Non-Equilibrium Systems and Irreversible Processes

Adventures in Applied Topology

Vol. 6

The universal effectiveness of Topological Thermodynamics

from a Perspective of Continuous Topological Evolution.

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0.1 Preface to Vol6a chapter 1

It is my writing style to include bits of history and motivation that led to my development of a topological perspective of applied physics. The ideas of Continuous Topological Evolution and its application to Thermodynamics started about 1964. The ideas were assembled into a set of four monographs after I retired in 1999 as Emeritus Professor of Physics from the University of Houston. Pieces of these monographs were published in various places (see http://www.cartan.pair.com), and a paperback collection of the 4 volumes appeared in 2004 [1]. Updated copies in pdf file format can be downloaded on request (toptorsion@aol.com). These monographs are now in the fourth and fifth editions (as of 2013), with some additions, and with most of the typos corrected. The monographs are not meant to be text books on topology. The idea is to use some of the elementary concepts of finite topology to formulate methods of understanding non-equilibrium thermodynamic systems and irreversible processes.

Chapter 1

THERMODYNAMICS FROM A TOPOLOGICAL PERSPECTIVE

While the writing of these monographs started before 1997, it was not until 2003 that I appreciated the fact that thermodynamic fluctuations and irreversible processes in non-equilibrium systems must be related to macroscopic isotropic complex Spinor direction fields (not real vector direction fields). The first four editions were modified to include these facts. These Cartan Spinor direction fields are those complex direction fields that have a zero quadratic form, such that the product norm vanishes, yet the factors of the product are not zero. They are to be associated with the eigendirection fields of real anti-symmetric matrices, or exterior differential 2-forms (see Chapters 2.3 and 4.1-4.2 of Vol.1 [1]). The Cartan Spinors describe realizations of conjugate minimal surfaces, which are solutions of complex partial differential wavelet diffusion equations. The wavelets are elements of a topology where some, if not all, of the set closures are indistinguishable.
The bottom line is that without Cartan Isotropic Spinors (without antisymmetric structures) there would be no chaotic systems, no irreversible thermodynamic dissipation, and no turbulence.

Classic analysis has focused attention on symmetric matrices (metric, stress, strain...) for which Spinors, as complex isotropic vectors (of zero quadratic form, or length), do not exist. Macroscopic Spinor direction fields are studied in much more detail in the fifth (2007) monograph (see Vol5 [1]). One of the surprising results was the proof by example that C2 smooth processes could be irreversible ($Q\hat{}dQ \neq 0$), yet their C1 approximations were not irreversible ($Q\hat{}dQ = 0$). From a "particle" point of view (where singlet closures are distinguishable), an arrow of time is defined by those processes that describe topological change where the number of distinguishable parts decreases, such as in a pasting process. In contrast, topological change that describes an increase in the number of distinguishable parts (by a cutting process) is not topologically continuous. For topologies that have indistinguishable closures, the concepts of an arrow of time are more complicated, and are yet to be fully understood.

It should be noted that not until early 2006 did I appreciated, fully, that the projective geometry concepts of correlations can be deduced from the Jacobian matrix of the coefficients of a 1-form intensity, A_k. The determinant of the correlation matrix can be interpreted as a topological equation of state equal to a polynomial in molar density, ρ, and with coefficients equal to the similarity invariants of the Jacobian correlation matrix.

On the other hand, projective concepts of collineations are related to the Jacobian matrix of the coefficients of a direction field, V^k, that encodes a thermodynamic process, and which can have a representation as a (N-1)-form density, or current. When the collineation is constrained to be a polarity, the collineation matrix is symmetric. The current density concepts of classical electromagnetic theory do not appear in classical hydrodynamics, where stress-strain relations have been constrained to admit only those expositions of stress-energy in terms of symmetric matrices (polarities).

When anti-symmetries inherent in exterior differential forms are allowed, then fluids also permit excitation 2-forms and current density 3-forms. The 3-form densities can be constructed in terms of the Adjoint of correlation matrix constructed in terms of the coefficients of the 1-form, A, which defines the thermodynamic system. With this anti-symmetric extension, the theories of electromagnetism and hydrodynamics become topologically equivalent. In addition, the concept of Topological Torsion (an intensity 3-form) and Topological Spin (an excitation 3-form density) can be formatted. These 3-form currents have found practical application in transport of collective spin states.

An important point, recognized before 1976 [207], was that the geometrical equivalences generated by diffeomorphisms, representing maps, φ, from one tensor domain of "coordinates" $\{\xi^k\}$ to another $\{x^k\}$, did not capture all of the features of exterior differential forms. For a tensor equivalence, both the map and its differential, defined as a Diffeomorphism, must exist:

$$: \quad \text{Diffeomorphism}, \tag{1.1}$$
$$\varphi, \varphi^{-1} \quad : \quad \{\xi^k\} \Leftrightarrow \{x^k\}, \tag{1.2}$$
$$d\varphi, d\varphi^{-1} \quad : \quad \{d\xi^k\} \Leftrightarrow \{dx^k\}. \tag{1.3}$$

The idea is that a tensor remains a tensor under diffeomorphic transformations of the "coordinate variables". In simplistic terms, geometric properties are preserved by diffeomorphisms. The concept of coordinate transformations and their invariants can be extended to include Lagrangian varieties that consider maps between the sets of coordinates, and their derivatives.

The class of invariants which depend only upon the map and its inverse (but do not necessarily depend upon the map of the differentials) are defined as homeomorphisms. In simplistic terms, topological properties are preserved by homeomorphisms. The map has an inverse, but the differential map does not have an inverse.

$$: \quad \text{Homeomorphism} \tag{1.4}$$
$$\varphi, \varphi^{-1} \quad : \quad \{\xi^k\} \Leftrightarrow \{x^k\}, \tag{1.5}$$

It was recognized that a better word description for the domain of exterior differential forms was a "differential variety", defined as a collection of coordinate functions, $\{x^k\}$, *and their differentials,* $\{dx^k\}$ (rather than the tensor domain of "coordinates", $\{x^k\}$, or the Lagrangian domain of coordinates ,$\{x^k\}$, and derivatives, $\{dx^k/dt\}$). The exterior differential forms are invariant in a retrodictive sense with respect to maps that do not have inverses, and differentials of those maps, which also do not have inverses. Such maps are not homeomorphisms, hence they can include topological change (evolution).

$$: \quad \text{Differential Variety} \tag{1.6}$$

$$\varphi \quad : \quad \{\xi^k\} \to \{x^k\}, \tag{1.7}$$

$$d\varphi \quad : \quad \{d\xi^k\} \to \{dx^k\}. \tag{1.8}$$

The concept of a differential variety fits nicely into the domain of Finsler spaces, and those structures which are homogeneous of degree zero, or of degree 1, and are used in classical thermodynamics. The processes generated by the Lie differential (Cartan's magic homotopy formula) are much more general than the processes generated by the various forms of derivatives. Thermodynamically, the derivatives are adiabatic, while the differentials need not be adiabatic and can encode topological fluctuations. More to the point, geometric constraints on the underlying topology force physical processes to be time reversal symmetric, and therefore diffeomorphic constraints are useless in describing irreversible processes in a non-phenomenological manner. In addition, a concept of topological continuity separates processes describing continuous evolution of disconnected topological domains to connected topological domains (the decay to equilibrium) from the discontinuous evolution of connected topological domains to disconnected topological domains (the emergence of ordered metastable sets far from equilibrium).

The few discussions in the literature that have employed a topological point of view for thermodynamics were created by physicists and engineers (see [26], [278], [320]), not Topologists. I must state that it was a serendipity event (1987) that led me to the idea that *any* exterior differential 1-form could be used to derive a topological structure of exterior differential forms. I called the resulting structure a Cartan-Kuratowski topology. After some 20 years of research development using a non-commuting set of ingredients (exterior differential forms), I decided to attend the meeting on topological applications held at Haceteppe (Ankara), Turkey in 2009 I hoped that some professional topologists (who use topologies based upon a commuting set of ingredients) could teach me more about thermodynamics and topology. It was at this meeting that I realized that the Cartan-Kuratowski topology of exterior differential forms was precisely equivalent to the Kolmogorov T0 (poset 3) topology of 4 (commuting) ingredients. The T0 separation axioms permitted "particle property" subsets to be distinguished in terms of their topological properties. My early intuition of a topological basis for non-equilibrium thermodynamics was justified. The name of the topology based on exterior differentials is now described as the Kolmogorov-Cartan T0, or the KCT0 topology.

At the Ankara meeting it became apparent that the major interest of the professional topologist is directed towards those topologies which are not finite. For a physicist, however, all measurement processes are finite. Hence the focus in this monograph is restricted to finite topological structures. For purposes of simplicity, the topologies of interest will be constructed in terms of exterior differential forms and differential form densities, constrained to spaces of (Pfaff) topological dimension ≤ 4.

The idea is that classic geometric (Lagrangian) methods can be used to distinguish n particles in terms of 2n+2 geometric dimensions and their differentials, but this system of differentiable geometric functions can be reduced to N $\leq$ 4 topological dimensions of particle-like properties (think mole number, charge, torsion, spin) that are distinguishable by topological (not metrical) methods

There are only two species of finite topological systems:

1. T0 topologies that have Kolmogorov separation axioms. Hence singlet subsets are distinguishable in a particle-like sense.

2. Not T0 topologies that do not have separation axioms. Hence some, if not all, of the subsets are indistinguishable in an ensemble, statistical-like sense.

It is remarkable that the particle-like topologies of distinguishable subsets have an extremal element, the Hausdorff $T2 \subset T0$ topology. All subsets of the Hausdorff topology are both open and closed. The subsets are discrete and disconnected. The Hausdorff T2 topology is the only finite topology to admit a real, metric based, geometric distance function with a Cartesion signature. The other finite T0 topologies are not metrizable.

The statistical-like Not-T0 topologies also have an extremal element that consists of the Top Set, X and the empty set, { }. All subsets are connected and topologically indistinguishable. At this early stage (2011) of development, it is conjectured that such topologies are applicable to statistical thermodynamic systems, or systems of exterior differential form densities, but should not be used to describe thermodynamic systems with distinguishable particle properties.

Conjecture 1 *It is remarkable that the Not-T0 topologies can admit a pseudo-metric with a Minkowski Signature. Hence, the importance of the Minkowski metric is a topological concept associated with wave phenomena and indistinguishable ingredients. In principle, it would seem inappropriate to use the Minkowski metric to describe particles,*

The Hacettepe (2009) meeting led me to construct numerous physical applications that are based upon the disconnected Kolmogorov-Cartan T0 (poset 3) and its application to non-equilibrium systems and processes. It is remarkable to me that out of the sixteen Kolmogorov T0 topologies that can be constructed on a set of 4 points, $X = \{a, b, c, d\}$, ten are connected topologies, five are disconnected topologies with isolated singletons, but only one is a disconnected topology without isolated singletons. That unique topology is the Kolmogorov-Cartan T0 (poset 3) topology – the Cartan-Kuratowski topology of exterior differential forms, discovered in a serendipity moment in 1987! One of the two disconnected components is applicable to equilibrium particle systems, and the other disconnected component is applicable to non-equilibrium particle systems.

Even more interesting to me, now (2011), is that there are 33 topologies that can be constructed from 4 ingredients, of which 16 have T0 separation axioms, but for which 17 are $Not-T0$ topologies, without separation axioms. In the first edition of this Vol 6, not much more than an introduction to the concept of Non-T0 topologies and indistinguishability appeared in the newly added Chapter 7. The KCT0 topology (which is one of 16 possible $T0$ topologies on 4 "points" or ingredients), with subset elements defined in terms of exterior differential forms, is used as the basis of a universal perspective of a Topological Thermodynamic system based upon distinguishable particle-like systems of set closures.

In this third edition (2013) Chapter 7 has been rewritten, and a new Chapter 8 has been added to consider the concept of a Category Theory of Topological Thermodynamics. More emphasis is placed on Not-T0 topologies, indistinguishable set closures, and the evolution of exterior form densities. The use of Cartan's concept of homotopic evolution applied to exterior differential form densities leads to a new fundamental equation related to complex diffusion-wave equations, on connected and disconnected topological spaces of indistinguishable sets. As a final addition to Vol 6 is the output of a Maple program that computes the lattice structure topologies for a finite number of ingredients. This program demonstrates that the boundary of a boundary is zero is a valid statement for Hausdorff T2 (metrizable) topologies, but is not valid for Kolmogorov T0 (poset 3) topologies.

The mantra of my research interests since 1967 has been to exclude geometric ideas from a "universal" approach to the physics of fields and particles. Most geometric ideas evolve from the concept of a distance, a size, and other features associated with a metric, such as groups of diffeomorphisms (a required constraint of tensor analysis). I found it unacceptable to believe that structural constraints (such as group symmetries and gauge constraints) that have inverses could explain irreversible processes and systems. The 1934 paper by van Dantzig [292] was an early source of stimulation, and led to my realization that exterior differential forms of E. Cartan could be used as subsets of a universal topological theory of electromagnetism without metric. These (geometric-free) results then were extended to non-equilibrium thermodynamics and hydrodynamics.

As geometric constraints lead to time reversibility, it is necessary to use concepts that are not geometrically constrained - if one wants to understand irreversible concepts such as biological aging. All that is necessary is to insist that the topological foundations of interest must not be limited by the Hausdorff T2 separation axioms (topologies which are metrizable). A universal theory of non-equilibrium thermodynamics of distinguishable particles must be based upon T0 topologies, which are not dependent on geometrical properties and constraints.

I do not deny that it is possible to use a metric, do computations, and obtain useful and interesting results. The over-used Hodge duality theorem is a case in point. However, few researchers remember that the results of interest to Hodge were then shown, by Hodge, to be independent from a choice of a metric used to define the duality! Without this last step, the Hodge duality, so often used in physical theories such as electromagnetism, need not be independent from geometrical ideas. Note, abstractly, that most gauge theories, symmetry concepts, or manifold assumptions are all based upon the topological foundation that satisfies the Hausdorff T2 axioms, and therefore carry a lot of geometrical (interesting, but not universal) baggage that destroys the possibility of universal behavior.

1.1 The Hour of Mystery

Now I am well aware of the fact that thermodynamics (much less, Topological Thermodynamics) is a topic often treated with apprehension. In addition, I must confess, that as undergraduates at MIT (1949) we used to call the required physics course in Thermodynamics, "The Hour of Mystery!"

Let me present a few quotations (taken from Uffink, [290]) that describe the apprehensive views of several very famous scientists:

> *Any mathematician knows it is impossible to understand an elementary course in thermodynamics* .... V. Arnold 1990.
>
> *It is always emphasized that thermodynamics is concerned with reversible processes and equilibrium states, and that it can have nothing to do with irreversible processes or systems out of equilibrium* ......Bridgman 1941
>
> *No one knows what entropy really is, so in a debate (if you use the term entropy) you will always have an advantage* ...... Von Neumann (1971)

On the other hand Uffink states:

> *Einstein, ..., remained convinced throughout his life that thermodynamics is the only universal physical theory that will never be overthrown.*

1.1.1 *Particles or Fields, Extensive or Intensive*

The original classical development of thermodynamics was phenomenological, but it became motivated - and then dominated -by the concept of microscopic "molecules" after the start of the 20th century. However, as Sommerfeld has written (without explicit reference to topology, but inferring that "microscopic molecules" are not of thermodynamic importance):

> *"The atomistic, microscopic point of view is alien to thermodynamics. Consequently, as suggested by Ostwald, it is better to use moles rather than molecules."* Arnold Sommerfeld p. 11 [262].

The classical theory of thermodynamics is often presented as a number of phenomenological "Laws" to be written in stone and taken on faith. Indeed, contrived experiments are conducted to demonstrate a measure of credence in the "Laws", but the universality of the "Laws" is always left a bit clouded and mirky. Part of the thermodynamic mystery is due to the fact that the thermodynamic variables are of two types:

1. Additive, extensive quantities, or "Excitations"such as Entropy, Internal Energy....

2. "Intensities" such as Pressure, Temperature .....

Remark 2 *Following Arnold Sommerfeld, think of the first category in terms of "sources (particles) of Excitation" and the second category in terms of fields of "Intensities". The first category is related to "Sources" leading to additive, extensive (discrete particle) properties which are homogeneous of degree 1. The second category is related to potentials and "fields of force or intensities" leading to properties which are not additive, and are homogeneous of degree 0.*

For systems constrained by diffeomorphic (tensor analysis) equivalences, the intensities behave like the components of a covariant vector, while the processes behave like the components of a contravariant vector. However, in terms of a topological perspective, such is not the case. The Jacobian matrix of the Intensities form a correlation matrix in the sense of projective geometry. The Jacobian matrix of the processes form a collineation matrix in the sense of projective geometry. The correlation matrix need not be a "polarity", which would require that the matrix is symmetric. In fact, it is the anti-symmetric parts of the correlation matrix that are of most importance to the topological theory of thermodynamics.

A major part of the "Mystery" of classical thermodynamics can be related to the fact that:

> ..."there are thermodynamic variables which are uniquely specified by the equilibrium state (independent from the past history of the system) and which are not conclusions deduced logically from some philosophical first principles. They are conclusions ineluctably drawn from more that two centuries of experiments" ... P. M. Morse p.8 in [175].

In addition, the "thermodynamic coordinates" are not well defined as functions on the usual 4D space time differential variety of measurement. In fact, most of classical thermodynamic theories define equilibrium when the Intensities are constants with zero differentials across a finite domain.

This monograph rectifies this geometric problem, by formulating the first principles in terms of topological ideas.

1.1.2 The "Laws"

Permit me to inscribe the Stone tablet with the following tongue-in-cheek "Laws" of thermodynamics:

1. Thou shall not destroy Energy.
2. Entropy must always flow uphill from lower entropy to higher entropy.
3. Heat must always flow downhill from high-temperature to low-temperature.
4. Thou shall not destroy Entropy.
5. Thou must not admit Negative Pressure.
6. Thou must not admit Negative Temperature.
7. The Laws of Nature must predict unique final data from unique initial data.
8. The Laws of Nature must be based upon geometry and symmetries.
9. The Laws of Nature must be explained in terms of microscopic quantum mechanics.
10. The Laws of Nature must have a probabilistic, statistical, foundation and interpretation.

As my wife says, Nature does not obey these laws.

1.1.3 A Topological Perspective

My objective has been to present a universal theory of Thermodynamics based upon Continuous Topological Evolution and Topological sets of Exterior Differential forms on low dimensional topologies. As a matter of faith, it will be presumed that measurement processes are defined by functions that must be evaluated, ultimately, in terms of a 4 dimensional space-time differential variety,

$$\{x, y, z, t\ ; dx, dy, dz, dt\} = \{x^k, dx^k\}. \tag{1.9}$$

This (pre-geometric) differential variety will not be constrained by metric or scales.

From this universal foundation, the meaning of the "Laws" are to be rationally deduced. Many of the mysteries of thermodynamics, especially those currently found in non-equilibrium thermodynamics, can be removed and be made transparent. Indeed, new practical applications can be devised as the result of a logical dynamical description of emergence. The evolutionary emergence of topological defects and other singularities is produced by processes acting on thermodynamic systems in the universal non-equilibrium topological environment. The topological environment is neither a vacuum, nor an empty set, but should be considered as a non-equilibrium thermodynamic system of topological dimension 4.

1.1.4 The Arrow of Time

Topology is the study of the number of disconnected parts, the number of connected parts, and the number of obstructions, topological defects, or "holes" in the connected parts. Topological evolution focuses on processes where these numbers change. Geometric evolution focuses on processes when these numbers stay the same. Processes of topological change can be identified with *topologically continuous* processes of "pasting" together disconnected parts, or with the deformation and pasting together certain parts of a boundary of a connected part to form a hole, or with the collapse and pasting together of a cyclic portion of boundary to destroy a hole All such continuous processes can create topological defects or "holes". Disconnected parts can continuously evolve into connected parts thereby causing topological changes (condensation of a vapor), but connected parts cannot evolve continuously into disconnected parts (evaporation of a liquid). Both processes involve topological change; but they can not be geometrical processes for which the topology is a continuous evolutionary invariant. Topological change is a necessary, but not sufficient, property of an irreversible process.

Paraphrasing Eddington [71]:

Remark 3 *Aging and the arrow of time have slipped through the net of geometric analysis.*

Yet it is the interplay of topological evolution and continuity (of disconnected sets into connected sets which can be continuous) that establishes the arrow of time. The reciprocal process of topological evolution (of connected sets into disconnected sets) can NOT be continuous. As a geometrical process preserves topology, a geometrical analysis cannot describe an irreversible process, which requires topological change.

1.1.5 Thermodynamic Systems and Continuous Processes

Topological thermodynamics has two distinct categories of exterior differential forms. The first category describes a *process* in terms of a vector (or spinor) direction field, $\mathbf{V}^k$, of 4 ordered functions that form the components of a differential N-1=3 form density, C, or current. The 3-form currents, C, are the topological analogues of the thermodynamic extensive properties. The second category describes a thermodynamic *system* in terms of an ordered array of of 4 functions, $\mathbf{A}_k$, that form the components of an exterior differential 1-form, A. The 1-forms are the topological analogues of the intensive variables in classical thermodynamics.

Ultimately, the exterior differential 1-forms will be defined on domains of Pfaff topological dimension 4 or less) ; the open domains of Pfaff dimension 4 are defined herein as the thermodynamic physical environment. The "particulate" matter is defined in terms of the topological closed defect structures, or collective states, of Pfaff topological dimension 3 that emerge from, and interact with, the thermodynamic physical environment. The process current that induces this emergence from the environment of Pfaff topological dimension 4 to Pfaff topological dimension 3, is dissipative and irreversible.

For example, the number of components is a topological concept that can change *continuously* by "pasting" together (or condensing) various components. The number of components can change discretely by the "pasting" process, but the process can be topologically continuous in a formal sense.

Theorem 4 *A map from topology $\tau 1$ to topology $\tau 2$ is topologically continuous iff the limit points of every subset in the domain, $\tau 1$, permute into the closure of the range, $\tau 2$.*

It can be demonstrated (see Vol 1 [1]) that the closure of a differential form, Σ, (in the KCT0 topology) is equivalent to the the differential ideal, $\{\Sigma \cup d\Sigma\}$, and limit points are equivalent to $d\Sigma$. Both Hydrodynamics and Electrodynamics (as well as almost any other of the physical specializations) have a foundation in terms of Topological Thermodynamics. The methods lead to precise methods for determining when a process, $V^k(x, y, z, t)$, applied to a specific thermodynamic "particle-like" system, $A_k(x, y, z, t)dx^k$, is thermodynamically irreversible or not, or Adiabatic or not, or Adiabatically irreversible or reversible.

1.2 Axioms of Topological Thermodynamics of Particles

In this section, a topological perspective will be used to construct those Axioms of Topological Thermodynamics that will enable discovery and practical utilization of physical properties and processes that otherwise are excluded by the imposition of geometrical constraints which, ultimately, are dependent upon a metric. For a differential variety, the dynamics is described by a process related to the action of the Lie differential acting on the topological structure. For the diffeomorphic variety, the dynamics is described by a process related to the action of the tensor covariant derivative acting on the metric structure. Note that a given differentiable variety may support many different topological structures; hence a given base may support many different coexistent physical systems with particle properties. A major success of the theory is that continuous non-homeomorphic processes of continuous topological evolution can be used to establish a logical basis for thermodynamic irreversibility of distinguishable particle-like T0 systems, and the arrow of time [237].

Remark 5 *Only recently (2011) has it been realized that Not-T0 topological systems can be used to describe the irreversible evolution of indistinguishable complex wave dif-*

fusion systems, where statistical-like ensembles of indistinguishable elements play a companion role to distinguishable particle-like systems in the theory of non-equilibrium thermodynamics. For example, it appears that entropy has two components, a statistical Not-T0 (indistinguishable wavelet) component and a T0 (distinguishable particle-like) component. These statistical and particle components can coexist. These concepts are not complete; but a few remarks have been added to this vol6 (see chapters 7 and 6).

The fundamental axioms and theorems utilized in the distinguishable "particle-like" T0 theory of non-equilibrium thermodynamics are:

Axiom 1. *Thermodynamic physical (particle-like) systems can be encoded in terms of a 1-form of Action potentials, $A_k(x^k)$, on a four-dimensional abstract differentiable variety of ordered independent variables, $\{x^k; dx^k\}$. The variety supports a non-zero differential volume element $\Omega_4 = dx\hat{}dy\hat{}dz\hat{}dt$.*

Axiom 2. *Every 1-form of Action, $A = A_k(x, y, z, t)dx^k$,* of an arbitrary number functions, k, *has an irreducible number of functions, N, required to encode its topological features. This minimum number, N, is defined as the Pfaff Topological Dimension, or class, of the 1-form, A. It is assumed (for symplicity) that the largest PTD of interest is PTD = 4, which corresponds to an Open non-equilibrium thermodynamic system. The PTD=4 Open system plays the role of the physical environment, (the aether). Open, Closed, Isolated and Equilibrium thermodynamic systems will be associated with 1-forms of PTD=4, PTD=3, PTD=2, and PTD=1.*

Axiom 3. *Thermodynamic processes are assumed to be encoded, to within a density distribution factor, $\rho(x, y, z, t)$, in terms of a direction field,* $\boldsymbol{V}_4$(x,y,z,t), *and/or a complex isotropic Spinor direction field, $\boldsymbol{S}_4$(x,y,z,t). The density distribution factor can be chosen such that the differential "mass volume" element, $\rho(x, y, z, t)dx\hat{}dy\hat{}dz\hat{}dt$ is an invariant of the process.*

Axiom 4. *Continuous topological evolution of the thermodynamic system can be encoded in terms of Cartan's magic formula (see p. 122 in [161]). The Lie differential, relative to a process direction field* $\boldsymbol{V}_4$(x,y,z,t), *when applied to a exterior differential 1-form of Action, $A = A_k dx^k$, is equivalent abstractly to the first law of thermodynamics.*

1.

$$\begin{aligned}
\textbf{The First Law} &: \textbf{of Thermodynamics} & (1.10)\\
\text{Cartan's Magic Formula } L_{(\rho\mathbf{V}_4)}A &= i(\rho\mathbf{V}_4)dA + d(i(\rho\mathbf{V}_4)A), & (1.11)\\
\text{A statement of cohomology} &: W + dU = Q, & (1.12)\\
\text{Inexact heat 1-form } Q &= W + dU = L_{(\rho\mathbf{V}_4)}A, & (1.13)\\
\text{Inexact work 1-form } W &= i(\rho\mathbf{V}_4)dA, & (1.14)\\
\text{Internal energy } U &= i(\rho\mathbf{V}_4)A. & (1.15)
\end{aligned}$$

Axiom 5. *Equivalence classes of systems and continuous processes can be defined and refined in terms of the Pfaff Topological Dimension of the 1-forms of Action, A, work, W, and heat, Q.*

Axiom 6. *$Q\hat{}dQ \neq 0$ (Pfaff Topological Dimension of Q is ≥ 3) is a necessary and sufficient condition for a process, $\mathbf{V}$, to be thermodynamically indeterministic. This axiom means that 1-form Q is not uniquely integrable, and cannot be embedded in R2.*

In a perhaps over simplistic comparison, it might be said that ubiquitous tensor diffeomorphic methods are restricted to geometric applications, while Cartan's homotopic methods can be applied directly to topological concepts as well as geometrical concepts. Cartan's theory of exterior differential systems is a topological theory not necessarily limited by geometrical constraints and the class of diffeomorphic transformations that serve as the foundations of tensor calculus. A major theme of this monograph is to show the usefulness of limit points, intersections, closed sets, continuity, connectedness and other elementary concepts of modern topology which are inherent in Cartan's theory of exterior differential systems.

Earlier intuitive results recognized that Cartan's exterior product could be used as an intersection, or meet, operator, and Cartan's exterior differential could be used as a limit point operator, acting on differential forms. With these facts, the Cartan topological structure can be evaluated in terms of the closure and intersection properties of an arbitrary exterior differential 1-form of Action, A. The remarkable result (1987) was that the topological structure based on these concepts is isomorphic to the Kolmogorov T0 (poset 3) finite topology of 4 points (2009). The combination of the Kolmogorov (poset 3) topology and the sets of exterior differential forms is defined herein as the Kolmogorov-Cartan T0 topology, or KCT0 for short. It is important to observe that the KCT0 topology is not necessarily a connected topology. A subset of the KCT0 topology can be connected, if the property of Topological Torsion vanishes; the PTD(A) ≤ 2. Thermodynamic irreversibility is a natural consequence of PTD(A)=4.

Details and examples using the Axioms above will occupy much of this monograph.

1.2.1 A Point Set Topology Example of KCT0

First, consider the features of the KCT0 topology based upon 4 points, $X = \{a, b, c, d\}, ..., \varnothing$. The open sets are

$$KCT0\ \{open\} : \{\varnothing, \{a\}, \{c\}, \{a, b\}, \{c, d\}, \{a, c\}, \{a, b, c\}, \{a, c, d\}, \{a, b, c, d\}\}. \tag{1.16}$$

By using the standard definitions for limit points, interior, exterior, boundary, and the two equivalent definitions for closure $= S + S_{limitpoints} = S^0 + \partial S$ leads to the following table of topological structure for KCT0:

Table 1. **A (Kolmogorov-Cartan) KCT0 Topology of 4 points**

$X = \{a, b, c, d\}$

Basis subsets $\{a\}, \{a, b\}, \{c\}, \{c, d\}$

$KCT0\{open\} : \{\varnothing, \{a\}, \{c\}, \{a, b\}, \{c, d\}, \{a, c\}, \{a, b, c\}, \{a, c, d\}, X\}$

$KCT0\{closed\} : \{X, \{b, c, d\}, \{a, b, d\}, \{c, d\}, \{a, b\}, \{b, d\}, \{d\}, \{b\}, \varnothing\}$

Subset S		**$\mathbb{L}$imit $\mathbb{P}$ts**	**$\mathbb{I}$nterior - $\mathbb{E}$xterior**		**$\mathbb{B}$oundary**	**$\mathbb{C}$losure**
$\varnothing$	*	$\varnothing$	$[\varnothing]$	$[X]$	$\varnothing$	$\varnothing$
$\{a\}$		$\{b\}$	$\{a\}$	$\{c, d\}$	$\{b\}$	$\{a, b\}$
$\{b\}$		$\varnothing$	$[\varnothing]$	$\{a, c, d\}\mathbb{D}$	$\{b\}$	$\{b\}$
$\{c\}$		$\{d\}$	$\{c\}$	$\{a, b\}$	$\{d\}$	$\{c, d\}$
$\{d\}$		$\varnothing$	$[\varnothing]$	$\{a, b, c\}\mathbb{D}$	$\{d\}$	$\{d\}$
$\{a, b\}$	*	$\{b\}$	$\{a, b\}$	$\{c, d\}$	$\varnothing$	$\{a, b\}$
$\{a, c\}$	**d**$\mathbb{D}$	$\{b\}, \{d\}$	$\{a, c\}\mathbb{D}$	$[\varnothing]$	$\{b, d\}\mathbb{D}$	X
$\{a, d\}$	$\mathbb{D}$	$\{b\}$	$\{a\}$	$\{c\}$	$\{b, d\}\mathbb{D}$	$\{a, b, d\}\mathbb{D}$
$\{b, c\}$	$\mathbb{D}$	$\{d\}$	$\{c\}$	$\{a\}$	$\{b, d\}\mathbb{D}$	$\{b, c, d\}\mathbb{D}$
$\{b, d\}$	$\mathbb{D}$	$\varnothing$	$\varnothing$	$\{a, c\}\mathbb{D}$	$\{b, d\}\mathbb{D}$	$\{b, d\}$ $\mathbb{D}$
$\{c, d\}$	*	$\{d\}$	$\{c, d\}$	$\{a, b\}$	$\varnothing$	$\{c, d\}$
$\{a, b, c\}$	**d**$\mathbb{D}$	$\{b\}, \{d\}$	$\{a, b, c\}$	$[\varnothing]$	$\{d\}$	X
$\{b, c, d\}$	$\mathbb{D}$	$\{d\}$	$\{c, d\}$	$\{a\}$	$\{b\}$	$\{b, c, d\}\mathbb{D}$
$\{a, c, d\}$	**d**$\mathbb{D}$	$\{b\}, \{d\}$	$\{a, c, d\}$	$[\varnothing]$	$\{b\}$	X
$\{a, b, d\}$	$\mathbb{D}$	$\{b\}$	$\{a, b\}$	$\{c\}$	$\{d\}$	$\{b, c, d\}\mathbb{D}$
$\{a, b, c, d\}$	*	$\{b\}, \{d\}$	$[X]$	$[\varnothing]$	$\varnothing$	X

*: subset is open and closed, **d:** subset set is dense in X, $\mathbb{D}$: subset is disconnected.

This KCT0 topology is quite interesting for many demonstrable reasons. First note that all of the singletons of the topology are not closed. This implies that the topology is NOT a metric topology, and NOT a Hausdorff T2 topology. Note that all closed sets contain all of their limit points. Some open sets can contain limit points, but some open sets do not contain their limit points. Some subsets have boundaries that are composed of their limit points. Some subsets have limit points which are not boundary points. Certain subsets have a boundary, but do not have limit points, and in other cases there are subsets that have limit points, but do not have a boundary. There are certain subsets with a boundary, but without an interior. There are certain subsets with an interior, but without a boundary. These situations, though topologically correct, are not always intuitive to those accustomed to metric based topological concepts, which impose a number of additional constraints on the sets of interest. Yet all of these topological ideas, including the non-intuitive ones, are easy to grasp from the simple example of the KCT0 point set topology.

One other very important observation is that there are subsets of the KCT0 topology, $\{a, b\}$ and $\{c, d\}$, (other than $\varnothing$ and X) which are both open and closed. The union of these two subsets $\{a, b\}$ and $\{c, d\}$ is X. Topologies with this property are said to be disconnected topologies. What is important is that it is possible to construct a **continuous** map from a disconnected topology to a connected topology, but it is impossible to construct a **continuous** map from a connected topology to a disconnected topology. If the mapping process is interpreted as an evolutionary process, these facts establish a logical or topological basis for the arrow of time [237]. This idea can be exploited to explain a non-statistical component of thermodynamic irreversibility. Note that the subsets $\{a, c\}$, $\{a, b, c\}$ and $\{a, c, d\}$ are dense. They are homeomorphic invariants relative to the KCT0 topology of the thermodynamic system, but when the topology is augmented (refined) by the constraint of *continuous* topological evolution and *processes* acting on thermodynamic systems, they can represent topological change.

1.2.2 An example of the Universality of Electromagnetism

Most geometric ideas evolve from the concept of a distance, a size, and other features associated with a metric, such as groups of diffeomorphisms (a required constraint of tensor analysis), and connections. The paper, "Electromagnetism independent of metrical geometry", by van Dantzig [292] was an early stimulation to the research presented herein. Van Dantzig developed the idea that Maxwell-Faraday electromagnetism was a system of partial differential equations that did not depend explicitly upon a geometric choice of metric. This result was interpreted as a theory that was independent from a choice of coordinates. Only now is it appreciated that the system of Maxwell PDE's is much more general, and can be deduced from an exterior differential system [34] of topological constraints.

For Maxwell-Faraday electromagnetic theory, the exterior differential system as a topological (not geometric) constraint is very simple: "There exists a C2 2-form, $F(x^k, dx^k)$, that is exact."

$$\begin{aligned}
\textbf{Maxwell EM} &: \quad \text{and } \textbf{Topological Constraints} && (1.17)\\
F - dA &= 0. \text{ The exterior differential system.} && (1.18)\\
d(F - dA) &= dF \Rightarrow 0. \text{ The Limit points of } F \text{ vanish.} && (1.19)\\
dF &= 0: \text{ a set of Maxwell-Faraday PDE's} && (1.20)
\end{aligned}$$

The coefficients of the 1-form, A, represent the co-vector and scalar electromagnetic potentials, $(\mathbf{A}, \phi)$. The coefficients of the 2-form, F, are composed of those functions classically defined as the **E** and **B** field intensities. The closure of this topological condition, $F - dA = 0$, involves the limit points of F, a 3-form, $dF = 0$ (where d is the exterior differential). This equation is satisfied, universally, by a set of PDE's known as the Maxwell-Faraday equations. The result is valid in *any* coordinate system, for *any* metric, and for *any* dimension! The Maxwell-Faraday PDE's are universal.

These (geometric-free) results have been extended to non-equilibrium thermodynamics and hydrodynamics. The intuition of the past can now be formulated by succinct statements (with details in the monographs):

Definition 6 *Topological Universality implies independence from any choice of geometric constraints formulated in terms of metric or connection.*

Conclusion 7 *To insure that physical properties are not excluded by some choice of geometric constraints, all that is necessary is to insist that the topological foundations of interest must not be globally Hausdorff T2+ (which are metrizable), but must contain T0 partitions, or contain partitions that are Not-T0 and have indistinguishable elements.*

Theorem 8 *Topological Torsion, $A\hat{}dA$, (not affine torsion or Frenet torsion) is a thermodynamic artifact of non-metrizability, and a necessary property of irreversible processes.*

To repeat a paragraph in the preface, it is not suggested that geometrical constraints are not useful. It is possible to use a metric, do computations, and obtain (often very interesting) physical results, but certain properties can be excluded if the results depend upon a choice of metric. Recall that Hodge theorems were computed in terms of a given metric and its dual., but it was the results that were independent from the choice of metric that interested Hodge.

1.3 Kuratowski Closure and Cohomology

1.3.1 Limit Points

My interest in a topological formulation of thermodynamics started in 1964, after I left Los Alamos to become a professor in a new graduate department of physics at the University of Houston. Over the next 35 years, these ideas matured and were assembled into a set of monographs [1], after I retired in 2000. Details and many examples of topics described herein are to be found in these monographs.

The primary, intuitive conclusion was that non-equilibrium thermodynamic systems can be encoded in terms of sets of Cartan's exterior differential forms. The topological structure constructed from the existence of an exterior differential 1-form of action, A, can be put into correspondence with the topological structure displayed by the Kolmogorov disconnected T0 topology of 4 ingredients (Poset 3). The combination of the Poset 3 topology with subsets defined in terms of exterior differential forms will be defined as the Kolmogorov-Cartan T0 structure, or KCT0 for short. It was realized that such a T0 topological foundation for thermodynamics had a content that was excluded by insistence on geometric (metric) constraints. One excluded property is the concept of Topological Torsion.

The KCT0 topological structure indicates that the exterior differential, d, acting on sets of exterior differential forms, Σ, is a limit point generator, and the Grassmann exterior "wedge" product, $\hat{}$, is related to an intersection (meet) operator. It follows for any exterior differential 1-form, A, that Kuratowski's closure formula is generated by the construction:

$$dA = \text{"Limit points of" } A, \tag{1.21}$$
$$(\mathbb{I} + d)\ A = \text{Kuratowski Closure of } A\,. \tag{1.22}$$

In classical point set topology there are two definitions of the closure of a set, S:

$$\text{Homology : Closure(S)} = \text{Exterior(S)} \cup \text{boundary(S)} \tag{1.23}$$
$$\text{Cohomology: Closure(S)} = \text{(S)} \cup \text{Limit points of (S)} \tag{1.24}$$

When the sets are exterior differential forms, the cohomology structures are easily evaluated as the exterior differential is a limit point generator. So, starting from an exterior differential 1-form, A, a number of topological concepts can be evaluated easily. An important topological property that is to be associated with non-equilibrium thermodynamics is the 3-form of Topological Torsion:

$$A\hat{}dA = \text{Topological Torsion of } A \tag{1.25}$$
$$A\hat{}dA = \varnothing \text{ implies } A \text{ is isolated.} \tag{1.26}$$

From the Frobenius theorem, when (in domains where) the Topological Torsion vanishes, $A\hat{}dA = \varnothing$, the 1-form A requires only 2 functions from which the topological features generated by the 1-form can be extracted. The ODE represented by the 1-form, A, is said to be uniquely integrable. There exists a map such that locally $A = \phi(x^k)d\psi(x^k)$.

Thermodynamic processes acting on such systems of 1-forms, $A \simeq$ Action, $W \simeq$ Work, $Q \simeq$ Heat, could be encoded in terms of the Lie differential, with respect to processes represented by vector and spinor direction fields, V^k. The extraordinary result is that these definitions result in a dynamical generalization of the First Law of Thermodynamics as a statement in cohomology theory: "The difference between the inexact 1-form of heat, Q, and the inexact 1-form of work, W, is an exact differential of the internal energy, U":

$$L_{(V)}A = W - dU = Q, \tag{1.27}$$
$$\text{or } dU = Q - W\ :\ \text{cohomology.} \tag{1.28}$$

From a topological view of smoothness, one of the extraordinary, non-intuitive results of the dynamical formulation is the proof that C1 segmented processes (that approximated C2 smooth processes) could be thermodynamically reversible, where the approximated C2 processes were thermodynamically irreversible. In this sense, C1 (digital) processes generate an approximation logic that is "fuzzy"; so be warned that computerized digital C1 formulations may not capture such things as biological aging.

1.3.2 Geometry vs Topology.

Where the classic homological ideas are based on geometrical concepts of {"points, edges, faces, volumes..."} and Hausdorff T2+ topologies, the use of sets of exterior differential forms (which are completely anti-symmetric), and the KCT0 topology, leads to *extended* cohomological concepts based upon {"1-forms, 2-forms, 3-forms, 4-forms..."}. The extended cohomology theory endows the system with a *new* computable topological property, defined as Topological Torsion, a concept that is related to the idea that the system can not be embedded in R2. The Topological Torsion is easily computed in terms of the "class", or Pfaff Topological Dimension, of those exterior differential 1-forms, A, associated with the KCT0 topology.

Both geometry and topology place constraints on sets that lead to certain evolutionary topological structures. In simple terms, geometric constraints specify limiting processes of ratios to obtain the concept of a derivative:

$$\text{Geometry} \quad : \quad d\mathbf{x}^m/dt = \mathbf{V}^m(x..., t), \qquad \text{derivatives,} \tag{1.29}$$

$$\text{Topology} \quad : \quad d\mathbf{x}^m - \mathbf{V}^m(x..., t)dt = 0, \quad \text{differentials.} \tag{1.30}$$

Topological constraints are defined in terms of exterior differential systems, and the differentials are more general than the derivatives for they can contain topological fluctuations. Geometric constraints are constructed on topologies that satisfy (at least) the Hausdorff T2 separation axioms. Such Hausdorff systems are defined to be "metrizable". Kolmogorov T0 topological structures on domains of sets that do not satisfy the Hausdorff axioms are "not metrizable". In other words, domains that are metrizable exclude topological features that exist in domains that are not metrizable. It appears that such excluded properties are those that describe non-equilibrium systems and irreversible processes.

1.3.3 Projective geometries and low dimensional topologies

Early on, an appreciation was developed for the fact that the projective geometry concepts of correlations are related to the Jacobian matrix of the coefficients of a 1-form intensity, $A_k(x^m)$, and projective concepts of collineations are related to the Jacobian matrix of the coefficients of a N-1 form process direction field, $V^k(x^m)$. Classical thermodynamics distinguishes between two categories of sets, defined as intensive and extensive variables. The intensive variables (like temperature and pressure) are homogeneous of degree zero. The extensive variables (like mole, number, volume and entropy) are additive and homogeneous of degree one.

The collineations of interest are homogeneous of degree one, and hence correspond to the extensive thermodynamic sets of additive variables, charge and current (so called) sources of the electromagnetic field quantities, **D** and **H**. In electromagnetic theory, it became apparent that the concept of Affine (not Topological) Torsion could be associated with the concept of field excitations, or sources, expressed as extensive vector arrays of 2-forms densities, $|G\rangle$. Such concepts of field excitation

2-forms, $|G\rangle$, have long been part of EM theory and act as the source of charge-current density 3-forms, $J = dG$, and the Maxwell-Ampere PDE's. Such charge current density 3-forms do not appear in classical hydrodynamics, where stress-strain relations have been constrained to admit only those expositions of stress-energy in terms of symmetric matrices. When anti-symmetries are allowed, then fluids also permit excitation 2-forms and current density 3-forms.

The 3-form densities can be constructed also in terms of the Adjoint matrix of the Jacobian correlation matrix constructed in terms of the coefficients of the 1-form, $A_k(x^m)dx^k$, which defines the thermodynamic system. With this concept, the partial differential equation theories of electromagnetism and hydrodynamics become topologically equivalent. In addition to the concept of Topological Torsion (an intensity 3-form), the different concept of Topological Spin (an excitation 3-form density) can be formulated for fluids. The latter have found practical application in transport of collective spin states, describe by the name "Spintronics" in Silicon Valley.

I must admit that my training as a physicist has installed a bit of dogma into my thinking. Physics is based upon measurement, and to me that means the fundamental variables ultimately must be three spatial dimensions and one time dimension. However, the idea of dimension is not restricted to geometry alone, but can incorporate features of topological dimension. I therefor consider low dimensional finite topologies (topological dimension ≤ 4) to be the most promising for developing a topological perspective of physics.

As mentioned above, the useful domain of exterior differential forms is determined in terms of equivalences between differential varieties,

$$\text{differential variety: } \{x^m,\ dx^m\}, \tag{1.31}$$

which go beyond the limitations of (tensor geometric) diffeomorphisms, and (topological) homeomorphisms. This basic idea distinguishes a thermodynamic *system* (as an exterior differential 1-form) from a thermodynamic *process* (as an exterior differential N-1 = 3-form density). It cannot be overemphasized that the cohomological dynamics is governed by the Lie differential with respect to a direction field – not just a covariant derivative, and not just a Lie derivative. The distinction between Lie differential and the Lie derivative (and the covariant derivative) is that the covariant derivative is an element of a (diffeomorphism) group, and the Lie derivative is the element of a pseudo-group, but the Lie differential can be an element of a semi-group. In other words, covariant derivative $\subseteq$ Lie derivative $\subseteq$ Lie differential. The processes generated by the Lie differential are much more general than the processes generated by the various forms of derivatives. Thermodynamically, the derivatives are related to processes that are adiabatic, while the differential processes need not be adiabatic. More to the point, subject to geometric constraints, physical processes are time reversal symmetric and reversible.

It is also possible to show that processes, $V^k(x^m)$which are C1 continuous can be conservative and fractal. Homogeneous processes lead to self-similar systems, where small sub-domains have recognizable emergent holographic features of the entire domain. In addition, integration with respect to closed p-chains of closed, homogeneous, but not exact p-forms, will lead to values that have rational ratios. This "topological quantization" of mole number (particles), charge number, and spin number is valid at all scales, from the micro-scales of quantum mechanics to the global scales of the cosmological universe. Applications to problems in non-equilibrium plasma systems and turbulent hydrodynamics can be based on the emergence of four topological currents. These currents represent the transport of charge, torsion, and two species of collective spins representing superconductivity (charge) and superfluidity (mass and inertia)

Conjecture 9 *It is remarkable that discrete partitions of Not-T0 topologies that are C0 might represent propagating outbound waves different from inbound waves which are topologically distinguishable in a chiral sense. Such topics are in their infancy as of Jan 2, 2012.*

1.3.4 The KCT0 Topology with sets of exterior differential forms

The Pfaff Sequence

What is even more remarkable is that properties of the KCT0 topology can be replicated in terms of the Pfaff sequence of exterior differential sets,

$$\text{Pfaff Sequence} : \{A, dA, A\hat{}dA, dA\hat{}dA...\}, \tag{1.32}$$

generated from any given 1-form of Action, A, or W, or Q, on the space time differentiable variety. The Pfaff sequence is readily computed, and will contain $M \leq N$ elements, where M is defined as the Pfaff Topological Dimension (or class) of the given 1-form, A, W, or Q. The topologies associated with W, or Q, are process dependent. The Topology associated with A is not dependent upon a process; the topology of the 1-form A will be described as an intrinsic realization of a thermodynamic system.

Topological Torsion and Parity

All that is necessary is to identify the "points" $\{a, b, c, d\}$ with the exterior differential forms For notational simplicity in this monograph the systems of p-forms will be assumed to consist of the single 1-form, A. Then the exterior differential of A is the 2-form $F = dA$, and the closure of A is the union of A and $F : K_{Cl}(A) = A_{\cup}F$. The other logical operation is the concept of intersection, so that from the exterior differential it is possible to construct the set $A\hat{}F$ defined collectively as $\mathrm{H} : \mathrm{H} = A\hat{}F$. The exterior differential of H produces the set defined as $\mathrm{K} = d\mathrm{H}$, and the closure of H is the union of H and $\mathrm{K} : K_{Cl}(\mathrm{H}) = \mathrm{H}_{\cup}\mathrm{K}$. These sets have been given the names:

$$\begin{aligned}
\text{Topological ACTION} &: \quad A && (1.33)\\
A &= A_\mu dx^\mu && (1.34)\\
\text{Topological VORTICITY} &: \quad F = dA && (1.35)\\
dA &= F_{\mu\nu} dx^\mu \hat{} dx^\nu && (1.36)\\
\text{Topological TORSION} &: \quad \mathrm{H} = A\hat{}dA && (1.37)\\
A\hat{}dA &= H_{\mu\nu\sigma} dx^\mu \hat{} dx^\nu \hat{} dx^\sigma && (1.38)\\
\text{Topological PARITY} &: \quad \mathrm{K} = dA\hat{}dA && (1.39)\\
dA\hat{}dA &= K_{\mu\nu\sigma\tau} dx^\mu \hat{} dx^\nu \hat{} dx^\sigma \hat{} dx^\tau. && (1.40)
\end{aligned}$$

First consider the domain of support of an exterior differential 1-form, A. Define this "point" by the symbol A. $A \Leftrightarrow a$ is the first open set of the KCT0 topology. Next construct the exterior differential, $F = dA$, and determine its domain of support. Next, form the closure of A by constructing the union of these two domains of support, $K_{Cl}(A) = A_\cup F$. $A_\cup F \Leftrightarrow b$ forms the second open set of the KCT0 topology. Next construct the intersection $\mathrm{H} = A\hat{}F$, and determine its domain of support. Define this "point" by the symbol H $\Leftrightarrow$ c, which forms the third open set of the KCT0 topology. Now follow the procedure established in the preceding paragraph. Construct the closure of H as the union of the domains of support of H and K $= d$H. The construction forms the fourth open set $\mathrm{H}_\cup\mathrm{K} \Leftrightarrow$ d of the KCT0 topology. Now fill in the point set topological structure above with the "points",

$$\{A,\ A_\cup F,\ \mathrm{H},\ \mathrm{H}_\cup\mathrm{K}\} \Leftrightarrow \{a, b, c, d\}, \tag{1.41}$$

to form the topological structure of the KCT0 topology.

Chapter 2
TOPOLOGICAL FEATURES OF EXTERIOR DIFFERENTIAL FORMS

2.1 Early Developments

2.1.1 Cartan's Exterior Differential forms on Differentiable Varieties

In the period from 1899 to 1926, Eli Cartan used his expertise in the differential geometry of exterior differential forms, and his experience with the theory of Lie groups, to formulate a theory of physical equations of motion appropriate to many physical systems. Cartan developed a theory of exterior differential systems [41], [42], which include the ideas of spinor systems [45] and the differential geometry of projective spaces and spaces with torsion [43].

Remark 10 *It is apparent that Cartan did not realize that his theory, and its universality, were due to the fact the theory was a topological formulation of thermodynamics. Cartan's theory was appreciated by only a few contemporary researchers, and made little impact on the main stream of the physical sciences until about the 1960's. Even specialists in differential geometry (with a few notable exceptions [52]) made little use of Cartan's methods until the 1950's. Even today, many physical scientists and engineers have the impression that Cartan's theory of exterior differential forms is just another formalism of fancy. That conclusion is wrong. The Cartan methods transcend the geometrical constraints in current vogue.*

Cartan's theory of exterior differential systems has several advantages over the methods of tensor analysis that were developed during the same period of time. The principle fact is that differential forms are well behaved with respect to functional substitution of C1 differentiable maps. Such maps need not be invertible even locally, yet differential forms are always deterministic in a retrodictive sense [207], by means of functional substitution. Such retrodictive determinism is not to be associated with contravariant tensor fields, if the map is not a diffeomorphism. Cartan's theory of exterior differential systems contains topological information, and admits non-diffeomorphic maps which can describe topological evolution.

A major success of the Cartan method is that continuous non-homeomorphic processes of topological evolution can be used to establish a logical basis for thermodynamic irreversibility, dissipation, and the arrow of time [237] — without the use of phenomenological balance equations. In effect, Cartan used his thesis experience with the theory of Lie groups to recognize that the Lie derivative could be used as a homotopy operator acting on differential forms and differential form densities to define "equations of motion" for many physical processes. The constraints of the Lie *derivative* imply that the vector fields considered are restricted to be elements of a single parameter group. (Now it is known that the underlying topology of the Lie derivative is metrizable Hausdorff T2.) In modern terms, the Lie derivative becomes the Lie *differential* based on a semigroup. The "equations of motion" then can describe irreversible, non-equilibrium, nondeterministic, features of topological evolution of physical systems.

Although the word "topology" had not become popular when Cartan developed his ideas (topological ideas were described as part of the theory of "analysis situs"), there is no doubt that Cartan's intuition was directed towards a topological development. For example, Cartan did not define what were the open sets of his topology, nor did he use, in his early works, the words "limit points, or accumulation points" explicitly, but he did describe the union of a differential form and its exterior differential as a differential ideal (now known as the "Kuratowski closure"* of the form. With this concept, Cartan effectively used the Kuratowski idea (thesis 1919, text book 1922) that the closure of a subset is the union of the subset with its topological limit points. What was never stated in the literature (known to the author before 1990) is the idea that the exterior differential is indeed a limit point generator relative to a Kolmogorov-Cartan T0 poset 3 topology. The union of the identity operator and the exterior differential satisfy the axioms of a Kuratowski closure operator [151], which can be used to define a topology. The other operator of the Cartan calculus, the exterior product, also has topological connotations when it is interpreted as a meet (intersection) operator.

In a perhaps over simplistic comparison, it might be said that ubiquitous tensor methods are restricted to geometric applications, while Cartan's methods can be applied directly to both topological concepts and geometrical concepts. Cartan's theory of exterior differential systems is a topological theory not necessarily limited by geometrical constraints, such as the class of diffeomorphic transformations that serve as the foundations (and limitations) of tensor calculus. It is possible to show how limit points, intersections, closed sets, continuity, connectedness and other elementary concepts of modern topology are inherent in Cartan's theory of exterior differential systems.

2.1.2 Differential Varieties vs Diffeomorphisms

The formulation of the Maxwell-Ampere equations given in Chapter 4.3 is relative to a choice of ordered array of independent "coordinate" variables $\{x, y, z, t\}$ and functions built on these "coordinate variables". The topological features of the

*The Kuratowski Closure of a subset is defined as the union of the set, S, and its limit points, dS. For sets of exterior differential forms, the limit points are equal to dS, where d is the exterior differential.

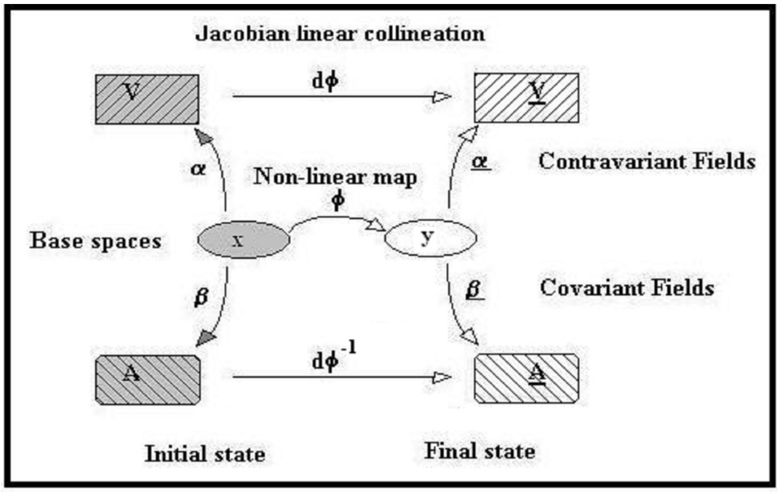
formalism are not immediately evident. Historically, the science community became enamored with the idea that the fundamental laws of nature should be the same for any observer, and in particular independent of how the observer defined (or chose) his "coordinate" frame for measurement purposes. The ideas culminated with the concepts of diffeomorphic equivalences of tensor analysis, which preserved metric based properties of systems and dynamical processes.

