as above, does generate charge-current free wave solutions. Negative ε appears to hold in metals and the Earth's ionosphere. Recent announcements indicate constructions that yield negative μ. [17]. However, for situations where ε or μ are negative, the Hopf wave solutions imply that the Spin angular momentum $A\hat{}G$ is not a deformation invariant (hence Spin angular momentum of the field is not conserved.)

2.5. Electromagnetism of Index one Hopf 1-forms

When the number of minus signs in the quadratic form is one (index 1), and the exponents are n = 4, p = 2, define

$$\lambda_1^2 = (x^2 + y^2 + z^2 - C^2t^2), \quad and \ \ \Lambda = \lambda_1^4 \tag{2.28}$$

(lower case letters will be used for Index one Hopf 1-forms). For his choice, it is remarkable that the derived 2-form has coefficients (**E** and **B**) that are proportional to different Hopf Maps. The Action 1-form is the same as above, but with a different denominator. This fact leads to the classic result that $\mathbf{E}^2 = C^2\mathbf{B}^2$, but now the **E** field is not collinear with the **B** field. Using the negative ambiguity (parity) sign leads to the fields:

$$
\begin{aligned}
F &= dA && (2.29)\\
\mathbf{B} &= curl\mathbf{A} && (2.30)\\
&= [2(Cty + xz), -2(-yz + Ctx), (-x^2 - y^2 + z^2 - (Ct)^2)](2/\lambda_1^6)\\
\mathbf{E} &= -grad\phi - \partial\mathbf{A}/\partial t && (2.31)\\
&= [2(Cty - xz), 2(-yz - Ctx), -(-x^2 - y^2 + z^2 + (Ct)^2)](2C/\lambda_1^6)
\end{aligned}
$$

The Spin current density for this first non-transverse wave example is evaluated as:

$$
\begin{aligned}
Spin \;:\; &\mathbf{S}_4 = && (2.32)\\
&[x(3\lambda_1^2 - 4y^2 - 4x^2), y(3\lambda_1^2 - 4y^2 - 4x^2),\\
&z(\lambda_1^2 - 4y^2 - 4x^2), t(\lambda_1^2 - 4y^2 - 4x^2)](2/\mu)/\lambda_1^{10},
\end{aligned}
$$

and has zero divergence. The Torsion current may be evaluated as

$$Torsion : \mathbf{T}_4 = -[x, y, z, t]2c/\lambda_1^8. \quad (2.33)$$

and has a non-zero divergence equal to the second Poincare invariant

$$Poincare\ 2 = -2\mathbf{E} \circ \mathbf{B} = +8c/\lambda_1^8. \quad (2.34)$$

The solution has magnetic helicity as $\mathbf{A} \circ \mathbf{B} \neq 0$ and is radiative in the sense that the Poynting vector, $\mathbf{E} \times \mathbf{H} \neq 0$.

Independent from any other constraints, it is possible to construct the 3-form of Topological Torsion, and its exterior derivative defined as Topological Parity. The Topological parity can be either positive, zero, or negative. For the example Hopf 1-form given above (using the negative ambiguity sign), the Topological Torsion is represented to within a factor by a position vector $[-x, -y, -z, -t]$ inbound in 4 dimensions, and having a negative divergence or parity. If the positive sign of the ambiguity factor is changed, then the parity of the form changes sign. For example, for the 1-form, $A = A_{(1+)}/\lambda = \{+yd(x) - xd(y) - Ctd(z) + zCd(t)\}/\lambda_1^4$, the 4 -form of topological parity is positive, and the topological torsion is represented by an outbound position vector (to within a factor).

Similar to the investigation described above for the index zero Hopf vectors, it is natural to ask if these **E** and **B** fields admit a Lorentz symmetry constitutive constraint such that vacuum state is charge current free. Again, such a condition implies that the fields are solutions of the vector wave equation. Direct computation of the Maxwell Ampere equations indicates that $dG = J = 0$ if the phase velocity constraint vanishes, $\varepsilon\mu C^2 - 1 = 0$. Hence the scaled Hopf Action, where the scaling is of index one, **does** describe a charge current free vacuum, for real positive values of ε, μ, and C.

It is some interest to give the charge current solutions to show how the "phase factor " $(\varepsilon\mu C^2 - 1) \Rightarrow 0$ establishes the vacuum charge free conditions. The example results for the components of the current density are: (note $\rho = 0$):

$$J^x = -(yx^2 + yz^2 + 5yC^2t^2 - 6zCtx + y^3)(\varepsilon\mu C^2 - 1)4/\lambda_1^6 \tag{2.35}$$

$$J^y = (x^3 + xy^2 + xz^2 + 5xC^2t^2 + 6zCty)(\varepsilon\mu C^2 - 1)4/\lambda_1^6 \tag{2.36}$$

$$J^z = -(2x^2 + 2y^2 - z^2 + C^2t^2)(\varepsilon\mu C^2 - 1)8Ct/\lambda_1^6 \tag{2.37}$$

It is conjectured that fluctuations of the "perfect" vacuum phase relations, where $\varepsilon\mu C^2 - 1 \neq 0$, are associated with ZPF. Note that there are possible charge-current free (singular wave solutions) that are governed by curves in space time. There curve are generated by the intersection of the three surfaces created by setting each of the coefficients of the current density equal to zero. These solutions are valid for any phase velocity and could be a source of "needle" radiation.