However, electromagnetism has a formulation in terms of Cartan's exterior differential forms where the topological features become more apparent. The differential forms are defined upon differential varieties, $\{x, y, z, t;\ dz, dy, dz, dt\}$, not diffeomorphic varieties. Both the differential varieties and diffeomorphic varieties admit differentiable maps $\{\phi, d\phi\}$ between varieties:

$$\begin{aligned} &: \ \{x^k, dx^k\} \Rightarrow \{y^m, dy^m\} && (2.1)\\ \phi &: \ \{x^k\} \Rightarrow \{y^m\} = \{y^m(x^k)\}, \text{ base coordinates} && (2.2)\\ d\phi &: \ \{dx^k\} \Rightarrow \{dy^m\} = \{(\partial y^m(x^k)/\partial x^k)dx^k\}, && (2.3)\\ &: \ = \mathbb{J}[(\partial y^m(x^k)/\partial x^k)] \circ \left|dx^k\right\rangle && (2.4) \end{aligned}$$

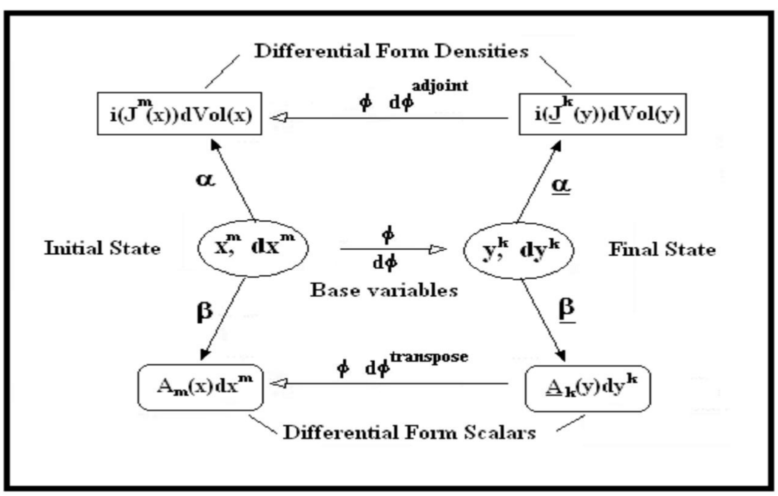
The diffeomorphic (geometric) varieties (which are specializations of differential varieties) require maps, ϕ, and Jacobians, $d\phi$, that have inverses. The differential varieties admit maps and Jacobians that do not have inverses. In this sense exterior differential forms defined on differential varieties do not depend upon a choice of coordinates. However, they are also independent from the symmetry constraints imposed by metric, gauge groups and connections. In such a formulation the equations of an electromagnetic system become recognized as consequences of topological constraints on a domain of independent variables, not geometric constraints on a domain.

The use of differential forms should not be viewed as just another formalism of fancy. The technique goes beyond the methods of tensor calculus, and admits the study of topological evolution, while tensor based theories do not. In classical tensor analysis, both the map, ϕ, and the Jacobian matrix, $\mathbb{J}$, of the map, $d\phi$, between *diffeomorphically* equivalent differential varieties, are constrained to have inverses. The constraints go farther than that. Consider maps from and initial diffeomorphic variety, x, to a final diffeomorphic variety, y. Let $\alpha(x)$ be a map from the initial diffeomorphic variety, x, to a contravariant vector space, V(x), and $\beta(x)$ be a map from x to a covariant vector space, A(x). Consider similar maps, $\underline{\alpha}$(y) and $\underline{\beta(y)}$ defined on the final diffeomorphic variety. (see the Figure "Diffeomorphic Collineations" below where the arrows define functional compositions. Hence, functional composition can predict $\underline{\beta}$ from β, or retrodict β from $\underline{\beta}$. Similar remarks apply to α and $\underline{\alpha}$. If the map is a diffeomorphism, covariant tensors (thermodynamic intensities) pullback retrodictively with respect to the transpose of the Jacobian matrix (of functions) and functional substitution, and contravariant tensor densities (thermodynamic excitations) pullback retrodictively with respect to the adjoint of the Jacobian matrix, and functional substitution [207].



Diffeomorphic collineations

The more general differential varieties need not be diffeomorphically constrained. Both functional prediction and functional retrodiction is not possible when both the map ϕ, and the Jacobian matrix of the map between differential varieties, $d\phi$, do not have to have an inverse. A glance at the Figure, Exterior Differential Forms on Differential Varieties, below, demonstrates that the functional definitions of the differential forms, β, on the initial base differential variety, can be deduced by functional composition in terms of the definitions of the differential forms, $\underline{\beta}$, on the final state; but it is not possible to deduce $\underline{\beta}$, on the final state, given β on the initial state by means of functional substitution when the inverses of ϕ and $d\phi$ do not exist. Functional retrodiction is permissible, but functional prediction is not possible.



Exterior Differential Forms on Differential Varieties

The differential forms (with coefficient intensities) on the initial variety are functionally well defined by the pullback mechanism, which involves algebraic composition with components of the Jacobian matrix transpose. The differential form densities (with coefficient excitations) on the initial variety are functionally well defined by the pullback mechanism, which involves algebraic composition with components of the Jacobian matrix adjoint. The transpose and the adjoint of the Jacobian exist, even if the Jacobian inverse does not, so the coefficients of the differential forms need not be tensors, and yet they are well defined in a functional sense.

2.1.3 Connected and Disconnected Topologies

An essential feature of the Kolmogorov-Cartan T0 (poset 3) topological structure on subsets of exterior differential forms is that it is a disconnected topology. If the only sets of a topology that are both open and closed are X and $\varnothing$, then the topology is a connected topology. You can get from point a to point b by a connected path. The two separated points are reachable on the connected topological domain.

However, it is possible that a topology will have pairs of subsets that are both open and closed, and these pairs are different from X and $\varnothing$. Such "clopen" subsets will be disjoint and also will have empty boundaries. Such topologies are said to be disconnected topologies. The concept of particle-like non-equilibrium thermodynamics requires that the underlying topology contains disconnected parts.

For the T0 topologies of 4 ingredients, in came to my attention in 2009 that there are 10 T0 topologies that are connected, and 6 T0 topologies that are disconnected. Details of the 16 topological structures are given in the appendix. Of the 6 disconnected topologies, 5 have isolated singletons. There is one disconnected T0 topology that does not have an isolated singleton, and that is the Poset (3) Kolmogorov T0 topology†. When the sets of this topology are expressed in terms of exterior differential forms, I now define this topology as the Kolmogorov-Cartan T0 topology of 4 ingredients.

The set of T0 topologies on a domain of N points consists of 3 partitions:

1. The set of T0 topologies which are connected spaces.
2. The set of T0 topologies that are disconnected topologies with isolated singular points.
3. The set of T0 topologies that are disconnected topologies without isolated singular points.

In the set of 16 T0 topologies on 4 points, The KCT0 poset 3 topology is the only disconnected topology without isolated points.

It is also of interest to note that all T0 topologies (connected and disconnected) on a domain of N-1 points can be embedded into the partition of connected topologies on domains of N points.

†The Poset (3) T0 topology was intuitively selected in 1987 to be the topological base for a topological perspective of thermodynamics — before I knew anything about posets

Two subsets of a topology are **disjoint** if $A \cap B = \varnothing$. A disconnected topology (space) has disjoint Open Subsets G and H such that

$$X = G \cup H, \text{ and } G \cap H = \varnothing. \tag{2.5}$$

If the only sets that are Open and Closed are X and $\varnothing$, then the Topology (and X) is connected.

Disconnected Subsets

A subset $A \subset X$ is said to be disconnected if there are open subsets, G and H, such that $(A \cap G)$ and $(A \cap H)$ are not empty, and

$$(A \cap G) \cap (A \cap H) = \varnothing, \tag{2.6}$$
$$(A \cap G) \cup (A \cap H) = A. \tag{2.7}$$

Note that G and H are not necessarily disjoint. If the subset is NOT disconnected, it is connected. In the Kolmorgorov-Cartan topology, the pairs of subsets that are not both Open and Closed, are disconnected.

Separated Subsets

Two (distinct) subsets A and B, are said to be separated if they are disjoint and satisfy the rules,

$$A \cap B = \varnothing, \tag{2.8}$$
$$A \cap \overline{B} \neq \varnothing, \tag{2.9}$$
$$\overline{A} \cap B \neq \varnothing, \tag{2.10}$$

where $\overline{A}$ is the symbol for the closure of A; $\overline{A} = A \cup dA$, the union of the subset A and its limit points.

Disconnected Exterior Differential Forms

From the Pfaff Sequence of an exterior differential 1-form, A, the elements can be grouped according to the rules given in Section 1.4. The result is that the "open sets" of exterior differential forms, identified as $K_{Cl}(A) = A \cup F$, and for $K_{Cl}(\mathrm{H} = A\hat{\ }F) = \mathrm{H}_{\cup}\mathrm{K}$ are disjoint and are both open and closed. Hence the KCT0 poset 3 topology on 4 ingredients composed of exterior differential forms is a disconnected topology.

2.1.4 Far Away vs Not Reachable

In non-metrizable T0 topologies, the possibility exists that the topology can consist of connected sets or disconnected sets. The idea of "nearby" or "far away", so commonplace in classic, phenomenological theories, of connected sets, must be replaced by the idea that a point b is *reachable* from a (by a continuous path which might be short or long) if both a and b are in the same connected set. In other words b is not reachable from a if a and b are in different disconnected sets. In fact, the most useful of Cartan's ideas do not depend explicitly upon the geometric ideas of a metric, distance, nor upon the choice of a differential connection between basis frames, as in fiber bundle theories. The topological theory of T0 thermodynamics explores the physical usefulness of those universal topological features of Cartan's methods which are independent from the constraints and refinements imposed by a connection and/or a metric.

There are other distinctions between algebraic and topological ideas. Note that a basic formal distinction between algebra and topology is that although the inverse of a one-one, onto group homomorphism is automatically a homomorphism again, The inverse of a one-one, onto continuous map can fail to be topologically continuous. It is a consequence that, amongst compact $T2$ spaces, this cannot happen. This is another example that the finite metrizable topologies, $\{T2\}$, exclude certain important features.

2.1.5 Continuous Topological Evolution

Continuous Topological Evolution (developed in the period 1974-1984) is encoded in terms of the Lie differential with respect to the process direction field, V, acting on that 1-form of Action, A, chosen to encode the thermodynamic system. For C2 functions it can be demonstrated that this formulation not only represents continuous topological evolution,

$$L_{(\mathbf{V}_4)}A = i(\mathbf{V}_4)dA + d(i(\mathbf{V}_4)A) = Q \tag{2.11}$$

$$Q_{heat} = W + dU, \tag{2.12}$$

$$W_{work} = i(\mathbf{V}_4)dA, \quad U_{internal\_energy} = i(\mathbf{V}_4)A, \tag{2.13}$$

but also represents the abstract, cohomological, dynamical, equivalent to the First Law of Thermodynamics (1984-2009). The dynamics of interest are those topological dynamics, not excluded by geometrical constraints. A prime example are the chaotic dynamics of self similarity, and the emergence dynamics where the disconnected components form connected aggregates. The dynamics of the KCT0 poset 3 thermodynamic system can be refined by the T0 Topological structures associated with the 1-forms of Q_{heat} and the 1-forms of W_{work} . The increment of heat is represented by the symbol Q, a differential 1-form that need not be closed much less exact. Similar results apply to the increment of Work, W.

2.1.6 Topological Spin 3-form densities

In 1969, it was recognized that Van Dantzig's concept [292] of a topological basis for electromagnetism required the additional imposition of a tensor density 2-form, G, to accompany the thermodynamic 1-form, A, and the process 3-form, V. The 2-form density, G, was necessary to account for discrete Charge and the Amperian charge current density found in experimental electrodynamic-thermodynamic systems. This suggested that the 3-form density, $A\hat{}G$, of a thermodynamic - electromagnetic theory would have useful physical application [199]. In fact, if the 3-form density was closed over certain subsets of the variety, then in that region, the closed integral of $A\hat{}G$ obeyed a conservation law (a transport theorem) equivalent to the First Poincare Invariant of electromagnetism. Based upon dimensional analysis, the 3-form density $A\hat{}G$ had the (suggestive) physical dimensions of Planck's constant (Spin angular momentum). The 2-form density, G, if without limit points, and integrated over closed 2-chains (which are not boundaries), represents discrete Charge (via deRham's theorems [197]).

The 3-form density, $A\hat{}G$, if without limit points and integrated over closed 3-chains, represents discrete Spins. All of this follows from deRham cohomology theory. In other words, the discrete features of charge and spin found in quantum mechanics are contained in the T0 topological approach. Again, a subtle feature shows up for non-metrizable topologies. The 3D period integrals can be factored if the Topology is metrizable:

$$\iint\limits_{3D\_cycle}\int A\hat{}G = \int\limits_{1D\_cycle} A \;\hat{}\int\limits_{2D\_cycle}\int G. \tag{2.14}$$

However, if the Topology is not metrizable, then the factorization is not permissible. This result points out the difference between topology and algebra, for in algebra, a map f : G ⇒ H of a group to another one is a homomorphism, if f(xy) = f(x)f(y) for any x, y ⊂ G. The function of an algebraic product, is an algebraic product of the functions. This factorization is defined as the Kunneth formula (see p.108) [177] in deRham cohomology theory, but to be valid the underlying topology must be Hausdorff T2+ in the category $\{> T0\}$. In category $\{= T0\}$, the period integral yields (rational) values not found in category $\{> T0\}$. Experimentally, these differences show up in the theory of the Quantum Hall effect versus the Hall Impedance associated with Meissner repulsion (see Vol3 and Vol4 [1]).

2.1.7 Topological Torsion 3-forms

It was only several years later that I appreciated [205] that the thermodynamic system also supported a 3-form (not a density) , $A\hat{}F$, which I called Topological Torsion (not equivalent to affine torsion in geometric systems). Null Topological Torsion, $A\hat{}dA, = 0$ defines integrable equilibrium and isolated thermodynamic systems. Non-zero Topological Torsion, $A\hat{}dA \neq 0$, defines non-equilibrium non-integrable systems. Topological Torsion is an artifact of topologies generated by exterior differential 1-forms that are not uniquely integrable. Such non-equilibrium systems lead to non-unique solutions, such as envelopes and edges of regression. Remarkably, the 3-form of Topological Torsion is defined entirely by the 1-form, A, that encodes the thermodynamic system, and its limit points, $dA = F$:

$$i(\mathbf{T}_4)dx\hat{}dy\hat{}dz\hat{}dt = A\hat{}dA. \tag{2.15}$$

Non-zero Topological Torsion is intrinsic to the choice of the thermodynamic system, and requires that the Topological Dimension of the 1-Form A is greater that 2.

2.1.8 Kuratowski Closure and Limit Points

It is extraordinary, but the struggle to understand fully the extraordinary properties of Cartan topological structure only came about recently (Feb-2009). After some 45 years of study, I became aware that the Cartan topology of a 1-form had the topological structure of a Kolmogorov $T0$ (poset 3) topology, where the topological subsets are exterior differential forms. I now call this topology, the Kolmogorov-Cartan T0 Topology, KCT0. An important feature of the KCT0 topology is that its subsets of exterior differential forms support the Kuratowski closure axioms, expressed in terms of the Identity operator and the exterior differential, d:

Theorem 11 *The closure of an exterior differential p-form, Σ, is the union of the p-form and its exterior differential, and is equivalent to the Cartan concept of a differential ideal:*

$$\textit{Relative to the KCT0 topology,} \qquad closure(\Sigma) = \Sigma \cup d\Sigma. \tag{2.16}$$

In effect, the exterior differential, d, is a Limit Point generator. The 2-form $F = dA$ represents the limit "points" of the 1-form of Action, A. It will be demonstrated that in terms of EM notation, the field intensities, $F(\mathbf{E}, \mathbf{B})$ are to be interpreted as the "Limit Points" of the vector and scalar potentials. A similar statement can be made for hydrodynamic systems. A first result is that both Electromagnetic and Hydrodynamic systems obey a Faraday induction law.

2.2 The Kolmogorov-Cartan T0 topology

2.2.1 A Brief Summary of prior work

A topological perspective can be used to extract those properties of physical systems and their evolution that are independent from the geometrical constraints of connections and/or metrics. Monographs Vol. 1 to Vol. 5 contain detailed examples of this development [1]. It is subsumed that the presence of a physical system establishes a *topological structure* on a differentiable variety of independent variables. The concept of the Cartan Topological Structure was developed 1984-1991. The main ideas were presented as a talk given in August, 1991, at the Pedagogical Workshop on Topological Fluid Mechanics held at the Institute for Theoretical Physics, Santa Barbara, UCSB. Part of the Cartan topological truth table was due to Phil

Baldwin. The recognition that the Cartan topology was a disconnected topology is due to the author. In 2009 it was determined that the Topology of (particle) Thermodynamics was a Kolmogorov T_0 topology. This concept is different from, but similar to, the geometric perspective of general relativity, whereby the presence of a physical system is presumed to establish a *metric* on a *diffeomorphic* variety of independent variables, and the dynamics is established in terms of a connection. The diffeomorphic constraints on a differentiable variety are avoided when a topological perspective, not a geometric perspective, is assumed. Note that a given differentiable variety may support many different topological structures simultaneously; hence a given differentiable variety may support many different coexistent physical systems.

For any given 1-form, A, on a 4D differential variety it is possible to construct the Pfaff Sequence $\{A, dA, A\hat{}dA, dA\}$. The Pfaff Topological Dimension, PTD(A), of the 1-form, A, that generates the Pfaff sequence is defined as the minimum number of non-zero terms in the Pfaff Sequence. The non-zero elements in the Pfaff Sequence can be used to define a topological basis of exterior differential forms. The details of such a construction are found in chapter 2.5. The outcome is that the topology of Cartan's exterior differential forms is a Kolmogorov T0 (poset 3) topology.

The Kolmogorov-Cartan T0 (KCT0) topology can be constructed explicitly for an arbitrary exterior differential 1-form, A. All elements of the KCT0 topology can be evaluated quickly. The limit points, the boundary sets and the closure of every subset can be computed abstractly. Earlier, intuitive results, which presumed that the exterior product could be interpreted as a meet (intersection) operator, and the exterior differential could be used as a limit point operator acting on differential forms, can be given formal substance in terms of the KCT0 topological structure. An important fact is that the Kolmogorov-Cartan T0 topology is a disconnected topology for non-equilibrium systems (PTD(A)=4, PTD(A)=3) and is a connected topology for equilibrium systems (PTD(A)=2, PTD(A)=1). A key artifact of non-equilibrium is the existence of Topological Torsion current 3-forms, Topological Spin current 3-forms, and Topological Adjoint interaction current 3-forms. All three 3-forms are similar to the charge-current 3-form densities, J, of electromagnetic theory, but differ in the sense that the exterior differential of each topological current need not be zero. The non-zero divergences of the 3-forms are related to different species of dissipative phenomena, which only occur in non-equilibrium systems.

2.2.2 Continuous Topological Evolution

Previous topological approaches to thermodynamics [26], [13] missed the point that the fundamental topological structure of particle-like thermodynamics is based upon the *disconnected* Kolmogorov T0 topology of 4 points. When written in terms of Cartan's theory of exterior differential forms, the First Law of Thermodynamics (see Chapter 1.3) becomes a dynamical statement of Cohomology, $Q = W + dU$: the difference between two non-exact 1-forms $(Q - W)$ is equal to an exact differential dU.

$$\text{The First Law} \quad W + dU = Q \tag{2.17}$$

The fundamental idea is that a topological analysis of a thermodynamic system can be based upon the Kolmogorov-Cartan T0 topological structure. This KCT0 topology will have subsets defined in terms of exterior differential forms. The KCT0 topological structure can be composed from a (any) 1-form of Action,

$$A(x^k, dx^k) = A_k(x^m)dx^k, \tag{2.18}$$

defined herein on a differential variety, of 4 components, say $\{x, y, z, t)$. The topological structure can be used to determine if an evolutionary mapping is continuous or not.

2.2.3 Continuity

It was noted that topological evolution of discrete collections could occur *continuously* if the limit points of any subset relative to the topology of the initial state was to be found within the closure of the subset relative to final state [151], [220],. For example, the number of disconnected components is a topological concept that can change continuously by "pasting" together (or condensing) various components. The number of components can change discretely by the "pasting" process, but the process is topologically continuous in a formal sense.

Theorem 12 *A map is topologically continuous iff the limit points of every subset in the domain permute into the closure of the subsets in the range.*

It is also possible to define C2 differentiable arrays (vectors and/or spinors) of ordered functions, $J = \rho V_4(x^k)$, that encode thermodynamic processes acting on the various thermodynamic systems. The KCT0 topological structure permits the definition of (topologically) continuous processes (See Vol. 1 [1]) to be evaluated in terms of the Lie exterior differential "propagator" acting on an exterior differential p-form, Σ,

$$\text{"Cartan's Magic Formula"} \quad L_{(\rho \mathbf{V}_4)}\Sigma = i(\rho \mathbf{V}_4)d\Sigma + d(i(\rho \mathbf{V}_4)\Sigma). \tag{2.19}$$

The name "Cartan's Magic Formula" is due to Marsden [161] .

Using the Lie differential, the continuous topological evolution of a p-form, Σ, will yield a p-form, Ξ.

$$L_{(\rho \mathbf{V}_4)}\Sigma = i(\rho \mathbf{V}_4)d\Sigma + d(i(\rho \mathbf{V}_4)\Sigma) = \Xi. \tag{2.20}$$

When applied to a 1-form of Action that defines a thermodynamic system,

$$L_{(\rho \mathbf{V}_4)}A = i(\rho \mathbf{V}_4)dA + d(i(\rho \mathbf{V}_4)A) = Q. \tag{2.21}$$

Using the notation, Work, $W = i(\rho\mathbf{V}_4)dA$ and Internal energy, $U = i(\rho\mathbf{V}_4)A$, it becomes apparent that continuous topological evolution is an abstract dynamical (cohomological) equivalent of the First Law of Thermodynamics. The 1-form of Action, A, "evolves" into the 1-form of Heat, Q, due to the process, $\rho\mathbf{V}_4$:

$$\text{The First Law}: \; L_{(\rho\mathbf{V}_4)}A = i(\rho\mathbf{V}_4)dA + d(i(\rho\mathbf{V}_4)A) = W + dU = Q. \qquad (2.22)$$

This topological perspective separates the concept of a thermodynamic system (defined in terms of a 1-form of Action, A) and the concept of a 4D process (defined in terms of an N-1 form current, J with $\mathbf{J}_4 = \rho\mathbf{V}_4$) acting on the thermodynamic system. For particular processes, $\rho\mathbf{V}_4$, the 1-form of Heat, Q, need not have the same Pfaff Topological Dimension as the 1-form of Action, A. Therefor it is apparent that Cartan's magic formula can encode topological change[‡]. Perhaps surprisingly, and not intuitively, such topological changes can appear to be discrete geometrically, but are topologically continuous.

2.2.4 Non Metrizable Topological Systems

As mentioned above, my mantra over the years had been to go beyond (that is, avoid, or ignore) techniques, or constraints, metric, tensor analysis with affine connections, diffeomorphic and group (symmetry) gauge constraints. Although each of these various geometric constraint disciplines have interesting features, I was convinced, early on, that such constraints impeded the development of physical understanding of thermodynamic irreversibility and biological non-equilibrium evolution to the ultimate state of equilibrium, and death.

Now the 45 years of effort seems to have born fruit, for the Kolmogorov-Cartan T0 (poset 3) topological structure means that the topology is NOT a metric topology, NOT a Hausdorff topology, is not necessarily constrained by symmetries, and is NOT the space of any topological group! I find it more remarkable and pleasing that such a topology apparently can find broad application to the science of thermodynamics, and therefor to other allied physical systems, To repeat, this KCT0 topology of thermodynamics is far afield from the metric based, diffeomorphically constrained, gauge theoretic, group theoretic symmetries found in many of the current physical theories, which, due to the constraints, inherent time reversibility, in disagreement with experiment. Topological change is a necessary condition for thermodynamic irreversibility.

It must be said that in 2010-2011 it became apparent to me that Not-T0 topologies of indistinguishable elements must be included as another component or partition in a universal non-equilibrium thermodynamic category. As will be shown below, the Not-T0 components will correspond to complex wave diffusion of indistinguishable, statistical-like, elements that can exist along side the T0 components

[‡]Note that the covariant differential, based upon the constraint of diffeomorphic processes, cannot describe topological change, as diffeomorphisms preserve topology.

that are associated with distinguishable particle-like elements of a universal thermodynamic system.

2.3 Why Topology?

For many centuries, physical sciences have used geometric mathematical methods to describe physical structures and interpret physical processes. Early on, the key method of measurement was made by means of visual detection and recording of electromagnetic interactions with stationary and moving objects. Little has changed over the centuries, for even though the human eye has been replaced by sophisticated instruments, most methods of measurement and recording still depend upon the use of electromagnetic interactions. Experiments have been designed to measure smooth collective effects (long term exposure of waves) on the one hand, and discontinuous properties of electromagnetic signals on the other.

Although geometric concepts were (and still are) fundamentally useful to an understanding of physical systems and the interpretation of physical processes, it has become apparent that simple experiments indicate that there are certain physical effects that do not rely on the geometric ideas of shape, size, distance, or other ideas related to metric (or to the geometrical specification of the observer's frame of reference). A key example of a collective measurement which is independent from geometrical constraints is given by the fact that all hot objects, at some uniform surface temperature, T, radiate electromagnetic radiation with an envelope distribution of frequencies given by Planck's "blackbody" radiation formula. The Planck distribution formula, as an envelope of electromagnetic radiation frequencies, is independent from the size, shape, or even the structural details of the hot glowing object. A stationary solid ball of gold, a twisted iron branding iron, a ceramic coffee cup, a graphite rod, small hole in a hot furnace, a star, all at the same uniform surface temperature, T, emit the same radiation frequency distribution, and this collective distribution pattern of frequencies depends only on the surface temperature of the object.

Claim 13 *The black body radiation distribution pattern is not dependent upon geometric issues. The understanding of such phenomena requires a topological basis, not a geometrical basis.*

A short list is displayed below of a number of measurable physical effects which - at least to first order - do not depend explicitly on size or shape; therefore, these phenomena must have a topological basis for their explanation. Key topological properties are the number of disconnected parts, the connectivity of the connected parts, and other qualities that involve additive counting (including quantization and topological dimension). Concepts of size and shape, metric, or connection can not enter into the explanation of such phenomena, except, perhaps, in an auxiliary way.

- **Planck's Radiation Formula**

  The distribution law is independent from the size and shape of the hot body.

- **Coaxial Wave-Guide Propagation**

  A hollow wave guide is a high-pass filter - low frequencies are not transmitted. However, a non-simply connected coaxial cable can pass DC as well as high frequencies.

- **Continuous Chaos does not occur in Dimension $\leq$ 2**

  A dynamical system (representing a process in the domain $\{x^k, t\}$), is a collection of ODE constraints of the form

$$\omega^k = dx^k - V^k(x^m, t)dt \Rightarrow 0. \tag{2.23}$$

  Associated with this zero set of differential constraints is a complement 1-form, A, (representing the thermodynamic systems) such that when $\omega^k \neq 0$, it follows that,

$$A\hat{}dA = \omega^1\hat{}\omega^2\hat{}\omega^3 = i[V^k, 1]dx\hat{}dy\hat{}dz\hat{}dt. \tag{2.24}$$

  If the 3-form $A\hat{}dA = 0$, the system of ODE,s admits a unique solution of topological dimension ≤ 2. Self-similarity and continuous chaos are artifacts of non-uniqueness.

- **Bohm-Aharanov effect and the Flux quantum (based on 1-forms)**

  Flux quanta in superconductors come in integer multiples of $\hbar/2$e.

- **Gauss Law and the charge quantum (based on 2-form densities)**

  The integral of the $\iint_{closed} \mathbf{D} \circ \mathbf{dS}$ over a bounding surface only depends upon the number, n, of electrons of charge e = 1.6 x $10^{-19}C$ in the interior of the closed surface, no matter what the size or shape of the closed surface may be.

- **Topological Torsion and Topological Spin (The Poincare quanta based on 3-forms)**

  The Bohm-Aharanov effect is a one-dimensional period integral. Gauss's Law is a two-dimensional period integral. There are two three-dimensional period integrals that have been little studied in science, but appear to have significance in non-equilibrium hydrodynamics and plasmas as robust coherent vortex-like helical structures - insensitive to deformations in space-time. Period integrals are rational valued functions independent from size and shape.

- **Quantum Transition Probability (cross ratios are independent from scale)**

  Fermi's Golden Rule demonstrates that the transition probability is a cross-ratio projective invariant, independent from scale.

- **Thermodynamic Irreversibility (PTD(Q) > 2)**

  The idea that thermodynamic irreversibility is defined by the failure of the Frobenius Theorem implies that Q and its limit points, dQ, are not isolated.

  The difference between a vapor and a liquid is that the liquid is apparently connected and the vapor consists of many disconnected parts. The number of components is a topological property. Condensation implies a change in topology takes place. Condensation is a gluing or pasting process which often can be described continuously. Vaporization, on the other hand, is a discontinuous or cutting process, and does not usually permit a continuous description.

- **The Law of Corresponding States**

  In chemistry there is the law of corresponding states, which demonstrates the universality of thermodynamics, independent from the size and configuration of the molecules under consideration.

2.4 Why Exterior Differential Forms?

In this monograph the emphasis will be placed on the use of Cartan's theory of exterior differential forms to describe topological features that are independent of scales, distance, size, shape, and other concepts that are dependent upon metric. Algebraic collections of differentials, dx^k, with C2 differential coefficients, $A_k(x^k)$ define exterior differential 1-forms. Exterior differential forms set equal to zero define topological constraints, for the concept of a derivative limit is not imposed necessarily on the equations of differential constraints.

Topology teaches that such systems which admit a derivative limit obey the Hausdorff separation axioms, T2, and are said to be metrizable. However there exist more general $T0$ topologies with separation axioms that do not satisfy the axioms (geometric constraints) of $T2$; these T0 topologies are not metrizable over the reals. Indeed, there also are topologies that do not support the $T0$ axioms. Simply said, $T0$ topologies are not metrizable, and therefor admit properties that do not depend upon geometrical constraints.

Claim 14 *The reason that non-equilibrium systems and irreversible processes are so poorly understood in terms of the classical physical theories is due to the imposition of geometric (metric) constraints on the underlying topology.*

2.4.1 Kolmogorov $T0$ Topologies

For low dimensional, $N \leq 4$, Kolmogorov $T0$ topologies of 3 ingredients, there are 3 $T0$ topologies that are connected topologies, and 2 $T0$ topologies that are disconnected topologies. The 2 disconnected topologies have 2 (poset 1), or 1 (poset 2), isolated singletons. (See the Appendix for the complete T0 topological structures of 3, 4 and 5 elements.

For Kolmogorov $T0$ topologies of 4 ingredients, there are 10 $T0$ topologies that are connected topologies, and 6 $T0$ topologies that are disconnected topologies, but only 1 of these 6 disconnected topological structures is without isolated singularities. It is this Kolmogorov poset (3) topology that is used to construct non-equilibrium thermodynamic systems in this monograph; the sets of this topological structure are composed of exterior differential forms. As mentioned above, the discovery of the Cartan topology of four ingredients (and its connection to topological non equilibrium thermodynamics and continuous topological evolution formulated in the period 1984 - 1991) was a serendipity event of intuition. Then, to realize in late 2008, that the Cartan topology of four ingredients was the Kolmogorov T0 poset (3) topology was an encouraging surprise.

2.4.2 Topological Features

Cartan's differential forms have amazing properties not contained in other mathematical entities:

1. Differential forms on the final state are well behaved with respect to functional substitution of differentiable maps from the initial to the final state, even though these maps are irreversible (without inverse)! This fact is known as the "pull-back" and is the cornerstone of the investigations about irreversible processes. By functional substitution, not only is it possible to compute *values* of a differential form on the initial state from values given on the final state, but also it is possible to compute the *functional form* of the differential forms on the initial state. The statement is true for both scalar differential forms as well as for differential form densities. Scalar differential forms are defined (retrodictively) on the initial state in terms of the pullback method of functional substitution, which is equivalent to multiplication of the coefficients of the differential form by the transpose of the Jacobian of the map from the initial to final state. Exterior differential form densities also enjoy the pullback property, but now the pullback is equivalent to multiplication of the coefficients by the adjoint of the Jacobian map from initial to final state. Both the transpose and the adjoint of the Jacobian matrix exist even thought the inverse does not.

2. Differential forms can carry information about singularities, and these singularities dictate much of the topological content of the differential form. The singularity to be studied is not of the type that blows up to infinity, necessarily, but instead is the more innocuous zero set. It is the zero points of functions and vector fields under the action of continuous maps that is of predominant interest herein. The zeros of a map become the infinities of the inverse map, but in this monograph the emphasis is on how much can be determined about processes that need not have an inverse.

3. Differential forms may have pre-images (on the initial state) which are not unique. It is precisely this multi-valuedness of an integral pre-image that al-

lows differential forms to single out the physically interesting characteristics, or wave-fronts, or shock-fronts, or defects of dynamical systems. These topological singularities or defects are point sets (which may or may not be stationary) upon which a unique solution to some system of partial differential equations describing the physical system cannot be analytically continued from a nearby neighborhood. As will be determined later, dimension and the number of components are topological properties, and the generation of a defect will correspond to a topological change of dimension, or the creation of multiple components, over some domain.

4. Continuous Topological Evolution, in terms of Vector and Spinor processes, operating on Thermodynamic systems, can be used to formally account for the "arrow of time". Topological evolution from disconnected sets to connected sets can be continuous (condensation), while topological evolution from connected sets to disconnected sets is discontinuous (evaporation). The combination of topological evolution, disconnected and connected sets, and continuity yield a unidirectional feature to an evolutionary process. Non-equilibrium thermodynamic systems are based on the unique KCT0 poset 3 topology of 4 points (ingredients) and domains where the Pfaff Topological Dimension of the these systems is ≥ 3. Equilibrium systems and isolated equilibrium systems are based on domains where KCT0 topology of 4 points becomes a connected topology. The non-equilibrium systems can decay to equilibrium continuously, but the topological evolution from equilibrium systems to non-equilibrium systems cannot be continuous. KCT0 poset 3 topology of 4 points (ingredients) is the unique topology (of the 16 possible T0 topologies) on 4 points which can be disconnected without isolated singletons. There are 5 other T0 topologies on 4 points that are disconnected topologies, but each has one or more isolated singletons.

5. It is possible to show that processes, V, which are C1 continuous, can be conservative and fractal, relative to the Lie differential acting on the exterior differential 1-form, A, that defines the system. Homogeneous processes lead to self-similar systems, where small sub-domains have recognizable emergent holographic features of the entire domain. In addition, integration with respect to closed p-chains of closed, homogeneous, but not exact p-forms, will lead to values that have rational ratios. This "topological quantization" of mole number (particles), charge number, or spin number is valid at all scales, from the micro-scales of quantum mechanics to the global scales of the cosmological universe. Applications to problems in non-equilibrium plasma systems and turbulent hydrodynamics can be based on the the emergence of 4 topological currents. These currents represent the transport of charge, torsion, and two species of collective spins representing superconductivity (charge) and superfluidity (mass and inertia)

6. For systems constrained by diffeomorphic equivalences (geometric tensor constraints), the coefficients of the p-form intensities behave like the components of a covariant vector, while the coefficients of the p-form density processes behave like the components of a contravariant vector. However, in terms of a topological perspective, such is not the case. The Jacobian matrix of the Intensities form a correlation matrix in the sense of projective geometry. The Jacobian matrix of the processes form a collineation matrix in the sense of projective geometry. The correlation matrix need not be a "polarity", which would require that the matrix be symmetric. In fact, it is the anti-symmetric parts of the correlation matrix that are of most importance to the topological theory of thermodynamics.

2.4.3 Extensive and Intensive Functions from a Topological Perspective

Topological thermodynamics has two distinct categories of exterior differential forms. The first category describes a *process* in terms of a vector (or spinor) direction field, $\mathbf{V}^k$, of 4 ordered functions that form the components of a differential N-1=3 form density, C, or current. The 3-form currents, C, are the topological analogues of the thermodynamic extensive properties. The second category describes a thermodynamic *system* in terms of an ordered array of of 4 functions, $\mathbf{A}_k$, that form the components of an exterior differential 1-form, A. The 1-forms are the topological analogues of the intensive variables in classical thermodynamics.

Ultimately, the exterior differential forms are presumed to be defined in open vector space domains of Pfaff topological dimension 4; these Open domains are defined as the thermodynamic physical environment. The "particulate" matter is defined in terms of the topological closed defect structures, or collective states, of Pfaff topological dimension 3 that emerge from, and interact with, the thermodynamic physical environment. The process current that induces this emergence is dissipative and irreversible.

For example, the number of components is a topological concept that can change continuously by "pasting" together (or condensing) various components. The number of components can change discretely by the "pasting" process, but the process is topologically continuous in a formal sense.

It can be demonstrated that the closure of a differential form, Σ, (in the KCT0 topology) is equivalent to the the differential ideal, $\{\Sigma \cup d\Sigma\}$, and limit points are equivalent to $d\Sigma$. An extended discussion of topological continuity is to be found in subsequent chapter where it is demonstrated that there is a common topological thread that links the sciences of Thermodynamics, Hydrodynamics, and Electrodynamics [1]. Both Hydrodynamics and Electrodynamics (as well as almost any other of the physical specializations) have a topological foundation in terms of Thermodynamics. As will be shown below, topological thermodynamics for particles can be built upon:

i: a 1-form of Action, $A(x, y, z, t, dx, dy, dz, dt)$, that encodes a specific Thermodynamic System, and

ii: a set of vector-spinor direction fields, $V(x, y, z, t)$, that define the Dynamic Processes acting on the specific Thermodynamic System.

iii: A Kolmogorov-Cartan T0 topology with subsets in terms of exterior differential forms.

The methods lead to precise, non-statistical, methods for determining when a process, $V(x, y, z, t)$, applied to a specific thermodynamic system encoded by a 1-form of Action, $A(x, y, z, t, dx, dy, dz, dt)$, is

1. Thermodynamically irreversible or not,
2. Adiabatic or not.
3. Adiabatically irreversible or reversible.

2.5 Closure, Cohomology and Homology

In this section, the properties briefly described above will be presented in a bit more detail and with a number of examples. The topological structure of interest is the Kolmogorov-Cartan T0 topology constructed in terms of sets of differential forms. As shown below, in this topology the exterior differential acting on an exterior differential p-form Σ generates the limit sets $d\Sigma$ (of exterior differential p+1forms) for the form, Σ. The (Kuratowski) closure of the p-form is defined as the Union of Σ and $d\Sigma$:

$$K_{Cl}(\Sigma) = \Sigma \cup d\Sigma. \tag{2.25}$$

Cartan never defined his topological structure explicitly, but he did refer to the closure of a p-form, Σ as the union of Σ and its exterior differential, $d\Sigma$. It is apparent that the use of exterior differential forms as the sets of a topological space emphasizes the concept of Cohomology, rather than concept of Homology. For an arbitrary topological space:

$$\begin{aligned}
\textbf{Cohomology} &: \quad \text{Closure of Set } \ Cl(\Sigma) = \Sigma \cup d\Sigma, &(2.26)\\
& \quad \{\text{Set}\} \cup \{\text{Limit points of the Set}\} &(2.27)\\
\textbf{Homology} &: \quad \text{Closure of Set } \ Cl(\Sigma) = \Sigma^0 \cup \partial\Sigma, &(2.28)\\
&= \{\text{interior of the Set}\} \cup \{\text{boundary of the Set}\} &(2.29)
\end{aligned}$$

The use of exterior differential forms as the topological sets of the KCT0 topology, demonstrates that the exterior differential is indeed a limit point generator.

It is important to note that the concepts cohomology and homology are distinct for metrizable properties, but the closures are equal for all three topological categories.

2.5.1 Ordered Arrays and Differential Form Densities

The exterior differential forms are presumed to be defined on a variety of 4 independent functions and their differentials, say $\{x, y, z, t; \quad dx, dy, dz, dt\}$. It is presumed that there exists a maximal system with a differential volume element $\Omega_4 = dx\hat{}dy\hat{}dz\hat{}dt$, upon which can be described a differential p-form density, $\rho_4(x^k)\Omega_4 = \rho_4(x^k)dx\hat{}dy\hat{}dz\hat{}dt$. The p-form density can be sensitive to a permutation of the ordering (˜orientation). The ordering is important, for $dx\hat{}dy\hat{}dz\hat{}dt = -(dy\hat{}dx\hat{}dz\hat{}dt$. Depending upon special properties of the density coefficient, $\rho_4(x^k)$, the differential p-form densities may, or may not, be sensitive to ordering (orientation). If the p-form density is sensitive to orientation it is described as an "impair-p-form density"; If the p-form is not sensitive to orientation it is described as an "pair-p-form density". For example, the 2-form density G associated with discrete charge in electromagnetic theory is impair, and exhibits both plus values and minus values of charge. The 2-form density associated with mole number (baryons) in fluid dynamics is pair, as "mass" is positive definite.

2.5.2 The Pfaff Sequence

As mentioned above, for any 1-form, A, on a differentiable variety, it is possible to construct a Pfaff Sequence:

$$\textbf{Pfaff sequence} \quad : \quad \{A, dA, A\hat{}dA, dA\hat{}dA\}, \tag{2.30}$$

$$= \quad \{A, F = dA, \mathrm{H} = A\hat{}F, \mathrm{K} = F\hat{}F\}. \tag{2.31}$$

Surprisingly, for a 1-form written in terms of the 4D base variables,

$$A = \sum_{k=1-4} A_k(x, y, z, t)dx^k, \tag{2.32}$$

the functional format of the coefficients, $A_k(x, y, z, t)$, will determine how many non-zero entries appear in the Pfaff sequence. This fact will be used to define the Pfaff Topological Dimension..

What became apparent to me (and Phil Baldwin, a post doc at the University of Houston) about 1990 was that it was possible to construct a topological structure in terms of the properties of the exterior differential form elements in the Pfaff Sequence (see Chapter 6, Vol. 1 [1]). The subsets of the Cartan topological space consist of all possible unions of the subsets that make up the Pfaff sequence. The Cartan topology was constructed from a topological basis which consists of the odd elements of the Pfaff sequence, and their closures,

$$\text{the Cartan topological basis} : \{A, K_{Cl}(A), A\hat{}dA, K_{Cl}(A\hat{}dA)\}. \tag{2.33}$$

Cartan referred to the union of Σ and $d\Sigma$ as the "closure" of Σ, which is agreement with the Kuratowski closure axioms. As mentioned above, the exterior differential can be considered to be a limit point generator:

$$\text{Kuratowski Closure} = K_{Cl}(\Sigma) = (I \cup d) \circ \Sigma = \Sigma + d\Sigma = \text{subset} + \text{limit points}. \tag{2.34}$$

A most important feature of the Cartan topological structure (detailed below), is that it turned out to represent a "Disconnected" topology – a surprise that startled me.

Remark 15 *The concept of the Cartan Topological Structure was developed in the period 1984-1991. The main ideas were presented as a talk given in August, 1991, at the Pedagogical Workshop on Topological Fluid Mechanics held at the Institute for Theoretical Physics, Santa Barbara UCSB. Part of the Cartan truth table was due to an assignment suggested to Phil Baldwin. The recognition that the Cartan topology was a disconnected topology is due to R. M. Kiehn.*

This fact allowed the concept of irreversibility to be well defined with respect to C2 continuous processes. The arrow of time is to be associated with the fact that Continuous Evolutionary Mappings of a disconnected topology to a connected topology are possible, but Continuous Evolutionary Mappings of a connected topology to a disconnected topology are **impossible**! Continuity and Topological Evolution are unidirectional.

For emphasis, I repeat a previous paragraph. Only recently, (Feb 2009), did I appreciate that the Topological Structure on 4 ingredients that Baldwin had worked out, and the Cartan Topological Structure that I suggested, - a topology whose sets were exterior differential forms - was in fact a Kolmogorov T0 space poset 3 with disconnected components! This means that the topology is NOT a metric topology, NOT a Hausdorff topology, does NOT satisfy the separation axioms to be a T1 topology, is not necessarily constrained by symmetries, and is NOT the space of any topological group! I find it more remarkable that such a primitive topology could find broad application to the science of thermodynamics. This T0 topology of thermodynamics is far afield from the metric based, diffeomorphically constrained, gauge theoretic, group theoretic symmetries found in many of the current physical theories.

2.5.3 Examples of the Pfaff Topological Dimension of a 1-form, A

Perhaps one of the most important topological tools to be used within the theory of the Kolmogorov-Cartan T0 spaces is the concept of Pfaff Topological Dimension. The PTD(A) is equal to the number of non-zero entries in the Pfaff Sequence. The maximum Pfaff Topological Dimension (or class of the form) is 4 on the 4D base variety of "coordinate functions".

For a given 1-form of Action,

$$A=\sum_{k=1-4}A_k(x,y,z,t)dx^k, \tag{2.35}$$

defined on the 4D base differentiable variety of $\{x,y,z,t;dx,dy,dz,dt\}$, it is possible to ask what is the irreducible minimum number of independent functions, $\theta^i(x,y,z,t)$, required to describe the topological features that can be generated by the specified 1-form, A. This irreducible number of functions gives topological importance to the

PTD(A). It is remarkable that the irreducible Pfaff Topological Dimension for any given 1-form A is readily computed by constructing the Pfaff Sequence of forms,

$$\textbf{Pfaff sequence for A} \quad = \{A, dA, A\hat{}dA, dA\hat{}dA\}, \tag{2.36}$$

and determining the number of non-zero entries in the sequence.

Example 1: PTD(A) = 1

For example, if only one C2 differentiable function, $\theta(x, y, z, t)$, is required to describe the Action:

$$\begin{aligned} A &= A_k dx^k \Rightarrow d\theta(x, y, z, t)_{irreducible}, & (2.37)\\ \text{such that } A_k &= \partial\theta(x, y, z, t)/\partial x^k, & (2.38)\\ A &= d\theta(x, y, z, t),\ dA = 0, \quad A\hat{}dA = 0,\ dA\hat{}dA = 0, & (2.39) \end{aligned}$$

then the Pfaff Sequence becomes

$$\text{Pfaff sequence} \quad = \{A, 0, 0, 0\} = \{d\theta, 0, 0, 0\}, \tag{2.40}$$

and has only 1 non-zero term. The PTD(A) = 1, even though the number of independent "coordinate" functions, $\{x, y, z, t\}$ is 4. Note that no metric or other geometric constraint is attached to the basis variety.

Example 2: PTD(A) = 2

For example, if the Pfaff sequence for a given 1-form A is $\{A, dA, 0, 0\}$ in a region $U \subset \{x, y, z, t\}$, then the Pfaff Topological Dimension of A is 2 in the region, U. The 1-form A, in the region U, then admits a topologically faithful description in terms of only 2, but not less than 2, independent variables, say $\{u^1, u^2\}$. For a differentiable map φ from $\{x, y, z, t\} \Rightarrow \{u^1, u^2\}$, the exterior differential 1-form defined on the target variety U of 2 pre-geometry dimensions as

$$A(u^1, u^2) = A_1(u^1, u^2)du^1 + A_2(u^1, u^2)du^2, \tag{2.41}$$

has a functionally well defined pre-image $A(x, y, z, t)$ on the base variety $\{x, y, z, t\}$ of 4 pre-geometric dimensions. This functionally well defined pre-image is obtained by functional substitution of u^1, u^2, du^1, du^2 in terms of $\{x, y, z, t\}$ as defined by the mapping φ. The process of functional substitution is called the pullback,

$$A(x, y, z, t) = A_k(x)dx^k \Leftarrow \varphi^*(A(u^1, u^2)) \Leftarrow \varphi^*(A_\sigma du^\sigma). \tag{2.42}$$

It may be true that the functional form of A yields a Pfaff Topological Dimension equal to 2 globally over the domain $\{x, y, z, t\}$, except for sub regions where the Pfaff dimension of A is 3 or 4. These sub regions represent topological defects in the almost global domain of Pfaff dimension 2. Conversely, the Pfaff dimension of A could be 4 globally over the domain, except for sub regions where the Pfaff dimension of A is 3, or less. These sub regions represent topological defects in the almost global domain of Pfaff dimension 4. Applications of both viewpoints will be described below. The important concept of Pfaff Topological Dimension also can be used to define equivalence classes of physical systems and processes.

Example 3: PTD(A) =3, or 4

When the 3-form $A\hat{}dA$ is not zero, the thermodynamic system is not uniquely integrable. If the limit points (sets), $d(A\hat{}dA) = 0$, then the PTD(A) =3. If solutions exist, they are not uniquely determined from a unique set of initial data. Such solutions are not deterministic. The non-unique solutions take the form of envelopes and edges of regression. These solutions have been called singular solutions, but it is apparent in the light of more modern physics that these non-unique solutions are "Collective states". I have defined the three form, $A\hat{}F = A\hat{}dA$, Topological Torsion. If the Limit points of the Topological Torsion 3-form are not zero, $d(A\hat{}dA) = F\hat{}F$, then the 4D volume element, $F\hat{}F = 2(\mathbf{E}\circ\mathbf{B})dx\hat{}dy\hat{}dz\hat{}dt \neq 0$ exhibits irreversible dissipation of expansion or contraction, and the PTD(A)=4. In hydrodynamics notation, the dissipation coefficient becomes the "bulk viscosity coefficient" [70].

2.5.4 Refinement Examples due to the PTD(W) of the Work 1-form,

It is also possible to examine the Pfaff Topological dimension for any 1-form. For example, the Work 1-form, W_{work}, and the Heat 1-form, Q_{heat}, are of interest. The thermodynamic system encoded by the Action 1-form, A, does not depend (explicitly) upon an evolutionary process, V. The 1-forms of Work and Heat depend upon the concept of a process, V, as well as the thermodynamic system, A. These 1-forms depend dynamically upon the process direction field, V. The topological properties of the 1-forms, W and Q, will "refine" the topology of the thermodynamic system.

Consider a special case where the Work 1-form, W_γ, is closed, but not exact, and constructed in terms of two independent functions, $\Phi(x, y, z, t)$ and $\Psi(x, y, z, t)$.

$$W_\gamma = i(V)dA = (\Phi d\Psi - \Psi d\Phi)/(a\Phi^2 + b\Psi^2), \tag{2.43}$$

$$dW_\gamma = 0 \quad mod \quad \text{zeros of } \lambda(\Phi, \Psi) = (a\Phi^2 + b\Psi^2), \tag{2.44}$$

$$W_\gamma\hat{}dW_\gamma = 0, \quad dW_\gamma\hat{}dW_\gamma = 0. \tag{2.45}$$

This representation for the 1-form W_γ is closed, but not exact, which requires that the (example) divisor, $(a\Phi^2 + b\Psi^2)$, to be homogenous of degree 2 in the independent functions, $\Phi(x, y, z, t)$ and $\Psi(x, y, z, t)$. The divisor, which makes the 1-form homogeneous of degree zero, has an infinite number of realizations, not just $\lambda = (a\Phi^2 + b\Psi^2)$. For example, $\lambda = (a\Phi^p + b\Psi^p)^{2/p}$ (which is a Holder Norm of degree 2) will generate similar results of closure, $dW_\gamma = 0$, for any choice of constants, a, b, p.

Note that for a closed but not exact 1-form, only the first term in the Pfaff sequence is non zero; hence the Pfaff Topological dimension of W_γ is PTD(W_γ) = 1.

Next, for example purposes, construct another representation for the Work 1-form , W, using W_γ plus other independent functions. There are three non-zero terms in the Pfaff sequence for this construction. Note that the 3-form $W\hat{}dW$ is not necessarily zero.

$$\widehat{W} = \Gamma(x,y,z,t)dW_\gamma + d\Theta, \tag{2.46}$$
$$d\widehat{W} = d\Gamma\hat{}W_\gamma \tag{2.47}$$
$$\widehat{W}\hat{}d\widehat{W} = d\Theta\hat{}d\Gamma\hat{}W_\gamma, \tag{2.48}$$
$$d\widehat{W}\hat{}d\widehat{W} = 0. \tag{2.49}$$

Hence the PTD($\widehat{W}$) for the modified example above is 3. Topological structures which are Pfaff Topological Dimension 3 are associated with Contact Manifolds (locally). (Topological structures of Pfaff Topological Dimension 4 are associated with Symplectic Manifolds - locally.)

Note that for thermodynamic systems (to be detailed below) the representation for the Work 1-form is dictated by the First Law of Thermodynamics. The work 1-form, W, (as well as the Heat 1-form, Q) depends upon **both** the 1-form of Action, A per unit source, and the process direction field, V:

$$\text{Thermodynamic Work 1-form, } W = i(V)dA. \tag{2.50}$$

The physical thermodynamic dimensions of A are Action (angular momentum) per "particle property". The physical thermodynamic dimensions of W are energy per "particle". To generalize a comment of A. Sommerfeld, "particle" is best expressed as mole number (which can be coagulates of particles, spin, and charge, representing nuclei, or molecules or even galaxies). The mole number is the number of coherent topological structures in the topology.

The 3-form of Work, $W\hat{}dW$, will be defined as the Topological Torsion 3-form of Energy, analogous to the definition of the 3-form, $A\hat{}dA$, as the Topological Torsion 3-form of Action. For the given example above, it is apparent that the induced 3-form of Topological Torsion for Energy consists of the exterior product of two exact 1-forms, and one closed, but not exact, 1-form. As such, the topological parity, $d\widehat{W}\hat{}d\widehat{W}$, of this energy-dynamic system is zero. The closed, but not exact, 3-form, $W\hat{}dW$, therefor represents a closed current density, but is not a 3-form monomial (volume element).

$$\text{Topological Torsion} \; : \; \text{based on Work 1-form} \tag{2.51}$$
$$\widehat{W}\hat{}d\widehat{W} = (\Phi d\Psi - \Psi d\Phi)\hat{}d\Theta\hat{}d\Gamma/(a\Phi^2 + b\Psi^2) \tag{2.52}$$
$$d\widehat{W}\hat{}d\widehat{W} = 0 \tag{2.53}$$

2.5.5 Limit cycles

Now consider the special case where $\Gamma = f(h)$ is some function of the (specific) divisor, $a\Phi^2 + b\Psi^2$. Then the topological torsion for energy 3-form for energy, $\widehat{W}\hat{}d\widehat{W}$, becomes

$$\begin{aligned} \widehat{W}\hat{}d\widehat{W} &= (\partial f/\partial h) d\Theta\hat{}dh\hat{}W_\gamma, && (2.54) \\ d\widehat{W}\hat{}d\widehat{W} &= 0. && (2.55) \end{aligned}$$

Although the example 3-form is not zero almost everywhere, the parity 4-form is zero globally. For special choices of the function, h, the 3-form of topological torsion for energy also vanishes. If, for example,

$$\begin{aligned} h^2 &= a\Phi^2 + b\Psi^2 && (2.56) \\ f(h) &= (b + h - h^3/3), && (2.57) \end{aligned}$$

then the zeros of $\partial f/\partial h$ generate an elliptical orbit in the two dimensional plane defined by Φ and Ψ. For $a = b = 1$, this orbit is a limit cycle with a circular orbit, of radius 1.

$$\begin{aligned} \partial f/\partial h &= 1 - h^2 \Rightarrow 0, && (2.58) \\ a\Phi^2 + b\Psi^2 &= 1 && (2.59) \end{aligned}$$

The Topological Torsion of energy vanishes on the limit cycle, which defines a subset of PTD(W) = 2.

Note that the limit cycle can be an attractive orbit or a repelling orbit depending upon the function $f(h)$. The fundamental idea is that the limit cycle is the evolutionary limit of a topological process that evolves with a topological change of PTD = 3 to PTD = 2.

If the function $f(h)$ has no zeros, the Space of Pfaff Topological dimension 3 is said to be a "tight" Contact structure; When limit cycles exist, the space of Pfaff Topological Dimension 3 is said to be an "overtwisted" Contact structure .

Conclusion 16 *The production of a thermodynamic limit cycle corresponds to the topological evolution of a system of Pfaff Topological Dimension 3 to a system of Pfaff Topological dimension 2.*

I will come back to this analysis and apply it to thermodynamic processes where the Heat 1-form Q is of Pfaff topological Dimension 2, but the Work 1-form, W, is of Pfaff topological dimension 3. In such systems, process paths which are C1 differentiable can exhibit behavior that is different from the behavior of C2 process paths. In fact the process paths that are C1 appear to be reversible, and the process paths that are smooth C2 appear to be irreversible.

2.6 Topological Torsion, and Topological Parity

The concept defined herein as the "Pfaff Topological Dimension" was developed more than 110 years ago (see page 290 of Forsyth [82]), and has been called the "class" of a differential 1-form in the mathematical literature. The term "Pfaff Topological Dimension" (instead of class) was introduced by me in order to emphasize the topological foundations of the concept. More mathematical developments can be found in Van der Kulk [255]. The method and its properties have been little utilized in the applied world of physics and engineering, where most classical analysis is only in "equilibrium regions" or uniquely integrable regions of PTD(A) <3.

Of key importance is the fact that the non-zero existence of the 3-form $A\hat{}dA$, or,

$$\textbf{Topological Torsion for } A, \quad \mathrm{H} = A\hat{}F, \tag{2.60}$$

implies that the Pfaff Topological Dimension of the region is 3 or more, and the non-zero existence of the 4-form of *Topological Parity*, $dA\hat{}dA = F\hat{}F$ implies that the Pfaff Topological Dimension of the region is 4. Either value is an indicator that the physical system (in the sub region) is NOT in thermodynamic equilibrium. It is also important to recall that non-zero values of Topological Torsion imply that the Frobenius unique integrability Theorem for the Pfaffian equation, $A = 0$, fails. The concept of *topological parity*, $F\hat{}F$, has its foundations in the theory of Pfaff's problem, with a recognizable four-dimensional formulation appearing in Forsyth [82] page 100. On a variety of 4 variables, the coefficient of the 4-form $F\hat{}F$ will be defined as the topological parity (or orientation) function, K, such that

$$\textbf{Topological Parity for } A, \quad \mathrm{K} = F\hat{}F = \sigma_4 dx\hat{}dy\hat{}dz\hat{}dt = \sigma_4\Omega_4. \tag{2.61}$$

It is possible to ascribe the idea of entropy production (due to bulk viscosity) to the coefficient σ_4 of the Parity 4-form.

The idea of *Topological Torsion*, $A\hat{}F$, has been associated with the idea of magnetic helicity density, a concept that apparently had its electromagnetic genesis with the study of plasmas in WWII. However, the concept of helicity density is but one component of the four-dimensional *Topological Torsion 4-vector.*

Recall that a space curve with non-zero Frenet-Serret torsion does not reside in a two-dimensional plane. Non-zero Frenet-Serret torsion of a space curve is an indicator that the *geometrical* dimension of the space curve is at least 3. The fact that the Pfaff Topological Dimension of the 1-form, A, is at least 3, when $A\hat{}F$ is non-zero, is the basis of why the 3-form, $A\hat{}F$, was called "Topological Torsion". The idea of a non-zero 3-form $A\hat{}F$ also appears in the theory of the Hopf Invariant [30].

The concept of $A\hat{}F$ has also appeared in the differential geometry of connections, where a matrix valued 3-form is known as the Chern-Simons 3-form. However, on varieties without connection or metric, the Chern-Simons concept is not well defined, but the Topological Torsion concept exists and is acceptable, for it does not depend upon the geometric features of metric and/or connection. The concepts can be extended to "pre-geometrical", and therefore topological, domains of dimension greater than 4. Pre-geometry implies that constraints of metric or connection have not been (necessarily) imposed on the base variety.

It is possible to define a "curvature" dimension (at a point) in terms of the number of non-null eigenvectors of the Jacobian matrix built from the partial derivatives of the C1 functional components that define the 1-form of Action. The "Curvature" dimension is always less than the dimension of the base variety. The implication is that the determinant of the shape matrix is zero. It is possible that the Pfaff Topological Dimension can exceed the "curvature" dimension.

The idea of the Pfaff Topological Dimension is analogous to the idea of the number of "essential parameters" in the theory of continuous semi-groups [73].

Chapter 3

THE KCT0 TOPOLOGY AND UNIVERSAL THERMODYNAMICS

3.1 Preface to Chapter 3

The bulk of this chapter is used to demonstrate how Kolmogorov-Cartan T0 poset 3 Spaces of Exterior Differential Forms can be used to form a universal foundation for both equilibrium and non-equilibrium thermodynamics. It will be demonstrated explicitly how any exterior differential 1-form, $A\{\xi^k, d\xi^k\}$, defined on a differential variety of N=4 points or less, $\{\xi^k, d\xi^k\}$, can be used to generate a topological structure.

The work will be based upon lattice structures of sets, which may or may not generate topologies. Those lattice structures that generate finite topologies may be of category I $\{= Not - T0\}$, category II $\{= T0\ \}$.

Each $T0$ topology of category II can be separated into 3 parts: a part A, that contains the connected $T0$ topologies; a part B, that contains disconnected $T0$ topologies with isolated singletons; and part C, that contains disconnected $T0$ topologies without isolated singletons.

Although the 4D $T0$ topology of interest to non-equilibrium thermodynamics is the unique disconnected $T0$ Topology without isolated singletons (conventionally labelled as Poset 3) it is useful to study all of the category II topologies and the category I topologies in detail. To this end, Appendix A includes the results of a Maple program used for computing the topological structure of T0 topologies from a given lattice structure built on the power set of N singletons. Results are given for N= 3, 4, and 5. An immediate result is to note that $T0$ topologies are nested in the sense that $N-1$ topologies of connected and disconnected topologies can be embedded subsets of those N topologies which are connected. In particular note that all N=3 $T0$ topologies (which consist of 2 Disconnected topologies and 3 Connected topologies) can be embedded in N=4, but in N=4, all of the embedded $T0$ topologies are now connected. Hence, the KCT0 poset 3, *disconnected*, lattice structure in N=4 can be embedded in N=5 and becomes a *connected* $T0$ topology in N=5. In addition the poset structure for this N=5 embedding, based upon distinct singleton closures in 5D, admits 3 Lattice branches (or loops).

There are 4 types of $T0$ structures depending on the Pfaff Topological Dimension, $n \leq N = 4$, of the 1-form, A. The $T0$ topological structures on N=3 points, can be embedded in 4D, but $T0$ topological structures which are Disconnected in 3D are

Connected in 4D. Isolated points in 3D can be connected in 4D. As the differential variety evolves, domains in the range can cause the PTD of a given exterior 1-form to change (topologically). In fact, disconnected patches in the 4D environment can be connected patches in 3D. Such is the foundation of the concept of Emergence, such as the formation of liquid droplets from a gas.

This topological structure of the KCT0 poset 3 whose sets are Exterior Differential Forms can be used to determine if processes are continuous or not. Recall that it is a topological theorem that processes between disconnected components to other connected components can be topologically continuous, while processes from a connected component to a disconnected component cannot be continuous.

It will be assumed that the differential variety of "measurement" is the ubiquitous 4D differential variety of three spatial and one temporal differentiable functions $\{x, y, z, t\ ;\ dx, dy, dz, dt\}$ which are pre-geometric in the sense that the functions are considered *not* to be constrained by scales, metric, or shape. The differential variety $\{\xi^k; d\xi^k\}$ may be of Dimension N ≥ 4, and if so it will be presumed that there exist C1 differentiable maps that connect the $\{x^k; dx^k\}$ variety to the $\{\xi; d\xi^k\}$ variety. These mapping functions are not invertible if the dimension N is greater than 4. However, the magic of using differential forms is that any p-form on the differential variety $\{\xi^k; d\xi^k\}$ is well-defined on the variety $\{x^k; dx^k\}$ by means of functional substitution (the Pullback in the parlance of differential forms). The method transcends the diffeomorphic equivalences used in tensor analysis, which do not admit topological change. The utility of the higher dimensions is that they sometimes they permit easier computations. For example any curved Riemannian variety can be mapped into a variety of higher dimension which is flat.

For purposes of simplicity attention will be focused on exterior differential forms defined on the 4D differential variety of physical measurement. A 1-form of Action, $A\{x^k; dx^k\}$ and its Pfaff Sequence of differential forms, $\{A,\ F = dA,\ H = A\hat{}F,\ K = F\hat{}F\}$ can be used to generate a basis for a topological structure. The exterior differential, d, acting on a p-form, Σ, of the topological structure, is a generator of the Limit Sets of the p-form, and is expressed in terms of the elements of a p+1-form, $d\Sigma$. Continuous Topological Evolution can be described in terms of the Lie differential with respect to a process Vector (or Macroscopic Spinor) direction field*, V, acting on the 1-form, A, which has been chosen to encode the thermodynamic system. The method develops a cohomological, universal, dynamical equivalent of the First Law of Thermodynamics:

$$L_{(\mathbf{V_4})}A = Q = i(\mathbf{V_4})dA + d(i(_\mathbf{4})A) = W + dU. \tag{3.1}$$

Recall that all of the variables, including the Heat and Work 1-forms and their coefficients, are well defined functions on the differential variety. The process is a 4 component direction field, $\mathbf{V_4}(x, y, z, t)$, the Action is a 1-form, $A\{x^k; dx^k\}$, the incremental Heat is a 1-form $Q\{x^k, dx^k\}$, the incremental Work is a 1-form, $W\{x^k; dx^k\}$,

*A direction field vector array of functions may or may not be an element of a semi-group.

the internal energy, U, is function of $\{x^k\}$. Every thing is well defined on the 4D differential variety of measurement, $\{x^k; dx^k\}$, without the geometric constraints of metric, connection, and gauge.

As such, the Kolmogorov-Cartan Topology gives credence to the idea that thermodynamics is one of the most fundamental physical laws, independent from the geometric constraints of metric, connection, scales and symmetry. It will be shown that C2 Macroscopic Spinor processes are the source of kinematic fluctuations, $d\xi^k - V^k dt \neq 0$, and thermodynamic irreversibility implies that $Q\hat{}dQ \neq 0$.

3.2 Algebraic and Differential Closure

The concept of closure is one of the most important ideas in Cartan's theory. His methods center on two procedures of closure, one algebraic, and one differential. Both processes are closed in the sense that when they operate on a subset of a set of exterior differential forms, the objects created are also subsets of the set of exterior differential forms. There are no surprises. Cartan utilized the exterior algebra over a variety of dimension N to construct a vector space of exterior differential forms of dimension 2^N. The N subspaces of this (Grassmann) space are vector spaces of dimension equal to N things taken p at a time. The elements of the subspaces are called p-forms. In four dimensions, the subspace sets are one dimensional, N = four dimensional, N(N+1)/2 = six dimensional, N = four dimensional, and one dimensional. The elements of the subspaces are often called scalars (0-forms), vectors (1-forms), tensors (2-forms), pseudovectors (3-forms), and pseudo-scalars† (4-forms) in relativistic physical theories. The Exterior (Grassmann) algebra has a finite 2^N-basis (equal to 16 elements in a space of 4 independent variables). The concept of closure means that the operations on elements of the 2^N-dimensional space yield results that are contained within the 2^N-dimensional space. When the operations are applied to elements of a subspace, the results usually are not contained in the same subspace, but they are contained within the 2^N-dimensional vector space of p-forms.

The exterior product (with symbol ^) takes elements of the 2^N-base space and multiplies them together in a manner such that the result is contained as an element of the 2^N-base space. This process of exterior multiplication is closed, for the action of the process on any subset of the 2^N-base space produces another subset of the 2^N-base space. However, the exterior product takes a p-form times a q-form into a (p+q)-form. The elements of the product can be from different or from the same vector subspaces, but the resultant is always a linear combination of the subspaces of the Exterior algebra.

Similarly the concept of exterior differentiation (with symbol d) is defined such that the operation produces a (p+1)-form from a p-form. This process of exterior differentiation is "closed", for the action of the process on any subset of the 2^N base

†Distinctions between differential form Scalars and differential form Densities will modify this terminology

space produces another subset of the 2^N base space. A differential ideal is defined as the union of a collection of given p-forms and their exterior differentials.

An "interior" product with respect to a direction field $\mathbf{V}$ (with symbol $i(\mathbf{V})$ and of dimension N) can be defined on the Grassmann algebra of exterior differential forms. The interior product takes a p-form to a (p-1)-form, and in this sense is another operation which is closed within the Grassmann algebra. The resultant product is still an element of the 2^N-base space. Where the exterior differential raises the rank of a p-form to a (p+1)-form, the inner product lowers the rank of a p-form to a (p-1)-form. (There are other useful operators that lower the rank of the exterior differential p-form, and involve integration.)

By composition of the exterior differential and the inner product operators, the Lie differential operator (with symbol $L_{(\mathbf{V})} = i(\mathbf{V})d + di(\mathbf{V})$) can be constructed, such that when the Lie differential operates on an exterior p-form, the resultant object is another p-form. For a 1-form of Action, A, the process reads,

$$L_{(\mathbf{V_4})}A = i(\mathbf{V_4})dA + d(i(\mathbf{V_4})A) = Q. \tag{3.2}$$

The resultant is not only closed relative to the Grassmann algebra, it also remains within the same Grassmann vector subspace. The Lie differential does not depend upon a metric nor upon a connection. When the Lie differential acting on a p-form vanishes, the p-form is said to be an invariant of the process, $\mathbf{V}$. When the Lie differential of a p-form does not vanish, the topological features of the resultant p-form permit the processes, $\mathbf{V}$, that produce such a result, to be put into equivalence classes, depending on the Pfaff dimension of the resultant heat 1-form. For example, if in the formula given above for a 1-form, A, yields a result Q such that $dQ = 0$, then the process $\mathbf{V}$ belongs to the class of process known as Hamiltonian processes in mechanics, and to the Helmholtz class of processes that conserve vorticity in Hydrodynamics. Of particular interest to this monograph are processes where Q is of Pfaff dimension greater than 2. The Pfaff sequence constructed from Q contains three or more elements. Such processes, V, that produce Heat 1-forms, Q, which are of Pfaff Topological Dimension 3, are thermodynamically irreversible.

The Lie differential will be used extensively in physical applications of Cartan's theory, especially to the study of processes that involve topological evolution. The perhaps more familiar covariant derivative, highly constrained by connection or metric assumptions, is a special case of the Lie differential. The use of the covariant derivative leads to useful, but limited, physical theories for which the description of topological evolution is awkward, if not impossible.

Even more remarkable in a thermodynamic sense is the comment made by Mason and Woodhouse (see p. 49 [176] and also [10]):

Remark 17 *"Then there is a Higgs field ϕ_V associated with each conformal Killing vector $V \in \mathfrak{h}$, (the Lie algebra of H) which measures the difference between the Covariant derivative along V and the Lie derivative along V."*

The implication is that the concept of a Higgs field represents the difference between a process that is not dependent upon the constraint of a gauge group (the Lie differential), and a process that is restricted to a specific choice of a connection defined by some gauge group, (the Covariant differential).

For the cases where $(i(fV)A) = 0$, (which corresponds to processes that do not change the internal energy, U) the two differentials are equivalent. It follows that

$$L_{(fV)}A = f\ L_{(V)}A + d(\ln f)\ (i(fV)A), \tag{3.3}$$
$$= f\ L_{(V)}A = f \cdot i(V)dA = f \cdot \nabla_{(V)}A = f\ Q. \tag{3.4}$$
$$\text{But then, }\ i(V)Q = f\ i(V)i(V)dA \Rightarrow 0. \tag{3.5}$$

Therefor the process, V, is a null orbit of the heat 1-form, $i(V)Q = 0$, which defines an adiabatic process [13]. The general adiabatic condition implies that all exchanges of Heat are transverse to the process. A strong adiabatic condition is defined when there is no heat exchange, $Q = 0$.

Theorem 18 *Hence, all covariant derivatives with respect to an affine connection have an equivalent representation as an adiabatic process!!! (Such covariant adiabatic processes need not be thermodynamically reversible.)*

Suppose that the covariant process satisfies the strong adiabatic condition,

$$L_{(V)}A = \nabla_{(V)}A = Q \Rightarrow 0. \tag{3.6}$$

Then,

$$d(L_{(V)}A) = L_{(V)}dA = dQ \Rightarrow 0, \tag{3.7}$$
$$Q\hat{}dQ = 0 \tag{3.8}$$

and it follows that the covariant (adiabatic) process is reversible. However, the strong covariant condition, $Q \Rightarrow 0$, is the equivalent to the condition of parallel transport:

$$L_{(V)}\omega \Rightarrow \nabla_{(V)}\omega = 0. \tag{3.9}$$

Theorem 19 *The remarkable conclusion is that the concept of parallel transport in tensor analysis is - in effect - an adiabatic, reversible process!!!*

3.3 The Kolmogorov-Cartan T0 Topology with sets that are differential forms

Cartan built his theory of exterior differential forms around an exterior differential system, Σ, which consists of a collection of 0-forms, 1-forms, 2-forms, etc. [44]. He defined the closure of this collection as the union of the original collection of differential forms with those differential forms which are obtained by forming the exterior differentials of every p-form in the initial collection. Note there are 16 Kolmogorov T0 topologies on the set of 4 points, but there is only one T0 topology that has a specialization order, that produces a disconnected topology. The symbol for such a Kolmogorov-Cartan topology is KCT0.

It is now appreciated via the KCT0 topological structure, that the exterior differential is a limit point generator. In general, the collection of exterior differentials will be denoted by $d\Sigma$, and the closure of Σ by the symbol, $K_{Cl}(\Sigma)$, where,

$$\text{Kuratowski Closure operator:} \quad K_{Cl}(\Sigma) = \Sigma_{\cup} d\Sigma. \tag{3.10}$$

For notational simplicity in this monograph the systems of p-forms will be assumed to consist of the single 1-form, A. Then the exterior differential of A is the 2-form $F = dA$, and the closure of A is the union of A and $F : K_{Cl}(A) = A_{\cup}F$. The other logical operation is the concept of intersection, so that from the exterior differential it is possible to construct the set $A\hat{}F$ defined collectively as $\mathrm{H} : \mathrm{H} = A\hat{}F$. The exterior differential of H produces the set defined as $\mathrm{K} = d\mathrm{H}$, and the closure of H is the union of H and $\mathrm{K} : K_{Cl}(\mathrm{H}) = \mathrm{H}_{\cup}\mathrm{K}$.

This ladder process of constructing exterior differentials, and exterior products, may be continued until a null set is produced, or the largest p-form so constructed is equal to the dimension of the space under consideration. The set so generated is defined as a Pfaff sequence. The largest rank of the sequence determines the Pfaff dimension of the domain (or class of the form, [255]), which is a topological invariant. The idea is that the 1-form A (in general the exterior differential system, Σ) generates on space-time four equivalence classes of points that act as domains of support for the elements of the Pfaff sequence, $A, F, \mathrm{H}, \mathrm{K}$. The union of all such points will be denoted by $X = A_{\cup}F_{\cup}\mathrm{H}_{\cup}\mathrm{K}$. The fundamental equivalence classes are given specific names [218]:

$$\begin{aligned}
\text{Topological ACTION} &: \quad A && (3.11)\\
A &= A_{\mu}dx^{\mu} && (3.12)\\
\text{Topological VORTICITY} &: \quad F = dA && (3.13)\\
dA &= F_{\mu\nu}dx^{\mu}\hat{}dx^{\nu} && (3.14)\\
\text{Topological TORSION} &: \quad \mathrm{H} = A\hat{}dA && (3.15)\\
A\hat{}dA &= H_{\mu\nu\sigma}dx^{\mu}\hat{}dx^{\nu}\hat{}dx^{\sigma} && (3.16)\\
\text{Topological PARITY} &: \quad \mathrm{K} = dA\hat{}dA && (3.17)\\
dA\hat{}dA &= K_{\mu\nu\sigma\tau}dx^{\mu}\hat{}dx^{\nu}\hat{}dx^{\sigma}\hat{}dx^{\tau}. && (3.18)
\end{aligned}$$

The Cartan topology is constructed from a basis of open sets, which are defined as follows. First consider the domain of support of A. Define this "point" by the symbol A. A is the first open set of the Cartan topology. Next construct the exterior differential, $F = dA$, and determine its domain of support. Next, form the Kuratowski closure of A by constructing the union of these two domains of support, $K_{Cl}(A) = A_{\cup}F$. $A_{\cup}F$ forms the second open set of the Cartan topology.

Next construct the intersection $\mathrm{H} = A\hat{\ }F$, and determine its domain of support. Define this "point" by the symbol H, which forms the third open set of the Cartan topology. Now follow the procedure established in the preceding paragraph. Construct the closure of H as the union of the domains of support of H and $\mathrm{K} = d\mathrm{H}$. The construction forms the fourth open set of the Cartan topology. In four dimensions, the process stops, but for $\mathrm{N} > 4$, the process may be continued.

Now consider the basis collection of open sets that consists of the subsets:

$$B = \{A,\ K_{Cl}(A),\ \mathrm{H},\ K_{Cl}(\mathrm{H})\} = \{A,\ A_{\cup}F,\ \mathrm{H},\ \mathrm{H}_{\cup}\mathrm{K}\}. \tag{3.19}$$

The collection of all possible unions of these base elements, and the null set, $\varnothing$, generate the Cartan topology of open sets:

$$T(open) = \{X,\ \varnothing,\ A,\ \mathrm{H}, A_{\cup}F,\ \mathrm{H}_{\cup}\mathrm{K},\ A_{\cup}\mathrm{H},\ A_{\cup}\mathrm{H}_{\cup}\mathrm{K},\ A_{\cup}F_{\cup}\mathrm{H}\}. \tag{3.20}$$

These nine subsets form the open sets of the Cartan topology constructed from the domains of support of the Pfaff sequence constructed from a single 1-form, A, in four dimensions. The complements of the open sets are the closed sets of the Cartan topology:

$$T(closed) = \{\varnothing,\ X,\ F_{\cup}\mathrm{H}_{\cup}\mathrm{K},\ A_{\cup}F_{\cup}\mathrm{K}, A_{\cup}F,\ \mathrm{H}_{\cup}\mathrm{K}, F_{\cup}\mathrm{K}, F,\ \mathrm{K}\}. \tag{3.21}$$

From the set of 4 "points" $\{A, F, \mathrm{H}, \mathrm{K}\}$ that make up the Pfaff sequence it is possible to construct 16 subset collections by the process of union. It is possible to compute the limit points for every subset relative to the Cartan topology. The classical definition of a limit point is that a point p is a limit point of the subset Y relative to the topology T if and only if for every open set which contains p there exists another point of Y other than p [151]. The results of this and other standard definitions are presented in Table 3.1.

Table 3.1 The Kuratowski T0 Topology of 4 ingredients

A 1-form in 4D: $A = A_k(x)dx^k$

$X = \{A, \quad F = dA, \quad \mathrm{H} = A\hat{}F, \quad \mathrm{K} = F\hat{}F\}$

Basis: Basis subsets $\{A,\ K_{Cl}(A),\ \mathrm{H},\ K_{Cl}(\mathrm{H})\} = \{A, A_\cup F, \mathrm{H}, \mathrm{H}_\cup \mathrm{K}\}$

$T(open) = \{X,\ \varnothing,\ A,\ \mathrm{H}, A_\cup F,\ \mathrm{H}_\cup \mathrm{K},\ A_\cup \mathrm{H},\ A_\cup \mathrm{H}_\cup \mathrm{K},\ A_\cup F_\cup \mathrm{H}\}$

$CT4\{open\} : \{X,\ \varnothing,\ A,\ \mathrm{H}, A_\cup F,\ \mathrm{H}_\cup \mathrm{K},\ A_\cup \mathrm{H},\ A_\cup \mathrm{H}_\cup \mathrm{K},\ A_\cup F_\cup \mathrm{H}\}$

$CT4\{closed\} : \{\ \varnothing, X, F_\cup \mathrm{H}_\cup \mathrm{K}, A_\cup F_\cup \mathrm{K},\ \mathrm{H}_\cup \mathrm{K},\ A_\cup F,\ F_\cup \mathrm{K},\ F, \mathrm{K}\}$

The Kuratowski T0 Topology

Subset S	**$\mathbb{L}$imit $\mathbb{P}$ts**	**$\mathbb{I}$nterior**	**$\mathbb{E}$xterior**	**$\mathbb{B}$oundary**	**$\mathbb{C}$losure**
$\varnothing$	$\varnothing$	$[\varnothing]$	$[X]$	$\varnothing$	$\varnothing$
$\{A\}$	$\{F\}$	$\{A\}$	$\{H_\cup K\}$	$\{F\}$	$\{A_\cup F\}$
$\{F\}$	$\varnothing$	$[\varnothing]$	$\{A_\cup H_\cup K\}$	$\{F\}$	$\{F\}$
$\{H\}$	$\{K\}$	$\{H\}$	$\{A_\cup F\}$	$\{K\}$	$\{H_\cup K\}$
$\{K\}$	$\varnothing$	$[\varnothing]$	$\{A_\cup F_\cup H\}$	$\{K\}$	$\{K\}$
$\{A_\cup F\}$	$\{F\}$	$\{A_\cup F\}$	$\{H_\cup K\}$	$\varnothing$	$\{A_\cup F\}$
$\{A_\cup H\}$	$\{F\}, \{K\}$	$\{A_\cup H\}$	$[\varnothing]$	$\{F_\cup K\}$	X
$\{A_\cup K\}$	$\{F\}$	$\{A\}$	$\{H\}$	$\{F_\cup K\}$	$\{A_\cup F_\cup K\}$
$\{F_\cup H\}$	$\{K\}$	$\{H\}$	$\{A\}$	$\{F_\cup K\}$	$\{F_\cup H_\cup K\}$
$\{F_\cup K\}$	$\varnothing$	$\varnothing$	$\{A_\cup H\}$	$\{F_\cup K\}$	$\{F_\cup K\}$
$\{H_\cup K\}$	$\{K\}$	$\{H_\cup K\}$	$\{A_\cup F\}$	$\varnothing$	$\{H_\cup K\}$
$\{A_\cup F_\cup H\}$	$\{F\}, \{K\}$	$\{A_\cup F_\cup H\}$	$[\varnothing]$	$\{K\}$	X
$\{F_\cup H_\cup K\}$	$\{K\}$	$\{H, K\}$	$\{A\}$	$\{F\}$	$\{F_\cup H_\cup K\}$
$\{A_\cup H_\cup K\}$	$\{F\}, \{K\}$	$\{A_\cup H, K\}$	$[\varnothing]$	$\{F\}$	X
$\{A_\cup F_\cup K\}$	$\{F\}$	$\{A_\cup F\}$	$\{H\}$	$\{K\}$	$\{F_\cup H_\cup K\}$
$\{A_\cup F_\cup H_\cup K\}$	$\{F\}, \{K\}$	$[X]$	$[\varnothing]$	$\varnothing$	X

By examining the set of limit points so constructed for every subset of the Cartan system, and presuming that the functions that make up the forms are C2 differentiable (such that the Poincare lemma is true, $dd\omega = 0, any\ p-form,\ \omega$), it is easy to show that for all subsets of the Cartan topology every limit set is composed of the exterior differential of the subset thereby proving the conjecture that the exterior differential is a limit point operator relative to the Cartan topology.

Theorem 20 *With respect to the Cartan topology, the exterior differential is a limit point generator.*

For example, the open subset, $A_\cup \mathrm{H}$, has the limit points that consist of F and K. The limit set consists of $F_\cup \mathrm{K}$ which can be derived directly by taking the exterior differentials of the elements that make up $A_\cup \mathrm{H}$; that is, $(F_\cup \mathrm{K}) = d(A_\cup \mathrm{H}) = (dA_\cup d\mathrm{H})$. Note that this open set, $A_\cup \mathrm{H}$, does not contain its limit points. Similarly for the closed set, $A_\cup F$, the limit points are given by F which may be deduced by direct application of the exterior differential to $(A_\cup F) : (F) = d(A_\cup F) = (dA_\cup dF) = (F_\cup \varnothing) = (F)$.

3.4 Topological Torsion, Connected vs. Non-Connected Cartan topologies

Topological torsion of a 1-form, A, is defined as the exterior product of the 1-form and its exterior differential, H $= A\hat{}dA$. Topological torsion is different from, but can be related to, the Frenet torsion of a space curve and the affine torsion of a connection. If non-zero, Topological torsion has important topological properties. The Cartan topology as given in Table 2.1 is composed of the union of two sub-sets which are both open and closed:

$$(X = K_{Cl}(A)_{\cup}K_{Cl}(\mathrm{H}) = \{A_{\cup}F\}_{\cup}\{\mathrm{H}_{\cup}\mathrm{K}\}), \tag{3.22}$$

a result that implies that the Cartan topology is not necessarily a connected topology[‡] [151]. An exception exists if the topological torsion, H $= A\hat{}dA$, and hence its closure, vanishes, for then the resultant PTD$(A) = 2$ Cartan topology is connected. This extraordinary result has broad physical consequences. The connected Cartan topology based on a vanishing topological torsion is at the basis of most physical theories of equilibrium. In mathematics, the connected Cartan topology corresponds to the Frobenius integrability condition for Pfaffian forms. In thermodynamics, the connected Cartan topology is associated with the Caratheodory concept of inaccessible thermodynamic states [103], and the existence of an equilibrium thermodynamic surface. If the non-exact 1-form, Q, of heat generates a Cartan topology of null topological torsion relative to the 1-form Q, $Q\hat{}dQ = \varnothing$, then the Cartan topology built on Q is connected. Such systems are "isolated" in a topological sense, and the Heat 1-form, Q, has a representation in terms of two and only two functions, conventionally written as, $Q = TdS$. Note again that a fundamental physical concept, in this case the idea of equilibrium, is a topological concept independent from geometrical properties of size and shape. Processes that generate the 1-form Q such that $Q\hat{}dQ = \varnothing$ are thermodynamically reversible. If $Q\hat{}dQ \neq \varnothing$, the process that generates Q is thermodynamically irreversible [175].

When the Cartan topology is connected, it might be said that all forces are extendible over the whole of the set, and that these forces are of "long range". Conversely when the Cartan topology is disconnected, the "forces" cannot be extended indefinitely over the whole domain of independent variables, but perhaps only over a single component. The components are not arc connected. In this sense, such forces are said to be of "short range", as they are confined to a specific component. Note that this notion of short or long range forces does not depend upon geometrical size or scale. The physical idea of short or long range forces is a topological idea of connectivity, and not a geometrical concept of "how far".

In an earlier article, these ideas were formulated intuitively in order to give an explanation of the "four forces" of physics. The earlier work was based upon

[‡] It should be noted that disconnected is not the same as separated, for the disconnected component boundaries could be in contact.

experience with differential geometry [203]. The features of the Pfaff sequence were used to establish equivalence classes for 1-forms constructed from known example metric field solutions, $g_{\mu v}$, to the Einstein field equations. The original ideas, based upon experience with systems in differential geometry, can now be given credence based upon differential topology. The construction of a 1-form, $A = g_{\mu 4} dx^{\mu}$, whose coefficients are the space-time components of a metric tensor, will divide the topology into equivalence classes depending upon the number of non-zero elements of its Pfaff sequence. This number has been defined above as the Pfaff Topological Dimension. Long range parity preserving forces due to gravity (Pfaff dimension 1) and electromagnetism (Pfaff dimension 2) are to be associated with a Cartan Topology that is connected (H $= A\hat{}F = A\hat{}dA = 0$). Both the strong force (Pfaff dimension 3) and the weak force (Pfaff dimension 4) are "short" range (H $\neq 0$) and are to be associated with a disconnected Cartan topology. The strong force is parity preserving (K $= 0$) and the weak force is not (K $\neq 0$). The fact that the Cartan topology is not necessarily connected is the topological (not metrical) basis that may be used to distinguish between short and long range forces. The methods also have applicability to the theory of black holes.

In much of our physical experience with nature, it appears that the disconnected domains of Pfaff dimension 3, or more, are often isolated as nuclei, while the surrounding connected domains of Pfaff dimension 2 or less appears as fields of charged or non-charged molecules and atoms. However, part of the thrust of this monograph is to demonstrate that such disconnected topological phenomena are not confined to microscopic systems, but also appear in a such mundane phenomena as the flow of a turbulent fluid. Physical examples of the existence of topological torsion (and hence a non-connected Cartan topology) are given by the experimental appearance of what appear to be coherent structures in a turbulent fluid flow.

To prove that a turbulent flow must be a consequence of a Cartan topology that is not connected, consider the following argument. First consider a fluid at rest. From a global set of unique, synchronous, initial conditions generate a vector field of flow. Such flows must satisfy the Frobenius complete integrability theorem, which requires that H $= A\hat{}dA = 0$. The Cartan topology for such systems is connected, and the Pfaff dimension of the domain is 2 or less. Such domains do not support topological torsion (the 3-form H $= 0$). Such globally laminar flows are to be distinguished from flows that reside on surfaces, but do not admit a unique set of connected sychronizable initial conditions.

Next consider turbulent flows which, as the antithesis of laminar flows, can not be integrable in the sense of Frobenius; such turbulent domains support topological torsion (H $= A\hat{}dA \neq 0$), and therefore a disconnected Cartan topology [205]. The components of the disconnected Cartan topology can be defined as the (topologically) coherent structures induced by the turbulent flow.

Note that a domain can support a homogeneous topology of one component and then undergo continuous topological evolution to a domain with some interior holes. The domain is simply connected in the initial state, and multiply connected in the final state, but still connected. However, consider the dual point of view where the originally connected domain consists of a homogeneous space that becomes separated into multiple components. The evolution to a topological space of multiple components is not continuous. It follows that the case of a transition from an initial laminar state (H $= A\hat{}dA = 0$) to the turbulent state (H $= A\hat{}dA \neq 0$) is a transition from a connected topology to a disconnected topology; therefore the transition to turbulence is NOT continuous. However, note that the decay of turbulence can be described by a continuous transformation from a disconnected topology to a connected topology. Condensation is continuous, gasification is not. It can be demonstrated (see chapter 7 in Vol 1, [1]) that relative to the Cartan topology all C2 differentiable, **V**, acting on C2 p-forms by means of the Lie differential are continuous. The conclusion is reached that the transition to turbulence must involve transformations that are not C2, hence can occur only in the presence of shocks or tangential discontinuities.

Chapter 4
APPLICATIONS OF THE KOLMOGOROV-CARTAN TOPOLOGICAL STRUCTURE

4.1 Continuous Processes

A topological structure is defined to be enough information to decide whether a transformation is continuous or not [85]. The classical definition of continuity depends upon the idea that every open set in the range must have an inverse image in the domain. This means that topologies must be defined on both the initial and final state, and that somehow (for this definition of continuity) an inverse image must be defined. Note that the open sets of the final state may be different from the open sets of the initial state, because the topologies of the two states can be different.

There is another definition of continuity that is more useful for it depends only on the transformation, and not its inverse, explicitly. A transformation is continuous if and only if the image of the closure of every subset is included in the closure of the image. This means that the concept of closure and the concept of transformation must commute for continuous processes. Suppose the forward image of a 1-form A is Q, and the forward image of the set $F = dA$ is Z. Then if the Kuratowski closure, $K_{Cl}(A) = A_{\cup}F$ is included in the closure of $K_{Cl}(Q) = Q_{\cup}dQ$, for all subsets, the transformation is defined to be continuous. The idea of continuity becomes equivalent to the concept that the forward image Z of the limit points, dA, is an element of the closure of Q [104]:

> **Definition** *A function that produces an image $f[A] = Q$ is continuous iff for every subset A of the Cartan topology, $Z = f[dA] \subset K_{Cl}(Q) = (Q_{\cup}dQ)$.*

The Cartan theory of exterior differential systems can now be interpreted as a topological structure, for every subset of the KCT0 topology can be tested to see if the process of closure commutes with the process of transformation. For the Cartan topology, this emphasis on limit points rather than on open sets is a more convenient method for determining continuity. A simple evolutionary process, $X \Rightarrow Y$, is defined by a map Φ. The map, Φ, may be viewed as a propagator that takes the initial state, X, into the final state, Y. For more general physical situations the evolutionary processes are generated by vector fields of flow, $\mathbf{V}$. The trajectories defined by the vector fields may be viewed as propagators that carry domains into ranges in the manner of a convective fluid flow. The evolutionary propagator of interest to this monograph is the Lie differential with respect to a vector field, $\mathbf{V}$, acting on differential forms, Σ [21].

The Lie differential has a number of interesting and useful properties.

1. The Lie differential does not depend upon a metric or a connection.
2. The Lie differential has a simple action on differential forms producing a resultant form that is decomposed into a transversal and an exact part,

$$L_{(\mathbf{V_4})}\omega = i(\mathbf{V_4})d\omega + di(\mathbf{V_4})\omega. \tag{4.1}$$

   This formula is known as "Cartan's magic formula". For those vector fields V which are "associated" with the form ω, such that $i(V)\omega = 0$, the Lie differential becomes equivalent to the covariant differential of tensor analysis. Otherwise the two differential concepts are distinct.

3. The Lie differential may be used to describe deformations and topological evolution and change. Note that the action of the Lie differential on a 0-form (scalar function) is the same as the directional or convective differential of ordinary calculus, $L_{(V)}\Phi = i(V)d\Phi + di(V)\Phi = i(V)d\Phi + 0 = \mathbf{V}\cdot grad\Phi$. It may be demonstrated that the action of the Lie differential on a 1-form will generate equations of motion of the hydrodynamic type. In fact Arnold calls the Lie differential the "convective" or "Fisherman's" differential.
4. With respect to vector fields and forms constructed over C2 functions, the Lie differential commutes with the closure operator. Hence, the Lie differential (restricted to C2 functions) generates transformations on differential forms which are continuous with respect to the Cartan topology.

The last statement requires a formal proof:

Proof First construct the closure, $\{\Sigma_{\cup}d\Sigma\}$. Next propagate Σ and $d\Sigma$ by means of the Lie differential to produce the decremental forms, say Q and Z,

$$L_{(\mathbf{V})}\Sigma = Q \quad and \quad L_{(\mathbf{V})}d\Sigma = Z. \tag{4.2}$$

Now compute the contributions to the closure of the final state as given by $\{Q_{\cup}dQ\}$. If $Z = dQ$, then the closure of the initial state is propagated into the closure of the final state, and the evolutionary process defined by $\mathbf{V}$ is continuous. However,

$$dQ = dL_{(\mathbf{V})}\Sigma = di(V)d\Sigma + dd(i(V)\Sigma) = di(V)d\Sigma\ , \tag{4.3}$$

as $dd(i(V)\Sigma) = 0$ for C2 functions; but,

$$Z = L_{(\mathbf{V})}d\Sigma = d(i(V)d\Sigma) + i(V)dd\Sigma = di(V)d\Sigma, \tag{4.4}$$

as $i(V)dd\Sigma = 0$ for C2 p-forms. It follows that $Z = dQ$, and therefore $\mathbf{V}$ generates a continuous evolutionary process relative to the Cartan topology. *QED*

It is to be noticed that this concept of a topological structure is developed in terms of the action of the Lie differential acting on a 1-form. The method does not depend upon metric or connection.

Certain special cases arise for those subsets of vector fields that satisfy the equations, $d(i(\mathbf{V})\Sigma) = 0$. In these cases, only the functions that make up the p-form, Σ, need be C2 differentiable, and the coefficient functions of the vector direction field need only be C1. Such processes will be of interest to symplectic processes, with Bernoulli-Casimir invariants, and to the analysis of tangential discontinuities.

By suitable choice of the 1-form of Action it is possible to show that the action of the Lie differential on the 1-form of Action can generate the Navier Stokes partial differential equations [201], [225]. The analysis above indicates that C2 differentiable solutions to the Navier-Stokes equations can not be used to describe the transition to turbulence. The C2 solutions can, however, describe the irreversible decay of turbulence to the globally laminar state.

4.2 A Hydrodynamic Example.

An abbreviated summary of the topological features of hydrodynamic systems is presented in the next subsection. A more detailed set of examples and applications to fluids is to be found in (Vol 3, [1]).

4.2.1 Euler flows and Hamiltonian fluids

Consider the Action 1-form, A, per unit source (in thermodynamics, the unit source is mole number, or sometimes "particle mass"), that defines the thermodynamic system for a fluid,

$$A = \mathbf{v} \circ \mathbf{dr} - \{\mathbf{v}.\mathbf{v}/2\}dt. \tag{4.5}$$

Compute the exterior differential dA and define the following functions,

$$\text{Vorticity} \quad \boldsymbol{\omega} = curl\ \mathbf{v} \quad and \tag{4.6}$$

$$\text{Acceleration} \quad \mathbf{a} = -\{\partial\mathbf{v}/\partial t + grad(\mathbf{v}\cdot\mathbf{v}/2)\}, \tag{4.7}$$

such that,

$$F = dA = \{\partial A_k/\partial x^j - \partial A_j/\partial x^k\}dx^j\hat{}dx^k = F_{jk}dx^j\hat{}dx^k \tag{4.8}$$

$$= \boldsymbol{\omega}_z dx\hat{}dy + \boldsymbol{\omega}_x dy\hat{}dz + \boldsymbol{\omega}_y dz\hat{}dx + \mathbf{a}_x dx\hat{}dt + \mathbf{a}_y dy\hat{}dt + \mathbf{a}_z dz\hat{}dt.$$

As the coefficients of the 1-form of action are presumed to be C2 differentiable, these vector fields always satisfy the Poincare-Faraday induction equations, $dF = ddA = 0$, or,

$$curl\ \mathbf{a} + \partial\boldsymbol{\omega}/\partial t = 0, \quad div\ \boldsymbol{\omega} = 0. \tag{4.9}$$

Consider a thermodynamic process created by the vector field, $\mathbf{V}_4 = [\mathbf{V}^x, \mathbf{V}^y, \mathbf{V}^z, 1]$ and use Cartan's magic formula,

$$L_{(\rho\mathbf{V}_4)}A = i(\rho\mathbf{V}_4)dA + d(i(\rho\mathbf{V}_4)A) = W + dU = Q, \tag{4.10}$$

to compute the thermodynamic Work 1-form, W,. The expressions for Work, W, and internal energy, U, become:

$$\begin{aligned} W &= i(\rho\mathbf{V}_4)dA = \rho\{\partial\mathbf{v}/\partial t + grad(\mathbf{V}\cdot\mathbf{v}/2) - \mathbf{V}\times\boldsymbol{\omega}\}\circ d\mathbf{r} \\ &\quad -\rho\mathbf{V}\circ\{\partial\mathbf{v}/\partial t + grad(\mathbf{v}\cdot\mathbf{v}/2)\}dt, \qquad (4.11) \\ U &= i(\mathbf{V}_4)A = \rho(\mathbf{V}\cdot\mathbf{v} - \mathbf{v}.\mathbf{v}/2). \qquad (4.12) \end{aligned}$$

At first, topologically constrain the thermodynamic Work 1-form to be of the Bernoulli class in terms of the exterior differential system:

$$W - dP = 0, \tag{4.13}$$

$$\text{with } i(\mathbf{V}_4)dP = 0. \tag{4.14}$$

The last equation states that the Bernoulli function is invariant along a trajectory, but can vary (transversely) from trajectory to trajectory. Assume that $\mathbf{V} = \mathbf{v}$ and set the coefficients of spatial components of the Bernoulli exterior differential system to zero. The result is the partial differential equations that represent the Lagrange-Euler fluid:

$$\{\partial\mathbf{v}/\partial t + grad(\mathbf{v}\cdot\mathbf{v}/2) - \mathbf{v}\times\boldsymbol{\omega}\} - grad(P)/\rho = 0. \tag{4.15}$$

This formula should be compared to the derivation of the Lorentz force in electromagnetic systems. Note that the Bernoulli pressure, P, is an evolutionary invariant along a trajectory. The flow is Hamiltonian (but not extremal Hamiltonian) and reversible, as Q is exact, of Pfaff Topological Dimension, 1, and $Q\hat{}dQ = 0$.

The time-like component of the exterior differential system $W - dP = 0$ leads to the equation,

$$\partial P/\partial t = -\rho\mathbf{v}\circ\{\partial\mathbf{v}/\partial t + grad(\mathbf{v}\cdot\mathbf{v}/2) = \rho(\mathbf{v}\circ\mathbf{a}). \tag{4.16}$$

It is apparent that if the velocity and the acceleration are orthogonal, then the time rate of change of the Bernoulli pressure is zero.

It also follows that the "Master" equation of electrodynamic theory is valid for certain hydrodynamic systems, with the only difference being that $curl\ \mathbf{v}$ is defined as $\boldsymbol{\omega}$, the vorticity of the hydrodynamic flow. The master equation becomes,

$$curl(\mathbf{v}\times\boldsymbol{\omega}) = \partial\boldsymbol{\omega}/\partial t, \tag{4.17}$$

and this equation is to be recognized as the equivalent of the Helmholtz equation for the conservation of vorticity. In the hydrodynamic sense, conservation of vorticity implies uniform continuity. In other words, the Eulerian flow is not only Hamiltonian, it is also uniformly continuous, and satisfies both the master equation and the conservation of vorticity constraints. In addition, it may be demonstrated that such systems are at most of Pfaff dimension 3, and admit a relative integral invariant which generalizes the hydrodynamic concept of invariant helicity. In the electromagnetic topology, the Hamiltonian constraint is equivalent to the statement that the "Lorentz force" vanishes, a condition that has been used to define the "ideal" plasma or "force-free" plasma state.

4.2.2 The Navier-Stokes fluid

From the theory of topological fluctuations, it must be true that the 1-form of Work must have a format of the type

$$W = i(\rho \mathbf{V}_4)dA = -dP + \varpi_j(d\mathbf{x}^j - \mathbf{v}^j dt), \tag{4.18}$$

such that for a pressure, P, of the Bernoulli class (4.14), the work done is transverse to the process trajectory,

$$(i(\rho \mathbf{V}_4)W = 0. \tag{4.19}$$

The coefficients, ϖ_j, of the velocity topological fluctuations act in the manner of Lagrange multipliers. If ϖ_j/ρ is defined (arbitrarily) as $\upsilon\, curl\, curl\, \mathbf{v}$ then the spatial components of the thermodynamic Work 1-form, W, are constrained to yield the partial differential equations for a constant density Navier-Stokes fluid:

$$\{\partial \mathbf{v}/\partial t + grad(\mathbf{v} \cdot \mathbf{v}/2) - \mathbf{v} \times \boldsymbol{\omega}\} = -grad(P)/\rho + \upsilon\, curl\, curl\, \mathbf{v}. \tag{4.20}$$

Density variations can be included by adding a term $\lambda div(\mathbf{V})$ to the potential $\{\mathbf{v} \cdot \mathbf{v}/2\}$ to yield:

$$\begin{aligned} \partial \mathbf{v}/\partial t + grad\{\mathbf{v}.\mathbf{v}/2\} - \mathbf{v} \times curl\ \mathbf{v} \;&=\; -gradP/\rho && (4.21)\\ &\quad +\lambda\{grad(div\ \mathbf{v})\} && (4.22)\\ &\quad +\upsilon\{curl\ curl\ \mathbf{v}\}. && (4.23) \end{aligned}$$

Classically, υ can be identified with the geometric kinematic shear viscosity, and $\lambda = \mu_B - \upsilon$. The coefficient μ_B can be identified with the topological (space-time) bulk viscosity. It is thereby demonstrated that the Navier-Stokes equations correspond to a refinement of the Cartan topology [201].

The Navier-Stokes constraint implies that the thermodynamic Work 1-form need not be closed. There are solutions to the Navier-Stokes equations that are thermodynamically irreversible.

The 1-form of Action will generate a 3-form of Topological Torsion, $A\hat{}dA = i(\mathbf{T}_4)dx\hat{}dy\hat{}dz\hat{}dt$, of the form,

$$\begin{aligned}\mathbf{T}_4 &= [\mathbf{a}\times\mathbf{v}+\{\mathbf{v}.\mathbf{v}/2\}\ curl\ \mathbf{v},(\mathbf{v}\circ curl\,\mathbf{v})], &(4.24)\\ &= [\mathbf{a}\times\mathbf{v}+\{\mathbf{v}.\mathbf{v}/2\}\ \boldsymbol{\omega},(\mathbf{v}\circ\boldsymbol{\omega})]. &(4.25)\end{aligned}$$

Use the Navier-Stokes equations (4.22) to solve for $\mathbf{a}$,

$$\begin{aligned}\mathbf{a} &= -[grad\{\mathbf{v}.\mathbf{v}/2\}+\partial\mathbf{v}/\partial t] &(4.26)\\ &= -\mathbf{v}\times curl\ \mathbf{v}+gradP/\rho\\ &\quad -\lambda\{grad(div\ \mathbf{v})\}+\upsilon\{curl\ curl\ \mathbf{v}\}, &(4.27)\end{aligned}$$

and then substitute in the expression for $\mathbf{T}_4$, to yield

$$\begin{aligned}\mathbf{T} &= [h\mathbf{v}-(\mathbf{v}\circ\mathbf{v}/2)curl\ \mathbf{v}-\mathbf{v}\times(gradP/\rho)\\ &\quad +\lambda\{\mathbf{v}\times grad(div\ \mathbf{v})\}-\upsilon\{\mathbf{v}\times(curl\ curl\ \mathbf{v})\},h], &(4.28)\\ h &= \mathbf{v}\cdot curl\,\mathbf{v}, &(4.29)\end{aligned}$$

which persists even for Euler flows, where $\upsilon = 0$, if the flow is baroclinic. The measurement of the components of the Torsion vector, $\mathbf{T}_4$, have been completely ignored by experimentalists in hydrodynamics.

By a similar substitution, the topological parity pseudo-scalar (equivalent to the second Poincare 4-form) becomes expressible in terms of engineering quantities as,

$$\begin{aligned}K &= \{2(\mathbf{a}\circ\boldsymbol{\omega})\}\Omega_4 = \{-2\{gradP/\rho\circ curl\ \mathbf{v}\\ &\quad -\lambda\{\ grad(div\ \mathbf{v})\circ curl\ \mathbf{v}\}\\ &\quad -\upsilon\{curl\ \mathbf{v}\circ(curl\ curl\ \mathbf{v})\}\}\Omega_4. &(4.30)\end{aligned}$$

The coefficient K is a measure of the space-time bulk dissipation coefficient (not λ), and it is the square of this number which must not be zero if the process is irreversible. Recall that turbulent dissipative irreversible flow is defined when the Pfaff dimension of the Action 1-form is equal to 4, which implies that $K \neq 0$. From this expression it is apparent that if the vorticity is of Pfaff dimension 2, then the last term vanishes, and the there is no irreversible dissipation due to shear viscosity. Other useful situations and design criteria for dissipation, or the lack thereof, can be gleaned from the formula. If the vector field is harmonic, then an irreversible process requires that,

$$K = \{2(\mathbf{a}\circ\boldsymbol{\omega})\}\Omega_4 = \{[2gradP/\rho-\mu_B grad(div\ \mathbf{v})]\circ curl\ \mathbf{v}\}\Omega_4 \neq 0. \qquad (4.31)$$

(Recall that harmonic vector fields are generators of minimal surfaces.) For fluids where $(\mu_B) \Rightarrow 0$, and if the pressure gradient is orthogonal to the vorticity and the flow field is harmonic, then there is no irreversible dissipation, and the flow is not turbulent. Note that for many fluids the bulk viscosity is much greater than the shear viscosity. When $K = 0$ no topological torsion defects are created; the acceleration and the vorticity of the Navier-Stokes fluid are colinear. The integral of K over {x,y,z,t} gives the Euler Index of the flow. See the discussions in Vol 3, [1] .

These results should be compared to those generated by Lamb and Eckart [70] for the fluid dissipation function, defined by the requirement that the dissipative flow has a (geometric) entropy production rate greater than or equal to zero.

4.3 An Electromagnetic Example

In this section, a more detailed summary of the topological features of electrodynamic systems is presented. An extensive set of examples and applications is to be found in Vol 4 [1]. Again, the terms in Cartan's magic formula are not whimsical, but instead demonstrate the formal deduction of historical concepts of electromagnetism from

4.3.1 The Thermodynamic Postulates of Electromagnetism

The 1-form of Potentials, A, defines the Thermodynamic System

To establish an initial level of credence in the terminology, consider the 1-form of Action, A, per unit charge, where the coefficient functions are the symbols representing the vector, $\mathbf{A}$, and scalar potentials, ϕ, of electromagnetic theory. The unit charge plays the role of thermodynamic unit mole number. The 1-form of Action, A , per unit charge, encodes the thermodynamic system of Topological Electromagnetism. This 1-form of Action potentials, A, is utilized to produce the 2-form, $F = dA$, of field intensities ($\mathbf{B}$ and $\mathbf{E}$), which are the limit sets of the Action, relative to the KCT0 topological structure.

The Amperian 3-form density, J, defines the Thermodynamic Process

Experimental experience requires the recognition of an Amperian charge-current density, whose components is an exact 3-form density $J = dG$. The exact 3-form of charge current density, $J = dG$, is defined in terms of the inexact 2-form density of thermodynamic field quantities, or "excitations", $G(\mathbf{D}, \mathbf{H})$, with physical units of $\hbar$. The charge-current density plays the role of a thermodynamic process.

The Postulates

These ideas lead to the formulation of two postulates (two exterior differential systems). The exact 2-form of thermodynamic field "intensities" $F(\mathbf{E}, \mathbf{B}) = dA$ is defined in terms of inexact 1-form of potentials, A, which has physical units of $\hbar/e$. The exact 3-form of charge current density, $J = dG$, is defined in terms of the inexact 2-form density of thermodynamic field quantities, or "excitations", $G(\mathbf{D}, \mathbf{H})$, with physical units of $\hbar$. The 2-form, F, historically is associated with forces, and the

2-form density, G, historically is associated with sources. The charge-current 3-forms define a class of processes.

The first postulate, the existence of a differentiable 1-form of Action, A, leads to a the thermodynamic concept of a Topological Torsion 3-form, $A\hat{}F$, whose non-zero value is an indicator of a non-equilibrium electrodynamic system. The second postulate, the existence of a differentiable 2-form density, leads to another 3-form of Topological Spin, $A\hat{}G$, whose non-zero value will also be an indicator of non-equilibrium. Each of these 3-forms have exterior differentials (divergences) producing 4-forms whose closed integrals have been defined as the Poincare-Bateman 4-forms. If the divergences are not zero, the non-zero values act as sources of topological defects. Both Topological Torsion and Topological Spin have been recognized as Chern-Simons 3-forms. The Topological Torsion 3-form is an intrinsic property of the thermodynamic system; the Topological Spin 3-form depends upon both the thermodynamic system and the process. Both Chern-Simons constructions may or may not have zero divergence. In fact it is possible that closed integrals of both 3-forms, if they are closed but not exact, can be topologically quantized, yielding values for Helicity and Spin that have rational ratios. The charge-current Amperian 3-form is exact, but closed integrals of its pre-image, G, on domains where G is closed, can give charges with rational ratios.

4.3.2 The classical Maxwell-Faraday and the Maxwell-Ampere equations

Using the notation and the language of Sommerfeld and Stratton [263] [269], the classic definition of an electromagnetic system is a domain of space-time $\{x, y, z, t\}$ which supports both the Maxwell-Faraday equations,

$$curl\ \mathbf{E} + \partial \mathbf{B}/\partial t = 0, \qquad div\ \mathbf{B} = 0, \tag{4.32}$$

and the Maxwell-Ampere equations,

$$curl\ \mathbf{H} - \partial \mathbf{D}/\partial t = \mathbf{J}, \qquad div\ \mathbf{D} = \rho. \tag{4.33}$$

This formulation of two sets of Partial Differential Equations makes no statement about metric, or connections with geometric constraints of "gauge".

The conservation of charge current

In every case, the charge current density for the Maxwell system satisfies the conservation law,

$$div\, \mathbf{J} + \partial \rho/\partial t = 0. \tag{4.34}$$

The charge-current densities are subsumed to be zero $[\mathbf{J},\ \rho] = 0$ for the vacuum state. For the Lorentz vacuum state, the field excitations, $\mathbf{D}$ and $\mathbf{H}$, are assumed to be linearly connected to the field intensities, $\mathbf{E}$ and $\mathbf{B}$, by means of the Lorentz (homogeneous and isotropic) constitutive relations:

$$\mathbf{D} = \varepsilon \mathbf{E}\,, \quad \mathbf{B} = \mu \mathbf{H}. \tag{4.35}$$

The two vacuum constraints imply that the solutions to the homogeneous Maxwell equations also satisfy the vector wave equation, typically of the form

$$grad\ div\ \mathbf{B} - curl\, curl\, \mathbf{B} - \varepsilon\mu\partial^2\mathbf{B}/\partial t^2 = 0. \tag{4.36}$$

The constant wave phase velocity, v_p, is taken to be

$$v_p^2 = 1/\varepsilon\mu \equiv c^2. \tag{4.37}$$

Similar results can be obtained for the solid state where the constitutive constraints can be more complex [222], and for the plasma state where the charge-current densities are not zero. It should be emphasized that the Lorentz constitutive equations form a severe topological (and not necessary) constraint on the general Maxwell electromagnetic system.

The existence of potentials

It is further subsumed that the classic Maxwell electromagnetic system is constrained by the statement that the field intensities (equivalent to the specification of an electromagnetic thermodynamic system) are deducible from a system of twice differentiable potentials, $[\mathbf{A}, \phi]$:

$$\mathbf{B} = curl\ \mathbf{A}, \qquad \mathbf{E} = -grad\ \phi - \partial\mathbf{A}/\partial t. \tag{4.38}$$

This constraint topologically implies that domains that support non-zero values for the covariant field intensities, $\mathbf{E}$ and $\mathbf{B}$, may *not* be compact domains with or without a boundary. It is this constraint that distinguishes classical electromagnetism from Yang Mills theories, which have propensity for topological manifolds that are compact without boundary. Two other classical 3 component-vector fields are of historical interest, the Poynting vector, $\mathbf{E} \times \mathbf{H}$, representing the flux of electromagnetic radiative energy, and the field momentum flux, $\mathbf{D} \times \mathbf{B}$ (in the sense that $\mathbf{D} \times \mathbf{B} = c^2\, \mathbf{E} \times \mathbf{H}$).

This independence from a choice of independent variables (or coordinates) for Maxwell's equations was reported early on by Van Dantzig [320]. It is a surprise to many, that the Maxwell differential system of PDE's is well defined in a covariant manner for both Galilean transformations as well as Lorentz transformations, or any other diffeomorphism. The singular solution sets to the Maxwell differential system of PDE's do *not* enjoy this universal property.

Maxwell's PDE's are topological statements that can be deduced from an exterior differential system. The two postulates are:

$$\textbf{The Postulate of Potentials} \quad F - dA = 0, \tag{4.39}$$

which defines the Maxwell Faraday exterior differential system, $F - dA = 0$, (leading to the concept of conserved flux), and

$$\textbf{The Postulate of Field Excitations:} \quad J - dG = 0, \tag{4.40}$$

which defines the Maxwell Ampere exterior differential system, $J - dG = 0$, (leading to the concept of conserved charge). The first postulate involves a 2-form of thermodynamic intensities whose coefficients transform as a covariant tensor when the maps between differential varieties are constrained to be diffeomorphisms. The second postulate involves a 2-form density of thermodynamic quantities [262] with coefficients that will transform as a contravariant tensor density when the maps between differential varieties are constrained to be diffeomorphisms. The topological properties are associated with the idea that the exterior differential forms are functionally well define, even though the maps between differential varieties are not diffeomorphisms, nor homeomorphisms.

Each postulate is to be recognized as an exterior differential system [34] constraining the topology of the functions defined in terms of the independent variables. For example, from Stokes theorem, the (two dimensional) domain of finite support for F can not, in general, be compact without boundary, unless the Euler characteristic vanishes. There are two exceptional cases for two-dimensional domains, the torus and the Klein-Bottle, but these situations require the additional topological constraint that $F\hat{}F \Rightarrow 0$. The fields in these exceptional cases must reside on these exceptional compact surfaces, which form topological coherent structures in the electromagnetic field.