The solution given above is not free of Topological Torsion, $A\hat{}F$, and there is a non-zero value of the second Poincare invariant, $\mathbf{E} \cdot \mathbf{B} \neq 0$. However, the Spin 3-form $A\hat{}G$ is also non-zero, but it has, subject to the phase constraint, a zero 4-divergence. (The first Poincare invariant is zero.) The divergence of the Spin 3-form, has 2 parts. The first part is twice the conventional Lagrange density of the fields, $(\mathbf{B} \cdot \mathbf{H} - \mathbf{D} \cdot \mathbf{E})$. The second part is the interaction between the potentials and the charge currents, $(\mathbf{A} \cdot \mathbf{J} - \rho\phi)$. When the divergence of the 3-form is zero, then the closed integrals of Topological Spin are deformation invariants, and have closed integrals with rational (quantized) ratios. That is, with regard to any singly parametrized vector field, V, describing an evolutionary process,

$$L_{(\beta V)} \int_{z3} (A\hat{\ }G) = \int_{z3} i(\beta V)d(A\hat{\ }G) + \int_{z3} d(i(\beta V)A\hat{\ }G) \qquad (2.38)$$
$$= 0 + 0 \supset \textit{evolutionary invariance.}$$

The function β is an arbitrary deformation parameter.

2.6. Lack of time reversal symmetry

It should be noted that if the Action 1-form in the above example is subject to the time reversal operation in its coefficients ($t \Rightarrow -t$), the new Action 1-form does **NOT** describe a charge current free vacuum, for real positive values of ε, μ, and C and the Lorentz constitutive constraint.

2.7. Twistors composed by superposing two index 1 Hopf 1-forms

By superposing (adding or subtracting) two different, index 1, Hopf 1-forms (which will be shown below to be equivalent to a Penrose twistor solution) it is possible to construct a vacuum (charge current free wave) solution to the Maxwell system, subject to the constraint that the phase speed satisfies the phase velocity equation, $(\varepsilon\mu C^2 - 1) = 0$.

As an example consider another Hopf 1-form of index 1 formulated as

$$A = \{Ctd(x) + zd(y) - yd(z) - xCd(t)\}/\lambda_1^4 \qquad (2.39)$$

Similar formulas for the field intensities can be determined as above. Note that the parity of the Hopf forms to be superposed can be the same or different. If the parity of the two superposed Hopf 1-forms are opposite, then without consideration of the phase constraint, the Topological Torsion of the "twistor" 1-form vanishes, $A\hat{\ }F = 0$. Yet the quantized topological spin3-form $A\hat{\ }G$ does not vanish, and moreover, subject to the phase constraint, the closed integrals of the Spin 3 form are conserved. This result implies that such a construction yields "quantized" values for the Spin integrals. These formulations can becompared with the Penrose twistor definitions in terms of differential forms [18]

2.8. The Classical Photon

When the spinor solution of two Hopf vectors of opposite parity are combined, the resulting wave solution is transverse magnetic (in the topological sense that $\mathbf{A} \circ \mathbf{B} = 0$). Not only does the second Poincare invariant vanish under the superposition, but so also does the Torsion 4 vector. Conversely, there can exist transverse magnetic waves which can be decomposed into two non-transverse waves. A notable feature of the superposed solutions is that the Spin 4 vector current does not vanish, hence the example superposition is a wave that is not transverse electric (in the topological sense that $\mathbf{A} \circ \mathbf{D} \neq 0$). For the superposed example, the first Poincare invariant vanishes, which implies that the Spin integral remains a conserved topological quantity, with values proportional to the integers. The non-zero Spin current density for the combined examples is given by the formula:

$$\begin{aligned} Spin \; &: \; \mathbf{S}_4 = [-2x(y+ct)^2, (y+ct)(x^2-y^2+z^2-2cty-c^2t^2), -2z(y+ct)^2, \\ &-(y+ct)(x^2+y^2+z^2+2cty+c^2t^2)](4/\mu)/\lambda_1^{10}, \end{aligned} \tag{2.40}$$

while the Torsion current is a zero vector, $A\hat{}F \Rightarrow 0$.

In addition, for the superposed example, the spatial components of the Poynting vector are equal to the Spin current density vector multiplied by γ, such that

$$\mathbf{E} \times \mathbf{H} = \gamma \; \mathbf{S}, \quad with \; \gamma = -(x^2+y^2+z^2+2cty+c^2t^2)/2c(y+ct)\lambda_1^2. \tag{2.41}$$

These results seem to give classical credence to the Planck assumption that the vacuum state of Maxwell's electrodynamics supports quantized angular momentum, and that the energy flux must come in multiples of the spin quanta. In other words, these combined solutions to classical electrodynamics have some of the experimental qualities of the quantized photon.

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