No constraints of geometrical connection or metric are required explicitly. Such geometric constraints can be used to refine the Maxwell topology for different specific physical systems. For example, constitutive equations of constraint between the two 2-forms F and G can be used to distinguish birefringent media from optically active media. The Maxwell-Faraday PDE's are not restricted to spaces of topological dimension N = 4. For an exterior differential system, $F - dA = 0$, on a space of any dimension $N > 3$, the closure conditions, $ddA = dF = 0$, always yield the same identical Maxwell-Faraday PDE's for the first 4 variables. Additional PDE's are also generated for $N > 4$, but the system of PDE's form a nested set, with the Maxwell-Faraday equations as topological kernel, of invariant format for any dimension N. A remarkable result is that Faraday induction is a topological idea, and does not depend upon metric or connection. The concept of Faraday induction applies to any system that satisfies the Postulate of Potentials, including the fundamental axiom of topological thermodynamics which encodes a physical system in terms of a 1-form of Action.

As demonstrated below, the Postulate of Potentials establishes the field of thermodynamic intensities, **E** and **B**, (think forces), and the Postulate of Field Excitations establishes the field of thermodynamic quantities, **D** and **H**, (think sources). The topological perspective subsumes that the two species are independent ideas.

The experimental justification of such ideas can be demonstrated with a simple parallel plate capacitor experiment. First connect the plates to a battery of constant potential and let it remain connected. Insert a slab of plastic dielectric halfway between the plates. Release the plastic slab. Does the slab remain motionless, or is there motion such that the slab is expelled or attracted? For a second experiment, attach the plates of the capacitor to a battery and then disconnect the battery after charging the capacitor. Now insert the plastic slab halfway, and release it. Does the slab remain motionless, or is there motion such that the slab is expelled or attracted? In the first case, the $\mathbf{E}$ field remains constant (the potential does not change), and motion of the dielectric slab causes the $\mathbf{D}$ field to change (the battery adjusts the charge distribution). In the second experiment, the charge distribution is constant, so that the $\mathbf{D}$ field remains constant, but the $\mathbf{E}$ field changes. Consider the simple constitutive constraint, $\mathbf{D} = \varepsilon\mathbf{E}$. In the first experiment, insertion would cause the average ε to increase, hence even though $\mathbf{E}$ remains constant, the $\mathbf{D}$ field would increase. However, the total energy density $\mathbf{D} \circ \mathbf{E}$ would decrease if the slab was expelled, and that is what happens. In the second experiment, motion of the slab would cause the $\mathbf{E}$ field to change. As the $\mathbf{D}$ field remains constant, the minimum energy density occurs when the slab is fully inserted.

Current electromagnetic dogma presents the idea that from a given charge current density distribution, $[\mathbf{J}, \rho]$, it is possible to deduce the $\mathbf{E}$ and $\mathbf{B}$ fields. However, the postulate of conserved Charge-Current densities indicates that it is $\mathbf{D}$ and $\mathbf{H}$ that are the related quantities, not $\mathbf{E}$ and $\mathbf{B}$. The Postulate of Potentials indicates that the field intensities $\mathbf{E}$ and $\mathbf{B}$ are deduced from the potentials $[\mathbf{A}, \phi]$. It takes some constitutive constraint to convert $\mathbf{D}$ and $\mathbf{H}$ into $\mathbf{E}$ and $\mathbf{B}$, or $[\mathbf{J}, \rho]$ into $[\mathbf{A}, \phi]$. However, it has been discovered that there exists an adjoint mapping that will produce a conserved charge current algebraically (see Chapter 9 Vol 4 [1]). Both types of constraints appear in the literature in great detail and variety. Such assumptions obscure the topological basis and differences between exterior differential forms and exterior differential form densities.

There are two types of differential forms considered in this monograph. The first type transforms as a scalar with respect to diffeomorphisms. The second type transforms as a scalar density, which is proportional to the determinant (or the magnitude of the determinant) of the diffeomorphism. The coefficients of the first type pullback with respect to the transpose of a differential Jacobian mapping, whether it is a diffeomorphism or not. The coefficients of the second type, the differential form densities, pullback with respect to the adjoint of a differential Jacobian mapping, whether it is a diffeomorphism or not.

The Postulate of Potentials indicates that the domain of support for the 2-form F is not compact without boundary. The postulate also demonstrates that magnetic monopoles are not compatible with the assumption of C2 differentiability. Such a statement does not apply to the density (N-2)-form, G, which can have closed and non-closed components. The closed but not exact components of G lead to the quantization of charge as a topological result. If G is a density with a pullback related to the magnitude of the determinant, it follows that quantized charge is a pseudo-scalar [190]. The historical assumptions of charge as a scalar are not compatible with the topological format. Experiments with piezo magnetic crystals indicate that volume deformations can cause electrical phenomena. If charge was not a pseudo scalar, the concept of magnetic permeability would have to vanish in crystals with centers of symmetry, a result not compatible with experiment [189].

4.3.3 The Maxwell Ampere system: $J - dG = 0$

The D H field excitations: Differential (N-2)-form densities

For example consider the exterior differential of the (N-1)-form density, D, in three dimensions, given by the expression,

$$\begin{aligned} dD &= d(D^x dy\hat{}dz - D^y dz\hat{}dx + D^z dx\hat{}dy) \\ &= div_3(D) dx\hat{}dy\hat{}dz \Rightarrow \rho(x,y,z) dx\hat{}dy\hat{}dz, \end{aligned} \tag{4.41}$$

where ρ has been defined as the resultant of the action of the exterior differential, $div_3(\mathbf{D})$. The usual interpretation of Gauss' law is that the field lines of the vector (density) $\mathbf{D}$ terminate (or have a limit or accumulation point) on the charges, Q. Gauss' law generates both the intuitive idea that sources are related to limit points, and demonstrates the novel concept that the exterior differential is a limit point operator. The exterior differential creates limits points when the operation is applied to a differential form. However, as demonstrated above, the concept that the exterior differential is a limit point operator relative to the Cartan topology is a general idea, and is not restricted to Gauss' law.

Extending this idea to four dimensions for the (N-2)-form density, G, of Maxwell excitations $(\mathbf{D}, \mathbf{H})$,

$$G = -D^x dy\hat{}dz + D^y dz\hat{}dx - D^z dx\hat{}dy + H^x dx\hat{}dt + H^y dy\hat{}dt + H^z dz\hat{}dt, \tag{4.42}$$

the exterior differential dG of G yields a 3-form, J, defined as the electromagnetic current 3-form,

$$J = J^x dy\hat{}dz\hat{}dt - J^y dx\hat{}dz\hat{}dt + J^z dx\hat{}dy\hat{}dt - \rho dx\hat{}dy\hat{}dz, \tag{4.43}$$

where in 3-vector language,

$$curl\ \mathbf{H} - \partial \mathbf{D}/\partial t = \mathbf{J} \qquad div\, \mathbf{D} = \rho. \tag{4.44}$$

The charge current density act as the "limit points" of the Maxwell field excitations. Note that $dJ \Rightarrow 0$ for C2 functions by Poincare's lemma.

However, consider the N-1 current, C (not necessarily equal to J as defined above), in four dimensions,

$$C = \rho\{V^x dy\hat{}dz\hat{}dt - V^y dx\hat{}dz\hat{}dt + V^z dx\hat{}dy\hat{}dt - 1 dx\hat{}dy\hat{}dz\}, \tag{4.45}$$

and its exterior differential as given by the expression,

$$dC = \{div_3(\rho\mathbf{V}) + \partial\rho/\partial t\} dx\hat{}dy\hat{}dz\hat{}dt \tag{4.46}$$
$$= R dx\hat{}dy\hat{}dz\hat{}dt = R\,\Omega_{4vol}. \tag{4.47}$$

When the 4-form with coefficient R vanishes, the resultant expression is physically interpreted as the "equation of continuity" or as a "conservation law". Over a closed boundary, that which goes in is equal to that which goes out (when $dC = 0$). Note that the concept of the conservation law is a topological constraint; the "limit points" of the "current 3-form" in four dimensions must vanish if the conservation law is to be true. If the RHS of the above expression (4.46) is not zero, then the current 3-form is said to have an "anomaly", or a source (or sink). The anomaly acts as the source of the otherwise conserved quantity. The limit points, R, of the 3-form, C, are generated by its exterior differential, $dC = \{div_3(\rho\mathbf{V}) + \partial\rho/\partial t\}\Omega_4$. When the RHS is zero, the current "lines" do not stop or start within the domain. (It is possible for the current lines to be closed on themselves in certain topologies.)

The conservation of charge

The Maxwell Ampere exterior differential system is a topological constraint for by Stokes theorem the support for G can be compact without boundary only if the domain is without charge-currents. The closure of the exterior differential system, $dJ = 0$, generates the charge-current conservation law. The integral of J over a closed three-dimensional domain is a relative integral invariant (a deformation invariant) of any process that can be described in terms of a vector direction field, V. The formal statement is given by Cartan's magic formula [161], which describes continuous topological evolution in terms of the action of the Lie differential, with respect to a vector field, acting on the exterior differential 3-form, J. The Maxwell-Ampere exterior differential system leads to the evolutionary Conservation of Charge theorem. Charge is defined as the integral of the 3-form, J, over a three-dimensional integration chain $M3$. Using Stoke's theorem,

$$\text{Charge} \;:\; \iiint_{M3} J = \iint_{\partial M3} G, \qquad dJ = ddG = 0, \tag{4.48}$$
$$L_{\beta\mathbf{V}_4}\iiint_{M3} J = \iiint_{M3} d\{i(\beta\mathbf{V}_4)J\} = \iint_{\partial M3} G \Rightarrow 0, \text{ if conserved.} \tag{4.49}$$

When the Lie differential of the integral vanishes, charge is an evolutionary invariant. Then there are three cases to consider:

$$i(\beta\mathbf{V}_4)J = 0, \tag{4.50}$$
$$i(\beta\mathbf{V}_4)J = d\Phi, \tag{4.51}$$
$$i(\beta\mathbf{V}_4)J = d\Phi + \zeta,\; d\zeta = 0. \tag{4.52}$$

If $i(\beta\mathbf{V}_4)J = 0$, then the integral vanishes (the extremal case) independent from the deformation function β. The process, $\beta\mathbf{V}_4$, is an extremal field relative to J, and the integral is a deformation invariant which represents a topological property.

4.3.4 The Maxwell-Faraday system: $F - dA = 0$

The E B Field Intensities: differential 2-forms

On a four-dimensional space-time of independent variables, (x, y, z, t) the 1-form of Action (constrained by the postulate of potentials, $F - dA = 0$) can be written in the form

$$A = \Sigma_{k=1}^{3} A_k(x, y, z, t)dx^k - \phi(x, y, z, t)dt = \mathbf{A}\circ d\mathbf{r} - \phi dt. \tag{4.53}$$

This 1-form of Action defines (part of) the physical system of electromagnetism. Subject to the constraint of the exterior differential system, the 2-form of field intensities, F, becomes,

$$\begin{aligned} F = dA &= \{\partial A_k/\partial x^j - \partial A_j/\partial x^k\}dx^j\hat{}dx^k = F_{jk}dx^j\hat{}dx^k \qquad (4.54) \\ &= \mathbf{B}_z dx\hat{}dy + \mathbf{B}_x dy\hat{}dz + \mathbf{B}_y dz\hat{}dx + \mathbf{E}_x dx\hat{}dt + \mathbf{E}_y dy\hat{}dt + \mathbf{E}_z dz\hat{}dt, \end{aligned}$$

where, in usual engineering notation,

$$\mathbf{E} = -\partial\mathbf{A}/\partial t - grad\phi, \qquad \mathbf{B} = curl\ \mathbf{A} \equiv \partial A_k/\partial x^j - \partial A_j/\partial x^k. \tag{4.55}$$

The closure of the exterior differential system, $dF = 0$, vanishes for C2 differentiable p-forms, to yield,

$$dF = ddA = \{curl\ \mathbf{E} + \partial\mathbf{B}/\partial t\}_x dy\hat{}dz\hat{}dt - .. + .. - div\,\mathbf{B}dx\hat{}dy\hat{}dz\} \Rightarrow 0. \tag{4.56}$$

Equating to zero all four coefficients leads to the Maxwell-Faraday partial derivative equations,

$$\{curl\ \mathbf{E} + \partial\mathbf{B}/\partial t = 0, \qquad div\,\mathbf{B} = 0\}. \tag{4.57}$$

This topological development of the Maxwell-Faraday equations has made no use of a connection nor of a metric.

Be aware that the engineering notation, where the six components of the second rank covariant tensor, F_{jk}, are grouped into two 3-component vectors, is deceptive, for the diffeomorphic transformational properties of the field intensities ($\mathbf{E}$ and $\mathbf{B}$) are not that of Cartesian rank 1 three-dimensional vectors, but are that of a second rank covariant tensor field. The component functions ($\mathbf{E}$ and $\mathbf{B}$) of the 2-form, F, transform as covariant tensor of rank 2. The topological constraint that F is exact, implies that the domain of support for the field intensities cannot be compact without boundary, unless the Euler characteristic vanishes. These facts distinguish classical electromagnetism from Yang-Mills field theories, where in addition it is assumed that G is dual or anti-dual to F. Moreover, the fact that F is subsumed to be exact and C1 differentiable excludes the concept of magnetic monopoles from classical electromagnetic theory on topological grounds.

Note that the Poincare lemma applied to the topological constraint ($ddA = dF = 0$) always leads to the Maxwell pair of (Faraday Induction) PDE equations (4.57). This result is actually true for a variety of any dimension $\geq$ 4 and for any set of covariant symbols. Even if the dimension of the ordered domain exceeds 4, the concept of $ddA = dF = 0$, always yields the same set of partial differential equations relative to the first four base variables. The first Maxwell set of equations forms a nested set of PDE's on varieties of pre-geometric dimension > 4. The addition of new independent base variables does not change the format of the first four Maxwell PDE equations, but just adds to the set new PDE equations involving field components defined over the new ordered set of base variables. The concept of Faraday Induction is universal, and should not restricted to the science of electromagnetism. It is valid for any physical system which can be described by a 1-form of Action with a Pfaff dimension 2 or larger, such as a fluid with vorticity.

The very existence of the $\mathbf{E}$ and $\mathbf{B}$ fields implies that the 2-form $F = dA$ does not vanish. Hence, the 2-form defines a symplectic manifold of at least Pfaff dimension 2. As the 2-form is exact, the symplectic 2-manifold cannot be compact without boundary. This result follows from Stokes theorem, with 2 exceptions, the Klein bottle and the torus. Except for these two exceptions, the exact symplectic domain of the electromagnetic field intensities, $\mathbf{E}$ and $\mathbf{B}$, must either be open, or if compact it has a boundary.

The conservation of Flux

The Maxwell Faraday exterior differential system is a topological constraint, for by Stokes theorem the support for F can not be compact without boundary. The closure of the exterior differential system, $dF = 0$, generates the flux conservation law. The integral of F over a closed two-dimensional domain is a relative integral invariant (a deformation invariant) of any process that can be described in terms of a vector direction field, V. The formal statement is given by Cartan's magic formula [161], which describes continuous topological evolution in terms of the action of the Lie differential, with respect to a vector field, acting on the exterior differential 2-form, F. The Maxwell-Faraday exterior differential system leads to the evolutionary Conservation of Flux theorem. Flux is defined as the integral of the 2-form, F, over a two-dimensional integration chain $M2$. Using Stokes' theorem,

$$\text{Flux} \quad : \quad \iint_{M2} F = \int_{\partial M2} A, \; dF = ddA = 0, \tag{4.58}$$

$$L_{\beta \mathbf{V}_4}(\iint_{M2} F) = \iint_{M2} d(i(\beta \mathbf{V}_4)F) = \int_{\partial M2} A \Rightarrow 0, \text{ if conserved} \tag{4.59}$$

When the Lie differential of the integral vanishes, flux is an evolutionary invariant. Then there are three cases:

$$\begin{aligned} \text{Extremal } W &= i(\beta \mathbf{V}_4)F = 0, && (4.60)\\ \text{Bernoulli } W &= i(\beta \mathbf{V}_4)F = d\Theta, && (4.61)\\ \text{Stokes } W &= i(\beta \mathbf{V}_4)F = d\Theta + \gamma, \quad d\gamma = 0. && (4.62) \end{aligned}$$

If $i(\beta\mathbf{V}_4)F = 0$, then the last integral in equation (4.59) vanishes (the extremal case) independent from the deformation function β. The process $\beta\mathbf{V}_4$ is an extremal field relative to F, and the integral is a deformation invariant which represents a topological property. Thermodynamically, the extremal case implies that the thermodynamic Work 1-form, $W = i(\beta\mathbf{V}_4)F$, is of Pfaff Topological Dimension 0, otherwise W is of Pfaff Topological Dimension 1. Note that the Pfaff Topological Dimension of the Action 1-form is not explicitly defined, but it must be 2 or greater to produce the 2-form, F.

It is remarkable that the Maxwell-Faraday differential system can also be used to encode hydrodynamic and thermodynamic systems, both of the equilibrium and non-equilibrium variety.

Limit Points

This now almost classic generation of the Maxwell field equations has another less familiar interpretation. The **E** and **B** field intensities are the topological limit "points" of the 1-form of potentials, $\{\mathbf{A}, \phi\}$, relative to the Cartan topology! The limit points of the 2-form of field intensities, F, are the null set. For C2 vector fields, the Cartan topology admits flux quanta, charge quanta, and spin quanta, but excludes magnetic monopoles. When the differential system of interest is built upon the forms A, F and G, it is possible to show that superconductivity is to be associated with the constraints on the limit point sets of A, $A\hat{}F$, *and* $A\hat{}G$ [221]. That is, superconductivity has its origins in topological, not geometrical, concepts. This remarkable idea that the exterior differential is a limit point operator is based upon Kuratowski's closure operator (see p.72 [151]) is equivalent to the union of the identity and the exterior differential.

4.3.5 The Lorentz force

Consider the 1-form of electromagnetic action, A, given above (4.53). In electromagnetic format, for all processes, it follows that

$$curl\ \mathbf{E} + \partial\mathbf{B}/\partial t = 0, \ div\ \mathbf{B} = 0. \tag{4.63}$$

Consider an abstract process, $\mathbf{V}_4 \Rightarrow \rho\{V^k, 1\} = \rho\{\mathbf{V}, 1\}$, on the space, $\{x^k, t\}$. Evaluate the evolution of the electromagnetic action in terms of Cartan's magic formula,

$$\begin{aligned} L(\mathbf{V}_4)A &= i(\mathbf{V})dA + d(i(\mathbf{V})A) = W + dU = Q, && (4.64)\\ W &= i(\mathbf{V})dA = -\{\rho(\mathbf{E} + \mathbf{V}\times\mathbf{B})_k dx^k - \rho(\mathbf{V}\circ\mathbf{E})dt\}, && (4.65)\\ U &= \rho(\mathbf{V}\circ\mathbf{A} - \phi). && (4.66) \end{aligned}$$

The covariant spatial components of the thermodynamic Work 1-form, W, are to be recognized as the Lorentz force per unit charge to within the parametrization factor, ρ,

$$\mathbf{f}_{(Lorentz)} = -\rho(\mathbf{E} + \mathbf{V} \times \mathbf{B}). \tag{4.67}$$

Note that the time-like component becomes the "local dissipative" power, $P = \rho(\mathbf{V} \circ \mathbf{E})$.

The classic formula for the Lorentz force has been *derived* on topological grounds as a consequence of the perspective of continuous topological evolution. It has not been injected into the theory as a constraint of special relativity.

4.3.6 Non-Equilibrium Features of Electromagnetism

The exterior differential forms that make up the electromagnetic system on a geometric domain of four dimensions consist of the primitive 1-form, A, and the primitive (N-2)-form density, G, their exterior differentials, and their algebraic intersections defined by all possible exterior products. The complete Maxwell system of exterior differential forms (the Pfaff sequence for the Maxwell system on 4 geometric dimensions) is given by the set:

$$\{A;\ F = dA, G;\ J = dG, A\hat{}F, A\hat{}G, A\hat{}J;\ F\hat{}F, G\hat{}G, F\hat{}G\}. \tag{4.68}$$

These differential forms and their unions may be used to form a topological base on the domain of independent variables. The Cartan topology constructed on this system of forms has the useful feature that the exterior differential may be interpreted as a limit point, or closure, operator in the sense of Kuratowski (see p. 72 in [151]). The exterior differential systems that define the Maxwell-Ampere and the Maxwell-Faraday equations above are essentially topological constraints of closure.

The complete Maxwell system of differential forms (which assumes the existence of A and G and C2 differentiability) also generates two other exterior differential systems,

$$(F\hat{}G - A\hat{}J) - d(A\hat{}G) = 0, \tag{4.69}$$

$$F\hat{}F - d(A\hat{}F) = 0, \tag{4.70}$$

which prolong the primary (exact) exterior differential systems,

$$F - dA = 0, \tag{4.71}$$

$$J - dG = 0. \tag{4.72}$$

The terms $A\hat{}F$ and $A\hat{}G$ are zero for equilibrium systems. The existence of these 3-forms are indicators that the electromagnetic system is NOT in equilibrium.

Each of the forms, A, G, $A\hat{}G$, $A\hat{}F$, can have closed but not exact components. The two 4-forms $(F\hat{}G - A\hat{}J)$ and $(F\hat{}F)$ are exact and have closed integrals which are evolutionary (relative) invariants of continuous deformations. The closed integrals therefore describe topological properties.

The first 3-form density, $A\hat{}G$, with physical units of $\hbar$, is called "Topological Spin" (or chirality) [209] and the second 3-form, $A\hat{}F$, with physical units of $(\hbar/e)^2$, is called "Topological Torsion" (or helicity) [218]. These two exterior 3-forms, $A\hat{}G$ and $A\hat{}F$ are not usually found in discussions of classical electromagnetism. The 3-forms are abstractly defined (on a space of 4 geometric dimensions with a volume element, $\Omega_4 = dx\hat{}dy\hat{}dz\hat{}dt$) in terms of exterior multiplication, but can be given realization in terms of 4-component engineering variables, $\mathbf{S}_4$ and $\mathbf{T}_4$:

$$\textbf{Topological Spin density} \tag{4.73}$$
$$A\hat{}G = i(\mathbf{S}_4)\Omega_4 \tag{4.74}$$
$$\mathbf{S}_4 = [\mathbf{S}, \sigma] = [\mathbf{A} \times \mathbf{H} + \mathbf{D}\phi, \mathbf{A} \circ \mathbf{D}], \tag{4.75}$$

$$\textbf{Topological Torsion vector} \tag{4.76}$$
$$A\hat{}F = i(\mathbf{T}_4)\Omega_4 \tag{4.77}$$
$$\mathbf{T}_4 = [\mathbf{T}, h] = -[\mathbf{E} \times \mathbf{A} + \mathbf{B}\phi, \mathbf{A} \circ \mathbf{B}]. \tag{4.78}$$

These constructions should be compared with the (closed and exact) 3-form of charge-current (4-vector) density, $J = dG$, which has a 4-component engineering representation as, $\mathbf{J}_4 = [\mathbf{J}, \rho]$. The concepts of the topological Spin density (current) and the topological Torsion vector have had almost no utilization in applications of classical electromagnetic theory. Each construction depends explicitly on the existence of the 1-form of Action-potentials, from which it follows that these constructions are further indications of the universality of topological thermodynamics.

Recall that the closed components of the 1-form of Action do not effect the components of the 2-form of intensities (or limit "points"), $F = dA = d(A_c + A_0) = 0 + d(A_0)$. However, these "gauge" additions do influence the topological dimension of the 1-form of Action. Consider a 3-form, $A = A_c + A_0$

For example, let A_0 be of Pfaff Topological Dimension 2, representing an equilibrium system, where $A_0\hat{}dA_0 = 0$, and $A_0\hat{}dA_0 = 0$. The addition of a closed component, A_c, to the original action, A_0, leads to a new 1-form of Action, $A = A_c + A_0$. The PTD of the composite PTD$(A = A_0 + A_c)$ could be 3, as

$$A\hat{}dA = (A_c + A_0)\hat{}dA_0 = A_c\hat{}dA_0 \neq 0. \tag{4.79}$$

However, the 4-form $dA\hat{}dA$ is not influenced by the (gauge) addition to the original 1-form of Action, as

$$dA\hat{}dA = dA_0\hat{}dA_0 = 0. \tag{4.80}$$

Hence the gauge addition, A_c,changes the thermodynamic system from an equilibrium system, PTD$(A_0) = 2$ to a non-equilibrium system, PTD$(A) = 3$.

4.3.7 The Poincare Topological 4-forms

The exterior differentials of the 3-forms of topological Spin and topological Torsion produce two exact 4-forms, $F\hat{}G - A\hat{}J$ and $F\hat{}F$, whose closed integrals are topological objects which generalize the conformal invariants [312] of a Lorentz system, as discovered by Poincare and Bateman. Note that these topological properties of invariance with respect to continuous deformations are valid even in the non-equilibrium domain of dissipative charge-currents and radiation. Note that the topological format is not constrained by the concept of dualism, which forms the basis of Yang-Mills formalisms.

In the format of independent variables $\{x, y, z, t\}$, with a volume element Ω_4, the exterior differential, acting on the 3-forms as a topological limit point generator, can be related to the classic 4-divergence of the 4-component Topological Spin and Topological Torsion vectors, $\mathbf{S}_4$ and $\mathbf{T}_4$:

$$\begin{aligned} \textbf{Poincare 1} &= d(A\hat{}G) = F\hat{}G - A\hat{}J \\ &= \{div_3(\mathbf{A} \times \mathbf{H} + \mathbf{D}\phi) + \partial(\mathbf{A} \circ \mathbf{D})/\partial t\}\ \Omega_4 \\ &= \{(\mathbf{B} \circ \mathbf{H} - \mathbf{D} \circ \mathbf{E}) - (\mathbf{A} \circ \mathbf{J} - \rho\phi)\}\ \Omega_4. \end{aligned} \tag{4.81}$$

$$\begin{aligned} \textbf{Poincare 2} &= d(A\hat{}F) = F\hat{}F \\ &= \{div_3(\mathbf{E} \times \mathbf{A} + \mathbf{B}\phi) + \partial(\mathbf{A} \circ \mathbf{B})/\partial t\}\ \Omega_4 \\ &= \{2\mathbf{E} \circ \mathbf{B}\}\ \Omega_4. \end{aligned} \tag{4.82}$$

The Poincare 4-forms are, in effect, the evolutionary source terms for the 3-forms of topological spin, $A\hat{}G$, and topological torsion, $A\hat{}F$. When the Poincare 4-forms are zero, the integrals over boundaries of the domains where electromagnetic 3-forms of $A\hat{}G$ and $A\hat{}F$ are closed, become topologically coherent, bounded configurations, invariant with respect to *all* evolutionary processes of continuous deformation.

The first term in the first Poincare 4-form has a coefficient function which represents twice the difference between the magnetic energy density and the electric energy density of the electromagnetic field in a Lagrangian sense,

$$\textbf{Topological Field Lagrangian}:\ F\hat{}G = (\mathbf{B} \circ \mathbf{H} - \mathbf{D} \circ \mathbf{E})\ \Omega_4. \tag{4.83}$$

The second term in the first Poincare 4-form has a coefficient function which is defined as the interaction energy density:

$$\textbf{Topological Interaction}:\ A\hat{}J = (\mathbf{A} \circ \mathbf{J} - \rho\phi)\ \Omega_4. \tag{4.84}$$

In Lagrangian variational methods, the 4-form $F\hat{}F$, which defines the second Poincare 4-form, has been related to the concept of Topological Parity:

$$\textbf{Topological Parity}:\ F\hat{}F = +\{2\mathbf{E} \circ \mathbf{B}\}\ \Omega_4. \tag{4.85}$$

4.3.8 Deformation Invariants as Topological Properties

Topological properties are defined as invariants with respect to homeomorphisms. A more mundane definition is that a topological property is an invariant of a continuous deformation. Certain integral properties of a thermodynamic system are deformation invariants with respect to those continuous evolutionary processes that can be described by a singly parameterized vector field. For the example of an electrodynamic thermodynamic system, the absolute deformation invariants lead to fundamental topological conservation laws, described in the physical literature of electromagnetism as the conservation of charge and the conservation of flux.

Recall the definitions used to describe processes of continuous topological evolution.

> **Definition** *A continuous process is defined as a map from an initial state of topology $T_{initial}$ into a final state of perhaps different topology T_{final} such that the limit points of the initial state are permuted among the limit points of the final state (see p. 97 et.seq. [151]). If the ordering of the limit points is invariant, the process is uniformly continuous. If the ordering (as in a folding of a boundary) or the number of the limit sets is changed, then the process is non-uniformly continuous.*

A simple description of a topological property (invariant of a homeomorphism) is an object that is a deformation invariant. Consider a rubber sheet with three holes. Stretch the rubber sheet. The holes may be deformed but the fact that there are three holes stays the same under small deformations. The concept of three holes is a topological property. It is remarkable that such topologically coherent objects (the holes) can be determined from those open and closed integrals which are deformation invariants.

A topological deformation invariant is defined as an integral of an exterior differential p-form over a p-dimensional manifold, or cycle, *zpd*, such that the Lie differential of the integral of the p-form ω with respect to a singly parameterized vector field, ρV^k, vanishes, for any choice of deformation parameter, ρ.

$$\textbf{Integral Deformation Invariant}: \quad L_{(\rho V^k)} \int_p \omega = 0 \quad \textit{any } \rho. \tag{4.86}$$

The requirements that a given p-form becomes a deformation invariant (and therefore a topological property, invariant with respect to homeomorphisms) is expressed in terms of certain topological constraints. Those objects that remain the same under continuous deformation represent topological, not geometric, properties. However, if the topological constraints required for continuous deformation are not satisfied, then topological change takes place. Topological change would require that the number of holes in the thin rubber sheet example were to change. Topological change can occur continuously or discontinuously. The focus in this monograph is on continuous topological change, and as will be demonstrated below, topological change is a necessary requirement for thermodynamic irreversibility [207].

4.3.9 Absolute Integral Invariants

There are two types of invariant integrals, Absolute and Relative integral invariants. If the exterior p-form that forms the integrand is exact, the Absolute integral invariant places conditions only on the boundary of the domain of integration. It is these types of objects (Absolute integral invariants) that give a formality to those thermodynamic concepts whereby a physical system reaches a steady state uniformly within its interior, and yet may couple with its exterior environment via fluxes across its boundary. In such cases, only effects related to the boundary are of consequence. For example, consider physical systems that can be defined by a 1-form of Action, A, such that the derived 2-form $F = dA$, is exact. It follows from Stokes theorem that the two-dimensional integral of F is an absolute integral deformation invariant with respect to *all* continuous processes that can be defined by a singly parameterized vector field, subject to a boundary condition that the net flux, $i(\rho V^k)F$, of F, across the one-dimensional boundary of M is zero:

$$L_{(\rho V^k)} \int\int_M F = \int\int_M i(\rho V^k) dF + \int\int_M d(i(\rho V^k)F) \tag{4.87}$$

$$= 0 + \int_{boundary\ of\ M} i(\rho V^k)F \Rightarrow 0. \tag{4.88}$$

This concept is at the basis of the Helmholtz theorems of vorticity conservation (or angular momentum per unit mass) in hydrodynamics, and the conservation of flux in classical electromagnetism. Herein, this concept of deformation invariance of a topologically coherent structure will be written in the form of an exterior differential system [34], $F - dA = 0$. The exterior differential system is to be recognized as a topological constraint. From Stokes theorem, the two-dimensional domain of finite support for F can not, in general, be compact without boundary, unless the Euler characteristic vanishes. There are two exceptional cases for absolute invariance of the integral, and they occur when the integration domain is compact without boundary. Such two-dimensional domains which have a zero Euler characteristic are the torus and the Klein-Bottle, but these situations require the additional topological constraint that $F\hat{}F \Rightarrow 0$. The fields in these exceptional cases must reside on these exceptional compact surfaces without boundary, which form topological coherent structures. Note that an evolutionary process could start with $F\hat{}F \neq 0$, and possibly evolve to a state with $F\hat{}F = 0$. If such residue states are compact without boundary, then they must be either tori or Klein bottles.

The same integration technique can be applied to non-exact but closed p-forms.

Relative Integral Invariants

If the integration of the exact 2-form, F, is over a *closed* two-dimensional integration chain, designated as a two-dimensional cycle, z2d (which may or may not be a two-dimensional boundary), then the integral is invariant for any deformation factor, ρ :

$$L_{(\rho V^k)} \int\int_{z2d} F = \int\int_{z2d} i(\rho V^k) dF + \int\int_{z2d} d(i(\rho V^k)F) = 0 + 0. \tag{4.89}$$

The two integrals on the right vanish, the first due to the fact that $dF = 0$, and the second due to the fact that the closed integral over an exact form vanishes. Closed integrals of exact p-forms are always relative deformation integral invariants. However, the same technique can be applied to non-exact but closed p-forms. For electromagnetism, there are several exact p-forms, each producing a relative deformation integral invariant. For example, the 3-form of charge-current density is exact, $J = dG$. The 4-forms that define the Poincare 4-forms are exact, $F\hat{}F = d(A\hat{}G)$ and $F\hat{}G - A\hat{}J = d(A\hat{}G)$ (see Chapter 4.3, [1].

If the conditions of relative integral invariance are applied to an arbitrary 1-form of Action, then the relative integral invariance condition becomes,

$$L_{(\rho V^k)} \int_{z1d} A = \int_{z1d} i(\rho V^k) dA + \int_{z1d} d(i(\rho V^k)A) \tag{4.90}$$

$$= \int_{z1d} i(\rho V^k) F + 0 \Rightarrow 0. \tag{4.91}$$

It follows that $i(\rho V^k)dA$ must be zero on the cycle $z1d$ for any deformation parameter ρ. Cartan has shown that this is the condition that implies the process ρV^k has a "Hamiltonian" representation [44] (see Chapter 4.5, [1]).

4.3.10 Holder Norms, Period Integrals and Topological Quantization

Besides the invariant structures considered above, the Cartan methods may be used to generate other sets of topological invariants. Realize that over a domain of Pfaff Topological Dimension n less than or equal to N, the Cartan criteria admits a submersive map to be made from N to a space of minimal dimension n. Assume the submersive map produces functions,

$$[V^1(x, y, z..), V^2(x, y, z..), ...V^n(x, y, z..)], \tag{4.92}$$

with a differential volume element, $\Omega_n = dV^1\hat{}dV^2\hat{}...\hat{}V^n$. Then construct the (n-1)-form,

$$C = i(V^1, V^2, ..., V^n)\Omega_n. \tag{4.93}$$

Define an integrating factor ρ in terms of the Holder norm,

$$\rho = 1/\lambda = 1/\{a(V^1)^p + b(V^2)^p + c(V^3)^p + ...\}^{m/p}. \tag{4.94}$$

Then multiply (n-1)-form, C, by ρ to produce (n-1)-form density (current) J as:

$$J = i(V^1, V^2, V^3, ...)\Omega = \rho\{V^1\ dV^2\hat{}dV^3... \ - V^2\ dV^1\hat{}dV^3... + V^3\ dV^1\hat{}dV^2... - ...\}. \quad (4.95)$$

Theorem 1. *The (n-1)-form J is closed, $dJ = 0$, for any choice of constants a,b,c.. and for any p, if the Holder homogeneity index satisfies the equation, $m = n$* (for proof, see Chapter 8, Volume 1 [1]).

It is remarkable that the "current" J so defined has a vanishing exterior differential, independent of the value of p for a given m (equal to the dimension of the submersive volume element), and for all values of the constants (plus or minus a,b,c...). All such "currents" thereby define a "conservation law". As the map defining the components of the vector field in terms of the base {x,y,z..} is presumed to be differentiable, then the (n-1)-form, J, has a well defined pullback on the base space (almost everywhere), and its exterior differential on the base space also vanishes everywhere mod the defects. That is, the (n-1)-form, J, is locally exact. The number of negative coefficients in set $\{a, b, c, d...\}$ determines the signature index of the Holder norm. The number m determines the homogeneity index. The Holder integrating factors are more familiar when $m = 1,\ p = 1$ which generates the barycentric coordinates $\{a, b, c, d...\}$ of Moebius, [31], and for $m = 1,\ p = 2,\ a = b = c...$, which is known as the Gauss map. Use for both of these special Holder constructions will be developed in that which follows.

The integrals of these closed currents, when integrated over closed (n-1)-dimensional chains, create deformation invariants, with respect to any evolutionary process that can be described by a vector field, because

$$L_{(\rho\mathbf{V})} \int_{z(n-1)d} J = \int_{z(n-1)d} i(\rho\mathbf{V})dJ + \int_{z(n-1)d} d(i(\rho\mathbf{V}))J) \Rightarrow 0 + 0 = 0. \quad (4.96)$$

These integral objects appear as "topological coherent" structures (which may have defects or anomalous sources, when the integrating factor $1/\lambda$ is not defined). The integration chain, $z(n-1)d$, is a $(n-1)$-dimensional (d) cycle z.

Remark 21 *The Holder norm, which creates closed forms when the homogeneity index $m = n$, is a topological idea independent from the metric signature.*

The complement to the zero sets of the function λ determine the domain of support associated with the specified vector field. The closed (n-1)-form, J, that satisfies the conservation law, $dJ = 0$, has integrals over closed domains that have rational fraction ratios. As this (n-1)-current is closed globally, it may be deduced on a connected local domain from a (n-2)-form, G. In every case J has a well defined pullback to the base variety, {x,y,z,t}. Note that the n functions

$[V^x(x,y,z..), V^y(x,y,z..), V^z(x,y,z..), ...]$ represent the minimum number of Clebsch variables that are equivalent to the original action, A, over the domain of support. As each of these integrals is intrinsically closed, the Lie differential with respect to any C1 vector field, $\rho\mathbf{V}$, is a perfect differential, such that (when integrated over closed domains that are p-1 boundaries) the evolutionary variation of these closed integrals vanishes. These n-1 integrals are relative integral invariants for any C1 evolutionary processes, or flows. The values of the integrals are zero if the closed integration domains are boundaries, or completely enclose a simply connected region. If the closed integration domains encircle the zeros of the function λ, then the values of the integrals are proportional to the integers; i.e., their ratios are rational.

In general, by deRham's theorems, the values of these period integrals, for different closed integration chains in domains where $dJ = 0$, have rational ratios [209]. When the evolution of a period integral is such that the integer changes, the process can describe the decay from a quantized stationary state of topological quantum number m to a state of topological quantum number n:

$$\textbf{Topological Quantization}: \ L_{(\rho\mathbf{V})} \int_{z(n-1)d} J = n \text{ constant.} \tag{4.97}$$

Note that each signature of λ must be investigated. For the elliptic (positive definite) signature, the singular points are the stagnation points, and the domain of support excludes those singularities. For the hyperbolic signatures, the domain of support excludes the hyperbolic singularities of lower dimension, such as the light cone. Further note that a given vector field may not generate real domains of support for all possible signatures of the quadratic form, λ.

4.4 Uniform Continuity, Frozen - in Fields, the Master Equation

All three topological theories, Thermodynamics, Electromagnetism, and Hydrodynamics are the same relative to the Kolmogorov-Cartan topology whose subsets are exterior differential forms. The thermodynamics is the same; what forms the differences between the concepts? The answer is related to the functional form of the processes, and their classes, represented by 3-form source-current densities that will refine the KCT0 topological structure. The classes of processes enable the discrete (quantum) properties of charge, spin, and baryon number to be incorporated into the topological perspective. The discrete quantities are to be evaluated in terms of deRham period integrals.

4.4.1 The master equation

A starting point for many discussions of the magnetic dynamo and allied problems in hydrodynamics starts with what has been called the "master equation" ,

$$\text{Master equation for electromagnetism} \quad Curl(\mathbf{V}\times\mathbf{B}) = \partial\mathbf{B}/\partial t. \tag{4.98}$$
$$\text{Master equation for hydrodynamics} \quad Curl(\mathbf{v}\times\boldsymbol{\omega}) = \partial\boldsymbol{\omega}/\partial t, \tag{4.99}$$

The ideas of the "master equation" and its applications are also valid for hydrodynamics. Using the Cartan methods it may be shown that this "master equation" is equivalent to the constraint of process "uniform" continuity relative to the Cartan topology. Moreover, it is easy to show these constraints generate symplectic processes which include Hamiltonian evolutionary systems, such as Euler flows, as well as a number of other evolutionary processes which are continuous, but not homeomorphic. In addition a criteria can be formulated to develop an extension of the "helicity" conservation law to a more general setting.

The proof of these results produces a nice exercise in use of the Cartan theory. Consider a 1-form A that satisfies the exterior differential system,

$$F - dA = 0, \tag{4.100}$$

where A is a 1-form of Action, with twice differentiable coefficients (potentials proportional to momenta) which induce a 2-form, F, of electromagnetic intensities ($\mathbf{E}$ and $\mathbf{B}$, related to forces). The exterior differential system is a topological constraint that in effect defines field intensities in terms of the potentials.

Now search for all vector fields that leave the 2-form F an absolute invariant of the flow; that is, search for all vectors that satisfy Cartan's magic formula,

$$L_{(\rho\mathbf{V})}F = i(V)dF + di(V)F = 0 + di(V)F = 0. \tag{4.101}$$

For C2 functions, the term involving dF vanishes, leaving the expression,

$$L_{(\mathbf{V})}F = di(V)F, \tag{4.102}$$

$$= d\{(\mathbf{E} + \mathbf{V} \times \mathbf{B}) \cdot d\mathbf{r} - (\mathbf{E} \cdot \mathbf{V})dt\}, \tag{4.103}$$

$$= \{curl(\mathbf{E} + \mathbf{V} \times \mathbf{B})\}_z dy\hat{}dz... \tag{4.104}$$

$$+ \{\partial(\mathbf{E} + \mathbf{V} \times \mathbf{B})/\partial t + grad(\mathbf{E} \cdot \mathbf{V})\} \cdot d\mathbf{r}\hat{}dt, \tag{4.105}$$

$$= 0. \tag{4.106}$$

Setting the first three factors to zero yields the constraint of C2 continuity,

$$curl(\mathbf{E} + \mathbf{V} \times \mathbf{B}) = 0. \tag{4.107}$$

From the Maxwell Faraday equations for C2 functions, $curl\mathbf{E} = -\partial\mathbf{B}/\partial t$, and when this expression is substituted into the above equation, the "master" equation given above is the result,

$$\text{Master equation} \;\; Curl(\mathbf{V} \times \mathbf{B}) = \partial\mathbf{B}/\partial t. \tag{4.108}$$

Now recall that dF generates the limit points of A, and if $F = dA$ is a flow invariant, then all limit points are flow invariants relative to the Cartan topology. This result implies that the vector fields, $\mathbf{V}$, that satisfy the constraints of the "master equation" are uniformly continuous evolutionary processes, as the limit points, $F =$

dA, of the 1-form A are flow invariants, and the lines of vorticity are "frozen-in" the flow.

Non-uniform continuity would imply that the limit points are not invariants of the process, but that the closure of the limit points of the target range includes the limit points of the initial domain. Such processes would correspond to a folding of the "lines" of vorticity, which preserve the limit points, but not their sequential order.

A second criteria for limit point invariance is given by the equation,

$$\{\partial(\mathbf{E}+\mathbf{V}\times\mathbf{B})/\partial t + grad(\mathbf{E}\cdot\mathbf{V})\} = 0. \tag{4.109}$$

This formula indicates that limit point invariance can occur in the presence of input-output power, $\mathbf{E}\cdot\mathbf{V} \neq 0$.

The criteria for frozen-in fields is established as a constraint of uniform continuity on the admissible vector fields,

$$\textbf{Uniform Continuity}: \quad di(V)dA = di(V)F = 0. \tag{4.110}$$

The solution vector fields, V, subject to this constraint can be put into three global categories:

1. Extremal (Hamiltonian) $\quad i(V)F = 0.$
2. Bernoulli-Casimir (Hamiltonian) $\quad i(V)F = d\theta.$
3. Helmholtz-Symplectic $\quad i(V)F = d\theta + \gamma_{harmonic}.$

The first category can exist only on domains of support of F which are of odd Pfaff dimension, but then the solution vector is unique to within a factor. In the other categories, the solution vector need not be unique. Vector fields that satisfy the equation for uniform continuity are said to be symplectic relative to the 1-form, A. Vector fields that belong to categories 1 and 2 have a Hamiltonian representation. Vector fields that belong to category 1, are said to be "extremal" relative to the 1-form, A.

When the concepts are applied to the integrals of the 2-form F, then the criteria for invariance of the flux integral depends on the topology of the integration domain. If the integration area of the 2-form is a boundary or a cycle of a three-dimensional domain, the flux integral over the closed boundary or cycle is always a flow invariant. If the integration area is bounded, then by Stokes theorem the flux integral depends only on the boundary conditions; F or $i(V)F$ must vanish, on the boundary, or when integrated over the boundary.

4.4.2 Non-uniform continuity and folds

Recall the definitions used to describe processes of continuous topological evolution.

> **Definition** *A continuous process is defined as a map from an initial state of topology $T_{initial}$ into a final state of perhaps different topology T_{final} such that the limit points of the initial state are permuted among*

the limit points of the final state (see p. 97 et.seq. [151]). If the ordering of the limit points is invariant, the process is uniformly continuous. If the ordering (as in a folding of a boundary) or the number of the limit sets is changed, then the process is non-uniformly continuous.

Remark 22 *Processes that are uniformly continuous,* $dQ = 0$*, can cause topological change, but non-uniform continuity,* $dQ \neq 0$*, is required to produce thermodynamic irreversibility.*

Chapter 5

SPINTRONICS WITH FALACO SOLITONS AS SURFACE SPIN PAIRS

To summarize the previous chapters: From a topological perspective, a differential 1-form of Action on a differential variety encodes a particle physical thermodynamic system. The Pfaff Topological Dimension of A will be limited to N $\leq$ 4. The coefficients of a differential N-1=3-form encodes a process vector (a direction field) acting on the thermodynamic system. The dynamics are given in terms of the Lie differential, with respect to a process vector direction field, acting on the system 1-form of Action potentials to produce the differential 1-form of Heat. The resulting formula is an abstract realization of the First Law of Thermodynamics. The 3-forms, to within a density factor, represent Currents in the thermodynamical system.

5.1 Current 3-forms

Process Vector fields are direction fields that can be related to the coefficients of a 3-form on the 4D variety. Vector direction fields $V^k(x,y,z,t)$ are not necessarily global generators of of a 1-parameter group. That is, Kinematic Perfection,

$$dx^k - V^k(x,y,z,t)dt \neq 0 \tag{5.1}$$

may not be true over a neighborhood. In terms of the 1-form, $A_{kinematic}$:

$$\begin{aligned}
\Delta_{kinematic} &= (dx^k - V^k dt), && (5.2)\\
d\Delta_{kinematic} &= dV^k\hat{}dt,\ and && (5.3)\\
\Delta\hat{}d\Delta_{kinematic} &= 0. && (5.4)
\end{aligned}$$

Hence kinematic perfection corresponds to a differential system of Pfaff Topological Dimension, PTD($\Delta_{kinematic}$) = 2. This constraint is a highly restrictive topological constraint that denies the viability of kinematic perfection to non-equilibrium systems, for which the Pfaff topological dimension is 3 or greater.

Typically a process vector field has functional components,

$$V = [V^1(\xi^k), V^2(\xi^k), V^3(\xi^k), V^4(\xi^k)], \tag{5.5}$$

relative to the ordered array of base variables, $\{\xi^1, \xi^2, \xi^3, \xi^4\}$, with an oriented differential volume element $\Omega_4 = d\xi^1\hat{}d\xi^2\hat{}d\xi^3\hat{}d\xi^4$. The volume element, or the vector field can be scaled by an arbitrary function, $\rho(\xi^k)$. The N-1=3-form Current, C, can be defined as

$$C = i(\rho V)\Omega_4 = \rho(V^1 d\xi^2\hat{}d\xi^3\hat{}d\xi^4 - V^2 d\xi^1\hat{}d\xi^3\hat{}d\xi^4 \quad (5.6)$$
$$+V^3 d\xi^1\hat{}d\xi^2\hat{}d\xi^4 - V^4 d\xi^1\hat{}d\xi^2\hat{}d\xi^3). \quad (5.7)$$

On the classic 4D differential variety $\{x, y, z, t;\ dx, dy, dz, dt\}$ the process vector direction field can be made homogenous, by dividing each coefficient by V^4.

$$\text{Homogeneous} \quad \mathbf{v} = [V^1/V^4, V^1/V^4, V^1/V^4, 1], \quad (5.8)$$
$$= [\mathbf{v}^1, \mathbf{v}^2, \mathbf{v}^3, 1] \quad (5.9)$$

Then by renaming the 3 spatial components as $\mathbf{J}_3 = \rho\mathbf{v}_3$, the familiar format of a current 3-form becomes recognizable:

$$J = i(\rho V)\Omega_4 = (\mathbf{J}^x dy\hat{}dz\hat{}dt - \mathbf{J}^y dx\hat{}dz\hat{}dt \quad (5.10)$$
$$+\mathbf{J}^z dx\hat{}dy\hat{}dt - \rho dx\hat{}dy\hat{}dz. \quad (5.11)$$

The exterior differential of the Current then defines the classic germ of a 4-divergence conservation law, when $dJ \Rightarrow 0$.

$$dJ = di(\rho V)\Omega_4 = \{div(\mathbf{J}_3) + \partial\rho/\partial t\}dx\hat{}dy\hat{}dz\hat{}dt. \quad (5.12)$$

However, unlike the Amperian current density (which is closed in the sense that $dJ_{Ampere} = 0$) the general process current is not closed.

There are several important 3-forms of current in a topological perspective:

i. The Amperian Current, (Maxwell 1861).
ii. The Spin Current (1969 [199]).
iii. The Torsion Current (1976 [205]).

The latter two currents have been ignored in most treatments of classical Physics

The Torsion Current is an indicator of non-unique predictability, and if its divergence is not zero, the system process is thermodynamically irreversible and dissipative.

The Spin Current is an indicator of collective states, and if its divergence is not zero, it is presumed to be an indicator of inertial energy.

The zero divergence conditions of Spin Current and Torsion Current are known as the first and second Poincare conservation theorems in Electromagnetism. From a topological thermodynamic perspective, the concepts are universal.

5.1.1 The Amperian Current

When the current is computed in terms of the Amperian 3-form, $J = dG$, the components of the Current define a process, $\mathbf{J}_4$

$$\begin{aligned}
\text{Amperian Current} \quad \mathbf{J}_4 &= [\mathrm{J}^1, \mathrm{J}^2, \mathrm{J}^3, \rho], && (5.13)\\
dG &= i(\mathbf{J}_4)\Omega_4, && (5.14)\\
ddG &= d(i(\mathbf{J}_4)\Omega_4) = (div4\mathbf{J}_4)\Omega_4 = 0. && (5.15)\\
L_{(\mathbf{T}_4)}A &= i(\mathbf{J}_4)dA + d(i(\mathbf{J}_4)A) && (5.16)\\
&= W_{Lorentz_force} + dU_{interaction_energy.} && (5.17)
\end{aligned}$$

Note that the Amperian Current 3-form is exact, and has Zero 4 Divergence. This Amperian Current charge-current 3-form is ubiquitous in electromagnetic systems.

5.1.2 The Torsion Current

When a current is computed in terms of the Torsion 3-form, $A\hat{}F$, then the components of the Torsion Current define a special process, $\mathbf{T}_4$

$$\begin{aligned}
\text{Torsion Current} \quad \mathbf{T}_4 &= [\mathrm{T}^1, \mathrm{T}^2, \mathrm{T}^3, \mathrm{T}^4], && (5.18)\\
A\hat{}F &= (i(\mathbf{T}_4)\Omega_4, && (5.19)\\
L_{(\mathbf{T}_4)}A &= i(\mathbf{T}_4)dA + d(i(\mathbf{T}_4)A). && (5.20)
\end{aligned}$$

The Torsion Current is a direction field that is not (necessarily) an element of a vector space group. Note that the 4-divergence (exterior differential) of the Torsion Current can be expressed as,

$$\begin{aligned}
d(i(\mathbf{T}_4)\Omega_4 &= (div4\mathbf{T}_4)\Omega_4 && (5.21)\\
&= d(A\hat{}F) = F\hat{}F\ , && (5.22)\\
\text{Poincare II} \quad d(i(\mathbf{T}_4)\Omega_4) &= F\hat{}F = \mathrm{K}\Omega_4, && (5.23)
\end{aligned}$$

where $F\hat{}F$ is the Topological Parity 4-form. The coefficient, K, when zero, is known as the second Poincare invariant.

$$\begin{aligned}
L_{(\mathbf{T}_4)}A\hat{}F &= (i(\mathbf{T}_4)d(A\hat{}F) + d((i(\mathbf{T}_4)(i(\mathbf{T}_4)\Omega_4, && (5.24)\\
&= (i(\mathbf{T}_4)(F\hat{}F) + 0. && (5.25)
\end{aligned}$$

$$\begin{aligned}
\text{However} \quad (i(\mathbf{T}_4)A)\Omega_4 &= A\hat{}A\hat{}dA = 0 && (5.26)\\
L_{(\mathbf{T}_4)}A &= i(\mathbf{T}_4)dA = \sigma A && (5.27)\\
L_{(\mathbf{T}_4)}dA &= di(\mathbf{T}_4)dA = d\sigma A + \sigma dA && (5.28)\\
Q\hat{}dQ &= \sigma^2 A\hat{}dA \neq 0 && (5.29)
\end{aligned}$$

Note that the 4-divergence of the Torsion current leads to an expanding or contracting 4D differential volume element, and can be associated with irreversible dissipation (bulk viscosity in the hydrodynamic case, and $\mathbf{E} \circ \mathbf{B} \neq \mathbf{0}$ in the electromagnetic case). Any process that has the direction field of $\mathbf{T}_4$ is an irreversible process, as $Q\hat{}dQ \neq 0$.

5.1.3 The Spin Current

When the current is computed in terms of the Spin 3-form, $A\hat{}G$, then the components of the Current define a process direction field, $\mathbf{S}_4$

$$\begin{aligned} \text{Spin Current} \quad \mathbf{S}_4 &= [\mathrm{S}^1, \mathrm{S}^2, \mathrm{S}^3, \mathrm{S}^4], && (5.30)\\ A\hat{}G &= i(\mathbf{S}_4)\Omega_4, && (5.31)\\ \text{Poincare I} \quad d(i(\mathbf{S}_4)\Omega_4) &= (div4\mathbf{S}_4)\Omega_4 = F\hat{}G - A\hat{}J. && (5.32) \end{aligned}$$

It is these features that imply that superfluidity and superconductivity are related to zero values of the Poincare invariants.

5.1.4 The London equation of superconductivity. $J_{London} = \chi A$

It the historical literature, the concept of superconductivity was conjectured to be described by a modification of Maxwell's equations, called London's equation. The fundamental assumption was that the superconductor Charge-Current 3-vector density, $\mathbf{J}$, was proportional to the 3-Vector potential, $\mathbf{A}$,

$$\text{London Supercurrent} \quad \mathbf{J}_{London} = \chi\mathbf{A}. \tag{5.33}$$

The implication is that Maxwell's equations must be modified to include the supercurrent. From a topological perspective, this conclusion is false. That is, the London equation is contained within the topological format of electromagnetism. Several examples are given in Vol 4 [1]. For a specific choice(s) of the electrodynamic potential(s) that encode the thermodynamic 1-form of Action, the Amperian current(s) $\mathbf{J}_4$ can be computed* and demonstrate that the 4 components of the Amperian Current, $\mathbf{J}_4$, are proportional to the 4 components of the electrodynamic potentials, $\mathbf{A}_4$.

$$\mathbf{J}_4 = [\mathrm{J}^x, \mathrm{J}^y, \mathrm{J}^z, \rho] = \chi\mathbf{A}_4 = \chi[A_x, A_y, A_z, -\phi] \tag{5.34}$$

The first 3-components yield the London equation.

5.1.5 The Inertial equation of Spin. $J_{spin} = \lambda W$

Another important equation that can be extracted from the topological perspective of electromagnetism (that does not seem to appear in the historic literature) is that for specific choice(s) of the electrodynamic potential(s) that encode the thermodynamic 1-form of Action, the Spin currents $\mathbf{S}_4$ can be computed and demonstrate that the 4 components of the Work 1-form, W, are proportional to the 4 components of the Spin current, $\mathbf{S}_4$.

$$\mathbf{W}_4 = [W_x, W_y, W_z, P] = \epsilon\chi\mathbf{S}_4 = \epsilon\chi[\mathrm{S}^x, \mathrm{S}^y, \mathrm{S}^z, \mathrm{S}^t] \tag{5.35}$$

Remark 23 *The Spin Current can be proportional to the Work 1-form, while the Amperian Current can be proportional to the Action 1-form. The physical impact of this Spin current formulation is under study.*

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*A Maple program was constructed for the somewhat tedious calculations.

5.2 Topological Insulators, Superconductors and the Topological Spin Hall effect

E. J. Post recognized that one of the most remarkable features of electrodynamics of the micro world compared to the macro world is that when written in terms of the MKS system of units, the fine structure constant, α, becomes,

$$\alpha = 2\pi e^2/4\pi\varepsilon\, hc = 1/2(\mu/\varepsilon)^{1/2}/(h/e^2). \tag{5.36}$$

This formula demonstrates that α is a ratio of two fundamental impedances, the free-space impedance, Z_0,

$$Z_0 = (\mu/\varepsilon)^{1/2} = 376.730313\Omega, \tag{5.37}$$

and the Hall impedance, Z_{Hall},

$$Z_{Hall} = h/e^2 = 25812.81491\Omega. \tag{5.38}$$

The relation between the quantum mechanical entities and the free space impedance as given by the equation,

$$\alpha = 1/2(Z_0/Z_{Hall}), \tag{5.39}$$

to 8 decimal places. The result elevates the importance of the free space impedance, Z_0.

Remark 24 *The Free Space Impedance, Z_0, must involve a topological argument related to the ratio of $(B \circ E)$ to $(D \circ H)$. The numerator is related to the 4-form $F\hat{}F$. What is the 4-form that is related to $(D \circ H)$?*

The objective is to define superconductivity in a topological manner which incorporates the "quantization" features of deRham cohomology theory. These topological defect structures can be evaluated in terms of period integrals, which are integrals over closed domains of closed integrands. Such integrals are topologically quantized in the sense that the values of the integrals are integers times a scaling constant. If the closed integration domain is a boundary, not a cycle, then the integer is zero, and the closed and bounded set is an invariant set with respect to any thermodynamic process. If the closed but not exact integrand is a 1-form, and the integration chain is a cycle, but not a boundary, then the period integral has been denoted as the Berry phase, but it was predicted long before by Bohm and Aharanov.

The idea of forming a ratio of period integrals to define an impedance was an idea of E. J. Post [190]. In fact, Post predicted that the Hall effect would exhibit rational fraction behavior two years before the experimental measurements were made. He did this without the assumption of fractional charge. The implication is that superconductivity is related to topological defect structures. The period integrals are exhibitions of the deRham theory of cohomology, and the quantization result is independent from scales.

There are three ways to construct an impedance Z (with physical dimensions (h/e^2) from period integrals [221]:

Ordinary Superconductors	Impedance $\mathsf{Z}_1 = \oint A / \iint_{z2} G$.
Anyon (High Tc ?)	$\mathsf{Z}_2 = \iiint_{z3} A\hat{}G / (\iint_{z2} G)^2$.
Fractional Hall	$\mathsf{Z}_3 = \iiint_{z3} A\hat{}F / \iiint_{z3} A\hat{}G$.

Post chose Z_1. In order to produce rational fractions, the closed integrals must be period integrals, where the closed but not exact integrands are closed in an exterior differential sense over the closed domains (cycles of one, two, or three dimensions) of integration. The closure condition on the first impedance Z_1 requires that $dA \Rightarrow 0$, which implies that the domain excludes the field intensities. This constraint is in agreement with the experimentally measured Meissner repulsion of the **B** field in ordinary superconductors, and the Bohm-Aharanov effect. The closure condition on the third impedance, Z_3 requires that both Poincare 4-forms must vanish, but **E** and **B** fields are permitted in the domain of integration (as is observed experimentally in the Hall effect). I believe, counter to Post, that Z_3 is the proper expression for the Hall effect, not Z_1. Z_3 is the ratio of the closed integrals of Topological Torsion, $A\hat{}F$, divided by the closed integrals of Topological Spin, $A\hat{}G$. Moreover, on thermodynamic grounds, the non-zero value of Topological Torsion, $A\hat{}F$, indicates that the Hall effect is an artifact of a non-equilibrium system.

Note that Ordinary Superconductors are such that the field intensities are expelled from the interior, and $dA = F \Rightarrow 0$. This criteria implies that this class of electromagnetic systems is of Pfaff Topological Dimension 1, which is an equilibrium system. The Fractional Hall superconductor admits a non-zero Topological Torsion, $A\hat{}F \neq 0$, and a non-zero Topological Spin, $A\hat{}G$. Hence such Hall effect superconductors are not equilibrium or isolated systems, but instead are non-equilibrium electromagnetic systems. However, the Poincare 4-forms vanish, enabling the fractional quantization, which implies that such systems are of Pfaff Topological Dimension 3. They are closed, not open, thermodynamic configurations.

5.2.1 Interaction Energy density and Topological Superconductivity

The conjecture to be explored herein is that a supercurrent corresponds to the case where the electromagnetic interaction energy density, $A\hat{}J$, vanishes in a topological sense. The motivation for such an assumption is founded upon the observation that if the 3-form of charge-current density, J, was proportional to either the 3-form of Topological Torsion, $J_{Torsion} = \gamma \cdot A\hat{}F$, or the 3-form of Topological Spin, $J_{spin} = \eta \cdot A\hat{}G$, then it follows that the interaction energy density of classical field theory will vanish, $A\hat{}J \Rightarrow 0$:

$$A\hat{}J_{Torsion} = A\hat{}(\gamma \cdot A\hat{}F) = \gamma \cdot (A\hat{}A\hat{}F) = 0, \tag{5.40}$$

$$A\hat{}J_{Spin} = A\hat{}(\eta \cdot A\hat{}G) = \eta \cdot (A\hat{}A\hat{}G) = 0. \tag{5.41}$$

Assume that a supercurrent contains components proportional to Topological Torsion 3-form, $\gamma \cdot A\hat{}F$, and the Topological Spin 3-form, $\eta \cdot A\hat{}G$. In order for *each* of the supercurrent components to be conserved,

$$d(J_{Torsion}) = d\gamma\hat{}A\hat{}F + \gamma \cdot F\hat{}F \Rightarrow 0, \quad (5.42)$$

$$d(J_{Spin}) = d\eta\hat{}A\hat{}G + \eta \cdot F\hat{}G \Rightarrow 0. \quad (5.43)$$

Simple solutions are found if both Poincare 4-forms vanish, and the functions γ and η are constants. More complicated solutions can occur if the sum of the two supercurrents has a zero divergence,

$$d(J_{\text{supercurrent}}) = d(J_{Torsion} + J_{Spin}) \Rightarrow 0. \quad (5.44)$$

In other words, the divergence of the torsion current and the spin current could compensate to form a total supercurrent which is conserved.

Another case would be to consider those situations where the 3-form charge-current density, J, has components proportional to those components of the 1-form of potentials which are elements of a spinor, $J = \lambda J_{spinor}$. The 3-form can always be multiplied by an integrating factor such that the rescaled spinor current has zero divergence. Similarly, suppose the 1-form A_{spinor} (to within a factor) also has the same spinor component functions. Then the interaction density vanishes, as,

Spinor London current

$$J_{spinor} = i(\lambda A_{spinor})\Omega, \quad (5.45)$$

Interaction Energy density

$$A\hat{}J = \lambda \langle A_{spinor} \circ A_{spinor} \rangle \ \Omega_4 \Rightarrow 0. \quad (5.46)$$

Hence, a charge-current 3-form composed of three parts, such that,

Total Supercurrent

$$J_{\text{supercurrent}} = J_{spinor} + A\hat{}F/\lambda + A\hat{}G/\eta, \quad (5.47)$$

With Interaction Energy density

$$A\hat{}J = A\hat{}J_{\text{supercurrent}} \Rightarrow 0, \quad (5.48)$$

is a candidate for a superconducting current, which intuitively has no interaction energy density.

If the Action 1-form is divided by a suitable quadratic Holder norm, then the Jacobian matrix of the Action 1-form can be computed. The matrix has a determinant zero, if the homogeneity index is 1. The matrix defines the equivalent to the Shape matrix in differential geometry, and its Cayley–Hamilton similarity invariants define the curvatures generated by the zero set of the Cayley-Hamilton characteristic polynomial. If the current $J_{adjoint}$ is defined as the product of the adjoint of the shape matrix times the homogeneous coefficients of the 1-form of Action, then the homogeneous interaction energy,

$$\text{Homogeneous Interaction energy} : A_{\text{homogeneous}}\hat{}J_{\text{adjoint}}, \tag{5.49}$$

is exactly equal to the cubic curvature similarity invariant.

5.2.2 Irreversible Evolutionary Processes (Pfaff Topological Dimension 4)

Assume that the Pfaff Topological Dimension of the domain of interest is 4, hence the space is symplectic. However, consider evolutionary fields that are not constrained to be symplectic such that $dW \neq 0$. Direct evaluation of the virtual work 1-form, $W = i(\mathbf{V}_4)dA$ yields (the Lorentz force),

$$W = i(\mathbf{V}_4)dA = -(\{\rho\mathbf{E} + \mathbf{J} \times \mathbf{B}\}_k dx^k - \{\mathbf{J} \bullet \mathbf{E}\}dt). \tag{5.50}$$

The obvious first choice for the evolutionary vector field has been based on the classic assumption that $\mathbf{V}_4$=[$\mathbf{J}$;ρ] $\Rightarrow \rho[\mathbf{V}; 1]$. The expression for virtual work becomes,

$$W = -\rho(\{\mathbf{E} + \mathbf{V} \times \mathbf{B}\}_k dx^k - \{\mathbf{V} \bullet \mathbf{E}\}dt). \tag{5.51}$$

However, another perhaps not so obvious a candidate for a solution vector field is the expression for the Torsion current. That is, examine the evolution along the unique four-dimensional vector field,

$$\mathbf{T}_4 = -\{(\mathbf{E} \times \mathbf{A} + \mathbf{B}\phi); \mathbf{A} \bullet \mathbf{B}\}. \tag{5.52}$$

The expression for virtual work becomes

$$W = i(\sigma\mathbf{T}_4)dA \tag{5.53}$$
$$= \sigma(\{(\mathbf{A} \bullet \mathbf{B})\mathbf{E} + (\mathbf{E} \times \mathbf{A}) \times \mathbf{B}\}_k dx^k - \{\mathbf{E} \bullet \mathbf{B}\phi\}dt) \tag{5.54}$$
$$= \sigma(\mathbf{E} \bullet \mathbf{B})A. \tag{5.55}$$

The torsion current is an associated field relative to the 1-form of Action, in the sense that

$$i(\sigma\mathbf{T}_4)A \Rightarrow 0. \tag{5.56}$$

Evolution in the direction of the Torsion vector does not produce any internal energy of interaction, even though the process is not extremal. The $\mathbf{T}_4$ process is thermodynamically locally[†] adiabatic. In Pfaff Topological Dimension 4, the Torsion vector is not extremal, but adiabatic. It is amazing that $\mathbf{T}_4$ can decay to a process which is characteristic; i.e., a process which is adiabatic and homogeneous of degree zero.

[†]Locally adiabatic means along a flow line, but not necessarily from flow line to flow line.

It follows that the Lie differential of the Action along the direction of the Torsion current is a special isovector process in the sense that

$$L_{(\sigma\mathbf{T}_4)}A = \Gamma A = \sigma(\mathbf{E}\bullet\mathbf{B})A = Q. \tag{5.57}$$

By direct computation,

$$L_{(\sigma\mathbf{T}_4)}dA = d\Gamma\hat{}A + \Gamma dA = dQ \tag{5.58}$$

from which it follows that

$$Q\hat{}dQ = \Gamma^2 A\hat{}dA. \tag{5.59}$$

If the topological parity $\Gamma = \sigma(\mathbf{E}\bullet\mathbf{B})$ does not vanish, then the Torsion current $\sigma\mathbf{T}$ represents an irreversible non-conservative process. For such processes the Heat 1-form, Q, does not admit an integrating factor.

The formula $L_{(\sigma\mathbf{T}_4)}A = \Gamma A$ was, in effect, the fundamental equation that I used in 1974 to describe "An Extension of Hamilton's Principle to Include Dissipative Systems" [200] [204, RMK 1975]. It was not known at that time the such processes implied the existence of a symplectic structure, nor the fact that these processes were not symplecto-morphisms.

5.3 Hedgehog B fields, Rotating plasmas, Accretion discs, Spin currents

Example 1a. A Plasma Accretion disc from Hedgehog B field solutions

It is possible to find a modification of a closed 1-form solution to Maxwell's equations that makes the magnetic field lines appear like the spines of a Hedgehog. All hedgehog 1-forms, A, lead to the existence of a London superconducting component, or order parameter, χ, It is also possible to demonstrate how such modifications of closed 1-forms make the z=0 plane of a rotating plasma a chiral attractor. Consider a 1-form that is based on a rotation about the z axis:

$$\text{Asume } \lambda = (x^2+y^2-\kappa z^2) \tag{5.60}$$
$$\text{Potentials } A = \alpha\varpi z/((x^2+y^2)\sqrt{\lambda})[-ydx, xdy, 0, 0]\,, \quad \text{Type 1} \tag{5.61}$$
$$\text{Scalar } \phi = 0. \text{ Note sign of } \kappa. \qquad \varpi = \pm 1 \tag{5.62}$$

These potentials induce the field intensities,

$$\mathbf{E} = [0,0,0], \tag{5.63}$$
$$\mathbf{B} = -\alpha\varpi\,[x,y,z]/(\lambda)^{3/2}. \text{ Cubic decay} \tag{5.64}$$

The $\mathbf{B}$ field is of the cubic format of the famous Dirac Hedgehog field often associated with "magnetic monopoles". However, the radial $\mathbf{B}$ field has zero divergence everywhere except at the origin, which herein is interpreted as a topological obstruction. The factor κ is to be interpreted as an oblateness factor associated with rotation of a plasma, and has values {-1,,,0,...+1}. Computation leads to the result that the helicity coefficient, $\mathbf{A}\circ\mathbf{B}$**,** vanishes, and the second Poincare 4-form coefficient, $\mathbf{E}\circ\mathbf{B}$, vanishes.

$$\mathbf{E}\circ\mathbf{B}=0 \qquad and \qquad \mathbf{A}\circ\mathbf{B}=0\,. \tag{5.65}$$

In fact, the 3-form of topological torsion vanishes identically (as $\phi=0$),

$$\mathbf{T}_4=[0,0,0,0]. \tag{5.66}$$

In this example, there is a non-zero value for the Amperian current density, even though the potentials are static. The Amperian Current Density 3-form, $\mathbf{J}_4$, has four components,

$$\mathbf{J}_4=(3\alpha\varpi z/\mu)\ (1+k)\ [y,-x,0,0]/(\lambda)^{5/2}, \tag{5.67}$$

which do not vanish if the system is "oblate". The constant, κ, has values between -1 and +1.. This current density has a sense of "circulation" about the z-axis, and is proportional to the vector potential – reminiscent of a 3 component London current, $\mathbf{J}=\chi\mathbf{A}$. The "order" parameter is

$$\chi=-(3/\mu)\ (1+\kappa)(x^2+y^2)/(\lambda)^2. \tag{5.68}$$

The Lorentz force can be computed as,

$$\mathbf{J}\times\mathbf{B}=(3\alpha^2\varpi^2 z/\mu)\ (1+\kappa)[-zx,-zy,z(x^2+y^2)]/(\lambda)^4. \tag{5.69}$$

The formula demonstrates that the Lorentz force on the plasma, for the given system of circulating currents, is directed radially away (centrifugally) from the rotational axis, and yet is such that the plasma is attracted to the $z=0$, x-y plane. The Lorentz force is divergent in the radial plane and convergent in the direction of the z-axis, towards the z=0 plane. This electromagnetic field, therefore, would have the tendency to form an accretion disk of the plasma in the presence of a central gravitational field.

Although the 3-form of Topological Torsion vanishes identically, the 3-form of Topological Spin, $\mathbf{S}_4$, is not zero:

$$\mathbf{S}_4=(\alpha^2\varpi^2 z/\mu(x^2+y^2))[-zx,-zy,(x^2+y^2),0]/(\lambda)^2, \tag{5.70}$$

The spatial components of the Lorentz force are proportional to the direction field of the Spin 3-form (in the sense of a radiation reaction). The components of the thermodynamic Work 1-form, W, are proportional to the components of the Topological Spin 3-form, with the ratio depending on the oblateness factor in the London coefficient,

$$\text{Lorentz force}=[F_x,F_y,0,0]=-\mu\chi\mathbf{S}_4. \tag{5.71}$$

It is also true that the divergence of the 3-form of spin is not zero, for the first Poincare 4-form is,

$$d(A\hat{}G) \;\Rightarrow\; P1 = (\alpha^2\varpi^2/\mu)(x^2+y^2+(4+3\kappa)z^2)/(\lambda)^3. \tag{5.72}$$

$$P2 \;=\; E\circ B = 0 \qquad \text{PTD} = 2 \tag{5.73}$$

For a more detailed discussion, see Vol 4, [1].

The potentials, the charge-current and the **B** field are sensitive to the rotation direction (orbital angular momentum), which is governed by the parameter, $\varpi = \pm 1$. The Topological Spin and the Lorentz force are not sensitive to the circulation sense, ϖ, as they are quadratic in the rotation parameter. For the isotropic case, $a = c$, there is no induced current density, no Lorentz force, and yet the Topological Spin density is non-zero. For any c/a ratio, the dissipative power, and the Poynting vector, vanish. The first Poincare invariant is not zero

It is important to note that the zero value for the charge density, ρ_{em}, implies no "net" charge. This will be defined as charge pairing; the volume contains the same number of plus charges and minus charges. The zero value for the charge density, ρ_{spin}, implies no net spin. This will be defined as spin pairing; the volume contains the same number of plus spins as minus spins. The argument establishes a topological foundation for the concept of Cooper pairs (spin pairing of electrons) and massive photons (spin pairing of Bosons).

The singularities for the potentials (as given in the example) are the z-axis and the origin (for positive anisotropy coefficients). If c is negative and a is positive, then the singular set for the denominator, $ax^2+ay^2-\kappa z^2 \Rightarrow 0$, generates a cone. The cone is centered on the z-axis, with its vertex at the origin. The cone is oblate when the ratio $|a/\kappa|$ is very small, and prolate when $|a/c|$ is very large. The prolate cone is remindful of jets, and the oblate cone is remindful of flat spirals and Cherenkov radiation.

The magnetic field in the example is "hedgehog radial". Although its divergence is zero everywhere, the lines of the field do not close on themselves. The **B** field starts and stops on points of a topological boundary of a not simply connected isolated-equilibrium thermodynamic system. Note that the rotational orientation of the flux circulation integral (whether ϖ is positive or negative) determines whether or not the **B** field is pointing inbound or outbound relative to the origin. The topological (radial) orientation is related to the rotational sense in this example. It is also true that the induced current **J** (linear in ϖ) depends upon the rotational orientation, and it is proportional to the vector potential, but the field is opposite in a rotational sense. However the Lorentz force, the Topological Torsion and the Interaction Energy do not depend upon the sense of the rotational circulation, as they are quadratic in ϖ.

The Hedgehog **B** field in this example has zero divergence everywhere except at the zeros of the denominator, which herein are interpreted as topological obstructions, or defects, to be excised from consideration.

Many such hedgehog structures can be generated by multiplying a closed, but not exact, 1-form with factors of the type $\Gamma(x, y, z, t)$. The first example, 1a above, uses a Type 1 factor. The next example will be of Type 2.

$$\begin{aligned} \text{Assume } \lambda &= (x^2 + y^2 \pm \kappa z^2) && (5.74) \\ \text{Type 1. } \Gamma(x, y, z) &= \alpha\varpi z/\sqrt{\lambda}. && (5.75) \\ \text{2. } \Gamma(x, y, z) &= \alpha\varpi(x^2 + y^2)/(\lambda). && (5.76) \\ \text{3. } \Gamma(x, y, z) &= \alpha\varpi z^2(x^2 + y^2)/(\lambda)^2. && (5.77) \\ \text{and } \phi &= 0. \text{ Note sign of } \kappa. \quad \varpi = \pm 1 && (5.78) \end{aligned}$$

Example 1b. Another Plasma Accretion disc from type 2 Hedgehog B field solutions

At first glance it would appear that there is a magnetic charge monopole similar to an electric charge monopole This interpretation is inconsistent with the Maxwell-Faraday postulate, $F - dA = 0$. Hence, the topology generated by the potentials is not simply connected. If the Topological Torsion tensor vanishes, then, thermodynamically, the domain is an isolated-equilibrium system.

A second hedgehog example uses a Type 2 factor $\Gamma(x, y, z)$:

$$\begin{aligned} \text{Asume } \lambda &= (x^2 + y^2 - \kappa z^2) && (5.79) \\ \text{Potentials } A &= \alpha\varpi/(\lambda)[-ydx, xdy, 0, 0]\text{, Type 2.} && (5.80) \\ \text{with } \phi &= 0. \text{ Note sign of } \kappa. \quad \varpi = \pm 1. && (5.81) \end{aligned}$$

The results are summarized as

$$\begin{aligned} \text{Hedgehog } \mathbf{B} &= -2\alpha\varpi[x, y, z]/(\lambda)^2, \text{ 4th power decay} && (5.82) \\ \mathbf{E} &= [0, 0, 0], && (5.83) \\ \mathbf{J} &= \chi\mathbf{A} \text{ London Super Current} && (5.84) \\ \text{London coefficient } \chi &= (2\kappa/\mu)(x^2 + y^2 + 4z^2 + 3kz^2)/\lambda^2 && (5.85) \\ \text{Top. Spin } A\hat{}G &= i(\mathbf{S}_4)\Omega_4, \quad \mathbf{S}_4 = [\mathbf{S}, \rho_{spin}] && (5.86) \\ \mathbf{S} &= (2\alpha^2\varpi^2 z\kappa/\mu)[-zx, -zy, (x^2 + y^2), 0]/\lambda^3, \ \rho_{spin} = 0. && (5.87) \\ \text{Poincare I } d(A\hat{}G) &= (\mathbf{B} \circ \mathbf{H} - \mathbf{D} \circ \mathbf{E}) - (\mathbf{A} \circ \mathbf{J} - \rho\phi), && (5.88) \\ & \text{see Maple program} && (5.89) \\ \text{Top. Torsion } A\hat{}F &= i(\mathbf{T}_4)\Omega_4 = 0, && (5.90) \\ \mathbf{T}_4 &= [0, 0, 0, 0], \text{ PTD } = 2 \text{ isolated equilibrium} && (5.91) \\ \text{Poincare II } d(A\hat{}F) &= 2(\mathbf{E} \circ \mathbf{B}) = 0, && (5.92) \end{aligned}$$

$$-\{\rho\mathbf{E}+\mathbf{J}\times\mathbf{B}\} = \text{Lorentz force}, \quad (5.93)$$
$$= 4k(3(\kappa-1)z\varpi^2/\mu)[zx, zy, -(x^2+y^2)]/(\lambda)^8, \quad (5.94)$$
$$= 3\chi\cdot\mathbf{S}_4, \quad \text{Spin Inertia !!!} \quad (5.95)$$
$$\text{Power } \mathbf{J}\circ\mathbf{E} = 0 \quad (5.96)$$
$$\text{Poynting Vector } \mathbf{E}\times\mathbf{H} = [0,0,0], \quad (5.97)$$
$$\text{Interaction Energy } A\hat{}J = 3z^2m^2(\kappa-1)/(\mu\lambda^3), \quad (5.98)$$
$$\text{Lagrange Field } F\hat{}G = z^2m^2(\kappa-1)/(\mu\lambda^3) \quad (5.99)$$
$$\text{Poincare II } d(A\hat{}F) = 2(\mathbf{E}\circ\mathbf{B}) = 0, \quad (5.100)$$
$$\text{Helicity } \mathbf{A}\circ\mathbf{B} = 0 \quad (\text{TTM mode as A\^{}F} = 0), \quad (5.101)$$
$$\text{Chirality } \mathbf{A}\circ\mathbf{D} = 0 \quad (\text{not a TTE mode as A\^{}G } \neq 0. \quad (5.102)$$

These potentials A induce the field intensities and charge-current densities below: Note that the **B** field is inbound for counterclockwise rotation, and outbound for clockwise rotation (negative ϖ). Also note that the Lorentz constitutive charge current is proportion to the 1-form of Action, $J = \chi A$. (This is to be recognized as the London current, $\mathbf{J} = \chi\mathbf{A}$).

The Lorentz force and the dissipative terms become:

$$-\{\rho\mathbf{E}+\mathbf{J}\times\mathbf{B}\} = \text{Lorentz force}, \quad (5.103)$$
$$= -(2k\alpha\varpi)^2(z\varsigma)[zx, zy, -(x^2+y^2)]/(\lambda)^5, \quad (5.104)$$
$$= 3\chi\cdot\mathbf{S}_4, \quad \text{Spin Inertia !!!} \quad (5.105)$$
$$\varsigma = ((x^2+y^2+(4+3\kappa)z^2 \quad (5.106)$$
$$\text{Poincare II } d(A\hat{}F) = 2(\mathbf{E}\circ\mathbf{B}) = 0, \quad (5.107)$$
$$\text{Helicity } \mathbf{A}\circ\mathbf{B} = 0 \quad (\text{TTM mode as A\^{}F} = 0), \quad (5.108)$$
$$\text{Chirality } \mathbf{A}\circ\mathbf{D} = 0 \quad (\text{but not a TTE mode as A\^{}G is not zero}) \quad (5.109)$$

Note that the current density is proportional to the 1-form of Action with a London order parameter equal to χ, and the Lorentz Force is proportional to the Spin current with the same factor times 3! (See Vol 4 for the details of TTM, TTE, TM, and TE modes.)

In this example, the non-zero plasma current density, **J** , has a sense of "circulation" about the z-axis, and is proportional to the vector potential representing a rotation. This deduced result has the format of a London current, $\mathbf{J} = \chi\cdot\mathbf{A}$. The "London" parameter , χ, due to the rotation is,

$$\chi = (2\kappa/\mu)(x^2+y^2+4z^2+3kz^2)/\lambda^2. \quad (5.110)$$

The London formula depends strongly upon the anisotropy, which is confirmed in superconductivity experiments. Indeed the hedgehog example in an anisotropic situation generates a non-dissipative current density with some of the attributes of a super current. Also note that the spatial components of the Topological Spin are proportional to the direction field of the spatial components of the Lorentz force. The proportionality factor is exactly equal to 3 times the (same!) London parameter, χ. It appears that the Spin current is related to inertia. Note that the Lorentz charge current density is closed, but the Spin current density is not closed: $d(A\hat{}G) \neq 0$. This result can be interpreted as the production of an energy density due to production of spin-paired Bosons (massive photons). The combination leads to the speculation that dynamic inertia can be associated with massive photons.

The formula for the Lorentz force demonstrates that the system of circulating currents is directed radially away (centrifugally) from the rotational axis, and yet is such that the plasma is attracted to the $z = 0, xy$ plane. Independent of the sense of rotation, but dependent upon the anisotropy, the Lorentz force is divergent in the radial plane and convergent in the direction of the z-axis, towards the $z = 0$ plane. The conjecture is that this electromagnetic field for the rotating plasma would have the tendency to form an accretion disk of plasma in the presence of a central gravitational field.

It is apparent that the helicity density and the second Poincare 4-form are zero. In fact, the 3-form of Topological Torsion vanishes identically. Although the 3-form of Topological Torsion vanishes identically, the 3-form of Topological Spin is not zero. It is also true that the divergence of the 3-form of the Topological Spin, $A\hat{}G$, is not zero, for the first Poincare 4-form is not zero.

If the system is spherical, then the deformation parameter vanishes, $\kappa = 1$, the Lorentz force vanishes, the charge-current 3-form vanishes, but the Topological Spin does not vanish and the first Poincare 4-form does not vanish. There is a residual field energy related to $\mathbf{B} \circ \mathbf{H}$. and a potential massive photon (spin paired Boson).

Example 1c. Other B field Hedgehog solutions

The famous Dirac monopole is considered to be a magnetic hedgehog solution similar to the ubiquitous Hedgehog **E** field distribution from an isolated charge. The Dirac radial **B** field solution given in the literature [89] decays with the inverse cube of the radial distance, r. In the previous example the Hedgehog **B** field decays with the inverse square of a length. The Dirac Hedgehog potentials were presumed to be imaginary, but that is not necessary.

The example 1c leads to the inverse cube hedgehog, consider the choice

$$\mathbf{A} = \Gamma(x,y,z)[-y,x,0]\ , \quad \text{CCW rotation} \tag{5.111}$$

$$with \quad \Gamma = (\alpha\varpi z^2)/\{\lambda^2\}, \quad \varpi = \pm 1, \ \text{Type 3.} \tag{5.112}$$

$$\phi = 0, \quad and \quad (x^2 + y^2 - \kappa z^2) = \lambda. \tag{5.113}$$

These potentials induce the field intensities below:

$$\text{1-form of potentials } A = (\alpha\varpi z^2/\lambda^2)[-ydx + xdy] \;\; \text{Type 3} \quad (5.114)$$
$$\phi = 0, \;\; \textit{and} \;\; (x^2 + y^2 - \kappa z^2) = \lambda. \quad (5.115)$$
$$\text{Hedgehog } \mathbf{B} = -2\alpha\varpi z(x^2 + y^2 + \kappa z^2)[x, y, z]/(\lambda)^3, \quad (5.116)$$
$$\mathbf{E} = [0, 0, 0], \quad (5.117)$$

The details of the computations are very complex, but can be found by use of a Maple program.

Example 2. An Electromagnetic Pump ?

A simple electromagnetic pump consists of an insulated rectangular section of steel pipe inserted into a circuit of pipe that contains liquid metal. The square sections will be labeled x and y, the liquid metal direction of flow will be labeled z. An external field **B** is applied to the x direction, and an external electric field **E** is applied to the y direction. Voila: the liquid metal moves in the direction of $\mathbf{E} \times \mathbf{B}$. Why?? The usual answer given is that if the applied **E** field induces a current flow, then the $\mathbf{J} \times \mathbf{B}$ component of the Lorentz force causes flow of the conductive material in the direction of $\mathbf{J} \times \mathbf{B}$. However, suppose the applied **E** field is prevented from producing an ohmic current, as in the field of a capacitor?

Consider that the vector and scalar potentials are given by the expressions:

$$A = [0, -(\mathbf{B}_x/2)z, +(\mathbf{B}_x/2)y, \mathbf{E}_y t], \quad (5.118)$$

The induced fields (assuming $\mathbf{D} = \varepsilon\mathbf{E}$, $\mathbf{B} = \mu\mathbf{H}$) are:

$$\mathbf{E} = [0, -\mathbf{E}_y, 0] \quad \mathbf{B} = [\mathbf{B}_x, 0, 0], \quad (5.119)$$
$$\text{Lorentz Current } \mathbf{J}_4 = [0, 0, 0, 0], \quad \rho_{em} = 0, \quad (5.120)$$
$$\text{Topological Spin } \mathbf{S}_4 = [\mathbf{S}, \rho_{spin}], \quad (5.121)$$
$$\mathbf{S} = [0, (\mathbf{B}_x^2 - 2\mathbf{E}_y^2\varepsilon\mu)y, \mathbf{B}_x^2 z,]/2\mu, \quad (5.122)$$
$$\rho_{spin} = (\mathbf{E}_y\mathbf{B}_x\varepsilon\mu)z/2\mu, \quad (5.123)$$
$$\textit{Poincare I} = (\mathbf{B}_x^2 - \mathbf{E}_y^2\varepsilon\mu), \quad (5.124)$$
$$\text{Top. Torsion } \mathbf{T}_4 = (\mathbf{E}_y\mathbf{B}_x/2)[-y, 0, 0, 0], \quad (5.125)$$
$$\textit{Poincare II} = 2(\mathbf{E} \circ \mathbf{B}) = \mathbf{0}, \quad (\mathbf{J} \circ \mathbf{E}) = 0, \quad (5.126)$$
$$\text{Lorentz Force} = [0, 0, 0], \quad (5.127)$$
$$\text{Poynting vector } \mathbf{E} \times \mathbf{H} = (\mathbf{E}_y\mathbf{B}_x/\mu)[0, 0, 1], \quad (5.128)$$
$$\textbf{Helicity } (\mathbf{A} \circ \mathbf{B}) = 0, \text{ but not } \textbf{TTM as T} \neq \mathbf{0}. \quad (5.129)$$
$$\textbf{Chirality } (\mathbf{A} \circ \mathbf{D}) = 0, \text{ but not } \textbf{TTE as S} \neq \mathbf{0}\ . \quad (5.130)$$

The first thing to note is that there is no Amperian current, $\mathbf{J}_4$. Is there any induced motion of say a gas or fluid within an insulated tube? It is apparent that there are both finite Topological Torsion currents and Topological Spin Currents, and that Spin Pairing effects can take place, especially if the region is such that the first Poincare 4-form is not zero. The on-shell photons occur when the First Poincare 4-form vanishes:

$$\text{On shell photons} \qquad Poincare\ I = (\mathbf{B}_x^2 - \mathbf{E}_y^2 \varepsilon\mu) \Rightarrow 0 \tag{5.131}$$

When the offshell photons are present they are propagated in the direction of the Poynting vector. Do these "heavy photons", which are embedded in the fluid or the gas , make it move. It appears that Current Quantum Field Theory and proponents of dark energy would have us to believe it is so. Come on, experimenters, give us an answer.

Example 3. Coulomb singularity

Consider the Potentials for the Coulomb 1/r potential:

$$A = [0, 0, 0, -(1/4\pi\varepsilon)q/r], \tag{5.132}$$
$$r = \sqrt{x^2 + y^2 + z^2}, \tag{5.133}$$
$$\lambda^2 = x^2 + y^2 \tag{5.134}$$

The induced fields (assuming $\mathbf{D} = \varepsilon\mathbf{E}$, $\mathbf{B} = \mu\mathbf{H}$) are:

$$\mathbf{E} = (1/4\pi\varepsilon)q[x, y, z]/r^3, \tag{5.135}$$
$$\mathbf{B} = [0, 0, 0], \tag{5.136}$$
$$\mathbf{J}_4 = [0, 0, 0, 0], \tag{5.137}$$
$$\rho_{em} = 0, \tag{5.138}$$
$$\text{Hedgehog Top. Spin } \mathbf{S}_4 = (1/4\pi\varepsilon)^2 \varepsilon q^2/r^4[x, y, z, 0], \qquad \rho_{spin} = 0, \tag{5.139}$$
$$Poincare\ I = -(1/4\pi\varepsilon)^2 \varepsilon q^2/r^4, \tag{5.140}$$
$$\text{Top. Torsion } \mathbf{T}_4 = (1/4\pi\varepsilon)q\varpi[-zx, -zy, \lambda^2, 0]/(r^3\lambda^2), \tag{5.141}$$
$$Poincare\ II = 2(\mathbf{E} \circ \mathbf{B}) = \mathbf{0}, \quad (\mathbf{J} \circ \mathbf{E}) = 0, \tag{5.142}$$
$$\mathbf{E} \times \mathbf{H} = [0, 0, 0], \tag{5.143}$$
$$\textbf{Helicity } (\mathbf{A} \circ \mathbf{B}) = 0, \text{ but not } \textbf{TTM as T} \neq \mathbf{0}. \tag{5.144}$$
$$\textbf{Chirality } (\mathbf{A} \circ \mathbf{D}) = 0, \text{ but not } \textbf{TTE as S} \neq \mathbf{0}\ . \tag{5.145}$$

This example demonstrates how the addition of a closed but not exact contribution to a 1-form of Action can influence the topological features of an electromagnetic system. The Topological Torsion depends upon the "rotation" or angular momentum term, ϖ, with a strength determined by the coefficient ϖ. Note that the Topological Spin density term does not vanish. The combined system is not in thermodynamic equilibrium as $A\hat{\ }G$ is not zero.

Example 4. Bateman-Whittaker solutions

In the modern language of differential forms, Bateman [16] (and Whittaker) determined that if two *complex* functions $\alpha(x, y, z, t)$ and $\beta(x, y, z, t)$ are used to define the 1-form of Action,

$$A = \alpha d\beta - \beta d\alpha \Rightarrow \mathbf{A} = \alpha \nabla\beta - \beta\nabla\alpha, \quad \phi = -(\alpha\partial\beta/\partial t - \beta\partial\alpha/\partial t), \tag{5.146}$$

then the derived 2-form $F = 2d\alpha\hat{}d\beta$ generates the complex field intensities,

$$\mathbf{E} = (\partial\alpha/\partial t)\nabla\beta - (\partial\beta/\partial t)\nabla\alpha, \tag{5.147}$$

$$\mathbf{B} = \nabla\alpha \times \nabla\beta, \tag{5.148}$$

which of course satisfy the Maxwell-Faraday equations. If in addition, the functions α and β satisfy the complex Bateman constraints:

$$\nabla\alpha \times \nabla\beta = \pm(i/c)[(\partial\alpha/\partial t)\nabla\beta - (\partial\beta/\partial t)\nabla\alpha], \tag{5.149}$$

then the complex field excitations, computed from the Lorentz Vacuum constitutive constraints, will satisfy the Maxwell-Ampere equations for the vacuum, without charge-currents. It is apparent immediately that the second Poincare 4-form is identically zero for such solutions. It is also apparent immediately that the Torsion vector is identically zero. What is not immediately apparent is that first Poincare 4-form and the Spin 4-vector vanish identically as well. In fact, the constrained complex solutions of the Bateman type are examples of topologically transverse (**TTEM**) waves. The Bateman solutions, like TTEM waveguide solutions, do not radiate!

As an explicit example, consider,

$$\alpha = (x \pm iy)/(z - r), \quad \beta = (r - ct), \quad r = \sqrt{x^2 + y^2 + z^2)}. \tag{5.150}$$

These functions satisfy the Bateman conditions (and, it should be mentioned, the Eikonal equation subject to the dispersion relation $\varepsilon\mu c^2 = 1$). The $\mathbf{E}$ and the $\mathbf{B}$ fields are complex (and complicated algebraically):

$$\begin{aligned}
\mathbf{B} &= [yx+\sqrt{-1}(z^2+y^2-rz), && (5.151)\\
&\quad -(z^2+x^2-rz)-\sqrt{-1}xy,\\
&\quad (r^2+z^2-2rz)/(r-z)\,)(y-\sqrt{-1}x)]2/(r(z-r)^2),\\
\mathbf{E} &= [-\sqrt{-1}yx+(y^2+z^2-rz), && (5.152)\\
&\quad \sqrt{-1}(x^2+z^2-rz)-xy,\\
&\quad (z-r\,)(x+\sqrt{-1}y)]2c/(r(z-r)^2),\\
\text{Top. Spin } \mathbf{S}_4 &= [0,0,0,0], && (5.153)\\
\text{Top. Torsion } \mathbf{T}_4 &= [0,0,0,0], && (5.154)\\
\mathbf{E}\times\mathbf{H} &= [0,0,0], \qquad \mathbf{D}\times\mathbf{B}=[0,0,0], && (5.155)\\
\mathbf{E}\circ\mathbf{E} &= 0, \qquad \mathbf{B}\circ\mathbf{B}=0, && (5.156)\\
(\mathbf{E}\circ\mathbf{B}) &= 0, \qquad (\mathbf{B}\circ\mathbf{H}-\mathbf{D}\circ\mathbf{E})=0\,. && (5.157)
\end{aligned}$$

The functions α and β that satisfy the Bateman condition may be used to construct an arbitrary function, $F(\alpha,\beta)$, and remarkably enough, that arbitrary function $F(\alpha,\beta)$ satisfies the Eikonal equation,

$$(\nabla F)^2-\varepsilon\mu(\partial F/\partial t)^2=0. \tag{5.158}$$

From experience with Eikonal solutions and wave equations, it might be thought that Eikonal solutions are sufficient. However, the Bateman conditions are necessary, for both the candidate solutions,

$$\alpha=(x\pm iy)/(z-ct),\quad \beta=(r-ct),\quad r=\sqrt{x^2+y^2+z^2)}, \tag{5.159}$$

satisfy the Eikonal equation, but not the Bateman conditions. They (for each sign, $\pm$) do not generate TTEM modes in the vacuum. For arbitrary functions the algebra can become quite complex. A Maple symbolic mathematics program for computing the various terms is available (see references below).

5.4 Cosmic Strings, Wheeler Wormholes and Falaco Solitons

5.4.1 Wheeler Wormholes and Falaco Solitons

Initially it was thought that the Falaco surface indentation (see Vol 2), immediately after creation, was in the form of a Rankine vortex (with regions of positive mean curvature, and positive Gauss curvature in a 3D Euclidean space), which then decayed (somehow) into a classic *minimal* surface (as shown in Figure 5.1) of zero mean curvature, but negative Gauss curvature. Admittedly, the extended conjugate catenoids of Figure 5.2 (a deformed Wheeler Wormhole with an open throat ?) have some of the features and appearance of the Falaco Solitons. However, the extended singular thread (without an open throat) between vertex singularities does not appear to be replicated by the minimal surface of negative Gauss curvature.

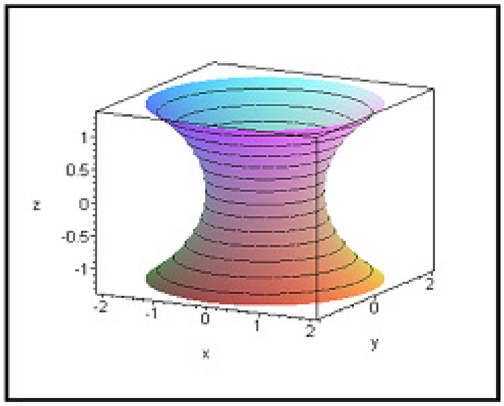


Figure 5.1 Soap film between rings

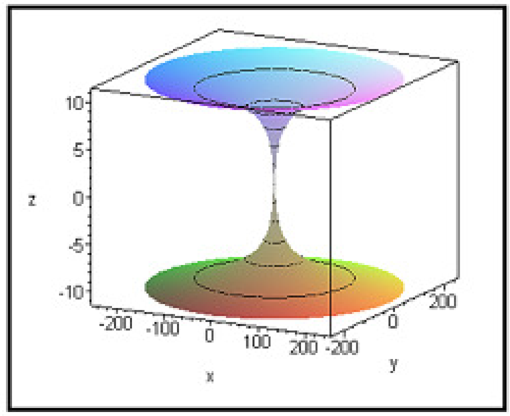


Figure 5.2 Deformed Wheeler Wormhole

Evolutionary processes that are solutions of the Navier-Stokes equations of fluid dynamics *in a rotating frame of reference* seem to mimic the features of the Falaco Solitons to a certain extent. However, the solutions to the Navier-Stokes dynamics seems to require that the connection (the string) between the Falaco dimple pairs has an open throat (like a Wheeler Worm hole).

For a "stationary" Euclidean soap film between two coaxial boundary rings, the system with negative Gauss curvature is stable only if the axial separation of the boundary rings is less than (approximately) 2.65 times the minimal throat diameter. The instability of a surface of the soap film type (Wheeler Wormhole with open throat) has been demonstrated analytically for a fluid flow in a rotating frame, where the zero set of the helicity function of the fluid flow has the appearance of a real minimal surface. As the bulk flow increases, the helicity function changes sign, and therefore represents a change in topology from a connected set to a disconnected set. With the change in sign, a torsion bubble (or a torsion burst) appears in the flow pattern [219] [1]. Such torsion bursts (incorrectly called vortex bursts?) have been observed by jet pilots in extreme maneuvers.

The stability argument makes it difficult to utilize the real minimal surface "soap film" concept, with negative Gauss curvature, as a model representing the topological structure of the Falaco Soliton of two dimpled surfaces of zero mean curvature connected by a one-dimensional thread.

Figure 5.3 Rotational Surfaces of Zero mean curvature in Euclidean and Minkowski 3 space

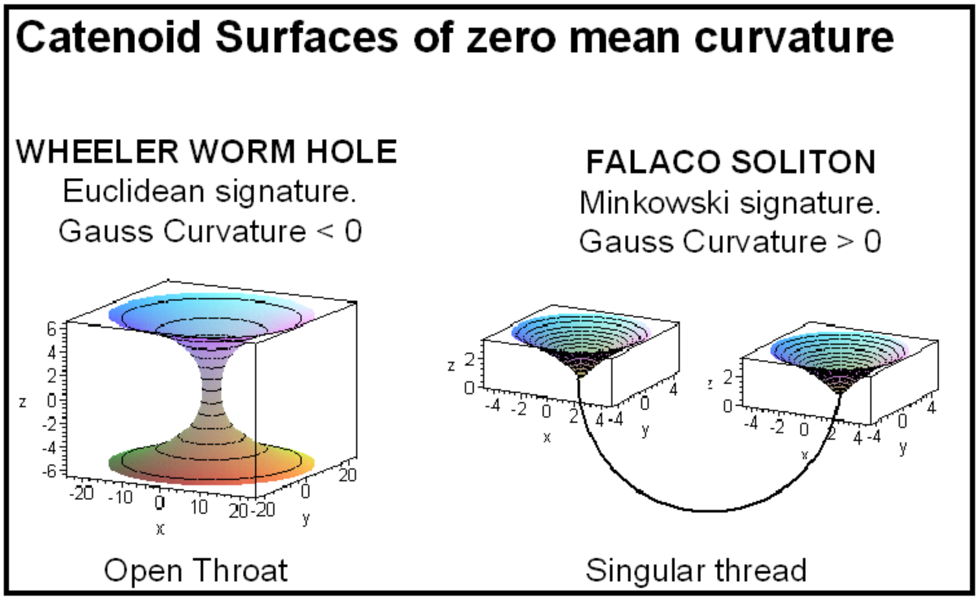


Figure 1

Remark 25 *The bottom line is that the remarkable features of creating a stable surface of zero mean curvature and positive Gauss curvature (the Falaco Soliton) is explained either by assuming that the usual 3D Euclidean Signature is rotationally dependent and can topologically evolve into a 3D Minkowski Signature; or, the Euclidean Signature is preserved, and a macroscopic evolutionary process described by complex Spinor direction fields (which are not the same as diffeomorphic vector fields) must be admitted on thermodynamic grounds.*

It is extraordinary, but the Falaco Solitons appear to be another form of a zero mean curvature surface structure connecting discontinuity surfaces. Such connections of one surface to another surface, or to two different regions of the same surface, either are related to macroscopic realizations of the Wheeler wormhole (soap film with possibly a very narrow, but open, throat), or to Spinor (complex) surfaces generated by complex eigendirection fields of infinitesimal rotations, producing pairs of dimpled conical zero mean curvature surfaces, connected by a 1-dimensional thread (a closed "throat").

The Wheeler wormhole structure was presented early on by Wheeler (1955), but was considered to be unattainable in a practical sense. To quote [321] p. 126 "The throat or "wormhole" (in a Kruskal metric) as Wheeler calls it, connects regions of the same physical space which are extremely remote from each other. (Zeldovich then gives a sketch that topologically is similar to the Falaco Soliton). Such a topology implies the existence of 'truly geometrodynamic objects' which are unknown to physics. Wheeler suggests that such objects have a bearing on the nature of elementary particles and anti particles and the relationships between them. However, this idea has not yet borne fruit; and there are no macroscopic "geometrodynamic objects" in nature that we know of. This quotation dates back to the period 1967-1971.

Now the experimental evidence (of Falaco Solitons) justifies Wheeler's intuition, which has so often been correct. Falaco Solitons are 'truly geometrodynamic objects'.

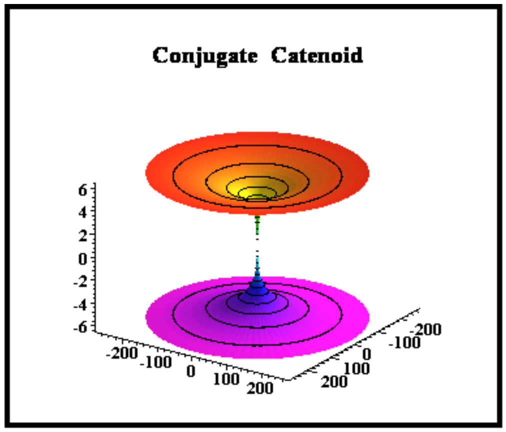


Figure 5.4 Falaco Soliton as a conjugate catenoid

The minimal surface solution to the Navier-Stokes equations given in the preceding chapter suggests a possible connection between Falaco Soliton topological defect structures and Wheeler worm holes. The Falaco Soliton endcaps can be made to be compact by adding a closed component to the 1-form of Action. In particular, the additional component gives a circulation (without vorticity) that can be adjusted to compensate for the vorticity induced circulation (due to bulk rotation of the fluid) at a fixed radius, r. Figure 5.4 approximates the minimal surface structure of a Falaco Soliton Pair. However, the throat, though relatively small, is open, and each of the conjugated catenoids is not compact.

5.4.2 Falaco Solitons as Strings between Branes

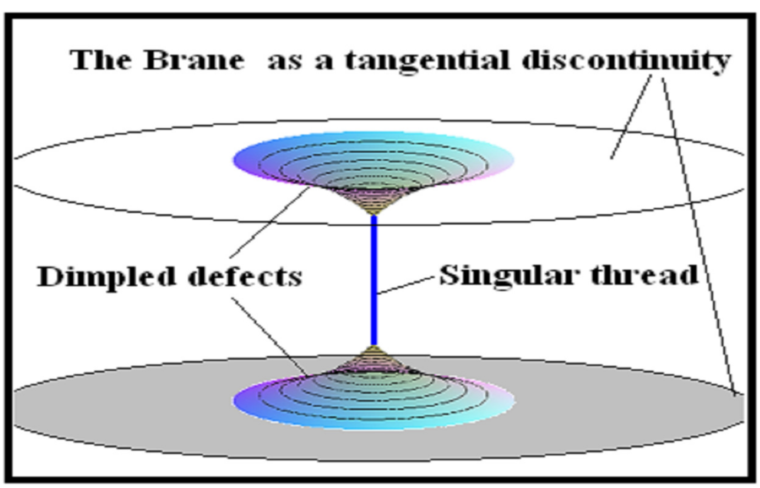


Figure 5.5 Falaco Solitons as connected dimples between Branes

The idea that the Falaco Solitons are related to strings connecting branes led to the thought that perhaps the modern advances in topology and string theory could yield a theoretical explanation of the formation and stability of Falaco Solitons. According, challenges and requests for help were sent out to many of the string theorists, asking for theoretical help to describe this "real life string connecting branes". The lack of response indicates that none of the string gurus seemed to think the effort was worthwhile. However, the theoretical work of Dzhunushaliev [69] seems to have many correspondences with the experimental facts of the Falaco Solitons. Figure 5.5 was abstracted from the work of Dzhunushaliev, by joining together the two minimal surface conoids in Minkowski 3-space as shown in Figures 5.1 and 5.2. It is not clear, experimentally, that the connecting "string" is a singular "vortex" thread (a 2D defect), or that the connecting string is a singular "circulation" thread (a 1D defect).

The Falaco Soliton endcap dimples (which are presumed to be surfaces of zero mean curvature and positive Gauss curvature) are related to Spinor eigendirection fields associated with antisymmetric matrices representing Symplectic spaces. If the Maximal surfaces appear as deformations in disconnected hypersurfaces of discontinuity, the topological structure has the appearance of "strings connecting branes", a concept touted by the string theorists. The new feature is that the "brane" surface of discontinuity is deformed by the Maximal surface dimple (which has been alluded to as a space-time foam [69]). In addition, the idea is related to the rotational structures of rotating Bose-Einstein condensates [293].

5.5 A Cosmological Conjecture based on Continuous Topological Evolution

The generation of almost flat spiral arm structures during the formation of the Falaco Solitons, and the idea that these structures could be macroscopic realizations of the Wheeler wormhole, suggests the possibility that topological defects are the fundamental causes of inhomogeneities in the night sky. Recall, that size and shape have no crucial significance in a topological theory. Therefore, topological things and effects that appear at the microscopic and macroscopic scale should also appear at a cosmological scale. The objective of this chapter is to examine topological aspects and defects of thermodynamic physical systems - especially non-equilibrium thermodynamic systems - and their possible continuous topological evolution, creation, and destruction on a cosmological scale. There are two ways of utilizing the concepts of continuous topological evolution; both are based upon the assumption that the universe can be modeled in terms of an exterior differential 1-form of Action, A. The first method examines the features of the Pfaff topological dimension and its evolution producing long-lived topological defects and topologically coherent structures. The second method exploits the properties that the Jacobian matrix of the coefficients of the 1-form of Action creates a universal thermodynamic phase function, Θ, in terms of the Cayley-Hamilton characteristic polynomial. That universal thermodynamic function contains a realization of the Universe as a universal, deformable van der Waals gas.

The cosmological creation and evolution of stars and galaxies will be interpreted herein in terms of a non-equilibrium thermodynamic system of Pfaff Topological dimension 4 subjected to irreversible processes. As explained below, based upon this single assumption it is possible to devise a model of the universe which can be approximated in terms of the non-equilibrium states of an extremely dilute van der Waals gas near its critical point. The stars and the galaxies are the topological defects, or topologically coherent - but not equilibrium - structures of Pfaff topological dimension 3 formed by irreversible dissipative processes in this non-equilibrium turbulent system of Pfaff topological dimension 4. The cosmology so constructed is opposite in viewpoint to those efforts which hope to describe the universe in terms of properties inherent in the quantum world of Bose-Einstein condensates, super conductors, and superfluids [293]. Both approaches utilize the ideas of topological defects, but thermodynamically the approaches are opposite. The quantum method involves, essentially, equilibrium systems, while the approach presented herein is based upon non-equilibrium systems.

The topological theory of the ubiquitous van der Waals gas leads to a more mundane explanation of negative pressure (dark energy), string tension (dark matter), irreversible dissipation due to expansion ("volume viscosity"), and a Higgs potential (space-time conformal inertial interaction). All of these concepts appear as natural consequences of a non-equilibrium thermodynamics and a deformable version of a van der Waals gas near its critical point. These ideas arise without invoking quantum mechanics explicitly, and without assuming microscopic quantum vacuum fluctuations. Fluctuations are important, but in the sense that they are topological fluctuations about a guiding fiber of Kinematic Perfection. Perhaps of more importance is the fact that these thermodynamic consequences explicitly do not depend upon the geometric constraints of metric or connection, and indeed impose a different perspective on the concept of gravitational interaction as a possible topological effect, rather than a geometrical idea.

For example, the concept of a fixed point can be identified with a null eigenvector of a Jacobian matrix. The Jacobian matrix of a 1-form has both symmetric, $\mathbb{S}$, and antisymmetric, $\mathbb{A}$, parts.

$$\mathbb{J}(\partial A_k/\partial x^m] = \mathbb{S} + \mathbb{A}. \tag{5.160}$$

The components of the antisymmetric part can be put into correspondence with the exterior differential of the 1-form of Action, dA. If the Pfaff topological dimension of dA is 4, such that $K = dA\hat{}dA \neq 0$, then there does not exist a null eigenvector such that $i(V_{null})dA = 0$. Written in matrix language, the statement that

$$\text{"Electromagnetic" Work } \ i(V_{null})dA = \mathbb{A} \circ |V_{null}\rangle \Rightarrow 0, \tag{5.161}$$

on a symplectic manifold of Pfaff topological dimension 4 cannot be true. However the concept of a fixed point implies that the similarity invariant of the fourth order Cayley Hamilton polynomial must vanish, $T_K = 0$. Hence:

$$\mathbb{J}\,|V_{null}\rangle = \mathbb{S}\,|V_{null}\rangle + \mathbb{A}\,|V_{null}\rangle = 0. \tag{5.162}$$

It follows that in such circumstances the symmetries must be balanced by the anti-symmetries in the sense that

$$\mathbb{S}\,|V_{null}\rangle = -\mathbb{A}\,|V_{null}\rangle\,. \tag{5.163}$$

The moral is that null vector constraints imply that symmetrical deformations $\mathbb{S}\,|V_{null}\rangle$ must be compensated by "electromagnetic" work $\mathbb{A}\,|V_{null}\rangle$ in the turbulent non-equilibrium domain of Pfaff topological dimension 4. The separation of "charge" is a possible remnant of "electromagnetic" work and occurs via irreversible processes in domains of Pfaff topological dimension 4.

If the domain is of Pfaff dimension 3, then there exist null eigenvectors of the 2-form dA and symmetrical deformations are not necessarily linked to "electromagnetic" work.

5.5.1 Landau's theory for interactions of fluctuations

The original motivation for the conjecture that the universe is a turbulent deformable van der Waals gas near its critical point is based on the classical theory of correlations of fluctuations presented in the Landau-Lifshitz volume on statistical mechanics [138]. When first reading Landau's ideas (about 1965), the present author made a written note in the textbook margin that Landau's ideas might be a method of explaining the fact that the night sky is not homogenous[‡], and instead is filled with objects (called stars) that appear to obey Newtonian gravitational attraction. However, the methods used in this chapter to describe cosmology of the universe are not statistical, not quantum mechanical, but instead are based on Cartan's methods of exterior differential forms and their application to the topology of thermodynamic systems and their continuous topological evolution ([1] or [220]). Landau and Lifshitz emphasized that real thermodynamic substances, near the thermodynamic critical point, exhibit extraordinary large fluctuations of density and entropy. In fact, these authors demonstrate that for an almost perfect gas near the critical point, the correlations of the fluctuations can be interpreted as a 1/r potential giving a 1/r^2 force law of attraction. Hence, as a cosmological model, the almost, but not, perfect gas - such as a very dilute van der Waals gas - near the critical point yields a reason for both the apparent granularity of the night sky and for the 1/r^2 force law ascribed to gravitational forces between massive aggregates. The stars are sparse topological defects in the otherwise

[‡]Counter to the then classic theory of a homogeneous cosmological universe

homogeneous cosmos. Landau also offers an argument for an inverse fourth power potential related to Bose-Einstein attraction or Fermi-Dirac repulsion (p. 373 [138]). It is remarkable that the law of force is essentially the famous Maxwell $1/\mathrm{r}^5$ law for non-equilibrium gases (p 238 [112])

5.5.2 The Universe as a Turbulent (Pfaff Dimension 4) van der Waals Gas near the Critical Point

A topological (and non-statistical) thermodynamic approach can be used to demonstrate how a four-dimensional variety can support a turbulent, non-equilibrium, physical system with universal properties that are homeomorphic (deformable) to a van der Waals gas [240]. The method leads to the necessary conditions required for the existence, creation or destruction of topological defect structures in such a non-equilibrium system. For those non-equilibrium physical systems that admit description in terms of an exterior differential 1-form of Action potentials of maximal rank, a Jacobian matrix can be generated in terms of the partial derivatives of the coefficient functions that define the 1-form of Action. When expressed in terms of intrinsic variables, known as the similarity invariants, the Cayley-Hamilton four-dimensional characteristic polynomial of the Jacobian matrix generates a universal phase equation as a 4th order polynomial in the (complex) eigen functions of the matrix. Certain topological defect structures can be put into correspondence with constraints placed upon those (curvature) similarity invariants generated by the Cayley-Hamilton four-dimensional characteristic polynomial. These constraints define equivalence classes of topological properties. It is assumed that the universe can be represented by such a 1-form of Action of Pfaff Topological dimension 4.

The characteristic polynomial, or Phase function, can be viewed as representing a family of implicit hypersurfaces. The hypersurface has an envelope which is related to a swallowtail bifurcation set of dynamical system theory when the hypersurface is constrained such that the linear similarity invariant vanishes (this constraint corresponds to the idea that the trace of the Jacobian matrix vanishes). The swallowtail defect structure is homeomorphic (can be deformed) to the Gibbs surface of a van der Waals gas.

Another possible defect structure corresponds to the implicit hypersurface constrained such that the quartic similarity invariant vanishes (this constraint corresponds to the idea that the determinant of the Jacobian matrix vanishes). The constraint implies that at least one eigenvalue is zero. On a four-dimensional variety (space-time) , this non-degenerate hypersurface constraint leads to a cubic polynomial that always can be put into correspondence with a set of non-equilibrium thermodynamic states whose kernel represents the equation of state of a van der Waals gas.

Hence the universal topological method for creating a universal phase function in terms of the Cayley-Hamilton theorem for the Jacobian matrix of a 1-form of Action, leads to a thermodynamic system that can be deformed into a van der Waals gas. Near the critical point, a low density turbulent non-equilibrium media leads to the setting examined statistically by Landau and Lifshitz in terms of classical fluctuations about the critical point.

The conjecture presented herein is that non-equilibrium topological defects in a non-equilibrium four-dimensional medium represent the stars and galaxies, which are gravitationally attracted singularities (correlations of fluctuations of density fluctuations) of a real gas near its critical point. Note that the Cartan methods (in contrast to metrical theories) do not impose (*a priori)* a constraint of a metric, connection, or gauge, but do utilize the topological properties associated with constraints placed on the similarity invariants of the universal phase function.

5.5.3 Results

A conjecture of a turbulent non-equilibrium thermodynamic cosmology can be constructed in terms of a dilute van der Waals gas near its critical point. The conjecture yields an explanation for:

a The granularity of the night sky as exhibited by stars and galaxies due to density fluctuation near the critical point, and the Newtonian law of gravitational attraction proportional to $1/r^2$ as a correlation between fluctuations (due to Lev Landau [138]).

b The conformal expansion of the universe as an irreversible phenomenon-associated with Quartic similarity invariants in the thermodynamic phase function, and conformally related to dissipative effects [204].

c The possibility of domains of negative pressure (explaining what has recently been called "*dark energy*") due to a *classical* "Higgs" mechanism for aggregates below the critical temperature

d The possibility of domains of negative temperature (explaining what has recently been called "*dark matter*") due to macroscopic collective states of ordered spins. The conjecture is that Positive temperature radiates, Negative temperature does not. The conjecture is that black holes could be negative temperature states of collective spins.

e The possibility of domains where gravitational effects (quadratic similarity invariants, or 2nd order Gauss curvature effects) appear to be related to entropy and temperature properties of the thermodynamic system, and where cubic curvature effects could impede gravitational collapse.

f Black Holes (generated by Petrov Type D solutions in gravitational theory [48]) are to be related to Minimal Surface solutions to the Universal thermodynamic 4th order Phase function of a van der Waals gas.

5.5.4 Fields versus Particles

The model of the universe, in terms of a van der Waals gas near its critical point, seems to indicate that there may be more to Gravity than just a metric field. The fluctuation condensates, with correlations that attract with a $1/r^2$ force law, appear to be distinguishable particles. Distinguishable particles imply that the topological foundations must include the classic separation axioms. If distinguishability can be accomplished by geometric methods, then the underlying topology must be (at least) Hausdorff, T2. All subsets, that are elements of the Hausdorff T2 discrete topologies, are not isolated in the sense that the subset has no non-zero intersection with its limit sets. On the other hand, some of the subsets of the T0 topologies (that are not T2) can be isolated, and some of the subsets are not isolated. The idea of isolation translates into a zero value for the 3-form of Topological Torsion, $A\hat{}dA$, where the physical system can be encoded in terms of a 1-form field (intensities) of Action potentials, A. Such systems cannot be reduced to a topology of two ingredients if they have non-zero Topological Torsion. Evolutionary processes for systems with non-zero $A\hat{}dA$ will not yield unique predictions from unique initial data. This result leads to the idea that a two body problem is uniquely soluble, where the three body problem is not uniquely soluble. $A\hat{}dA=0$ corresponds to equilibrium or isolated-equilibrium thermodynamics; $A\hat{}dA \neq 0$ corresponds to non-equilibrium thermodynamics.

But what about those topologies that do not obey separation axioms, so that some, if not all, of the particles cannot be distinguished? It would seem that the Non-T0 topologies form the topological domain of statistics, where "wavelets" are presumed to be indistinguishable. In particular, the QM particles (Bosons, Fermions) are believed to be indistinguishable. Are the collective states of Bosons, or Fermions, related to black holes? Note that all three topologies, Not-T0, T0 and T2, can coexist as partitions of the Hausdorff T2 topology of a power set.

My intuition tells me that there are two species, or classes of topology, the T0 topologies of distinguishable particle-like properties, and the Not-T0 topologies of indistinguishable wave-like or diffusion-like properties. Rather than trying to explain thermodynamics in terms of particles *or* waves it would seem that a topological thermodynamics should study features of particles *and* waves. Each species can contribute non-equivalent features of thermodynamics. For example, does wave phenomena contribute an entropy component that is distinct from an entropy component generated by particle phenomena? These comments will be under study in the future. In particular, differences between the concept of particles, and collective particle properties of Mass, Charge, Spin, need to be studied. In the following chapters, the concept of collective (coherent) behavior of topological Spin will be studied from the perspective of continuous topological evolution.

Chapter 6

SPINODAL DECOMPOSITION, PHASE COEXISTENCE AND THE NON-EQUILIBRIUM VAN DER WAALS GAS

The first part of this chapter will review chapter 3 of Vol1. A law of corresponding states will be generated from topological principles of thermodynamics on a differential variety, $\{\xi^m, d\xi^m\}$. The functions $A_m(\xi^n)$ on the differential variety $\{\xi^m, d\xi^m\}$ that form the 1-form of Action, $A = A_m(\xi^n)d\xi^m$ are well defined by functional substitution and the pullback to a differential variety $\{x^k, dx^k\}$. The two differential varieties are constrained by mappings $\{\phi, d\phi\}$ of the form:

$$
\begin{aligned}
&: \{x^k, dx^k\} \Rightarrow \{\xi^m, d\xi^m\} && (6.1)\\
\phi &: \{x^k\} \Rightarrow \{\xi^m\} = \{\xi^m(x^k)\}, \text{ base coordinates} && (6.2)\\
d\phi &: \{dx^k\} \Rightarrow \{d\xi^m\} = \{(\partial\xi^m(x^k)/\partial x^k)dx^k\}, && (6.3)\\
&: = [\mathbb{J}(\partial\xi^m(x^k)/\partial x^k)] \circ \left|dx^k\right\rangle = [\mathbb{J}^m_k]. && (6.4)
\end{aligned}
$$

These mappings need not be invertible. The Jacobian matrix is said to define a *collineation*. If the mappings are diffeomorphisms (with ϕ invertible, with the Jacobian matrix invertible), are such that the Jacobian matrix is symmetric, the mapping is called a polarity. The Jacobian matrix of the polarity is necessarily a symmetric matrix. The root structure can be applied to classical thermodynamic situations. Consider the example of a van der Walls gas. Gas phase corresponds to a cubic Jacobian polynomial with one real root zero and a spinor pair of complex roots; the Vapor phase of Clouds corresponds to 3 distinct roots; the liquid phase corresponds to three distinct real roots; the solid phase corresponds to all three real roots are the same.

The thermodynamic idea is to study the Jacobian *correlation* matrix of the coefficients of the 1-form or Action, A, that encodes the thermodynamic system,

$$[\mathbb{J}_{km}] = [\partial A_k/\partial x^m]. \tag{6.5}$$

Every square matrix with distinct eigenvalues, ξ, will satisfy the Cayley-Hamilton theorem, and will satisfy a characteristic polynomial equation, $\Theta(\xi) = 0$. For a 4x4 matrix, the polynomial equation is of 4th degree, and of the form,

$$\text{Cayley-Hamilton polynomial} = \xi^4 - X_M\xi^3 + Y_G\xi^2 - Z_A\xi + T_K = 0. \tag{6.6}$$

The Cayley-Hamilton polynomial equation defines a family of (implicit) functions, X_M, Y_G, Z_A, T_K, in the space of real variables, (x, y, z, t). The functions X_M, Y_G, Z_A, T_K are the real similarity invariants of the Jacobian matrix, even though the eigenvalues may be complex. This means these functions are universal relative to similarity transformations of the Jacobian matrix. If the eigenvalues are distinct, then the similarity invariants are given by the expressions:

$$\begin{aligned} X_M &= \xi_1 + \xi_2 + \xi_3 + \xi_4 = Trace\,[\mathbb{J}_{jk}]\,, & (6.7)\\ Y_G &= \xi_1\xi_2 + \xi_2\xi_3 + \xi_3\xi_1 + \xi_4\xi_1 + \xi_4\xi_2 + \xi_4\xi_3, & (6.8)\\ Z_A &= \xi_1\xi_2\xi_3 + \xi_4\xi_1\xi_2 + \xi_4\xi_2\xi_3 + \xi_4\xi_3\xi_1, & (6.9)\\ T_K &= \xi_1\xi_2\xi_3\xi_4 = \det\,[\mathbb{J}_{jk}]\,. & (6.10)\end{aligned}$$

The eigenvalues of the real matrix can be complex numbers, but as the similarity coefficients, $\{X_M, Y_G, Z_A, T_K\}$ are all real, classic analysis yields the result that eigenvalues form three equivalence classes:

1. 4 real eigenvalues.

2. 2 real eigenvalues, and 1 complex eigenvalue and its 1 complex conjugate.

3. 2 complex eigenvalues, and their 2 complex conjugates.

If $T_K = 0$, then the Cayley-Hamiltonian equation becomes,

$$\text{Cayley-Hamilton polynomial} = (\xi^3 - X_M\xi^2 + Y_G\xi^1 - Z_A)\xi = 0, \tag{6.11}$$

and the similarity coefficients are related to the "Curvatures" of the implicit surface induced by the molar density. The first (cubic) factor can be put into direct correspondence with the Classic van der Waals equation

$$\begin{aligned} \text{Van der Waals }\ \xi &= \widetilde{\rho}, & (6.12)\\ \text{Linear } &: \\ \xi_1{+}\xi_2{+}\xi_3{+}\xi_4 &= X_M \Rightarrow 3, & (6.13)\\ \text{Quadratic } &: \\ \xi_1\xi_2{+}\xi_2\xi_3{+}\xi_3\xi_1{+}\xi_1\xi_4{+}\xi_2\xi_4{+}\xi_3\xi_4 &= Y_G \Rightarrow (8\widetilde{T} + \widetilde{P})/3 & (6.14)\\ \text{Cubic } &: \\ \xi_1\xi_2\xi_3{+}\xi_1\xi_2\xi_4{+}\xi_2\xi_3\xi_4{+}\xi_3\xi_1\xi_4 &= Z_A \Rightarrow \widetilde{P} & (6.15)\\ \text{Quartic } &: \\ \xi_1\xi_2\xi_3\xi_4 &= T_K. & (6.16)\end{aligned}$$

In the above formulas the $\xi_1, \xi_2, \xi_3, \xi_4$ are the local eigenvalues of the Jacobian matrix.

Forces and energies associated with the Linear curvature are typical of surface tension effects. It becomes apparent that forces and energies associated with the Cubic similarity invariant represent the Pressures of interactions. The Gauss quadratic similarity invariant is dominated by temperature, with a pressure contribution.

A $\widetilde{P}, \widetilde{\rho}$ projection of the implicit universal van der Waals surface is given in the next figure. The diagram displays a critical isotherm that separates a single phase (the gas) from the different topological domains that can be interpreted as liquids and vapor. The shape of the critical isotherm should be remembered, for above the critical isotherm, there exists a unique value for the pressure, and below the critical isotherm there is more than one value for the pressure. This feature represents a topological property of the van der Waals gas, and will have importance in the study of non-equilibrium systems.

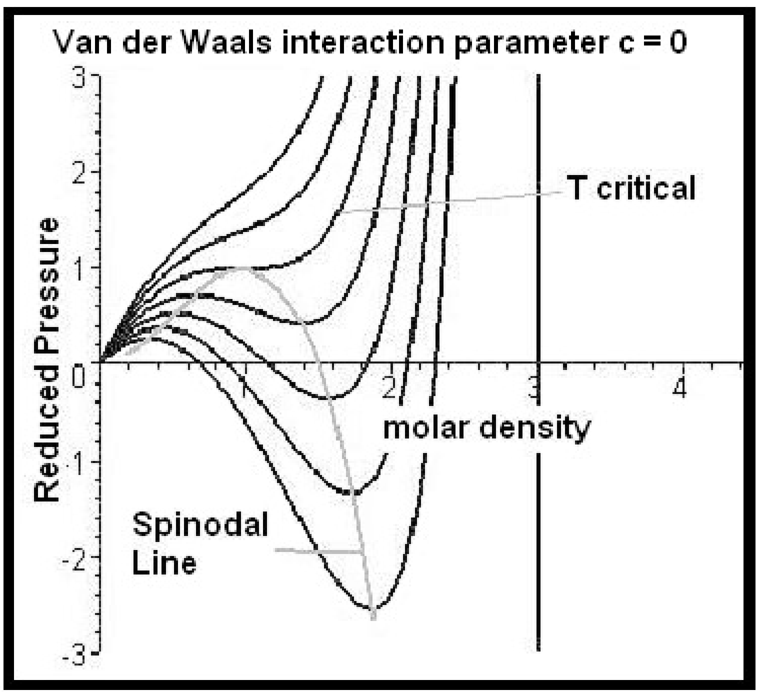


Figure 6.0a: **Note the regions of Negative Pressure**

Of interest to cosmologists, the pressure for the van der Waals gas, for values below the critical isotherm, can take on negative values. In addition, the envelope of the Phase function below the critical isotherm has the shape of a Higgs (quartic) potential (see Figure 6.9 below).

There exists a dual surface to the equation of state as defined by a Legendre transformation to the Gibbs function, $g = u - Ts + Pv$. The implicit surface defined by the Gibbs function (for a van der Waals gas) is not single valued, and appears as a deformation of a swallowtail bifurcation set. The actual Gibbs surface for the van der Waals gas can be numerically computed and is presented in the next Figure. An accurate drawing of the 3D Gibbs surface appears only occasionally in thermodynamic text books. Most presentations, if in 3D, are given by sketches, and not by actual computations. For example, in [4] p.196, the Gibbs surface misses the fact that the Spinodal line forms a cusp at the critical point. In Figure 2.7, below, the salient features are displayed by numeric computation of the Gibbs surface for the van der Waals gas. Remarkably, the dual Gibbs surface displays the envelope features of the Universal Phase function. Recall that the envelope is an element of the Renormalization Group [132]. The cuspoidal critical point singularity, the winged cusp representing the Spinodal line, and the Binodal self intersection are universal topological features of the discriminant (envelope) hypersurface. In the figure below, the white region is where the temperature is above the critical isotherm and represents the pure gas. The other sectors are below the critical isotherm, and are influenced by the "Higgs" features of the Phase potential. The dark gray sector represents the fluid phase, and the light gray sector represents the vapor phase. The light "blue" sector represents the unstable mixed phase region.

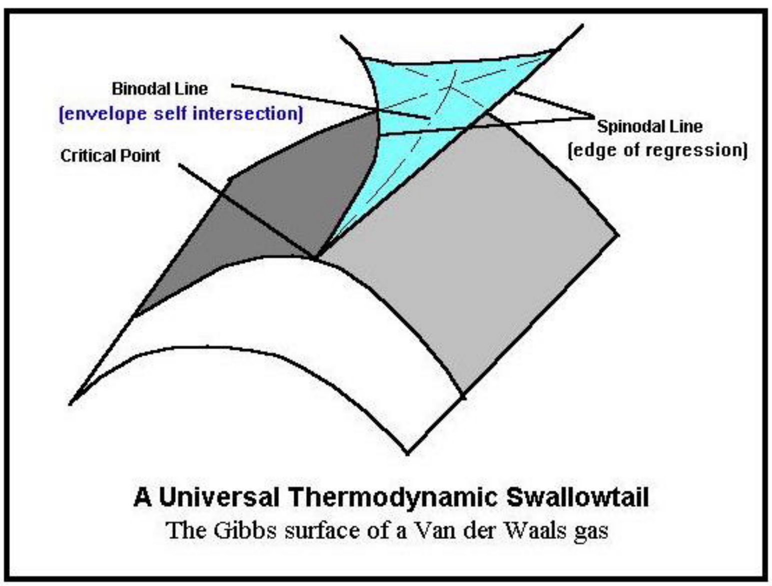


Figure 6.0b: Topological defects in the Gibbs surface

Conjecture *The topological features of the van der Waals gas are universal features (deformation invariants for all physical systems that admit a realization over 4D space-time variety. The van der Waals gas is one element of a topological equivalence class.*

Historically the two implicit surfaces defined by the reduced van der Waals equation became quite useful to chemical engineers and led to the law of corresponding states. If properties of a gas near its critical point could be measured, then the law of corresponding states permits estimates to be made for the properties of the gas by comparison to the universal van der Waals model. The topological results were independent of the geometric parameters of size, b, and interaction, a. In this and following sections, the universal topological features of the Phase function and the Gibbs surface of the generalized* van der Waals gas will be developed and applied to non-equilibrium systems. The "generalization" consists of adding a contribution to the reciprocal volume used in the interaction term. Recall that non-equilibrium requires that the Pfaff topological dimension of the Action 1-form is 3 or greater in certain regions. Non-equilibrium systems can exist in "stationary" states where there topological coherence properties are evolutionary invariants.

The principle (universal) topological defect structure of a van der Waals gas is the existence of a critical *point*. When expressed in terms of reduced coordinates, $\{\widetilde{P}, \widetilde{T}, \widetilde{\rho}\}$, the critical point of the implicit surface representing the equation of state, is where the reduced (dimensionless) functions all have the common value unity. The topological significance of the critical isotherm, which passes through the critical point, has already been mentioned above.

Another important topological defect structure is the existence of a Spinodal *line*, of ultimate phase stability, consisting of two parts that meet in a cusp at the critical point. The Spinodal line will be established by an edge of regression in the Gibbs surface.

Yet another topological defect structure is exhibited by the Binodal line, defining portions of a ruled *surface* representing the region of mixed phases. The Binodal line can be described by a deformation of a pitchfork bifurcation emanating from the critical point, and line which outlines the domain of mixed phases. The domain of mixed phase is related to regions where the Pfaff topological dimension of the encoded physical system (the 1-form of Action) is at least 3. The domains of isolated single phase are related to regions where the Pfaff topological dimension is 2 or less.

A lot can be learned from the van der Waals example, for its features are experimentally verifiable. The universal qualities are obtained in terms of variables that represent deformations and non-equilibrium extensions of the van der Waals properties. The van der Waals internal energy is a Lagrangian (phase) function in terms of extensive variables. In the language of classical mechanics, the Lagrange function is a function of the base variables, q^k, and their first derivatives, v^k, or velocity "extensive" functions. A Legendre transformation leads to a Hamiltonian function in terms of intensive variables, the momenta, p_k. The classic van der Waals phase function defines a hypersurface in the space of extensive variables of entropy, S, volume, V, and energy, U. A Legendre transformation produces a "Gibbs-Hamiltonian" function of intensive variables, temperature T, pressure, P, and Gibbs free energy, $Gibbs$.

*The "generalization" consists of adding a contribution to the reciprocal volume used in the interaction term.

The zero sets of certain algebraic combinations of the similarity curvature invariants of these hypersurfaces define universal topological features of the physical system, which are of value to the study of both equilibrium and non-equilibrium systems. Rather that formulating the non-equilibrium universal phase equation in a phenomenological manner, it will be demonstrated that such a universal phase function can be generated as the Cayley-Hamilton polynomial equation of the Jacobian matrix for the 1-form of Action, A, that represents the physical system. The topological Pfaff dimension of A permits the delineation between those phase functions that represent non-equilibrium systems and those that do not.

The following subsections first will discuss the ideas associated with Extensive and Intensive variables. Then the classic van der Waals expression for a Phase equation will be used to define an internal energy surface in terms of intensive variables. A dual construction will be used to create the Gibbs energy in terms of intensive variables. The Gibbs surface is deformably (topologically) equivalent to the swallowtail discriminant, or envelope of the classic phase equation. After this review of classical theory in the language of topological evolution, the theory will be extended to include non-equilibrium systems of the closed and open types.

6.0.5 The Phase function for a van der Waals Gas

In the classical development of thermodynamics, the van der Waals gas is often used as a cornerstone example. However, the phase function, Θ, given in many textbook treatments is not explicitly homogeneous of degree 1 in the extensive variables. A homogeneously correct formulation, to within a constant, is given by the relation:

$$\Theta\{...S,V,n;U\} = n[e^{\frac{S}{nC_v}}(\frac{V}{n} - b)^{-\frac{R}{C_v}} - \frac{a}{(\frac{V}{n} + cb)} - \frac{U}{n}] \Rightarrow 0. \tag{6.17}$$

The constant b is a representative size of the "particles" that make up molar quantities of the gas. Currently, it is usual to consider the "molar" quantities to be microscopic molecules, but the molar quantities from a topological perspective can be any size, ranging from nuclei to stars. To repeat Sommerfeld's statement:

> "The atomistic, microscopic point of view is alien to thermodynamics. Consequently, as suggested by Ostwald, it is better to use moles rather than molecules." (see p. 11 [262].)

The constant a is representative of the interaction forces between the molar quantities. The term $a/(\frac{V}{n})^2$ has been described by Sommerfeld as representing the "forces (or energy) of cohesion" (p. 58 in [262]). Note that a correction factor, cb, has been added to the historical collision term $a/(V/n) \to a/(V/n + cb)$ in order to account for the finite interaction size (or an effective scattering wavelength, or coherence length, cb) of the interacting molar particles. The coefficient c can be adjusted to give a better fit of the van der Waals gas equation to the experimental data of $\Omega_c = (nRT_c/P_cV_c)$ at the critical point.

This equation for $\Theta\{S, V, n; U\}$ satisfies the Euler condition for homogeneity of degree 1, with respect to the *extensive* variables $\{S, V, n; U\}$:

$$U\partial\Theta/\partial U + V\partial\Theta/\partial V + S\partial\Theta/\partial S + n\partial\Theta/\partial n - \Theta = 0. \tag{6.18}$$

The partial derivatives of the phase function, Θ, with respect to the extensive variables may be used to define *intensive* variables, (P, T, μ, β),

$$(P = -\partial\Theta/\partial V,\ T = \partial\Theta/\partial S,\ \mu = -\partial\Theta/\partial n, \beta = -\partial\Theta/\partial U). \tag{6.19}$$

From the phase function (6.17), partial differentiation yields:

$$T = \frac{\partial}{\partial S}(\Theta) = (e^{\frac{S}{nC_v}}(\frac{V}{n} - b)^{-\frac{R}{C_v}})/C_v, \tag{6.20}$$

$$P = -\frac{\partial}{\partial V}(\Theta) = \frac{nRT}{V - bn} - a\frac{n^2}{(V + cbn)^2}. \tag{6.21}$$

Differentiating P with with respect to V yields

$$\partial P/\partial V = -\frac{nRT}{(-V + bn)^2} + 2a\frac{n^2}{(V + cbn)^3}, \tag{6.22}$$

and differentiation again leads to

$$\partial^2 P/\partial V^2 = -2\frac{nRT}{(-V + bn)^3} - 6a\frac{n^2}{(V + cbn)^4}. \tag{6.23}$$

The classic argument, used to define the critical point, sets these partial derivatives of P with respect to V equal to zero. The values of the thermodynamic variables at the critical point are:

$$V_c = bn(2c + 3), \qquad T_c = \frac{8a/27}{bR(c + 1)}, \quad P_c = \frac{a/27}{b^2(c + 1)^2}. \tag{6.24}$$

Note that if the critical molar density is defined as $\rho_c = n/V_c$, then the previous equations lead to a universal constant, Ω_c, which is independent from the geometrical parameters $\{a, b\}$:

$$\Omega_c = R(\rho_c T_c/P_c) = nRT_c/P_c V_c = 8\frac{c + 1}{2c + 3}. \tag{6.25}$$

The reciprocal of Ω_c is often defined as the compressibility, $Z = 1/\Omega_c$. For the van der Waals gas ($c = 0$), Ω_c=1/.375, but for many real gases, the experimental value is closer to Ω_c=1/.27. This result is in good agreement with the value of $c = 4$.

For the classic van der Waals gas ($c = 0$), a rescaled equation of state can be obtained in terms of the dimensionless variables, scaled by their values at the critical point.

$$\widetilde{\rho} = \rho/\rho_c \quad \widetilde{T} = T/T_c \quad \widetilde{P} = P/P_c \tag{6.26}$$

$$0 = \widetilde{\rho}^3 - 3\widetilde{\rho}^2 + \{(8\widetilde{T} + \widetilde{P})/3\}\widetilde{\rho} - \widetilde{P}. \tag{6.27}$$

At the critical point, $\widetilde{\rho} = 1,\ \widetilde{T} = 1,\ \widetilde{P} = 1$. What is remarkable is that the coefficients a and b introduced to better account for the properties of the "particles" cancel out in the rescaled formulas. It is this feature that makes the van der Waals gas formulas have a universal appeal, and leads to the idea of "corresponding states".

The classic rescaled van der Waals formula leads to a critical isotherm that topologically separates the pure gas phase from those regions that admit liquid, or vapor, coexistent mixed phases. The "universal shape of the critical isotherm is given in the figure below. It is a topological invariant and is to be recognized by its distinctive shape.

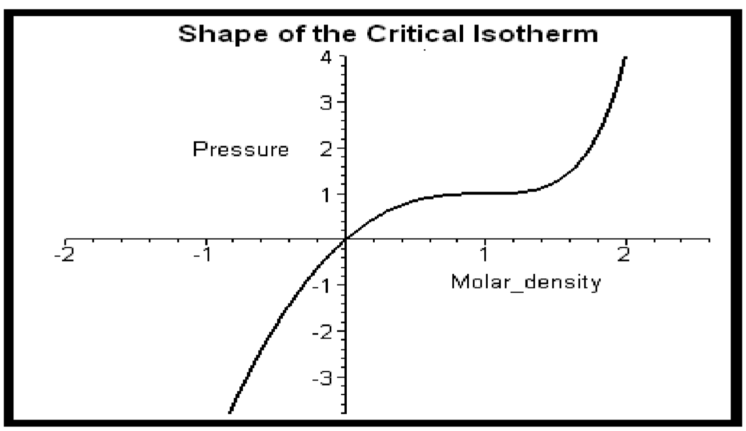


Figure 6.1 Universal Critical Isotherm for phase changes

For arbitrary coefficient c, the cubic formula for the reduced equation of state is also independent from the van der Waals parameters a and b, but is of a somewhat more complicated format:

$$\begin{aligned}\Theta &= (8\widetilde{T}c^3 + (27 + 8\widetilde{T} + \widetilde{P})c^2 + 54c + 27)\rho_c^3 \\ &\quad + ((-27 - \widetilde{P} + 16\widetilde{T})c^2 + (-54 + 16\widetilde{T} + 2\widetilde{P})c - 27)\rho_c^2 \\ &\quad + ((-2\widetilde{P} + 8\widetilde{T})c + 8\widetilde{T} + \widetilde{P})\rho_c) - \widetilde{P} \Rightarrow 0, \end{aligned} \tag{6.28}$$

$$\rho_c = \widetilde{\rho}/(2c + 3). \tag{6.29}$$

6.0.6 The Jacobian Matrix of the Action 1-form

The Cartan topological methods of exterior differential forms emphasize the antisymmetric features of a physical system, especially through the antisymmetric matrix that encodes the 2-form dA. The more geometric formulation of the van der Waals gas, as described in the previous section, can also be obtained from the symmetrical differential properties of the coefficients of 1-form of Action, A. It will be assumed that the 1-form of Action, A, that encodes the physical system, is of Pfaff topological dimension 4, except on certain subspaces of the four independent variables.

It should be noted that from the point of view of dimensional analysis each term in the Action 1-form is presumed to be of the same "physical" dimension. The coefficients are conjugate to the differentials. For projective realizations, the next step assumes that the coefficients are all of the same physical dimension, and the differentials are all of the same "physical" dimension. This latter assumption is stronger than the idea that the coefficients are intensive and the differentials are extensive.

The idea to be exploited in that which follows is that the Jacobian matrix $\mathbb{J} = [\partial A_k/\partial x^j]$ of partial derivative functions, created from the coefficients of the 1-form of Action, satisfies a Cayley-Hamilton matrix polynomial equation, and a complex algebraic polynomial equation in terms of the eigenvalues, ξ, of the Jacobian matrix.

$$\text{Cayley Hamilton polynomial} = \xi^4 - X_M\xi^3 + Y_G\xi^2 - Z_A\xi^1 + T_K = 0. \tag{6.30}$$

The coefficients of the polynomials $\{X_M, Y_G, Z_A, T_K\}$ are invariant with respect to similarity transformation of the Jacobian matrix, and in this (restricted) sense the method is universal. The Jacobian matrix contains both symmetric and antisymmetric components, where the 2-form, dA, emphasizes the antisymmetric features of the partial derivatives of the 1-form coefficients. The symmetric similarity properties are more related to Euclidean geometric properties of the physical system, but it should be realized that congruence (size) and distance are additional requirements necessary for a Euclidean geometry. Similarity transformations are special projective transformations that preserve parallelism and orthogonality (or better said, preserve points at infinity and the special point that defines the origin.)

It is assumed that this characteristic equation, as a polynomial of 4th degree, is in effect a Universal Thermodynamic Phase function, $\Theta(x, y, z, t;\ \xi)$:

$$\text{Cayley-Hamilton polynomial} = \Theta(x, y, z, t;\ \xi) = 0. \tag{6.31}$$

The Phase function is distinct for different categories of coefficient functions that make up the 1-form of Action, A, but all such Phase functions are related to the deformation equivalence classes that include the classic van der Waals gas. The Universal Phase function defines a family of implicit hypersurfaces in the space of "universal" coordinates defined in terms of the similarity invariants, $\{X_M, Y_G, Z_A, T_K\}$. It will be demonstrated how and when the similarity invariants can be related to "curvatures" of the universal implicit hypersurface. However, no metric is used explicitly to define the "curvatures".

The non-equilibrium extensions of the van der Waals gas (of Pfaff topological dimension 4) are to a certain extent encoded in the third and fourth order similarity invariants, Z_A and T_K, and the possibilities that the polynomial can have complex roots. In order to describe topologically isolated or equilibrium systems it is necessary (but not sufficient) these third and fourth order similarity invariants vanish. The similarity invariants are in effect symmetric averages of eigenvalues, which ignore the possible system antisymmetries. It is this difference that characterizes the failure of geometric concepts, (the quadratic metric form) and theories built on such symmetric constraints, to capture thermodynamic irreversibility.

In the special isolated-equilibrium cases, the topological features of a universal thermodynamic critical point, and a Spinodal line of ultimate phase stability have realizations in terms of topological constraints on the phase function implicit hypersurface that represents the universal equilibrium van der Waals gas. When written in terms of curvatures it can be demonstrated that the zero set of the quadratic similarity invariant (the Gauss curvature) represents the Spinodal line, or the edge of regression in the Gibbs surface, of a van der Waals gas. The thermodynamic critical point occurs when both the Mean curvature and the Gauss curvature of the equilibrium surface vanish. It is this universality that gives credence to the idea that the four-dimensional universe could be represented as a non-equilibrium van der Waals gas near its critical point [240]. These concepts will be extended to the non-equilibrium systems in that which follows.

6.0.7 The Non-Equilibrium Characteristic Phase Function

The 1-form of Action, used to encode a physical system, contains other useful topological information, as well as geometric information. Reconsider the details of an open thermodynamic system generated by a 1-form of Action, A, of Pfaff topological dimension 4. The component functions of the Action 1-form can be used to construct a 4x4 Jacobian matrix of partial derivatives, $[\mathbb{J}_{jk}] = [\partial A_j/\partial x^k]$. In general, this Jacobian matrix will be a $4\ x\ 4$ matrix that satisfies a 4th order Cayley-Hamilton characteristic polynomial equation, $\Theta(x, y, z, t;\ \xi) = 0$, with four perhaps complex roots representing the four perhaps complex eigenvalues, ξ_k, of the Jacobian matrix.

$$\Theta(x, y, z, t;\ \xi) = \xi^4 - X_M\xi^3 + Y_G\xi^2 - Z_A\xi^1 + T_K \Rightarrow 0. \tag{6.32}$$

The eigenvalues of the real matrix can be complex numbers, but the similarity coefficients, $\{X_M, Y_G, Z_A, T_K\}$ are all real. To repeat, a classic analysis yields the result that eigenvalues form three equivalence classes:

1. 4 real eigenvalues.

2. 2 real eigenvalues, and 1 complex eigenvalue and its 1 complex conjugate.

3. 2 complex eigenvalues, and their 2 complex conjugates.

These equivalence classes can be put into correspondence with the gas, vapor, and liquid regions of a van der Waals gas.

It should be noted that the characteristic polynomial is constructed from the "symmetric" properties of the Jacobian matrix, in the sense that the similarity coefficients are real combinations of complex numbers. On the other hand the antisymmetric components of the Jacobian matrix are emphasized by the Pfaff topological dimension constructed from antisymmetric differential forms, whose coefficients are the components of the antisymmetric parts of the Jacobian matrix. For example, there exist 1-forms of Action that are of Pfaff topological dimension 4, yet the 4th order symmetric similarity invariant $T_K \Rightarrow 0$. Similarly, there are examples where Z_A and T_K both go to zero, but the system defined by the 1-form of Action, A, is of Pfaff dimension 3, and therefore defines a non-equilibrium system. If the Quartic term T_K vanishes, then there exists a null eigenvector for the Jacobian matrix. If the Pfaff topological dimension of the 1-form A is equal to 4, then there cannot exist a null eigenvector for the antisymmetric part of the Jacobian matrix. The null eigenvector for the Jacobian must have non-zero results for both the symmetric and antisymmetric parts of the Jacobian matrix, separately. In particular, the 1-form of Work, $W = i(V_\xi)dA$ evaluated for the null eigenvector of the Jacobian matrix, can not be zero. The effect of the null eigenvector on the symmetric parts of the Jacobian must cancel the effects of the 1-form of Work. Hence, as noted above for symplectic systems:

$$\mathbb{J}\,|V_{null}\rangle = \mathbb{S}\,|V_{null}\rangle + \mathbb{A}\,|V_{null}\rangle = 0. \tag{6.33}$$

It follows that in such circumstances the symmetries must be balanced by the antisymmetries in the sense that

$$\mathbb{S}\,|V_{null}\rangle = -\mathbb{A}\,|V_{null}\rangle\,. \tag{6.34}$$

The moral is that null vector constraints imply that symmetrical deformations $\mathbb{S}\,|V_{null}\rangle$ must be compensated by "electromagnetic" work $\mathbb{A}\,|V_{null}\rangle$ in the turbulent non-equilibrium domain of Pfaff topological dimension 4. The separation of "charge" is a possible remnant of "electromagnetic" work and occurs via irreversible processes in domains of Pfaff topological dimension 4.

From the theory of strings and surface tension, the X_M term is - in a sense - a linear deformation contribution to the "energy" of the system. The coefficient Y_G can be related to the Gauss (quadratic) curvature of the system, and is related to an area deformation contribution. The coefficient Z_A can be related to the Interaction (Cubic) curvature of the system, and is related to a volume deformation contribution (a pressure) to the "energy". The last term T_K is a quartic contribution and can be related to an expansion or contraction of the four-dimensional volume element.

Symbolically, multiply the phase function by u/ξ^4 and consider u/ξ to be a length deformation, δ_{Length}, u/ξ^2 to be an area deformation, δA_{rea}, u/ξ^3 to be a volume deformation, δ_{Vol}, and u/ξ^4 to be a space-time expansion deformation, δ_{Exp_xyzt}. The suggestive formula becomes

$$\Theta = u - X_M \cdot \delta_{Length} + Y_G \cdot \delta_{Area} \quad (6.35)$$

$$-Z_A \cdot \delta_{Vol} + T_K \cdot \delta_{Exp\_xyzt} \quad (6.36)$$

and by comparison with a van der Waals gas,

$$X_M \approx \text{"String or Surface\_tension"} \quad (6.37)$$

$$Y_G \approx \text{"Temperature - Entropy"} \quad (6.38)$$

$$Z_A \approx \text{"Pressure - Interaction"} \quad (6.39)$$

$$T_K \approx \text{xyzt- "Higgs" Expansion - Rotation} \quad (6.40)$$

Automatically, the phase function incorporates string or surface tension effects through X_M, where X_M can be related to a mean four-dimensional curvature expression. Gravity effects, related to the 4D Gauss curvature, $G = Y_G/6$ are "area" related. From the idea that the entropy of a gravitational black hole is related to an area, and the fact that the phase formula for a van der Waals gas implies that Y_G is dominated by the temperature (see eq(6.12)), the universal phase formula suggests that the idea of gravity (and the Gauss curvature) is a temperature - entropic concept, contributing energy of the type TS. The phase formula for a van der Waals gas implies that the Z_A coefficient is related to pressure (which can be both negative or positive), and the energy contribution is of the type PV. The last term represents a 4D xyzt expansion, which from the topological theory of thermodynamics presented above can be related to irreversible dissipation.

It is sometimes more convenient to express the similarity invariants in terms of their averages, where the average is determined by dividing by the number of non-zero eigenvalues. This leads to a sequence of maps from the original variety of independent variables, $\{x, y, z, t\} \Rightarrow \{X_M, Y_G, Z_A, T_K\} \Rightarrow \{M, G, A^\star, K\}$. (Note that the symbol $A^\star$ is used for the Adjoint cubic average in order to eliminate confusion with the 1-form A.) When the averaged similarity invariants are treated as generalized coordinates, then the characteristic polynomial becomes a Universal Phase function, and will be used to encode universal thermodynamic properties.

A similar procedure can be applied to domains of lesser dimension. For example, suppose the dimension of the domain is reduced from Pfaff dimension 4 to Pfaff dimension 3 by the constraint that the determinant T_K vanishes (this corresponds to the reduction of a non-equilibrium turbulent system to a non-equilibrium non-turbulent system which can support steady states). The Phase equation must have one null eigenvalue, that represents a null eigenvector, or fixed point of the Jacobian matrix. The Phase equation with one eigenvalue = to zero (say $\xi_4 = 0$) reduces to

$$\Theta(x, y, z, t;\ \xi) = (\xi^3 - X_M\xi^2 + Y_G\xi^1 - Z_A)\xi \Rightarrow 0, \tag{6.41}$$
$$\text{with } X_M = (\xi_1 + \xi_2 + \xi_3), \tag{6.42}$$
$$Y_G = (\xi_1\xi_2 + \xi_2\xi_3 + \xi_3\xi_1), \tag{6.43}$$
$$Z_A = \xi_1\xi_2\xi_3. \tag{6.44}$$

Conjecture *An objective herein is to exploit the striking similarity between the cubic factor of the 3D phase equation (eq. 6.41), and the cubic equation of the rescaled van der Waals gas given by equation (6.27). The fundamental assumption is that the eigenvalue of the Cayley-Hamilton characteristic polynomial for the Jacobian matrix,* $[\mathbb{J}_{jk}] = [\partial(A)_j/\partial x^k]$*, plays the role of the rescaled molar density* ρ/ρ_c *in thermodynamics.*

The critical point in 3D occurs for the set $\{X_M = 3,\ Y_G = 3,\ Z_A = 1,\ T_K = 0\}$ with $\xi \Leftrightarrow \widetilde{\rho} = [1, 1, 1, 0]$. A comparison of the universal equation and the van der Waals gas equation yields

$$3 = X_M, \tag{6.45}$$
$$\{8\widetilde{T} + \widetilde{P}\}/3 = Y_G, \tag{6.46}$$
$$\widetilde{P} = Z_A. \tag{6.47}$$

For the classic van der Waals gas, it is apparent that the (linear) similarity invariant (which is composed of the sum of molar density eigenvalues) is at its critical point value, $X_M = 3$. The (quadratic) similarity invariant is equal to $Y_G = (\{8\widetilde{T} + \widetilde{P}\}/3)$ and is composed of both temperature and pressure terms. The Adjoint (cubic) interaction similarity invariant is equal to $Z_A = \widetilde{P}$, the rescaled pressure. It is remarkable that for the van der Waals gas, the linear terms (representing string or surface tension effects, have been fixed at their "Critical Point" values. These concepts are presented in more detail in Vol 1.

If a further reduction in dimension occurs to Pfaff dimension 2, (with 2 null eigenvalues) the Phase equation with $\{M, G\}$reduces to

$$\Theta(x, y; z, t\ \xi) = (\xi^2 - X_M\xi^1 + X_G)\xi^2 \Rightarrow 0, \tag{6.48}$$
$$\text{with } X_M = (\xi_1 + \xi_2), \tag{6.49}$$
$$Y_G = (\xi_1\xi_2). \tag{6.50}$$

The critical point in 2D occurs for the set $\{X_M = 2,\ Y_G = 1,\ Z_A = 0, T_K = 0\}$ with $\xi_k = [1, 1, 0, 0]$.

The reduced Phase function

There exists a well known transformation of complex variables which will reformulate the characteristic polynomial. Substitute $\xi = s+M/4$. The result is a new "reduced" Phase polynomial $\Phi(x,y,z,t;\ s) = \Theta(x,y,z,t;\ \xi)_{reduced}$ of the form

$$\begin{aligned}
\Phi(x,y,z,t;\ s) &= s^4 + gs^2 - as + k = 0. && (6.51)\\
g &= (-3X_M^2/8 + Y_G), && (6.52)\\
a &= (X_M/2)^3 - Y_G X_M/2 + Z_A), && (6.53)\\
k &= T_K - Z_A(X_M/4) + Y_G(X_M/4)^2 &&\\
&\quad -3(X_M/4)^4, && (6.54)\\
s &= \xi - X_M/4. && (6.55)
\end{aligned}$$

The "reduced" Phase function is not the same as the "rescaled" Phase function. The coefficients $\{g, a, k\}$ are constructed from the real numbers $\{X_M, Y_G, Z_A, T_K\}$, of the reduced Phase polynomial. Polynomial analysis implies that the eigenvalues of the reduced Phase function belong to 3 equivalence classes of root structures discussed above.

For a van der Waals gas $(X_M = 3,\ T_K = 0)$, the reduced coefficients become

$$\begin{aligned}
&\text{Van der Waals gas} && (6.56)\\
g &= -27/8 + Y_G = -27/8 + \{8\widetilde{T} + \widetilde{P}\}/3, && (6.57)\\
a &= -27/8 + \{8\widetilde{T} - \widetilde{P}\}/2, && (6.58)\\
k &= -243/256 - 9/16\widetilde{P} + 3/2\widetilde{T}, && (6.59)\\
s &= \xi - 3/4. && (6.60)
\end{aligned}$$

The critical point has been moved to $s = 1/4$ for the van der Waals gas, as one of the eigenvalues is presumed to be zero. The reduced formula has eliminated the cubic term in the universal phase function by displacing the critical point to the origin in terms of the variable s, if all eigenvalues are not zero.

Consider the reduced Phase formula, and its derivatives with respect to the family parameter, s.

$$\begin{aligned}
\Phi &= s^4 + gs^2 - as + k = 0, && (6.61)\\
&\therefore\ k = -(s^4 + gs^2 - as), && (6.62)\\
\Phi_s &= \partial\Phi/\partial s = 4s^3 + 2gs - a = 0 && (6.63)\\
&\therefore\ a = 4s^3 + 2gs, && (6.64)\\
\Phi_{ss} &= \Phi_s = \partial^2\Phi/\partial s^2 = 12s^2 + 2g = 0 && (6.65)\\
&\therefore\ g = -6s^2. && (6.66)
\end{aligned}$$

Replacing the parameter a (from the envelope condition, $\Phi_s = 0$) in the equation for k yields

$$\text{Thermodynamic Higgs Potential} \quad k = s^2(3s^2 + g). \tag{6.67}$$

A plot of the equation for k is given below for various g and s,. The 4th order shape of the function motivates the name "Thermodynamic Higgs Potential".

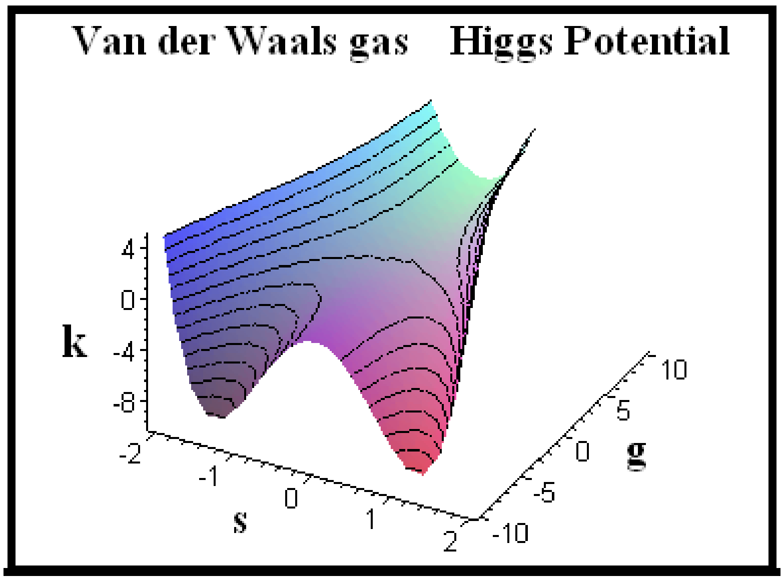


Figure 6.2 Reduced Envelope function, k(s,g)

In the general case with $T_K \neq 0$, then g = 0, s=0, k=0, represents the "critical point". Note that the set k=0 defines a pitchfork bifurcation. For g (∼reduced temperature) values below the critical point, the function k is a polynomial of 4th degree, but above the "critical temperature" the function k is quadratic. It is evident that below the critical isotherm, the "expansion" term k can have both negative and positive values. The formula for the 4D expansion coefficient therefore can also have positive or negative values. The quartic "potential" is reminiscent of the "Higgs" potential in relativistic field theories and the "Landau" potential in mean field theories. Note that these properties have been obtained without explicit use of a metric or connection, nor quantum mechanics.

From the van der Waals theory, the first partial derivative of the classic phase function yields the pressure. For the universal Phase polynomial the pressure is determined by the equation $\Phi_s = 0$, Indeed, the formula $a = 4s^3 + 2gs$ yields the universal equation for a (the pressure) in terms of the molar density "s". A plot of a (pressure) versus s (molar density) at fixed g (temperature) gives the familiar cubic shape, deformably equivalent to the van der Waals gas.

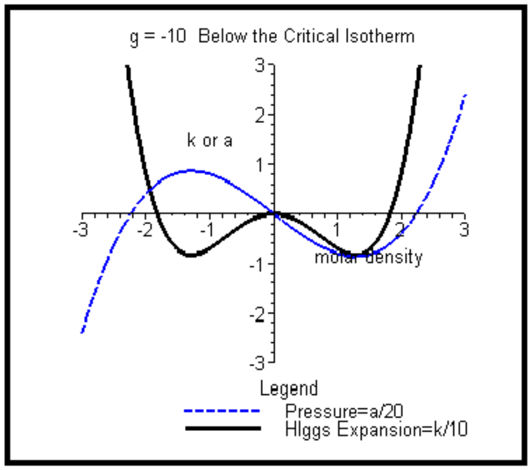


Figure 6.3 Higgs section at constant g = -10

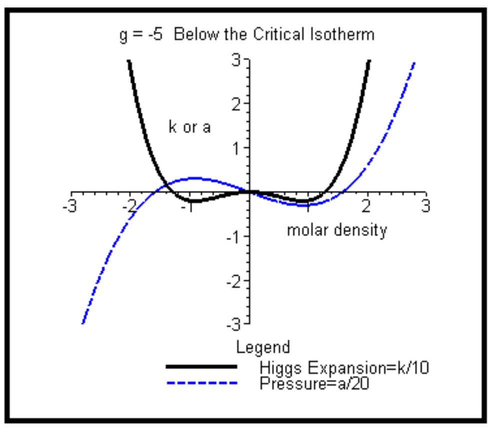


Figure 6.4 Higgs section at constant g = -5

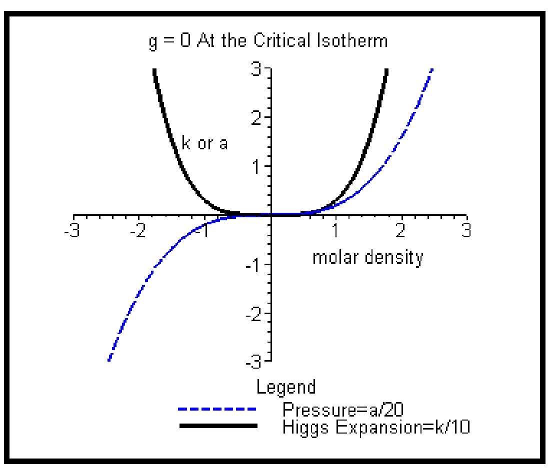


Figure 6.5 Higgs section at constant g = 0

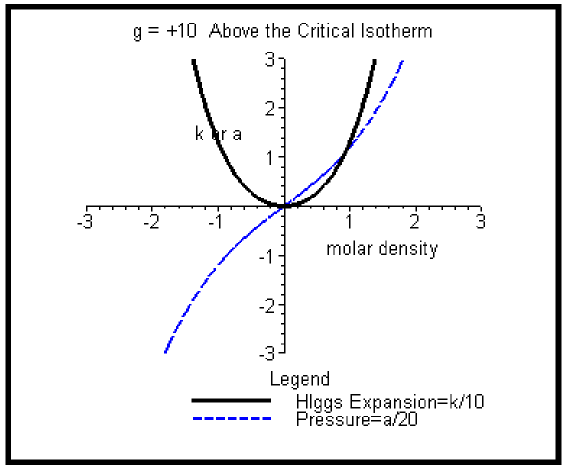


Figure 6.6 Higgs section at constant g = +10

Both k (Expansion in dashed blue) and a (pressure in solid black) are presented in the following diagrams as constant g (∼temperature) slices above and below the critical point

The Critical Isotherm

The topology of the quartic phase (potential) function is separated by a critical isotherm into two sectors. For temperatures below the critical temperature, the quartic formula yields a Higgs-like sector where expansion properties k are negative, and where liquid and vapor phases can coexist. Above the critical temperature the 4th order expansion properties k are positive, and the sector has lost its Higgs-like properties. The critical isotherm, $g = 0 = (-3X_M^2/8 + Y_G)$ defines a line of singularities separating the two sectors. For the critical temperature ($g = 0$) the shape of the critical isotherm is exactly the same as for the critical isotherm of the van der Waals gas. The shape of the curve mimics the dividing center of hysteresis phenomena.

The Binodal line

The zero sets of the "reduced pressure (a)" occur only for temperatures below the critical point and are described by the solution formula, $0 = 4s^3+2gs$. Along with the solution $s = 0$ coming from the second factor in the general phase formula for zero k, $\Phi = s^4+gs^2-as = 0$, a plot of the zeros of the "reduced pressure (a)" in the $s-g$ plane yield the Binodal line as a pitchfork bifurcation, with the transition occurring at the critical temperature. From the van der Waals gas model, the Binodal line delineates the single phase from the mixed phase regions. The Pitchfork is essentially the line of zero first partial derivatives of the Higgs sector of the universal phase function. This result appears to be the first non-phenomenological derivation of the Binodal line. These Pitchfork features are readily seen in the previous figure giving a 3D version of the Higgs - van der Waals gas potential.

The Spinodal Line

A second piece of topological information can be obtained from those points where the partial derivative of the pressure vanishes. These points are given by solutions to the equation $\Phi_{ss} = 12s^2 + 2g = 0$. Again only for temperatures g below the critical point will the formula give a set of points that describes classically what has been called the Spinodal line. In van der Waals theory the Spinodal line defines the "limit" of single phase stability and can only be realized transiently, in the absence of fluctuations. Both the Spinodal line (black) and the Binodal line (blue) are plotted in the next figure.

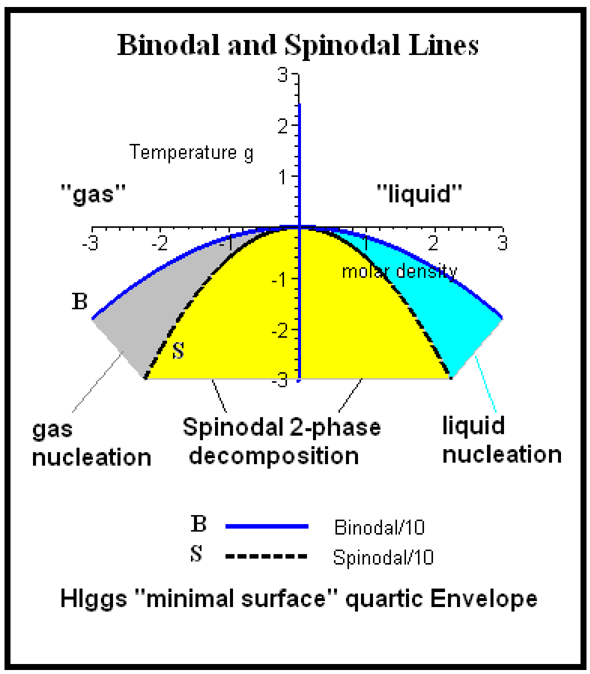


Figure 6.7 Binodal and Spinodal lines for Higgs envelope

The Binodal line and the Spinodal line can be related to homology invariants of projective transformations [314].

6.0.8 Oscillations and the Hopf bifurcation

When the eigenvalues of the characteristic polynomial are pure imaginary, Hopf oscillations can occur. Suppose that the complex eigenvalues are represented as $\{i\alpha, -i\alpha, i\beta, -i\beta\}$. Then the similarity invariants are $X_M = 0, Y_G = \alpha^2 + \beta^2, Z_A = 0, T_K = \alpha^2\beta^2$. Hence the criteria for a double Hopf oscillation frequency requires that the algebraically odd similarity invariants vanish and the algebraically even similarity invariants are positive definite. (Recall that in the 3D theory of minimal 2D surfaces, the mean curvature is related to the linear similarity invariant, $X_M \Rightarrow 0$). For a single dominant Hopf oscillation frequency ($\beta \Rightarrow 0$), the Hopf conditions are: $X_M = 0, Y_G \Rightarrow \alpha^2 > 0, Z_A = 0, T_K \Rightarrow 0$. These conditions can be computed relatively easily, and will be demonstrated in the examples below. Note that the minimal hypersurface condition $X_M \Rightarrow 0$ may be satisfied by states with $Y_G < 0$ in accord with the examples of soap films. Such conditions are related to non-oscillatory solitons which form "stationary states", but are globally stabilized far from equilibrium (see Vol 2. on Falaco Solitons).

6.0.9 Minimal surfaces

The Universal Phase function, Θ, may be considered as a family of implicit hypersurfaces in the four-dimensional space, $\{X_M, Y_G, Z_A, T_K\}$ with a complex family (order) parameter, ξ. Moreover, it should be realized that the Universal Phase Function is a holomorphic function, $\Theta = \phi + i\chi$ in the complex variable $\xi = u + iv$. That is

$$\Theta(X_M, Y_G, Z_A, T_K;\ \xi) \Rightarrow \phi + i\chi, \tag{6.68}$$

where

$$\begin{aligned} \phi &= u^4 - 6u^2v^2 + v^4 - X_M(u^2 - 3v^2)u \\ &\quad + Y_G(u^2 - v^2) - Z_A u + T_K, \qquad (6.69) \\ \chi &= 4(u^2 - v^2)uv - X_M(3u^2 - v^2)v + 2Y_G uv - Z_A v. \qquad (6.70) \end{aligned}$$

As such, in the 4D space of two complex variable pairs, $\{\phi + i\chi, u + iv\}$, according to the theorem of Sophus Lie, any such holomorphic function produces a pair of conjugate *minimal* surfaces in the four-dimensional space $\{\phi, \chi, u, v\}$. It follows that there exists a sequence of maps,

$$\{x, y, z, t\} \Rightarrow \{X_M, Y_G, Z_A, T_K\} \Rightarrow \{\phi, \chi, u, v\}, \tag{6.71}$$

such that the family of hypersurfaces can be decomposed into a pair of conjugate minimal surface components.

For a phase function generated by the constraints, $X_M = Z_A \Rightarrow 0$, the minimal surface functions become defined by the equations

$$\begin{aligned} \phi &= u^4 - 6u^2v^2 + v^4 + Y_G(u^2 - v^2) + T_K, \qquad (6.72) \\ \chi &= 4(u^2 - v^2)uv + 2Y_G uv. \qquad (6.73) \end{aligned}$$

For the Hopf Map the eigenvalues are pure imaginary, hence

$$\begin{aligned} \phi_{Hopf} &= +v^4 + Y_G(-v^2) + T_K, \qquad (6.74) \\ \chi &= 0. \qquad (6.75) \end{aligned}$$

It is important to realize that the similarity invariant $X_M \Rightarrow 0$ does not define a minimal surface unless the Jacobian matrix of the 1-form is scaled by the Gauss map.

Examples of conjugate pairs of minimal surfaces

The idea is that the complex position vector, $\mathbf{V} = [U, V, W]$, whose real or imaginary parts will map out a minimal surface in 3D, can be generated from the Weierstrass representation [180] in terms of the holomorphic function $H(\varpi) = \phi + i\chi$,

$$X(\varpi) = \int (1 - \varpi^2) H(\varpi) d\varpi, \quad (6.76)$$

$$Y(\varpi) = \int (1 + \varpi^2) H(\varpi) d\varpi, \quad (6.77)$$

$$Z(\varpi) = \int (2\varpi) H(\varpi) d\varpi. \quad (6.78)$$

Rewriting $H(\varpi)$ in the form

$$H(\varpi) = (b - ia)/2\varpi^2, \quad with \quad \varpi = -i\ exp(\eta + i\xi), \quad (6.79)$$

and substituting into the Weierstrass formulas yields the position vector to a family of minimal surfaces of the form

$$X = a\ sinh(\eta)\ cos(\xi) - b\ cosh(\eta) sin(\xi), \quad (6.80)$$

$$Y = a\ sinh(\eta) sin(\xi) - b\ cosh(\eta) cos(\xi), \quad (6.81)$$

$$Z = a\ \eta + b\ \xi. \quad (6.82)$$

For $a = 0$ the surface is a catenoid; for $b = 0$ the surface is a helicoid. (see p.70 in [178]). For a and b non-zero, the minimal surface so generated consists of two conjugate minimal surfaces intertwined (the example has $a = b = .5$)

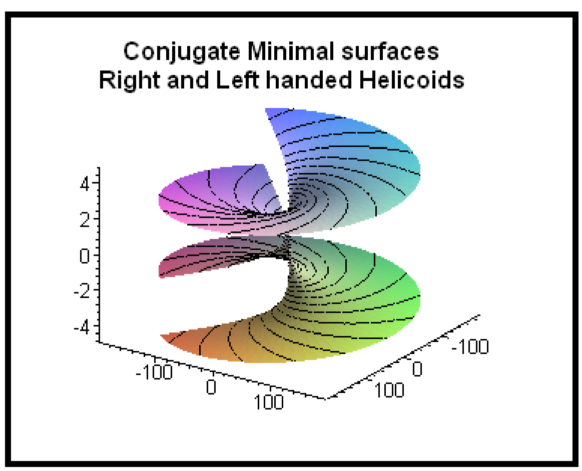


Figure 6.8 Chiral pairs of conjugate minimal surfaces

Note that the conjugate pairs have different chirality. Other examples of such conjugate pairs are displayed below.

Example of a fractal minimal surface

As a second example of the Sophus Lie theorem, consider the Holomorphic function and its functional iterates

$$H_1(\varpi) = (\varpi^2 - D), \quad H_2(\varpi) = ((\varpi^2 - D)^2 - D), \ ... \tag{6.83}$$

According to the minimal surface theorem, this Holomorphic function represents a one (complex) parameter family of minimal surfaces in four dimensions. It follows that the Mandelbrot set , which is given by the values of D for which the function $H_1(\varpi)$ fails to iterate the origin ($\varpi = 0$) to infinity is the fractal envelope of a family of minimal surfaces in four dimensions parameterized by $D = a+ib$. The complement to the Mandelbrot set is a minimal surface with a fractal boundary where all functional sequences iterate to infinity. Hence the "fractal" minimal surface is complete. The non-intuitive conclusion is that a minimal surface can be fractal!

The "Gibbs entropy" minimal surface

As another surprising example, consider those functions of a complex variable such that $H(\varpi) = (\partial F(\varpi)/\partial\varpi)^3$. All functions $F(\varpi)$ that have the form,

$$F(\varpi) = \{\alpha \ \varpi \ ln(\varpi) + C \ \varpi\} + (B - D \ \varpi^2) \tag{6.84}$$

$$= \{Gibbs \ Entropy\} + (Mandelbrot \ generator), \tag{6.85}$$

generate the same Weierstrass function,

$$H(\varpi) = (\partial^3 F(\varpi)/\partial\varpi^3) = 2\alpha/\varpi^2. \tag{6.86}$$

The format of $F(\varpi)$ is strikingly reminiscent of those formulas that appear in the literature to describe the Gibb's entropy. The coefficients α, B, C and D are presumed to be complex constants. Rewriting $H(\varpi)$ in the form,

$$H(\varpi) = (b - ia)/2\varpi^2, \quad with \quad \varpi = -i \ exp(\eta + i\xi), \tag{6.87}$$

and substituting into the Weierstrass formulas yields the position vector to a family of minimal surfaces. When α is real then extremal minimal surface is a catenoid again; when α is imaginary, the minimal surface is a helix. When α is complex, the result is a pair of conjugate helicoids.

The interesting features are :

1. All complex wave functions are related to a minimal surface by this technique.

2. The primitive function $F(\varpi)$ is related to the Helmholtz free energy, and it is the "entropy" term, $\alpha \ \varpi \ ln(\varpi)$ that generates the family of minimal surfaces known as the conjugate helicoids. (The topic of conjugate helicoids will be revisited below.)

3. The resulting minimal surface is independent of the linear term $C\ \varpi$ and the "Mandelbrot germ", $(B - D\ \varpi^2)$.

4. The Petrov type D classifications (which yield the only known black hole solutions to the Einstein gravity theory [48]) are related to minimal surfaces.

Envelopes

The theory of implicit hypersurfaces focuses attention upon the possibility that the Universal Phase function has an envelope. The existence of an envelope depends upon the possibility of finding a simultaneous solution to the two implicit surface equations of the family:

$$\Theta(x, y, z, t;\ \xi) = \xi^4 - X_M\xi^3 + Y_G\xi^2 - Z_A\xi + T_K \Rightarrow 0, \tag{6.88}$$

$$\partial\Theta/\partial\xi = \Theta_\xi = 4\xi^3 - 3X_M\xi^2 + 2Y_G\xi - Z_A \Rightarrow 0, \tag{6.89}$$

For the envelope to be smooth, it must be true that $\partial^2\Theta/\partial\xi^2 = \Theta_{\xi\xi} \neq 0$, and that the exterior 2-form, $d\Theta\hat{}d\Theta_\xi \neq 0$ subject to the constraint that the family parameter is a constant: $d\xi = 0$. The envelope as a smooth hypersurface does not exist unless both conditions are satisfied.

The envelope is determined (to within a factor) by the discriminant of the Phase Function polynomial, which, as a zero set, is equal to a universal implicit hypersurface, $DISC\Theta \Rightarrow 0$, in the four-dimensional space of similarity variables $\{X_M, Y_G, Z_A, T_K\}$. This function can be written in terms of the similarity "coordinates" (suppressing the subscripts) as :

$$\begin{aligned} DISC\Theta = {} & 18X^3ZYT - 27Z^4 + Y^2X^2Z^2 - 4Y^3X^2T \\ & +144YX^2T^2 + 18XZ^3Y - 192XZT^2 \\ & -6X^2Z^2T + 144TZ^2Y - 4X^3Z^3 \\ & -27X^4T^2 - 4Y^3Z^2 + 16Y^4T \\ & -128Y^2T^2 + 256T^3 - 80XZY^2T. \end{aligned} \tag{6.90}$$

The discriminant (envelope) has eliminated the family order parameter, ξ.

An alternate formulation describes the discriminant of the Reduced Phase phase function, $\Phi = \Theta_{reduced}$:

$$DISC\Phi := -27a^4 + 4(-g^2 + 36k)ga^2 + 16k(4k - g^2)^2. \tag{6.91}$$

The hypersurface defined by the discriminant of the phase function $\Theta_{reduced} = \Phi$ yields the (symmetrized) version of the universal swallowtail hypersurface. A plot of the universal envelope $\Phi = 0$ (in terms of the coordinates $(g,\ a,\ k)$ is given in Figure 6.9.

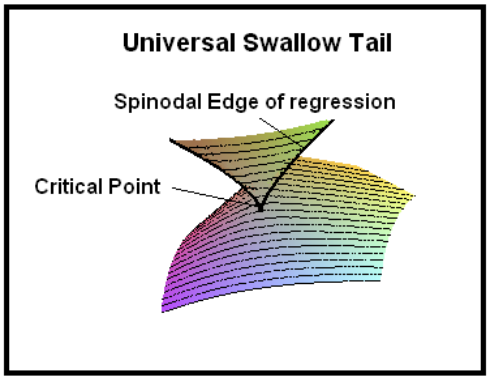


Figure 6.9 Universal topological defects

It is apparent that the van der Waals gas is a deformation of the universal swallowtail hypersurface formed as the envelope of the reduced phase function, $\Theta_{reduced}$. It is remarkable that the Discriminant Envelope of the universal phase function, Θ, and the Discriminant Envelope of the reduced phase function, Φ, are the same (in the space of coordinates $\{X_M, Y_G, Z_A, T_K\}$).

Remarkably, choosing the constraint condition in terms of the hypothetical condition that the Mean similarity invariant Curvature vanishes, $X_M \Rightarrow 0$, leads to a domain in the 4D space where the reduced discriminant defines a universal swallowtail surface homeomorphic (deformable) to the Gibbs surface of a van der Waals gas (subscripts
suppressed):

$$\begin{aligned} \text{Minimal Surface} &: \mathbf{Universal\ Swallowtail\ Envelope}\quad X_M \Rightarrow 0 \\ DISC\Theta_{(X_M=0)} &= -27Z^4 + 144TZ^2Y - 4Y^3Z^2 + 16Y^4T \\ &\quad -128Y^2T^2 + 256T^3, \qquad (6.92) \\ &\approx DISC\Theta_{reduced} = DISC\Phi \Rightarrow 0. \qquad (6.93) \end{aligned}$$

It must be remembered that this Minimal surface is a hypersurface in the space of Pfaff topological dimension 4. Examples are given in that which follows. In other words, the Gibbs function for a van der Waals gas is a universal idea associated with minimal hypersurfaces, $X_M = 0$, of thermodynamic systems of Pfaff topological dimension 4. The similarity coordinate T_K plays the role of the Gibbs free energy, in terms of the pressure ($\sim Z_A$) and the temperature ($\sim Y_G$). The Spinodal line as a limit of phase stability, and the critical point are ideas that come from the study of a van der Waals gas, but herein it is apparent that these concepts are universal topological concepts that remain invariant with respect to deformations

Another choice would be to constrain the envelope such that it resides in a domain where the 1-form of Action is of Pfaff topological dimension 3. The physical system is closed, but it is not necessarily in equilibrium. An equilibrium or isolated physical system consists of a single topological component, or phase (the Cartan topology is a connected topology). Domains where the Pfaff topological dimension represent mixed phases imply more than 1 topological component, and are to be associated with regions where the Pfaff topological dimension is ≥ 3. The case of Pfaff dimension 3 would correspond to regions where the 3-form of Topological Torsion is not zero (the Cartan topology becomes a disconnected topology [1]). Such non-equilibrium domains correspond to the situation where the determinant of the 4×4 Jacobian matrix vanishes. That is, set $T_K = 0$, to obtain the (3D constrained) envelope $DISC_{(T_K=0)}$:

$$DISC_{(T_K=0)} = Z^2\{-4X^3Z + 18XZY + Y^2X^2 - 4Y^3 - 27Z^2\} \Rightarrow 0, \tag{6.94}$$

It is remarkable that the bracketed formula (in X, Y, Z coordinates) is precisely the Cardano cubic formula that separates the topological features of the generalized cubic equation. It is important to recognize that the development of a universal non-equilibrium van der Waals gas has not utilized the concepts of metric, connection, statistics, relativity, gauge symmetries, or quantum mechanics.

The Edge of Regression and Self Intersections

The envelope is smooth as long as $\partial^2\Theta/\partial\Psi^2 = \Theta_{\xi\xi} \neq 0$, and that the exterior 2-form, $d\Theta\hat{}d\Theta_\xi \neq 0$ subject to the constraint that the family parameter is a constant: $d\xi = 0$. If $d\Theta\hat{}d\Theta_\xi \neq 0$, but $\Theta_{\xi\xi} = 0$, then the envelope has a self intersection singularity. If $d\Theta\hat{}d\Theta_\xi = 0$, but $\Theta_{\xi\xi} \neq 0$, there is no self intersection, and no envelope.

If the envelope exists, further singularities are determined by the higher order partial derivatives of the Universal Phase function with respect to ξ.

$$\partial^2\Theta/\partial\xi^2 = \Theta_{\xi\xi} = 12\xi^2 - 6X_M\xi + 2Y_G, \tag{6.95}$$
$$\partial^3\Theta/\partial\xi^3 = \Theta_{\xi\xi\xi} = 24\xi - 6X_M. \tag{6.96}$$

When $\partial^3\Theta/\partial\xi^3 = \Theta_{\xi\xi\xi} \neq 0$, and $d\Theta\hat{}d\Theta_\xi\hat{}d\Theta_{\xi\xi} \neq 0$, the envelope terminates in a edge of regression. The edge of regression is determined by the simultaneous solution of $\Theta = 0, \Theta_\xi = 0$ and $\Theta_{\xi\xi} = 0$. Solving for ξ in $\Theta_{\xi\xi} = 0$ yields $Y_G = \xi(3X_M - \xi)$.

Reduced Phase Functions

Reconsider the reduced phase function, Φ, in terms of coordinate coefficients $\{g, a, k\}$, and its partial derivatives with respect to the family parameter, s :

$$\Phi = s^4 + gs^2 - as + k = 0, \tag{6.97}$$
$$\Phi_s = \partial\Phi/\partial s = 4s^3 + 2gs - a, \tag{6.98}$$
$$\Phi_{ss} = \partial^2\Phi/\partial s^2 = 12s^2 + 2g, \tag{6.99}$$
$$DISC\Phi = -27a^4 + 4(-g^2 + 36k)ga^2 + 16k(4k - g^2)^2. \tag{6.100}$$

The reduced formula is more tractable for, if the family parameter is fixed, then the equation represents a implicit surface in the space of coordinates, $\{g, a, k\}$. A representation for this implicit surface $DISC\Phi = 0$ was given in the previous figure. It is an obvious deformation equivalent to the Gibbs function for a van der Waals gas. The edge of regression is given by the zero set of $\Phi_{ss} = 0$ or $g = -6s^2$. Using this value in $\Phi_s = 0$ permits a solution for a in terms of s. Using these values for a and g in $\Phi = 0$ gives the three components of a position vector $\mathbf{R}$ =$[-6s^2, -8s^3, -3s^4]$ in $\{g, a, k\}$ space for the edge of regression. The result for the edge of regression in the $g - a$ plane is plotted below:

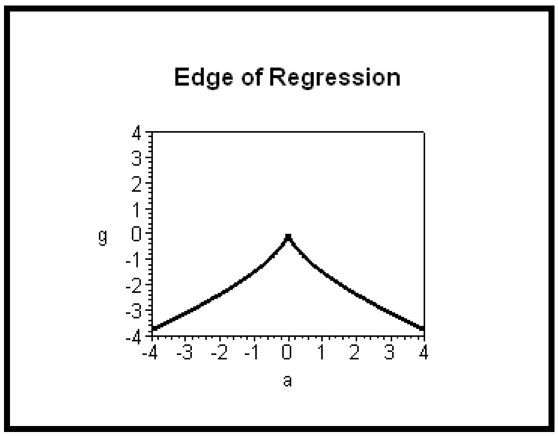


Figure 6.10 Edge of regression defects

The same function is plotted as the edge of regression for the Universal Swallowtail in the previous Figure.

Universal Phase Function Minimal Surfaces

For the minimal surface representation of the Gibbs surface for a van der Waals gas, the edge of regression defines the Spinodal line of ultimate phase stability. The edge of regression is evident in the Swallowtail figure (Figure 2.1) describing the Gibbs function for a van der Waals gas. If $\Theta_{\xi\xi} = 0$, then for $X_M = 0$ the envelope has a self intersection. It follows from $\Theta_{\xi\xi} = 0$, that $\xi^2 = -Y_G/6$, which when substituted into

$$\Theta_\xi = 4\xi^3 + 2Y_G\xi - Z_A \Rightarrow 0, \tag{6.101}$$

yields the

$$\textbf{Universal } (\mathbf{X_M}\textbf{=0}) \textbf{ Gibbs Edge of Regression : } Z_A^2 + Y_G^3(8/27) = 0, \tag{6.102}$$

which defines the Spinodal line, of the *minimal* surface representation for a universal non-equilibrium van der Waals gas, in terms of "similarity" coordinates.

Within the swallowtail region the "Gibbs" surface has 3 real roots; outside the swallowtail region there is a unique real root. The edge of regression furnished by the Cardano function defines the transition between real and imaginary root structures. The details of the universal non-equilibrium van der Waals gas in terms of envelopes and edges of regression with complex molal densities or order parameters will be presented elsewhere. These systems are not equilibrium systems for the Pfaff dimension is not 2. Of obvious importance is the idea that a zero value for both Z_G and T_K are required to reduce the Pfaff dimension to 2, the necessary condition for an equilibrium system.

Ginsburg Landau Currents

With a change of notation ($\xi \Rightarrow \Psi$), the Universal Phase function can be solved for the determinant of the Jacobian matrix, which is equal to the similarity invariant T_K,

$$T_K = -\{\Psi^4 - X_M\Psi^3 + Y_G\Psi^2 - Z_A\Psi\}. \tag{6.103}$$

The similarity invariant T_K represents the determinant of the Jacobian matrix. All determinants are in effect N-forms on the domain of independent variables. All N-forms can be related to the exterior differential of some (N-1)-form or current, J. Hence

$$dJ = T_K\Omega_4 = (div\mathbf{J} + \partial\rho/\partial t)\Omega_4 = -(\Psi^4 - X_M\Psi^3 + Y_G\Psi^2 - Z_A\Psi)\Omega_4. \tag{6.104}$$

For currents of the form

$$\mathbf{J} = grad\ \Psi, \tag{6.105}$$

$$\rho = \Psi, \tag{6.106}$$

the Universal Phase function generates the universal Ginsburg Landau equations

$$\nabla^2\Psi + \partial\Psi/\partial t = -(\Psi^4 - X_M\Psi^3 + Y_G\Psi^2 - Z_A\Psi). \tag{6.107}$$

6.0.10 Singularities as defects of Pfaff dimension 3

The family of hypersurfaces can be topologically constrained such that the topological dimension is reduced, and/or constraints can be imposed upon functions of the similarity variables forcing them to vanish. Such regions in the four-dimensional topological domain indicate topological defects or thermodynamic changes of phase. It is remarkable that for a given 1-form of Action there are an infinite number rescaling functions, λ, such that the Jacobian matrix $\left[\mathbb{J}_{jk}^{scaled}\right] = [\partial(A/\lambda)_j/\partial x^k]$ is singular (has a zero determinant). For if the coefficients of any 1-form of Action are rescaled by a divisor generated by the Holder norm,

$$\textbf{Holder Norm}: \ \lambda = \{a(A_1)^p + b(A_2)^p + c(A_3)^p + e(A_4)^p\}^{m/p}, \tag{6.108}$$

then the rescaled Jacobian matrix

$$\left[\mathbb{J}_{jk}^{scaled}\right] = [\partial(A/\lambda)_j/\partial x^k] \tag{6.109}$$

will have a zero determinant, for any index p, any set of isotropy or signature constants, a, b, c, e, if the homogeneity index is equal to unity: $m = 1$. This homogeneous constraint implies that the similarity invariants become projective invariants, not just equi-affine invariants. Such species of topological defects can have the image of a three-dimensional implicit characteristic hypersurface in space-time:

$$\textbf{Singular hypersurface in 4D}: \ \det[\partial(A/\lambda)_j/\partial x^k] \Rightarrow 0. \tag{6.110}$$

The singular fourth order Cayley-Hamilton polynomial of $[\mathbb{J}_{jk}]$ then will have a cubic polynomial factor with one zero eigenvalue.

For example, consider the simple case where the determinant of the Jacobian vanishes: $T_K \Rightarrow 0$. Then the Phase function becomes (for Pfaff Dimension 3):

$$\textbf{Universal Equation of State} \tag{6.111}$$

$$\Theta(\{X_M, Y_G, Z_A, T_K = 0\};\ \xi) \tag{6.112}$$

$$= \ \xi(\xi^3 - X_M\xi^2 + Y_G\xi - Z_A) \Rightarrow 0. \tag{6.113}$$

The space has been topologically reduced to three dimensions (one eigenvalue is zero), and the zero set of the resulting singular Universal Phase function becomes a universal cubic equation that is homeomorphic to the cubic equation of state for a van der Waals gas.

When the rescaling factor λ is chosen such that $p = 2, a = b = c = 1, m = 1$, then the Jacobian matrix, $[\mathbb{J}_{jk}]$, is equivalent to the "Shape" matrix for an implicit hypersurface in the theory of differential geometry (see Vol1[1]). Recall that the homogeneous similarity invariants can be put into correspondence with the linear Mean curvature, $X_M \Rightarrow C_M$, the quadratic Gauss curvature, $Y_G \Rightarrow C_G$, and the cubic Adjoint curvature, $Z_A \Rightarrow C_A$, of the hypersurface. The characteristic cubic polynomial can be put into correspondence with a nonlinear extension of an ideal gas *not necessarily* in an equilibrium state.

6.0.11 The Adjoint Current and Topological Spin

From the singular Jacobian matrix, $\left[\mathbb{J}_{jk}^{scaled}\right] = [\partial(A/\lambda)_j/\partial x^k]$, it is always possible to construct the Adjoint matrix as the matrix of cofactors transposed:

$$\text{Adjoint Matrix}: \ \left[\widehat{\mathbb{J}}^{kj}\right] = adjoint\left[\mathbb{J}_{jk}^{scaled}\right]. \tag{6.114}$$

When this matrix is multiplied times the rescaled co-vector components, the result is the production of an adjoint current,

$$\text{Adjoint current}: \ \left|\widehat{\mathbf{J}}^k\right\rangle = \left[\widehat{\mathbb{J}}^{kj}\right] \circ \left|\mathbf{A}_j/\lambda\right\rangle. \tag{6.115}$$

It is remarkable that the construction is such that the Adjoint current 3-form, if not zero, has zero divergence globally:

$$\widehat{J} = i(\widehat{\mathbf{J}}^k)\Omega_4 \quad (6.116)$$

$$d\widehat{J} = 0. \quad (6.117)$$

From the realization that the Adjoint matrix may admit a non-zero globally conserved 3-form density, or current, $\widehat{J}$, it follows abstractly that there exists a 2-form density of "excitations", $\widehat{G}$, such that

$$\textbf{Adjoint current}: \widehat{J} = d\widehat{G}. \quad (6.118)$$

$\widehat{G}$ is not uniquely defined in terms of the adjoint current, for $\widehat{G}$ could have closed components (gauge additions $\widehat{G}_c$, such that $d\widehat{G}_c = 0$), which do not contribute to the current, $\widehat{J}$.

From the topological theory of electromagnetism [238] [232] there exists a fundamental 3-form, $A\hat{\ }G$, defined as the "topological Spin" 3-form,

$$\textbf{Topological Spin 3-form} : A\hat{\ }G. \quad (6.119)$$

The exterior differential of this 3-form produces a 4-form, with a coefficient energy density function that is composed of two parts:

$$d(A\hat{\ }G) = F\hat{\ }G - A\hat{\ }\widehat{J}. \quad (6.120)$$

The first term is twice the difference between the "magnetic" and the "electric" energy density, and is a factor of 2 times the Lagrangian usually chosen for the electromagnetic field in classic field theory:

$$\textbf{Lagrangian Field energy density}: F\hat{\ }G = \mathbf{B} \circ \mathbf{H} - \mathbf{D} \circ \mathbf{E}. \quad (6.121)$$

The second term is defined as the "interaction energy density"

$$\textbf{Interaction energy density}: A\hat{\ }\widehat{J} = \mathbf{A} \circ \widehat{\mathbf{J}} - \rho\phi. \quad (6.122)$$

For the special (Gauss) choice of integrating denominator, λ with ($p = 2, a = b = c = 1, m = 1$) it can be demonstrated that the Jacobian similarity invariants are equal to the classic Mean, Gauss, and Adjoint curvatures:

$$\{X_M, Y_G, Z_A, T_K\} \Rightarrow \{4M_{(linear)}, 6G_{(quadratic)}, 4A^{\star}_{(cubic)}, 0\}. \quad (6.123)$$

It can be demonstrated (with the use of Maple) that the interaction density is exactly equal to the Adjoint curvature energy density [235]:

$$\textbf{Interaction energy } A\hat{\ }\widehat{J} = 4A^{\star}_{(cubic)}\Omega_4 \quad \text{(Adjoint Cubic Curvature)}. \quad (6.124)$$

The conclusion reached is that a non-zero interaction energy density implies the thermodynamic system is not in an equilibrium state (but it could be in a "steady state" far from equilibrium).

However, it is always possible to construct the 3-form, $\widehat{S}$:

$$\textbf{Topological Spin 3-form} : \widehat{S} = A\hat{}\widehat{G}. \tag{6.125}$$

The exterior differential of this 3-form leads to a cohomological structural equation similar to the first law of thermodynamics, but useful for non-equilibrium systems. This result, now recognized as a statement applicable to non-equilibrium thermodynamic processes, was defined as the "Intrinsic Transport Theorem" in 1969 [199] :

$$\begin{aligned} \textbf{Intrinsic Transport Theorem} &: \\ \textbf{(Spin)} \quad d\widehat{S} &= F\hat{}\widehat{G} - A\hat{}\widehat{J}, \qquad (6.126) \\ \textbf{First Law of Thermodynamics} &: \\ \textbf{(Energy)} \quad dU &= Q - W. \qquad (6.127) \end{aligned}$$

If one considers a collapsing system, then the geometric curvatures increase with smaller scales. If Gauss quadratic curvature, $6G_{(gauss\_quadratic)}$, is to be related to gravitational collapse of matter, then at some level of smaller scales a term cubic in curvatures, $4A^{\star}_{(adjoint\_cubic)}$, would dominate. It is conjectured that the cubic curvature produced by the interaction energy effect described above could prevent the collapse to a black hole. Cosmologists and relativists apparently have ignored such cubic curvature effects associated with non-equilibrium thermodynamic systems.

6.0.12 Non-Equilibrium Examples

In order to demonstrate content to the thermodynamic topological theory, two algebraically simple examples are presented below. (The algebra can become tedious for the rescaled Action 1-forms. Maple programs can be found in Vol 7 [1].

See http://www22.pair.com/csdc/maple/hopfphase.mw
and http://www22.pair.com/csdc/maple/holder4d.mw .

The first corresponds to a Jacobian characteristic equation that has a cubic polynomial factor, and hence can be identified with a van der Waals gas. The second example exhibits the features associated with a Hopf bifurcation, where the characteristic equation has a quadratic factor with two pure imaginary roots, and two null roots.

Example 1: van der Waals properties from rotation and a contraction

In this example, the Action 1-form is presumed to be of the form,

$$A_0 = \Omega(ydx - xdy) - \Psi(tdz + zdt). \tag{6.128}$$

The 1-form of Potentials depends on the (constant) coefficients Ω and Ψ. The similarity invariants of the Jacobian matrix , $\mathbb{J}\left[(A_0)\right]$, formed from A_0, are:

$$\begin{aligned} \textbf{Based on} &\quad : \quad \textbf{the 1-form } \mathbf{A}_0 \\ X_M &= 0, && (6.129) \\ Y_G &= \Omega^2 - \Psi^2, && (6.130) \\ Z_A &= 0, && (6.131) \\ T_K &= -\Omega^2\Psi^2. && (6.132) \end{aligned}$$

The eigenvalues of the Jacobian matrix are global constants: $\pm\Psi$, $\pm\sqrt{-1}\Omega$.

If the 1-form of Action is rescaled by the quadratic Gauss map,

$$\textbf{Gauss Map} \qquad (6.133)$$

$$\lambda_{Gauss} = \sqrt{a(\Omega x)^2 + a(\Omega y)^2 + (\Psi t)^2 + (\Psi z)^2} \qquad (6.134)$$

$$A_0 \Rightarrow A = A_0/\lambda_{Gauss}, \qquad (6.135)$$

then the Jacobian matrix becomes the equivalent of the shape matrix, and the similarity invariants of the shape matrix are related to the *average* curvatures of the implicit Phase hypersurface, in a space of 1 less dimension.

The computations for the given 1-form of Action yield the results:

$$\begin{aligned} \textbf{Based on} &\quad : \quad \textbf{the 1-form } A/\lambda_{Gauss} \\ \text{Linear Mean curvature} &\quad : \quad C_M = \Psi^3 tz/(\lambda^3), && (6.136) \\ \text{Quadratic Gauss curvature} &\quad : \quad C_G = \Omega^2\Psi^2\{-a(x^2+y^2) \\ &\qquad +(z^2+t^2)\}/(\lambda^4), && (6.137) \\ \text{Cubic Adjoint curvature} &\quad : \quad C_A = 2\Omega^2\Psi^3 tz/\lambda^5, && (6.138) \\ \text{Quartic Curvature :} &\quad\quad C_K = 0. && (6.139) \end{aligned}$$

The Determinant (4th order curvature) vanishes by construction of the renormalization in terms of the Gauss map. This null result does not mean the Pfaff dimension of A is less than 4 globally, but the constraint defines a singular set upon which there is a closed Current. This current is the Adjoint current of the previous section.

The Gauss map permits the construction of the "Adjoint conserved current", which combined with the components of the Action 1-form yield an interaction energy density exactly equal to the cubic curvature C_A. The interaction density depends upon the sign of the scalar potential (d(t) coefficient) in the 1-form of Action. It is classic to choose the negative sign to agree with the definitions of EM field intensities in terms of the vector and scalar potentials. To be consistent the sign of the coefficient, e, in λ_{Gauss} must be chosen properly to make the interaction energy C_A consistent. In the classic case, $C_A = 0$, $C_X = 0$, $e = -1$.

$$\begin{aligned} \text{Adjoint Current } \mathbf{J}_s &= \left(-\Omega^2\Psi^2[x,y,z,t]\right)/(\lambda^4), && (6.140) \\ div4(\mathbf{J}_s) &= 0 && (6.141) \end{aligned}$$

The rescaled Jacobian matrix has 1 zero eigenvalue and 3 non-zero eigenvalues. Hence, the cubic polynomial will yield an interpretation as a van der Waals gas. The Adjoint current, $\mathbf{J}_s$, represents a contraction in space-time, while the flow associated with the 1-form has a rotational component, Ω, about the z-axis.

However, the rescaled 1-form A is still of Pfaff dimension 4 and has a non-zero topological torsion 3-form and a non-zero topological parity 4-form:

$$\text{Torsion vector } \mathbf{T}_4 = 2\Omega\Psi \cdot [0, 0, -z, t]/\lambda^2, \tag{6.142}$$

$$\text{Parity 4-form } dA\hat{\ }dA = (4\Psi^3\Omega(z^2 - t^2)/\lambda^4)\ \{dx\hat{\ }dy\hat{\ }dz\hat{\ }dt\}. \tag{6.143}$$

Example 2: A Hopf 1-form linkage

In this example,the Hopf 1-form is presumed to be of the form

$$A_{Hopf} = \Omega(ydx - xdy) + \Gamma(tdz - zdt). \tag{6.144}$$

The 1-form of Potentials depends on the chirality coefficients Ω and Γ. There are two cases corresponding to left-handed and right-handed "polarizations": $\Omega = \Gamma$ or $\Omega = -\Gamma$. The results of the topological theory are:

$$\textbf{Based on} : \textbf{the 1-form } A_{Hopf}$$

$$X_M = 0, \tag{6.145}$$

$$Y_G = \Omega^2 + \Gamma^2, \tag{6.146}$$

$$Z_A = 0, \tag{6.147}$$

$$T_K = \Omega^2\Gamma^2, \tag{6.148}$$

$$\text{Eigenvalues} : \pm\sqrt{-1}\Gamma, \pm\sqrt{-1}\Omega, \tag{6.149}$$

$$\text{Torsion vector } \mathbf{T}_4 = 2\Omega\Gamma[x, y, z, t], \tag{6.150}$$

$$\text{Parity 4-form} = 8\Omega\Gamma dx\hat{\ }dy\hat{\ }dz\hat{\ }dt. \tag{6.151}$$

The 4 eigenvalues come in two imaginary pairs. The elements of each pair are equal and opposite in sign.

What is remarkable for this Action 1-form is that both the linear similarity invariant X_M and the cubic similarity invariant Z_A of the implicit phase hypersurface in 4D vanish, for any real values of Ω or Γ. The quadratic similarity invariant is non-zero, positive real and is equal to $\Omega^2 + \Gamma^2$. The quartic similarity invariant T_K is non-zero, positive real and is equal to $\Omega^2\Gamma^2$. The 1-form also supports a Topological Torsion current, $i(\mathbf{T}_4)dx\hat{\ }dy\hat{\ }dz\hat{\ }dt$, with a non-zero divergence, $8\Omega\Gamma$.

However, if the 1-form A_{Hopf} is scaled by the Gauss map, the resulting Hopf implicit surface is a single 4D imaginary *minimal* two-dimensional hyper surface in 4D and has two non-zero imaginary curvatures, but a positive Gauss curvature! This is a most unusual result, for the usual 2D minimal surface has equal and opposite real curvatures, with a negative Gauss curvature.

$$\begin{aligned}
\textbf{Based on} &: \textbf{the 1-form } A_{Hopf}/\lambda_{Gauss} \\
\lambda &= \sqrt{a(\Omega x)^2 + a(\Omega y)^2 + e(\Gamma z)^2 + e(\Gamma t)^2}, && (6.152) \\
r &= \sqrt{a(x^2+y^2)+e(z^2+t^2)}, && (6.153) \\
\text{Linear Mean curvature} &: C_M = 0, && (6.154) \\
\text{Quadratic Gauss curvature} &: C_G = +\Omega^2\Gamma^2\{r^2\}/(\lambda^4), && (6.155) \\
\text{Cubic Adjoint curvature} &: C_A = 0, && (6.156) \\
\text{Quartic Curvature} &: C_K = 0, && (6.157) \\
\text{Eigenvalues} &: [0, 0, +\sqrt{-1}, -\sqrt{-1}](\Omega\Gamma r/\lambda^2). && (6.158)
\end{aligned}$$

Strangely enough the charge-current density induced by the Adjoint current is not zero, but it is proportional to the Topological Torsion vector that generates the 3-form $A\hat{}F$.

$$\begin{aligned}
\text{Adjoint Current} &: \mathbf{J}_s = (\Omega^2\Gamma^2[x,y,z,t])\ /(r^4), && (6.159) \\
\text{interaction energy density:} &\quad \mathbf{A}\circ\mathbf{J}_s - \rho\phi = C_A = 0. && (6.160)
\end{aligned}$$

The topological Parity 4-form is not zero (Parity = $(4\Omega\Gamma/\lambda^2)\{dx\hat{}dy\hat{}dz\hat{}dt\}$ and depends on the sign of the coefficients Ω and Γ. In other words the 'handedness' of the different 1-forms determines the orientation of the normal field with respect to the implicit surface. It is known that a process described by a vector proportional to the topological torsion vector in a domain where the topological parity is non-zero $4\Omega\Gamma/(x^2+y^2+z^2+t^2) \neq 0$ is thermodynamically irreversible.

However, the rescaled 1-form A is still of Pfaff dimension 4 and has a non-zero topological torsion 3-form generated by a torsion vector, $\mathbf{T}_4$, and a non-zero topological parity 4-form:

$$\begin{aligned}
\text{Torsion vector } \mathbf{T}_4 &= -2\Omega\Gamma\ \cdot [x,y,z,t]/\lambda^2, && (6.161) \\
\text{Parity 4-form } dA\hat{}dA &= (-4\Gamma\Omega/\lambda^2)\ \{dx\hat{}dy\hat{}dz\hat{}dt\}. && (6.162)
\end{aligned}$$

6.1 The Cosmological van der Waals Gas

The concepts of a universal phase function generated from a 1-form of Action A for a non-equilibrium system (Pfaff Topological dimension > 2) will be applied to the construction of a cosmological model. It will be demonstrated how such a universal non-equilibrium van der Waals gas offers alternate explanations for the properties of our cosmological universe. In particular, the current "unexplained" concepts of dark energy and dark matter have a more classical foundation than is currently appreciated. Negative pressures are an enigma to many physicists, but are features well recognized by engineers who understand steam engines. Current relativity theories of gravity are based upon symmetric features of space-time and in a sense "overlook" the antisymmetric features of non-equilibrium thermodynamic systems.

From 1974 to the present, it has been the preoccupation of the present author to investigate the physical applications of irreversible topological evolution [200], [199], [209], [220]. This topic goes beyond the diffeomorphic equivalences of tensors, which can represent linearly connected processes that preserve the topology of the initial state during a transition to the final state, but which *cannot* be used to describe, deterministically, the thermodynamic irreversible processes of every day macroscopic reality. It became evident (due to inherent linearity restrictions) that the tensor analysis was inadequate to study irreversible topological evolution [207]. However, it was also noted that certain progress could be made by using methods inherent in Cartan's theory of exterior differential systems. In the theory of electromagnetism, it was known that ubiquitous tensor tools of metric and affine connection are useful, but not necessary, concepts [292]. Electromagnetism is indeed a topological theory, and has a universal expression in terms of two topological constraints on a set of exterior differential forms.

$$\text{Maxwell-Faraday:} \qquad F - dA = 0, \tag{6.163}$$

$$\text{Maxwell-Ampere:} \qquad J - dG = 0. \tag{6.164}$$

The resulting PDE's are covariant in form for any coordinate frame and in any number of dimensions greater than three [238].

The theory of thermodynamics is also a topological theory [278], independent from scales and deformations. Indeed the first law of thermodynamics is best understood as a topological constraint of cohomology, similar to the topological constraints that can be used to formulate Electromagnetism. The first law is a statement that the non-exact 1-form of heat, Q, minus the non-exact 1-form of work, W, is a perfect differential, dU :

$$\text{First Law: } Q - W = dU. \tag{6.165}$$

To explain irreversible evolutionary processes, Lagrangian extremal methods are to be replaced by Cartan's Magic formula of continuous topological evolution acting upon physical systems that admit description in terms of exterior differential forms [222]. Exterior differential forms can carry global, topological information, and their use has led to definite progress in the understanding of thermodynamic irreversible turbulent flow, including the evolutionary creation of topological defects, or coherent structures, in irreversible dissipative hydrodynamic processes. These macroscopic continuous "condensation" concepts have both a micro and a cosmological realization. One of the most vivid experimental examples of such topological structures is given by the creation of Falaco Solitons in a fluid surface of (density) discontinuity [213]. The visual evolutionary appearance of the swimming pool experiments leads to a suggestion that the creation of almost flat spiral arm galaxies from a non-linear dissipative cosmological fluid is also a feature of continuous topological evolution.

Topological evolution can take place by both continuous (cutting) and discontinuous (pasting) processes. The improper linear transformations (determinant = -1) of tensor analysis (such as mirror reflections) are not continuous about the identity. However, if the concept of tensor linear uniqueness is replaced by multivalued (but continuous about the identity) spinor transformations, the linear discontinuous but unique concepts admit equivalent descriptions in terms of continuous but non-unique topological evolution. A fundamental theme utilized herein is to replace the idea of discontinuous but unique with the concept of non-uniqueness but continuous. The spirit of the idea is similar to the extension from the real line to the complex plane, where a (zero) point obstacle on the real line (yielding a discontinuity between positive or negative decrements) can be circumvented by a continuous (but multi-valued) right-handed or left-handed circuit about the (zero) obstacle in the complex plane. Note that a hole can be produced in a deformable disc by discontinuously cutting a hole and separating the parts, or by deforming the disc into the shape of the letter C and then (continuously) pasting the ends together. Two or more holes can be formed by discontinuously cutting a second hole, or by squeezing one hole to form the outline of a figure 8, and continuously pasting together the central region.

Remarkably, the fact the exterior differential forms could be homogeneous and evolve in a self similar manner permitted fractal structures to be admitted to the possible process descriptions of continuous topological evolution. The fact that the exterior differential systems may not be uniquely integrable (hence not in equilibrium) and yet could evolve into long lived states far from equilibrium became a mathematical fact, not just a philosophical dream. Moreover, it became possible to distinguish between chaos (which can be thermodynamically reversible) and turbulence (which is thermodynamically irreversible). Indeed, it became evident that thermodynamic irreversibility was an artifact of topological dimension 4 [231].

Irreversible Processes and Topological Bulk Viscosity

When the Action for a physical system is of Pfaff dimension 4, there exists a unique direction field, $\mathbf{T}_4$, defined as the Topological Torsion 4-vector, that can be evaluated *entirely* in terms of those component functions of the 1-form of Action which define the physical system. To within a factor, this direction field† has the four components

†A direction field is defined by the components of a vector field which establish the "line of action" of the vector in a projective sense. An arbitrary factor times the direction field defines the same projective line of action, just reparameterized. In metric based situations, the arbitrary factor can be interpreted as a renormalization factor.

of the 3-form $A\hat{}dA$, with the following properties:

Properties of the Topological Torsion vector T_4

$$\begin{aligned}
i(\mathbf{T}_4)\Omega_4 &= A\hat{}dA, && (6.166)\\
W &= i(\mathbf{T}_4)dA = \sigma\ A, && (6.167)\\
U &= i(\mathbf{T}_4)A = 0, && (6.168)\\
L_{(\mathbf{T}_4)}A &= \sigma\ A, && (6.169)\\
Q\hat{}dQ &= L_{(\mathbf{T}_4)}A\hat{}L_{(\mathbf{T}_4)}dA = \sigma^2 A\hat{}dA \neq 0, && (6.170)\\
dA\hat{}dA &= (2!)\ \sigma\ \Omega_4. && (6.171)
\end{aligned}$$

Note that a $\mathbf{T}_4$ process is locally adiabatic, but not reversible.

Hence, evolution in the direction of $\mathbf{T}_4$ is thermodynamically irreversible, when $\sigma \neq 0$ and A is of Pfaff topological dimension 4. The kernel of this vector field is defined as the zero set under the mapping induced by exterior differentiation. In engineering language, the kernel of this vector field are those point sets upon which the divergence of the vector field vanishes. The Pfaff topological dimension of the Action 1-form is 3 in the defect regions defined by the kernel of $\mathbf{T}_4$. The coefficient σ can be interpreted as a measure of space-time volumetric expansion or contraction. It follows that both expansion and contraction processes (of space-time) are related to irreversible processes. It is here that contact is made with the phenomenological concept of "bulk" viscosity = $(2!)\sigma$. For symplectic systems of higher Pfaff dimension $m = 2n + 2 \geq 4$, the numeric factor becomes $(m/2)!$.) It is important to note that the concept of an irreversible process depends on the square of the coefficient, σ. It follows that both expansion and contraction processes (of space-time) are related to irreversible processes. It is tempting to identify σ^2 with the concept of entropy production.

Topological Evolution to Minimal Surfaces, Wakes and Spinors

During the period 1982 - 1995, [227], [229], [226], it also became apparent that long lived fluid dynamic wakes were related to minimal surfaces of tangential discontinuities. The argument was based on the fact that the dissipative Navier-Stokes equations could lead to long lived solutions, where the non-harmonic components of an initial velocity field would decay through viscous dissipation, leaving as residues, the harmonic components of the velocity field. This dissipative decay to long lived states far from equilibrium turns out to be a generic process of thermodynamic irreversibility [224]. The dissipative terms in the Navier-Stokes equations (neglecting compressibility) were proportional to the product of a viscosity coefficient times the vector Laplacian of the velocity field. As the harmonic components of the velocity field were precisely those components such that the vector Laplacian vanished, then no matter what the viscosity, the dissipation or decay of the harmonic components would be (almost) zero. Hydrodynamic wakes are essentially topological limit sets.

Experience with differential geometry brought to mind the notion that the generator of a minimal surface was a harmonic vector field. Therefore wakes and minimal surfaces must be related concepts. The fact that isotropic null vectors were related to minimal surfaces was known to me before 1986 when I became aware of Falaco Solitons, but the concept was reinforced when the Falaco Spots projected to circles on the pool floor. Struik had taught me that such an effect was related to refraction from a minimal surface.

A more recent re-reading (in the spring of 2000) of Cartan's book on Spinors [45] (and Chandrasekhar's book on Black Holes [48]) lead to the thought that minimal surfaces and spinors were also related ideas – via the concept of an isotropic complex null vector. It is remarkable to me that both Cartan and Chandrasekhar do not mention the fact that an isotropic (complex null) vector is related to the generator of a Minimal Surface [180]. This is surprising to me, as Cartan was a differential geometer who knew about minimal surfaces. Cartan defined the "Spinor" as a *mapping* of a complex pair, $\{\alpha, \beta\}$ to a special three component complex vector, $\Sigma = [\sigma 1, \sigma 2, \sigma 3]$, in such a way that its quadratic form (sum of squares of the three components) is zero: $(\sigma 1)^2 + (\sigma 2)^2 + (\sigma 3)^2 = 0.$. This relationship of Spinor maps to minimal surfaces has been and still is ignored by many other authors, as well as Cartan and Chandrasekhar. For example, a recent personal communication with Rindler (2000) also indicates that he also was not aware of the connection of these ideas. Evidently the idea of connecting Spinors and minimal surfaces was noticed by Dennis Sullivan (the topologist) about 1989. This reference I found (after I had stumbled on the idea independently) by an internet search which yielded the more recent article by R. Kusner and N. Schmitt [134].

There is another even earlier publication (1986) due to Paul Budinich and his collaborators [36], relating spinors and minimal surfaces. However, I did not become aware of the Budinich article until much later (2004) and did not study his works until just recently (2006). Budinich champions the idea that Spinors are useful objects at all physical scales (an idea that I support), but he apparently missed the idea (in his pioneering work) that eigendirection fields (with non-zero eigenvalues) of antisymmetric matrices are pure Spinors. This relationship between Spinors and antisymmetric 2-forms is studied and utilized throughout this series of monographs.

Remark 26 *The fundamental new idea is that Spinors are necessary components of thermodynamic processes which involve topological fluctuations, chaos and induce thermodynamic irreversibility in non-equilibrium systems.*

Indeed, it would appear that many physicists and most engineers are not aware of the connection between spinors and minimal surfaces, and also their relationship to wakes and tangential discontinuities. The concept of a minimal surface yields an interesting and useful physical interpretation of spinors, especially as the interpretation does not depend explicitly upon quantum mechanical ideas, nor relativistic ideas, nor concepts of scale. The bottom line is that spinors have application to the engineering sciences at all scales, as well as to the microscopic world of Fermions and Bosons. Spinors behave a bit differently from tensors. Better said, the concept of spinors is more related to a continuous topological idea, and not a discontinuous geometrical idea.

To quote Cartan, [45] p.151

> *"With the geometric sense we have given to the word spinor it is* ***impossible*** *to introduce fields of spinors into the classical Riemannian technique; that is having chosen an arbitrary system of coordinates x^i for the space, it is impossible to represent a spinor by any finite number N of components, u_α, such that the u_α have covariant derivatives of the form,*
>
> $$u_{\alpha,i} = \partial u_\alpha / \partial x^i + \Lambda^\beta_{\alpha i} u^\beta, \tag{6.172}$$
>
> *where the $\Lambda^\beta_{\alpha i}$ are determinate functions of x^h."*

The problem that Cartan states above has to do with the lack of uniqueness for the covariant transplantation rule when the connection, $\Lambda^\beta_{\alpha i}$, admits affine torsion of the non-integrable variety: $\Lambda^\beta_{\alpha i} - \Lambda^\beta_{i\alpha} \neq 0$. In a Riemannian space with a given metric, the connection coefficients of "parallel" transport are uniquely determined in terms of the Christoffel Symbols. Tensors restricted by neighborhood linearity and the General Linear group admit discrete (discontinuous but unique) transplantation laws about the identity. Spinors, on the other hand, are associated with a certain amount of multi-valuedness, and admit transplantation laws that are continuous about the identity, but are not uniquely defined. When the connection admits affine torsion, there are (at least) two methods of transplantation relating to "right-handed" or "left-handed" spinors. The multi-valuedness is also a property to be associated with systems that are not *uniquely* integrable in the Frobenius sense, and are characteristic of Huygen envelopes in wave propagation, Cherenkov radiation, and polarization states in Electromagnetism.

The bottom line is that Spinors permit a continuous but not unique evolution about the identity that is related to the unique but discontinuous linear tensor Vector transformations of negative determinant.

6.2 The Hopf Map, Spinors, and Minimal Surfaces

There is a close relationship between the Hopf map, minimal surfaces and Spinors. However, the historical lack of reference to these facts indicates that the relationship of Spinor Maps to the Hopf map and minimal surfaces has been ignored by many researchers. Recall that the Hopf map is a (non-linear) map from a vector of four components to a vector (the Hopf vector) of three components, such that the sum of squares of the three components is the square of the sum of squares of the four components. The map is ambiguous to within a sign (plus or minus one). If the components of the Hopf vector in 3D-space are presumed to be the dimensionless ratios (x/ct,y/ct/,z/ct), then the Hopf map can be viewed as a map from R4 to a projective 3D-space. Fixing the value of the sum of squares of the four components to a constant (say unity) generates the equation of the light cone in R4. There are three versions of the (real) Hopf vector, all with the same value for the sum of squares, which can be arranged such that they are mutually orthogonal. The implication is that there are at least three distinct constraints that can represent the light cone.

The correspondence between the Spinor map and the Hopf map will be investigated below, where it will be demonstrated that the rudimentary Cartan definition of a Spinor map is a complex three-dimensional "vector" whose real and imaginary components are both Hopf vectors. Each of the two Hopf vectors that make up the Cartan spinor are mutually orthogonal. As mentioned above, it is possible to construct three linearly independent Hopf vectors that are mutually orthogonal, and when these orthogonal Hopf vectors are combined in complex pairs, it is possible to construct six independent spinors. Integration of a given complex null vector leads to a complex "position" vector. The real and imaginary parts of the "position" vector separately describe a pair of (conjugate) minimal surfaces. In differential geometry, the null spinor is called an isotropic vector, and the pair of minimal surfaces are called conjugate surfaces. Linear combinations of the two conjugate components of the "position" vector also generate a minimal surface. The analog in physics can be described in terms of the optics of polarization. One extreme minimal surface is linear polarization, while the other extreme is circular polarization. A linear combination of the minimal surfaces is analogous to elliptical polarization. Each of the polarizations is ambiguous with respect to a sign (right handed vs. left handed, horizontal vs. vertical)

The Hopf map also appears embedded in the classical physics literature. It is latent in the classical optics theory of partial polarization [179]; in the classical electromagnetic theory of Bateman and Whittaker [16]; in the theory of hydrodynamic wakes [227], [229], [226], in the examples of electric wave singular solutions that give the appearance of breaking time reversal symmetry [222]. Yet these classic examples, and many others, do not focus attention on the fact that Hopf vector fields, so constructed in terms of ordered complex pairs, are related to spinors,. It took relativistic quantum theory to focus popularity on spinors, leading to a popular (but false) opinion that spinors were something of a "quantum mechanical"origin. It is now of interest to demonstrate the thermodynamic and cosmological importance of the Hopf map as encoded in terms of the "adjoint" 1-form. The adjoint Hopf 1-form, A_{Hopf}, is of Pfaff topological dimension 4, and has a non- zero Topological Torsion vector, $\mathbf{T}_4$, which corresponds to an expansion of space-time. Motion in the direction of $\mathbf{T}_4$ is thermodynamically irreversible. If the expanding universe was modeled in terms of A_{Hopf} then the system would be a turbulent non-equilibrium system of Pfaff dimension 4. However, the evolutionary processes could proceed to domains of Pfaff topological dimension 3, representing condensations, or coherent topological defect structures, (stars - galaxies) that would admit a non-equilibrium, but not dissipative, Hamiltonian evolution, modulo topological fluctuations. For these reasons it is of some importance to study Hopf maps and structures composed of Hopf maps

6.2.1 Hopf Maps and Hopf Vectors

A particularly useful and interesting 1-form of Action, A, that exhibits local stability can be deduced from the Hopf map. The Hopf map[‡] is a rather remarkable projective map from 4 to 3 (real or complex) dimensions that has topological properties related to links and braids and other forms of entanglement. As will be demonstrated, the adjoint 1-form to the Hopf map satisfies the criteria of Local Stability, and yet is not an integrable system. The 1-form deduced from the Hopf map is of Pfaff Topological Dimension 4, and admits irreversible dissipation for processes in the direction of the Topological Torsion vector.

Consider the map from R4[X,Y,Z,S] to R3[u,v,w] given by the formulas,

$$\mathbf{H}1 = [u1, v1, w1] = [2(XZ + YS), 2(XS - YZ), (X^2 + Y^2) - (Z^2 + S^2)]. \quad (6.173)$$

The components $[u1, v1, w1]$ can be considered as the velocity components of a dynamical system. These formulas define the format of a Hopf map. The 3-component Hopf vector $\mathbf{H}1$ is real, and has the property,

$$\mathbf{H}1 \cdot \mathbf{H}1 = (u1)^2 + (v1)^2 + (w1)^2 = (X^2 + Y^2 + Z^2 + S^2)^2. \quad (6.174)$$

Hence a real (and imaginary) four-dimensional sphere maps to a real three-dimensional sphere. If the functions $[u1, v1, w1]$ are defined as $[x/ct, y/ct, z/ct]$, then the 4D sphere $(X^2 + Y^2 + Z^2 + S^2)^2 = 1$, implies that the Hopf map formulas are equivalent to the 4D light cone.

In this section the complex notation will be used to define Hopf vectors, $\mathbf{H}1$, $\mathbf{H}2$, $\mathbf{H}3$. Consider the map from C2(α, β) to R3$(u1, v1, w1)$, as given by the formulas

$$\begin{aligned} \mathbf{H}1 &= [u1, v1, w1] \quad (6.175) \\ &= [\alpha \cdot \beta^* + \beta \cdot \alpha^*, i(\alpha \cdot \beta^* - \beta \cdot \alpha^*), \alpha \cdot \alpha^* - \beta \cdot \beta^*. \end{aligned}$$

The variables α and β can be viewed also as two distinct complex variables defining ordered pairs of the four variables $[X, Y, Z, S]$. For example, the classic format given above for $\mathbf{H}1$ can be obtained from the expansion,

$$\alpha = X + iY, \qquad \beta = Z + iS. \quad (6.176)$$

[‡]Also see Chapter 12 in [297]

Other selections for the ordered pairs of (X, Y, Z, S) (along with permutations of the three vector components) give distinctly different Hopf vectors. For example, the ordered pairs,

$$\alpha = X + iZ\,, \qquad \beta = Y + iS, \tag{6.177}$$

give

$$\begin{aligned} \mathbf{H}2 &= [u2, v2, w2] \\ &= [\alpha \cdot \beta^* + \beta \cdot \alpha^*, \alpha \cdot \alpha^* - \beta \cdot \beta^*, i(\alpha \cdot \beta^* - \beta \cdot \alpha^*)] \\ &= [2(YX - SZ), X^2 + Z^2 - Y^2 - S^2, -2(ZY + SX)]. \end{aligned} \tag{6.178}$$

which is another Hopf vector, a map from R4 to R3, but with the property that $\mathbf{H}2$ is orthogonal to $\mathbf{H}1$:

$$\mathbf{H}2 \cdot \mathbf{H}1 = 0. \tag{6.179}$$

Similarly, a third linearly independent orthogonal Hopf vector $\mathbf{H}3$ can be found

$$\begin{aligned} \mathbf{H}3 &= [u3, v3, w3] \\ &= [\alpha \cdot \alpha^* - \beta \cdot \beta^*, -(\alpha \cdot \beta^* + \beta \cdot \alpha^*), -i(\alpha \cdot \beta^* - \beta \cdot \alpha^*)] \\ &= [X^2 + Y^2 - Z^2 - S^2, -2(YX + SZ), 2(-ZX + SY)], \end{aligned} \tag{6.180}$$

such that

$$\mathbf{H}2 \cdot \mathbf{H}1 = \mathbf{H}3 \cdot \mathbf{H}2 = \mathbf{H}2 \cdot \mathbf{H}3 = 0, \tag{6.181}$$

$$\mathbf{H}1 \cdot \mathbf{H}1 = \mathbf{H}2 \cdot \mathbf{H}2 = \mathbf{H}3 \cdot \mathbf{H}3 = (X^2 + Y^2 + Z^2 + S^2)^2. \tag{6.182}$$

The three linearly independent Hopf vectors can be used as a basis of R3 excluding the origin.

Each Hopf vector can be differentiated with respect to the variables (X, Y, Z, S) forming a gradient field on R4. That is, the mapping functions (u, v, w) can be differentiated with respect to (X, Y, Z, S) to produce a set of three exact 1-forms. The matrix formed by the three rows of these 4-component gradient fields has an adjoint matrix of coefficients (composed of the matrix of cofactors) which may be adjoined to construct a 4×4 basis Frame for R4, excluding the origin. The exterior product, $d(u1)\hat{}d(v1)\hat{}d(w1)$, produces a 3-form, whose components are proportional to those of the adjoint matrix. These components may be used to construct a non-integrable "adjoint" 1-form, A. The three exact 1-forms and the non-integrable 1-form also can be used as a basis for the space. The exterior differentials of the basis frame produce the usual Cartan connection, but the Cartan connection so defined is not free of affine torsion. By this mechanism the differential structure of R4 as induced by the Hopf map is determined.

From another point of view, each of the four functions X, Y, Z, S can be considered as complex variables, so that the Hopf map has a realization from C4 to C3.

6.2.2 Isotropic Vectors and Minimal surfaces in 3D

Remark 27 *It is now (2011) recognized that Cartan Isotropic Spinors can be used to encoded the subsets of indistinguishable ingredients in Not-T0 topologies. Complex wave and diffusion equations have solution components that consist of statistical ensembles of such indistinguishable Spinor ingredients.*

Along with Cartan, define a rudimentary Spinor as an isotropic (or null) vector of three complex components, $\Sigma = [\sigma 1, \sigma 2, \sigma 3]$ such that

$$(\sigma 1)^2 + (\sigma 2)^2 + (\sigma 3)^2 = 0. \tag{6.183}$$

Next consider the two lemmas given in R. Osserman's book " A Survey of Minimal Surfaces" [180]

Lemma 8.1 (Osserman p 63) *Let D be a domain in the complex z-plane, $g(z)$ an arbitrary meromorphic function in D and $f(z)$ an analytic function in D having the property that at each point where $g(z)$ has a pole of order m, $f(z)$ has a zero of order at least 2m. Then the functions*

$$\sigma 1 = f(1-g^2)/2, \tag{6.184}$$
$$\sigma 2 = i\, f(1+g^2)/2, \tag{6.185}$$
$$\sigma 3 = \mp f g, \tag{6.186}$$

will be analytic in D and satisfy the equation of an "isotropic" null vector:

$$(\sigma 1)^2 + (\sigma 2)^2 + (\sigma 3)^2 = 0. \tag{6.187}$$

The only exception is for $\sigma 3 = 0$, $\sigma 1 = i\, \sigma 2$.

Next consider the theorem

Lemma 8.2 (Osserman) *Every simply connected minimal surface in E3 can be represented in the form of a position vector*

$$\mathbf{R}_{\text{real}} = [X1(u,v), Y1(u,v), Z1(u,v)], \tag{6.188}$$

where $\varpi = (u + i\,v)$. A conjugate minimal surface can be constructed from the imaginary components of the integral formulation,

$$\mathbf{R}_{\text{imag}} = [X2(u,v), Y2(u,v), Z2(u,v)]. \tag{6.189}$$

The position vector is computed from an isotropic complex 3-component vector by means of the formulas:

$$X1(u,v) = \mathrm{Re}\int \sigma 1(\varpi)d\varpi + constant\,, \tag{6.190}$$

$$X2(u,v) = \mathrm{Im}\int \sigma 1(\varpi)d\varpi + constant, \tag{6.191}$$

$$Y1(u,v) = \mathrm{Re}\int \sigma 2(\varpi)d\varpi + constant\,, \tag{6.192}$$

$$Y2(u,v) = \mathrm{Im}\int \sigma 2(\varpi)d\varpi + constant, \tag{6.193}$$

$$Z1(u,v) = \mathrm{Re}\int \sigma 3(\varpi)d\varpi + constant\,, \tag{6.194}$$

$$Z2(u,v) = \mathrm{Im}\int \sigma 3(\varpi)d\varpi + constant. \tag{6.195}$$

Either (real or imaginary) component of the complex position vector, or any linear combination of the components, may be used to induce a 2D real metric, whose Gaussian curvature is negative and whose mean curvature is zero. Hence it follows that a Cartan Spinor (isotropic 3-vector, Σ) generates (two) minimal surfaces.

It is unfortunate that the historic word isotropic is used to describe the "null " vector, for in engineering practice, the word isotropic is usually interpreted as meaning the same in all directions. Technically the word isotropic used for the null vector is correct, for no matter what direction the null vector points in C3, its quadratic form, as a sum of squares of the three components, is zero.

An equivalent formulation for an isotropic (null) vector was given by Cartan in terms of $\alpha(z)$ and $\beta(z)$, as follows.

$$\sigma 1 = \alpha^2 - \beta^2, \tag{6.196}$$

$$\sigma 2 = i\,(\alpha^2 + \beta^2), \tag{6.197}$$

$$\sigma 3 = \mp 2\alpha\beta. \tag{6.198}$$

The ambiguity in sign can be related to the concept of polarization.

Evidently D. Sullivan noticed that these formulas of Cartan could be related to minimal surfaces in 1989 (hence predates my own recent independent appreciation (2000) of this fact). The formulas can also be interpreted in terms of the sequence of maps from the 2D space $\{\varpi = u + iv\}$ to the 8D space $\{\alpha = X(\varpi) + iY(\varpi), \beta =$

$Z(\varpi) + iS(\varpi)\}$ to the 6D space $\{\sigma 1(\varpi), \sigma 2(\varpi), \sigma 3(\varpi)\}$. The quadratic form of an arbitrary vector on C3, $(\sigma 1)^2 + (\sigma 2)^2 + (\sigma 3)^2$, can be complex, real, or zero. However, the spinor construction given above always produces an *isotropic* or null vector: the quadratic form vanishes. The mapping described above is the original definition of a Cartan Spinor. A Cartan Spinor is in fact, not the pair of functions, $\alpha(\varpi)$ and $\beta(\varpi)$, but the map to the isotropic complex three component vector, Σ, such that

$$(\sigma 1)^2 + (\sigma 2)^2 + (\sigma 3)^2 = 0. \tag{6.199}$$

The isotropic (null) condition imposes two constraints on the 6D space of three complex variables reducing the dimension to a 4D space of two complex variables. In the examples below, for a simple choice of the functions α and β, the catenoid of revolution occurs as the real part of the integration and the helix is determined from the imaginary part of the integration. A linear combination of the two "conjugate" components is used to form the helicat, which is yet another minimal surface. Each of the functions defined above is ambiguous to a factor of ± 1. The mean curvature vanishes (the minimal surface condition) for all combinations of plus or minus signs.

As mentioned above, the real and imaginary parts of the minimal surface position vector correspond to extremes in "polarization". The interesting fact is that if ψ is a complex "constant" of the type $\psi = A\exp(i\theta)$, then each component of the complex position vector,

$$\mathbf{R} = A\exp(i\theta)[\mathbf{R}_{\text{real}} + i\mathbf{R}_{\text{imag}}], \tag{6.200}$$

also generates a minimal surface (of mixed polarization)

For example consider the position vector in 3D space parameterized by the two variables u and v.

$$\begin{aligned} \mathbf{R} &= [a\ sinh(v)cos(u) - b\ cosh(v)sin(u), && (6.201) \\ & \quad a\ sinh(v)sin(u) + b\ cosh(v)cos(u), && (6.202) \\ & \quad au + bv]. && (6.203) \end{aligned}$$

The space curve $\mathbf{R}$ generates a conjugate pair of minimal surface components for various choices of a and b. For $a = b = 0.5$, the minimal surface consists of two components which are conjugate pairs of opposite handedness. The conjugate pairs can be in a sense "mixtures" of catenoids and helices, and are termed helicoids. The helicoids can also be pure helices or pure catenoids. The situation is remindful of the concept of pure and mixed polarization states in electromagnetic waves. Whether the helicoid is right-handed or left-handed depends upon whether z is increasing or decreasing. In the first set of three Figures, z is increasing, and $a = b = 0.5$. In the second set of three Figures, $a = 0, b = 1$ and generates the helix structure. In the last set of three Figures, a=1,b=0 which generates the catenoid structure.

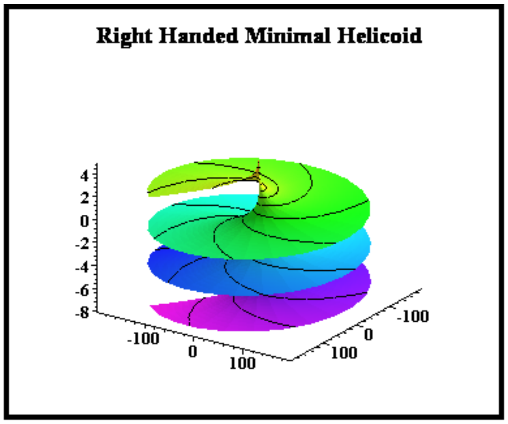


Figure 6.11

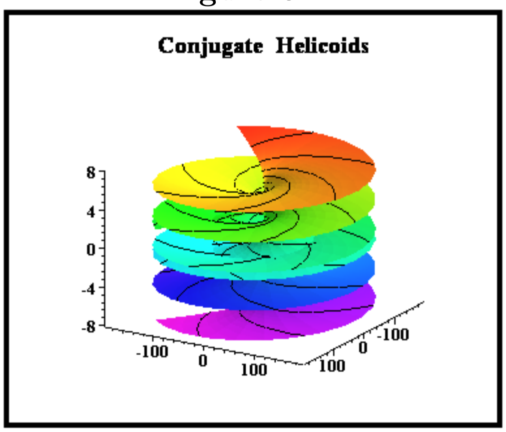


Figure 6.12

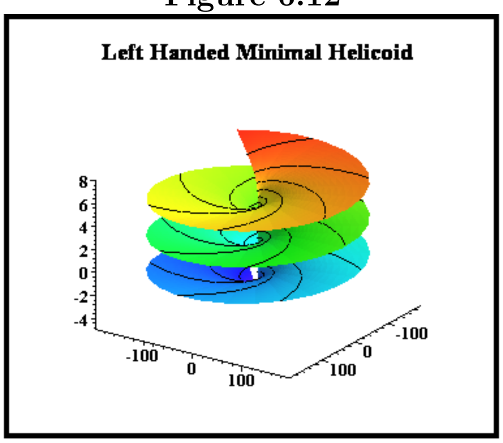


Figure 6.13
Chiral Pairs of Minimal Helicoids (Set 1)

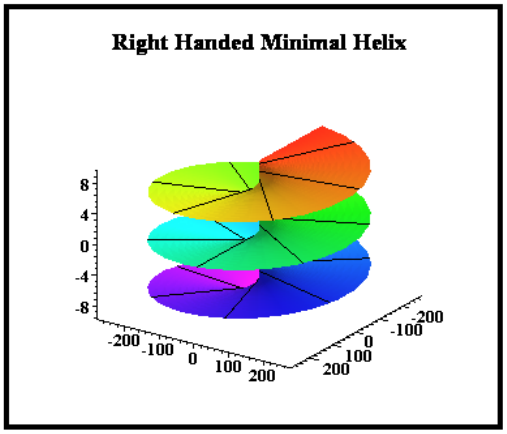


Figure 6.14

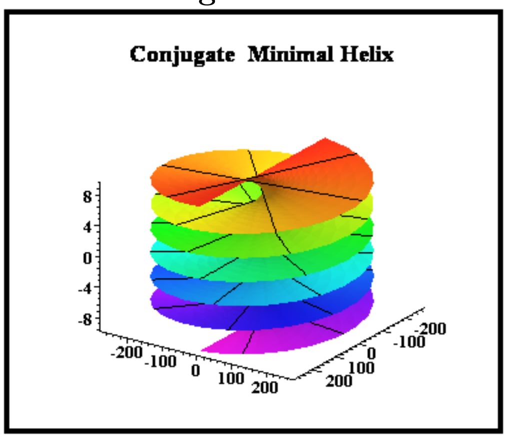


Figure 6.15

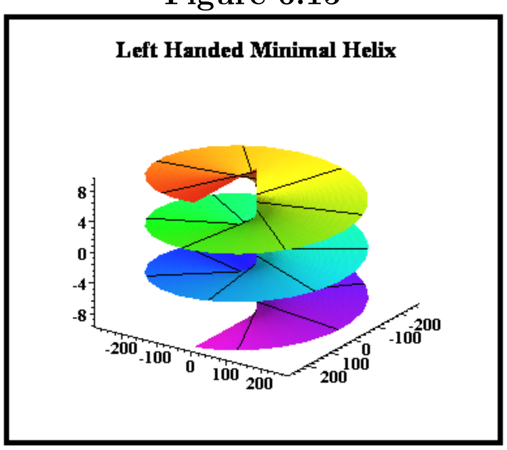


Figure 6.16
Chiral pairs of minimal Helices (Set 2)

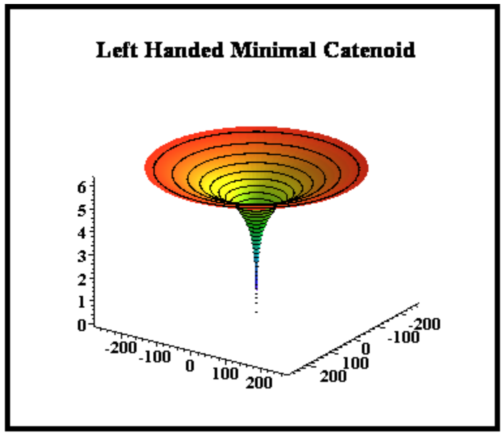


Figure 6.17

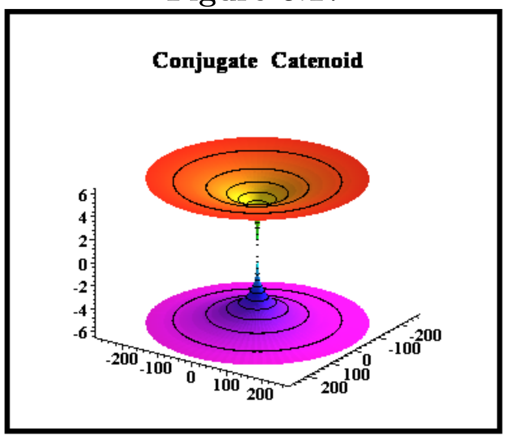


Figure 6.18

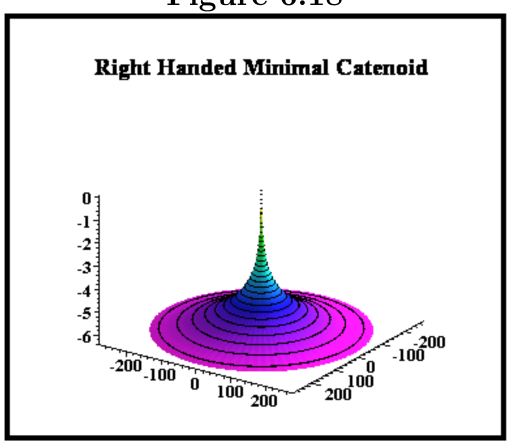


Figure 6.19
Chiral pairs of minimal catenoids (Set 3)

It is obvious that this last case of conjugate catenoids has deformable visual features of the Falaco Solitons.

6.2.3 Complex Curves

A theorem of Sophus Lie states that in 4D every complex holomorphic function generates a minimal surface. This is a rather remarkable result that has not been utilized fully in application to understanding space-time evolutionary processes. If an evolutionary process starts with a non-holomorphic representation and evolves into a holomorphic representation, then physically it would be expected that dissipative processes would be minimized, and tangential discontinuities (wakes) would be created. The normal field to a minimal surface is harmonic.

For Navier-Stokes like fluids, the viscous shear dissipation is a viscous coefficient times the vector Laplacian of the velocity field. As the vector Laplacian vanishes for a harmonic vector field it follows that such flows do not experience viscous dissipation due to shears. Consider vector fields that are composed of a harmonic part and a non-harmonic part. Viscous dissipation will cause the non-harmonic part to decay. What is left is the Harmonic part, which generates a minimal surface as the (measurable) wake.

Consider a complex curve defined in terms of $\xi = u + iv$ as

$$R = [\xi, \Theta(\xi)] = [u + iv, \mathrm{Re}(\Theta(\xi) + i\,\mathrm{Im}(\Theta(\xi)] \tag{6.204}$$

$$\Rightarrow [u, v, \Phi, \Psi]. \tag{6.205}$$

Osserman (p. 19 [180]) demonstrates how this four-dimensional position vector satisfies the minimal surface equation. The Minimal surface generated by a complex curve, does not admit a single implicit real function in 3D for its description. Such minimal surfaces are artifacts of 4D space-time.

Consider an evolutionary system like a fluid in space-time. Consider a complex holomorphic curve. This complex curve induces a two-dimensional subspace of space-time. The subspace is a minimal surface. However, there does exist a parametrization of such a minimal surface, and that is what the Weierstrass method is all about. The method represents a parametric version, 2D into 3D, but not implicit version, 3D to a constant. The "parametric" vector to the surface can be used to describe a vorticity field (or a velocity field). Such a vector is harmonic.

For thermodynamic systems that can be encoded by a 1-form of Action, of Pfaff Topological dimension 4 (the canonical example is the Hopf map), the Universal Phase function is a holomorphic function of the complex eigenvalue ξ

$$\Theta(x, y, z, t;\ \xi) = \xi^4 - X_M\xi^3 + Y_G\xi^2 - Z_A\xi + T_K \Rightarrow 0, \tag{6.206}$$

$$\mathrm{Re}(\Theta(z)) = u^4 - 6u^2v^2 + v^4 - X_M(u^2 - 3v^2)u \tag{6.207}$$

$$+Y_G(u^2 - v^2) - Z_A u + T_K, \tag{6.208}$$

$$\mathrm{Im}(\Theta(\xi) = 4(u^2 - v^2)uv - X_M(3u^2 - v^2)v + 2Y_G uv - Z_A v. \tag{6.209}$$

Hence the Universal Phase function defines a complex curve in terms of the similarity coefficients. A cosmological universe of Pfaff topological dimension 4 can be put into correspondence with conjugate minimal surfaces.

6.2.4 Spinors and the Hopf map

The isotropic Complex position vector, $[z1, z2, z3]$ can be decomposed into a real and imaginary part, such that both have the same sum of squares, and are orthogonal. In other words, the Cartan Isotropic Spinor can be represented as

$$|\boldsymbol{\sigma}_{12}\rangle = |\mathbf{H}1\rangle + i\ |\mathbf{H}2\rangle \qquad with \quad \langle\boldsymbol{\sigma}_{12}| \circ |\boldsymbol{\sigma}_{12}\rangle = 0. \tag{6.210}$$

Two other Cartan spinors are represented by the combinations.

$$|\boldsymbol{\sigma}_{23}\rangle = |\mathbf{H}2\rangle + i\ |\mathbf{H}3\rangle \qquad with \quad \langle\boldsymbol{\sigma}_{23}| \circ |\boldsymbol{\sigma}_{23}\rangle = 0, \tag{6.211}$$

$$|\boldsymbol{\sigma}_{31}\rangle = |\mathbf{H}3\rangle + i\ |\mathbf{H}1\rangle \qquad with \quad \langle\boldsymbol{\sigma}_{31}| \circ |\boldsymbol{\sigma}_{31}\rangle = 0. \tag{6.212}$$

These formulas can be obtained from the Cartan representation for the isotropic 3D vector. As an example consider the permuted form,

$$z1 = \alpha^2 - \beta^2, \tag{6.213}$$

$$z2 = -2\alpha\beta, \tag{6.214}$$

$$z3 = i\ (\alpha^2 + \beta^2). \tag{6.215}$$

Make the substitutions $\{\alpha = X + iZ, \quad \beta = Y - iS\}$ to obtain the equations

$$|\boldsymbol{\sigma}_{31}\rangle = \left| \begin{array}{l} X^2 + S^2 - Y^2 - Z^2 + i\,2(ZX + SY) \\ -2(YX + SZ) + i\,2(-ZY + SX) \\ 2(-ZX + SY) + i\,(X^2 + Y^2 - Z^2 - S^2) \end{array} \right\rangle = |\mathbf{H}3\rangle + i\ |\mathbf{H}1\rangle\ . \tag{6.216}$$

6.2.5 The Adjoint field to the Hopf Map

The Hopf Map, as characterized by the equations:

$$[u1, v1, w1] = [2(XZ + YS), 2(XS - YZ), (X^2 + Y^2) - (Z^2 + S^2)], \tag{6.217}$$

can be used to generated three linear independent 1-forms on R4, by forming the gradient with respect to $[X, Y, Z, S]$ of each of the three functions that define the map. These three covariant, four component, vectors may be used in the construction of a frame matrix on R4. A fourth linearly independent vector is needed, to complete the basis frame. This fourth vector can be constructed from the adjoint operation (on matrices or differential forms) to within an arbitrary scaling factor,$1/\lambda$. The linearly independent 1-forms are therefore,

$$\begin{aligned} d(u1) &= 2Zd(X) + 2Sd(Y) + 2Xd(Z) + 2Yd(S), & (6.218)\\ d(v1) &= 2Sd(X) - 2Zd(Y) - 2Yd(Z) + 2Xd(S), & (6.219)\\ d(w1) &= 2Xd(X) + 2Yd(Y) - 2Zd(Z) - Sd(S), & (6.220)\\ A_{Hopf} &= \{-Yd(X) + Xd(Y) - Sd(Z) + Zd(S)\}/\lambda. & (6.221) \end{aligned}$$

The Frame Matrix so generated is given by the expression:

$$F = \begin{bmatrix} Z & S & X & Y \\ S & -Z & -Y & X \\ X & Y & -Z & -S \\ -Y/\lambda & X/\lambda & -S/\lambda & Z/\lambda \end{bmatrix}, \qquad (6.222)$$

$$Det[F] = \left(Z^2 + S^2 + Y^2 + X^2\right)^2/\lambda. \qquad (6.223)$$

It is of some interest to examine the properties of the adjoint 1-form, A_{Hopf},defined hereafter as the Hopf 1-form. For $\lambda = 1$, it follows that the Hopf 1-form is of Pfaff dimension 4.

It is also of interest to consider factors λ that are of the format of the Holder norm, where n and p are integers, and (a,b,k,m) are arbitrary constants.

$$\lambda = (aX^p + bY^p + kZ^p + mS^p)^{n/p}. \qquad (6.224)$$

The exponents n and p determine the homogeneity of the resulting 1-form, which is given below an ambiguous format (the plus of minus sign)

$$A_\pm = A_\pm/\lambda = \{\pm(Yd(X) - Xd(Y)) - Sd(Z) + Zd(S)\}/\lambda. \qquad (6.225)$$

For example, for n=p=2, the scaling factor becomes related to the classic quadratic form. The scaled Hopf 1-form, A, is then homogeneous of degree zero.

For arbitrary n and p, the 3-form of topological (Hopf) torsion

$$\begin{aligned} &\textbf{Topological Torsion}\\ &= (A_\pm)\hat{}d(A_\pm) = i(\pm\mathbf{T}_4)d(X)\hat{}d(Y)\hat{}d(Z)\hat{}d(S) \qquad (6.226) \end{aligned}$$

generates a direction field defined as the four component Torsion vector, $\mathbf{T}_4$.

$$\mathbf{T}_4 = \pm[X, Y, Z, S]/\lambda. \qquad (6.227)$$

The factor β depends upon the integers n and p as well as the constants (a,b,k,m).

The Topological Parity 4-form, whose coefficient is the 4D divergence of the Torsion vector, $\mathbf{T}_4$, becomes

$$\begin{aligned} & \text{Topological Parity } d(A_{\pm})\hat{}(d(A_{\pm}) \\ = \; & -4(\pm\lambda)^{(-2n/p)})(n-2)d(X)\hat{}d(Y)\hat{}d(Z)\hat{}d(S). \end{aligned} \quad (6.228)$$

It is most remarkable that for n=2, any p and any (a,b,k,m), the topological parity vanishes and the scaled Hopf 1-form is of Pfaff dimension 3, not 4. In such cases the ratios of the integrals of the topological torsion 3-form over various closed manifolds are rational, and the closed integrals of the 3-form are topological deformation invariants (coherent structures).

Also note that if the scaling factor is restricted to values such that $n = 4,\ p = 2,\ a = b = k = m = 1,$ then the Frame matrix is unimodular, and the scaled Hopf 1-form is homogeneous of degree -2, relative to the substitution $X \Rightarrow \gamma X$, *etc.* (A somewhat different definition of homogeneity relative to the volume element will be given below.) For this constraint, the 2-form, $F = dA$, has two components in analog to the **E** and **B** fields of electromagnetism. The two "blades" of three components are identical only when all of the coefficients are equal to unity. A finite value for the quadratic form leads to a sphere in 3D of coordinates {u1,u2,u3}.

Electromagnetism of Index zero Hopf 1-forms

Guided by prior investigations, it is of interest to use the scaled Hopf 1-form as the generator of electromagnetic field intensities. The coefficients of the scaled Hopf 1-form can be put into correspondence with the classic vector and scalar potentials, $[\mathbf{A}, \phi]$ (using $S = CT$). The Action for the first examples is then of the format,

$$A_{\pm,0} = A_{\pm}/\lambda_0 = \{\pm(+Yd(X) - Xd(Y)) - CTd(Z) + CZd(T)\}/\lambda_0. \quad (6.229)$$

When the number of minus signs in the quadratic form is zero (index 0), and the exponents are n=4, p=2, such that

$$\lambda_0 = (X^2 + Y^2 + Z^2 + S^2)^2, \quad (6.230)$$

then it is remarkable that the derived 2-form has coefficients (**E** and **B**) that are proportional to the same Hopf Map with the classic result that $\mathbf{E}^2 = C^2\mathbf{B}^2$, Using the minus ambiguity (parity) sign, the **E** field is anti-parallel to the **B** field. If the positive ambiguity (parity) sign is used, the **E** and **B** fields are parallel:

$$F = dA, \tag{6.231}$$
$$\mathbf{B} = curl\mathbf{A} = \tag{6.232}$$
$$[2(CTY + XZ) - 2(-YZ + CTX) +(-X^2 - Y^2 + Z^2 + (CT)^2)](2/(\lambda_0)^{3/2}), \tag{6.233}$$
$$\mathbf{E} = -grad\phi - \partial\mathbf{A}/\partial T = \tag{6.234}$$
$$[-2(CTY + XZ) + 2(-YZ + CTX) -(-X^2 - Y^2 + Z^2 + (CT)^2)](2C/(\lambda_0)^{3/2}). \tag{6.235}$$

It is natural to ask if these **E** and **B** fields admit a Lorentz symmetry constitutive constraint such that the vacuum state is charge current free. Recall that a constitutive constraint is a relationship between a 2-form, F, and a 2-form density G, such that the coefficients of $G(\mathbf{D}, \mathbf{H})$ are related to the coefficients of $F(\mathbf{E}, \mathbf{B})$. A Lorenz vacuum condition implies that the fields are solutions of the vector wave equation. The question becomes, "If is presumed that $\mathbf{D} = \varepsilon\mathbf{E}$ and $\mathbf{B} = \mu\mathbf{H}$, do the Maxwell Ampere equations generate a zero 3-form of charge current? ". Direct computation of the index zero Hopf 1-form indicates that $dG = J \neq 0$, unless $\varepsilon\mu C^2 + 1 = 0$. Hence the scaled Hopf Action, where the scaling is of signature zero, does **not** describe a charge current free vacuum, for real positive values of ε, μ, and C. On the other hand, if it is presumed that the domain is such that say μ,or ε, is negative, then the Hopf Map, scaled as above, does generate charge-current free wave solutions. Negative ε appears to hold in metals and the Earth's ionosphere. Recent announcements indicate constructions that yield negative μ [185]. However, for situations where ε or μ are negative, the Hopf wave solutions imply that the Spin angular momentum $A\hat{}G$ is not a deformation invariant (hence the Spin angular momentum of the example field is not conserved.)

Electromagnetism of Index one Hopf 1-forms

When the number of minus signs in the quadratic form is one (index 1), and the exponents are n=4, p=2, such that (using lower case letters for Index one Hopf 1-forms),

$$\lambda_1 = (x^2 + y^2 + z^2 - c^2t^2)^2, \tag{6.236}$$

then it is remarkable that the derived 2-form has coefficients (**E** and **B**) that are proportional to different Hopf Maps. The Action 1-form is the same as above, but with a different denominator.

$$A_{\pm,1} = A_{(\pm)}/\lambda_1 = \{\pm(yd(x) - xd(y)) - Ctd(z) + zCd(t)\}/\lambda_1. \tag{6.237}$$

The fact leads to the classic result that $\mathbf{E}^2 = C^2\mathbf{B}^2$, but now the $\mathbf{E}$ field is not collinear with the $\mathbf{B}$ field. Using the negative ambiguity (parity) sign leads to the fields:

$$\begin{aligned}
F &= dA && (6.238)\\
\mathbf{B} &= curl\mathbf{A} = && (6.239)\\
&[2(Cty + xz) - 2(-yz + Ctx)\\
&+(-x^2 - y^2 + z^2 + (Ct)^2)](2/(\lambda_1)^{3/2}), && (6.240)\\
\mathbf{E} &= -grad\phi - \partial\mathbf{A}/\partial t = && (6.241)\\
&[2(Cty - xz) + 2(-yz - Ctx)\\
&-(-x^2 - y^2 + z^2 + (Ct)^2)](2C/(\lambda_1)^{3/2}). && (6.242)
\end{aligned}$$

Independent from any other constraints, it is possible to construct the 3-form of Topological Torsion, and its exterior differential defined as Topological Parity. The Topological parity can be either positive, zero, or negative. For the example Hopf 1-form given above (using the negative ambiguity sign), the Topological Torsion is represented to within a factor by a position vector$[-x, -y, -z, -t]$ inbound in four dimensions, and having a negative divergence or parity. If the positive sign of the ambiguity factor is changed, then the parity of the form changes sign. For example, for the 1-form,

$$A1 = A1_+/\lambda_1 = \{+yd(x) - xd(y) - Ctd(z) + zCd(t)\}/\lambda_1, \qquad (6.243)$$

the 4 -form of topological parity is positive, and the topological torsion is represented by an outbound position vector (to within a factor).

Similar to the investigation described above, it is natural to ask if these $\mathbf{E}$ and $\mathbf{B}$ fields admit a Lorentz symmetry constitutive constraint such that the vacuum state is charge current free. Again, such a condition implies that the fields are solutions of the vector wave equation. Direct computation of the Maxwell Ampere equations indicates that $dG = J = 0$ if the phase velocity constraint vanishes, $\varepsilon\mu C^2 - 1 = 0$. Hence the scaled Hopf Action, where the scaling is of index one, **does** describe a charge current free vacuum, for real positive values of ε, μ, and C.

It is some interest to give the charge current solutions to show how the "phase factor" $(\varepsilon\mu C^2 - 1) \Rightarrow 0$ establishes the vacuum charge free conditions:

$$\begin{aligned}
J^x &= -(yx^2 + yz^2 + 5yC^2t^2 - 6zCtx + y^3)(\varepsilon\mu C^2 - 1)4/\lambda^2, && (6.244)\\
J^y &= (x^3 + xy^2 + xz^2 + 5xC^2t^2 + 6zCty)(\varepsilon\mu C^2 - 1)4/\lambda^2, && (6.245)\\
J^z &= -(2x^2 + 2y^2 - z^2 + C^2t^2)(\varepsilon\mu C^2 - 1)8Ct/\lambda^2, && (6.246)\\
\rho &= 0. && (6.247)
\end{aligned}$$

Note that there are possible charge current free (wave solutions) that are governed by curves in space-time generated by the intersection of the three surfaces created by setting the coefficients of the current density equal to zero. These solutions are valid for any phase velocity.

The given solution above is not free of Topological Torsion, $A\hat{}F$, and there is a non-zero value of the second Poincare invariant, $\mathbf{E}\cdot\mathbf{B}\neq 0$. However, the Spin 3-form $A\hat{}G$ is also non-zero [199] [238], but it has, subject to the phase constraint, a zero 4-divergence. (The first Poincare invariant is zero.) The divergence of the Spin 3-form, has 2 parts. The first part is twice the conventional Lagrange density of the fields, $(\mathbf{B}\cdot\mathbf{H}-\mathbf{D}\cdot\mathbf{E})$. The second part is the interaction between the potentials and the charge currents, $(\mathbf{A}\cdot\mathbf{J}-\rho\phi)$. When the divergence of the 3-form is zero, then the closed integrals of Topological Spin are deformation invariants, and have closed integrals with rational (quantized) ratios. That is, with regard to any singly parametrized vector field, V, describing an evolutionary process,

$$\begin{aligned} L_{(\beta V)}\int_{z3}(A\hat{}G) &= \int_{z3} i(\beta V)d(A\hat{}G)+\int_{z3} d(i(\beta V)A\hat{}G) \qquad (6.248)\\ &= 0+0\supset \textit{evolutionary invariance.} \end{aligned}$$

The function β is an arbitrary deformation parameter.

Twistors composed by superposing two index 1 Hopf 1-forms

By superposing (adding or subtracting) two different, index 1, Hopf 1-forms (which will be shown below to be equivalent to a Penrose twistor solution) it is possible to construct a vacuum (charge current free wave) solution to the Maxwell system, subject to the constraint that the phase speed satisfies the phase velocity equation, $(\varepsilon\mu C^2-1)=0$.

As an example consider another Hopf 1-form of index 1 formulated as

$$A2 = A2_+/\lambda_1 = \{Ctd(x)+zd(y)-yd(z)-xCd(t)\}/\lambda_1. \qquad (6.249)$$

Similar formulas for the field intensities can be determined as above. Note that the parity of the Hopf forms to be superposed can be the same or different. If the parity of the two superposed Hopf 1-forms are opposite, then without consideration of the phase constraint, the Topological Torsion of the "twistor" 1-form vanishes, $A\hat{}F=0$. Yet the quantized topological Spin 3-form $A\hat{}G$ does not vanish, and moreover, subject to the phase constraint, the closed integrals of the Spin 3-form are conserved. This result implies that such a construction yields "quantized" values for the Spin integrals.

In this "twistor" case, note the vector represented by the vector $R=[x,y,z,t]$ in R4, is orthogonal to the 1-form of Action. It follows that for a twistor Action,

$$A = A1_- + A2_+, \tag{6.250}$$
$$i(R)A = 0, \textit{ and } L_{(R)}A = 2A, \tag{6.251}$$
$$i(R)dA = 2A, \textit{ and } L_{(R)}dA = 2dA. \tag{6.252}$$

Note that the Hopf 1-form, A and the derived 2-form, $F = dA$, are both homogeneous of degree 2, with respect to R.

The "twistor" Action created by superposing Hopf 1-forms of different parity (but not the general Hopf action) is integrable in the sense of Frobenius,

$$\text{Topological Torsion } H = A\hat{}F = 0. \tag{6.253}$$

The implication is that the 4-forms of Topological Parity, or the second Poincare invariant, (which does not depend upon constitutive properties) is also zero for the twistor 1-form:

$$\text{Second Poincare invariant PII} \tag{6.254}$$
$$= d(A\hat{}F) = F\hat{}F = 2\mathbf{E}\cdot\mathbf{B}dx\hat{}dy\hat{}dz\hat{}dt \Rightarrow 0. \tag{6.255}$$

Classically, one would say that the second Poincare invariant vanishes for this twistor Action.

From the constitutive relations, there exists a 3-form (density) S, [212] defined as the Spin 3-form,

$$S := A\hat{}G \text{ such that } A\hat{}S = 0. \tag{6.256}$$

The Action of the Lie differential on the Spin 1-form, S, is such that,

$$L_{(R)}S = (L_{(R)}A)\hat{}G + A\hat{}L_{(R)}G = 2A\hat{}G + A\hat{}(2G) = 4S, \tag{6.257}$$
$$L_{(R)}dS = 4dS. \tag{6.258}$$

The Spin 3-form, S, and its divergence 4-form, dS are homogeneous of degree 4 relative to the vector R. Subject to the phase constraint, the divergence of the Spin 3-form vanishes, which indicates that the closed integrals of the spin 3-form are conserved as period integrals.

These results are to be compared with the Penrose twistor definitions in terms of differential forms [184] The energy flow $\mathbf{E}\times\mathbf{H}$ of such a solution is collinear with the spatial components of the Spin, $\mathbf{S}$.

6.3 Some Conjectures

6.3.1 Maximal Surfaces in Cosmological Minkowski space

A *minimal* surface in a space with a Euclidean metric is defined as a surface with zero mean curvature and negative Gauss curvature. A *maximal* surface is defined as a surface in a 3D Minkowski space with zero mean curvature and positive Gauss curvature. The concept of a Minkowski space with a "Lorentz metric" forces attention to the concept that the Falaco Solitons (Maximal surfaces) are related to wave-bulk phenomena, and not to shear viscosities. Shear viscosity effects and minimal surfaces in Euclidean spaces are characteristics of affine translations that preserve parallelism and are without fixed points (transitive). The maximal surfaces in Minkowski space are to be associated with non-affine rotations and expansions which admit fixed points. These conical points of the maximal surface are the singularities at which the metric 3D Gauss curvature goes to infinity as the determinant of the metric goes to zero. Are these objects related to black holes?

Maximal surfaces are generated by immersive maps from a two-dimensional space into a three-dimensional space with a Lorentz metric [77]. The maximal surface is defined in terms of a space like immersion with positive Gauss curvature and with zero mean curvature. Such surfaces are related to minimal surfaces in a space with a Euclidean metric, but minimal surfaces in Euclidean space have negative Gauss curvature. Maximal Surfaces can admit isolated, or "conical", singularities, where Minimal surfaces can not. Maximal surfaces can mimic catenoidal and helical surfaces of Euclidean theory, but may exhibit singular subsets of points.

To repeat, consider a space with a Minkowski - Lorentz metric of the form,

$$(ds)^2 = (dx)^2 + (dy)^2 - (dz)^2. \tag{6.259}$$

The immersion,

$$R(s, \sigma) = (1/a)[(\sinh(a\sigma + b)\cos(s), (\sinh(a\sigma + b)\sin(s), (a\sigma + b)], \tag{6.260}$$

generates a space-like maximal surface in a space with a Minkowski metric. The surface is of zero mean curvature, but the metric vanishes at the conical singular point, and the Gauss curvature becomes infinite. The surface is similar to the minimal surface Catenoid in Euclidean geometry, but here, unlike the Euclidean catenoid, the Minkowski catenoid has a singular point at the dimple vertex.

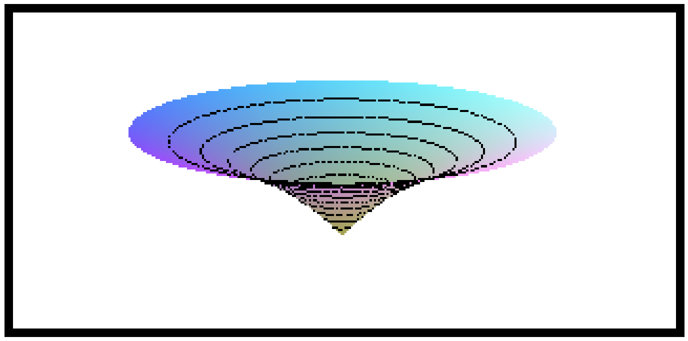


Figure 6.20 A Maximal (Catenoidal) Surface with Conical Singularity in a 3D Minkowski space. The Dimple of the Falaco Soliton

The zero mean curvature surfaces, with a singular point, can be formed experimentally in a fluid. The experimental evidence is presented below. The idea that three-dimensional space may or may not be Euclidean challenges a dogmatic precept of modern physics, where it is rarely perceived that physical 3D space can be anything but Euclidean. However, as discussed in the following section, the occurrence of long lived rotational structures in the free surface of a body of water, which have been described as Falaco Solitons, exhibit the features of maximal surfaces in a Lorentz - Minkowski space. The Falaco Solitons are topological defect structures easily replicated in an experimental sense. Optical measurements indicate that the surface defect structures have a zero mean curvature. In addition, the surface defect structures have an apparent conical singularity which is an artifact of the signature of a maximal space-like surface in Minkowski space.

6.3.2 Point Particles as Real and Complex Spheres of "zero radii"

A point particle is typically modeled as a three-dimensional Euclidean real ball with a vanishingly small radius vector. The length of the radius vector squared is defined by the sum of squares of its real components. The surface area of the real ball tends to zero as the length of the radius shrinks. However if a "point" particle is defined as a (complex) sphere of vanishingly small radius, then complex point particles could be represented by a complex isotropic vector, whose "length" squared is defined, in the same Euclidean manner as for real vectors, as the sum of the squares of its components. In a Euclidean space (where the signature of the fundamental quadratic form is zero) the isotropic complex vector is not realized in terms of real variables. In Minkowski space, where the signature of the fundamental form is 1, the isotropic vectors (of null length) can be represented by real vectors, relative to the pseudometric. It is suggested herein that a physical "point" in real Euclidean space be extended to include complex Euclidean space, and/or Minkowski space. The surface area of a real "point" is zero in real Euclidean space, but the surface area of a complex "point" can be finite, even though its "diameter" is zero. This result follows from the fact that an isotropic null vector can be used as the generator of a surface of zero mean curvature. The surface of zero mean curvature is not of zero area; its Gauss curvature is positive (not negative). The idea is to study a "point" volume of "zero" real radius that is bounded by two minimal (perhaps conjugate) minimal surfaces. The concept of a spinor is interpreted as the topological realization of a "point" particle of finite area, positive Gauss curvature, and therefore positive mass, but of zero diameter, or length.

6.3.3 More on Surfaces of Zero Mean Curvature

It is extraordinary that the Hopf Adjoint vector, when suitably normalized to have coefficients homogeneous of degree zero, can be used to define a minimal surface in 3D, where the Gauss curvature (sums of product pairs of curvatures) is real and positive. Real minimal surfaces in 3D have a Gauss curvature (sums of product pairs of curvatures) which is negative.

Consider the Hopf Adjoint vector, which is of the form,

$$A_0 = b(ydx - xdy) + a(tdz - zdt). \tag{6.261}$$

The 1-form of Potentials depends on the coefficients a and b which are presumed to take on values ± 1. There are two cases corresponding to left-handed and right-handed "polarizations": $a = b$ or $a = -b$. (There actually are 6 cases to consider, by cyclically permuting the variables, and these can be combined to represent spinor solutions.)

Next normalize the 1-form by dividing through by a Holder norm such that the coefficients of the renormalized 1-form are homogeneous of degree zero. Then construct the similarity invariants of the Jacobian matrix determined from the coefficients of the renormalized 1-form. What is remarkable for this example, is that both the Mean curvature (sum of curvatures), the Adjoint (Cubic curvature = sum of all curvature triples), and quadratic curvature (determinant of the Jacobian matrix = product of all curvatures) of the implicit hypersurface in 4D vanish, for any choice of a or b. The Gauss curvature (sum of all pairs of curvatures) is non-zero, positive, real and is equal to the square of the radius of a 4D Euclidean sphere. The cubic interaction energy density is zero:

$$\begin{aligned} Mean &= 0, & (6.262)\\ Gauss &> 0, & (6.263)\\ Cubic &= 0, & (6.264)\\ Top\_Torsion &\neq 0, & (6.265)\\ J_s &\neq 0. & (6.266) \end{aligned}$$

This situation occurs when the three curvatures of the implicit 3-surface are $\{0, +i\omega, -i\omega)$. This Hopf surface is therefore a 3D imaginary *minimal* two-dimensional hyper surface in 4D and has two non-zero pure imaginary curvatures! Strangely enough the charge-current density is not zero, but it is proportional to the topological Torsion vector that generates the 3-form, $A\hat{}F$. The topological Parity 4-form is not zero, and depends on the sign of the coefficients, a and b. In other words the 'handedness' of the different 1-forms determines the orientation of the normal field with respect to the implicit surface.

It is also possible to deduce a closed 3-form of "Charge-Current density", J_s, for such 3D hypersurfaces. The coefficients of $A\hat{}J_s$ are exactly equal to the "Cubic" curvature similarity invariant. The spatial scalar product of A and J is balanced by the product, $\rho\phi$. It is known that a process described by a vector proportional to the topological torsion vector in a domain where the topological parity ($4ba$) is non-zero is thermodynamically irreversible.

It is also possible to construct combinations of Hopf Adjoint 1-forms of different chirality to find what are called Instanton solutions [177] [89].

6.3.4 Bulk Viscosity and Cosmology

In Volume 4 [297], it is demonstrated that (in the language and symbolism of a Navier-Stokes fluid) the topological parity pseudo-scalar (equivalent to the coefficient of the second Poincare 4-form, $K = dA\hat{}dA$) becomes expressible in terms of engineering quantities as,

$$\begin{aligned} K &= 2\{-\{grad(P)/\rho + \mu_B\ grad(div\ \mathbf{v})\} \circ curl\ \mathbf{v} \\ &\quad -\nu\{curl\ \mathbf{v} \circ (\nabla^2 \mathbf{v})\}\}\Omega_4. \end{aligned} \tag{6.267}$$

The coefficient K is a measure of a turbulent open thermodynamic state that admits thermodynamically irreversible evolution in the direction of the topological torsion vector. Recall that turbulent dissipative irreversible flow is defined when the Pfaff dimension of the Action 1-form is equal to 4, which implies that $K \neq 0$. From this expression it is apparent that the shear (affine) viscosity coefficient, ν, has no dissipative contribution, if the fluid velocity is Harmonic, $\nabla^2\mathbf{v} \Rightarrow 0$. Further recall that a harmonic vector field is a requirement for a surface of zero mean curvature, tangential discontinuities, and Wakes. Shear viscosity eliminates the non-harmonic components of a velocity field, thereby producing a Wake. The "minimal-maximal" surface is in velocity space, for a Harmonic fluid flow. The resultant expression for the topological parity 4-form is:

Harmonic fluid dynamics - Wakes

$$\nabla^2\mathbf{v} \Rightarrow 0, \tag{6.268}$$

$$K = \{2\{grad(P)/\rho - \mu_B\ div\ \mathbf{v}) \circ curl\ \mathbf{v}\}\Omega_4 \neq 0. \tag{6.269}$$

The formulas demonstrate the importance of bulk viscosity, μ_B , to the concept of irreversible dissipation. It is also to be noticed that bulk viscosity may compensate for pressure effects, giving further speculation to the present controversy of dark matter and dark energy in the Universe. A Google search (October 2004) yields over 5000 articles that utilize the concepts of Bulk Viscosity in a General Relativistic treatment of dark matter, and almost 3,000,000 articles on dark matter and dark energy. In all cases the theories are more or less phenomenological. It is apparent that the authors do not realize that Bulk Viscosity is a topological effect related to the divergence of the Topological Torsion tensor and the expansion-contraction or rotational (non-affine) shears that may occur in a 4D space-time variety. For a cosmological universe encoded in terms of a 1-form of Action of Pfaff dimension 4, the Bulk Viscosity coefficient is an important component of the Topological Parity 4-form, (related to $2(\mathbf{E} \circ \mathbf{B})$ in EM notation). Dissipation (or the lack thereof) of irreversible processes depends upon Topological Parity of contraction or expansion, or the chirality of rotation. From this point of view, dark matter could be an artifact of irreversible topological evolution.

6.3.5 The Four Forces and Differential Topology

Almost twenty years ago [203], an argument was presented to show how the properties of the four forces in physics could be deduced from the features of the four distinct Pfaffian equivalence classes of differential geometry that can be constructed on a space of four dimensions. The four equivalence classes were determined from the metric solutions, $g_{\mu\nu}$, to the Einstein field equations, by constructing a 1-form of action, A, in terms of the space-time, $g_{4\nu}$, components of the metric field: $A = g_{4\nu}dx^{\nu}$. The methods of Pfaff reduction can be used to generate four equivalence classes in terms of the Pfaff Topological dimension, or class, of this 1-form. Summarizing the previous results, the methods lead to:

1. (Newtonian Force) The equivalence class of Pfaff Topological dimension 1 will support long range gravitation (mass) and is parity preserving.

2. (Coulomb Force) The second equivalence class of Pfaff Topological dimension 2 will support both gravity (mass) and electromagnetism (charge) and is to be associated with long range parity preserving forces.

3. (Strong Force) The third equivalence class of Pfaff Topological dimension 3 will support both mass and charge, but the forces - although parity preserving– are of short range.

4. (Weak Force) The last equivalence class of Pfaff Topological dimension 4 involved short range interactions that can violate time reversal and symmetry breaking .

Examples were given in terms of known solutions to the Einstein field equations.

Solution to Einstein equations	Pfaff Topological dimension of $A = g_{4\nu}dx^{\nu}$
Schwarzschild	1
Riessner-Nordstrom	2
Godel	3
Kerr Taub Nut	4

Although the previous methods were motivated by ideas of differential geometry, it is now known that the concepts used to generate the four equivalence classes associated with the four forces are not of a geometrical nature, but instead are better expressed in terms of equivalence classes which have their foundations in the topological property of Pfaff Topological dimension. Indeed, the older analysis concluded that two of the equivalence classes are to be associated with forces that are long range, in the sense of having "distance" limits that extended to infinity The other two equivalence classes are to be associated with forces that are of short range. However, the concept of "distance" is more of a geometrical idea, not a topological idea.

Now it is perceived that the true nature of the equivalence classes is based on the topological issue of connectedness, and does not reflect the geometrical idea of distance necessarily. Following the work of Baldwin [12], two of the equivalence classes belong to a connected topology (Pfaff Topological dimension 1 and 2), and the other two equivalence classes belong to a disconnected topology (Pfaff dimension 3 and 4). Hence the topological features of the strong and the weak forces do not involve short range, but instead reflect the concepts of accessibility. That is, the Cartan topology of the "long range" forces is connected, while the Cartan topology of the "short range" forces is disconnected. The topological idea of connectedness is to be exchanged for the geometrical idea of "long range or distance". There is a difference between the concepts of whether or not, from point a, the point b is not "reachable" by a continuous process and not "reachable" in a finite time. Moreover, the Pfaff topological dimension can be associated with Thermodynamic features. Systems of Pfaff Topological dimension greater than 2 are not in thermodynamic equilibrium.

These ideas are most readily understood in terms of the Cartan topology built on a Pfaffian system, and its differential closure. These sets have global properties, and therefore carry topological significance. These concepts of Pfaff equivalence classes have application not only to the microcosm of atoms and elementary particles, as well as the cosmological arena of galaxies, but also to the mundane physics of hydrodynamics. Such methods have been used recently to obtain a better understanding of the production of wake patterns, and the creation and decay of turbulence in fluids.

6.3.6 Signature Symmetry Breaking

However, over the years a new feature of the analysis has appeared, and it is to this new feature that this section is directed. Note that in 1975 reference [203], [202], the signature of the quadratic form was taken to be $\{+,+,+,-\}$. The question now arises: Is there a symmetry to be broken if one considers the often used but opposite signature $\{-,-,-,+\}$. The idea is that the wave equation,

$$+\partial^2\psi/\partial x^2 + \partial^2\psi/\partial y^2 + \partial^2\psi/\partial z^2 = +(1/c^2)\partial^2\psi/\partial t^2, \tag{6.270}$$

has a set of characteristics which satisfy the partial differential system:

$$+(\partial\psi/\partial x)^2 + (\partial\psi/\partial y)^2 + (\partial\psi/\partial z)^2 = +(1/c^2)(\partial\psi/\partial t)^2. \tag{6.271}$$

Hence, there are two ways to write this constraint as an algebraic variety (a null set):

$$+(\partial\psi/\partial x)^2 + (\partial\psi/\partial y)^2 + (\partial\psi/\partial z)^2 - (1/c^2)(\partial\psi/\partial t)^2 = 0, \tag{6.272}$$

or

$$-(\partial\psi/\partial x)^2 - (\partial\psi/\partial y)^2 - (\partial\psi/\partial z)^2 + (1/c^2)(\partial\psi/\partial t)^2. \tag{6.273}$$

Each quadratic form is the complete mirror symmetry (the negative) of the other, but it turns out that the signatures are intrinsically different from a topological point of view in the neighborhood of the null variety.

The analytic question that remains is: Does this symmetry of space-time signatures have distinguishable consequences? The physical question is: Are there experiments that can be done to distinguish the symmetry breaking between $\{-,-,-,+\}$ and $\{+,+,+,-\}$?

The analytic answer, based on the idea that the Clifford Algebras of such systems are not isomorphic to one another [13], is yes! The mathematical argument is similar to that used to distinguish the two species of angular momentum algebras in quantum mechanics, an argument which is based on the different signatures of the raising or lowering operators (commutator or anti-commutator brackets) for Bosons vs. Fermions. The fact that the differences in angular momentum signature are physically observable implies that the differences in space-time signatures may also be measurable.

Consider the Clifford Algebra with signature $\{+,+,+,-\}$. As discussed in reference [13], this algebra is isomorphic to the algebra of 4×4 matrices with real numbers as matrix elements. This matrix algebra is the usual representation used for waves in four dimensions. Next consider the Clifford Algebra with signature $\{-,-,-,+\}$. This algebra is isomorphic to the algebra of 2x2 matrices with quaternions as matrix elements. The non-abelian quality of the quaternions makes this algebra have extraordinary differences from the algebra of 4×4 matrices over the real numbers.

This positive analytic result which breaks the symmetry between the two space-time signatures implies there must be a physical difference between the two types of space-time, one with signature $\{+,+,+,-\}$, and the other with signature $\{-,-,-,+\}$. These differences imply that there exist two species of waves. What are they? A possible answer was first given by Schultz [257] who found exact quaternion solutions to Maxwell's equations that indicated that the speed of propagation in the inbound and outbound directions would be different for such waves. This result was in agreement with the ring laser experiments of Sanders [210]. These sets of experiments indicated that the electromagnetic four-fold degeneracy of the Lorentz equivalence class could be broken such that all four waves of left - right polarizations and of to $\Leftrightarrow$ fro propagation directions would propagate at four distinct speeds. A further more general analysis on the macroscopic parity and time reversal symmetry breaking effects in electromagnetic systems was presented in reference [222]. The question of whether or not these waves, or the effects of $\{+,+,+,-\}$ vs. $\{-,-,-,+\}$ signatures, produce any quantum or hydromechanical effects is open.

6.3.7 Dark matter, Dark energy (Negative Pressure), Energy Balance and Curvatures

Astronomical measurements over the last 20 years have been interpreted to imply that the constituents of our Universe are ordinary matter (∼5%), dark matter (25%) and dark energy (70%). Statements that dark matter and dark energy compose more that 95 % of the energy of the universe have been quite surprising. As mentioned above a Google search gives over 3 million articles on dark matter and dark energy.

In terms of a metrically based gravitational theory, the presence of dark matter has been inferred from the observed dynamics of cosmic objects, particularly from fast rotation of hydrogen clouds far outside the luminous disc of spiral galaxies, as well as high-velocity dispersion of galaxies in clusters. Very little (at present) is known about dark energy and dark matter, even though it would seem from the recent conjectures and interpretations that more than 95% of the universe is these forms. A rash of publications involving quantum virtual states and interactions coupled with general relativity concepts have been offered to "explain" the interpretations of astronomical data. Could there be a more mundane theory that would offer less "far out" explanations?

That being said, dark energy has the following defining properties:

1. Dark energy emits no light.
2. Dark energy has large, negative pressure, $p = -\rho c^2$.
3. Dark energy is approximately homogeneous.

Apparently dark energy does not cluster significantly with matter on scales at least as large as clusters of galaxies. Because its pressure is comparable in magnitude to its energy density, it is more "energy-like" than "matter-like" (matter being characterized by $p << \rho c^2$). Dark energy is qualitatively very different from dark matter. The 3-forms can be summarily classified by how their energy densities change with a cosmic scale factor a: ordinary and dark matter behave as a^{-3}, radiation, which interacts with ordinary matter, behaves as a^{-4}, while dark energy, at least in the simplest models thereof, is independent of the scale, a.

The ordinary matter content can be deduced from observations at e.g. visible or radio frequencies. The presence of dark matter can be inferred from the observed dynamics of cosmic objects, particularly from fast rotation of hydrogen clouds far outside the luminous disc of spiral galaxies, as well as high-velocity dispersion of galaxies in clusters. Dark energy, or quintessence, is equivalent to a non-zero cosmological constant in Einstein's equations; and also there is recent support for a non-zero cosmological constant from redshift observations. All of these direct measurements can be compared with theoretical cosmology and the observed angular structure of the CMB. The theorists claim that a sequence of peaks should indeed arise from coherent acoustic oscillations in the baryon-photon fluid during an early epoch. Their amplitudes and relative positions provide another series of tests of cosmological models, and put a different series of constraints on the parameters of such models,

The attempts to explain the fundamental "cause" of such surprising "results" (that 95% of cosmological matter is unknown) seem to verge on the ludicrous. Perhaps a review of some older physical ideas may be of value. In particular, the theory of real gases, as modeled by the universal van der Waals gas in space-time, is a theory that can be expressed not only in terms of Gauss curvature (a quadratic curvature property of implicit surfaces) but also in terms of molar interaction and cohesion Cubic curvature effects (pressures which can be both positive and negative - see A. Sommerfeld), and linear Mean curvature effects (that dominate surface tension and string theories).

None of these concepts (similarly to most thermodynamic concepts) depend upon size – hence metric is not explicitly required. Curvatures are generated from the similarity invariants of a 4th order Cayley Hamilton polynomial equation, deducible from the Jacobian matrix of the 1-form of Action used to encode a specific physical system in terms of the topological structure of 4 "points". There is a fundamental universal equation that links all of the different "energy - curvature" terms.

From a topological point of view:

1. Pressure is related to volume (a "3D" thing).

2. Temperature is related to a surface (a "2D" thing).

3. Tension is related to a length (a "1D" thing).

and all of these terms appear in the equations describing a van der Waals gas. The concepts are universal in that they are deformation invariants. The universal Phase function can always be deformed into a van der Waals gas. There is always a critical point, a spinodal line of ultimate stability (a winged cusp, or swallowtail bifurcation), a binodal line (Pitchfork bifurcation) that defines phase transitions, a line of critical temperature (Hysteretic bifurcation) that separates gas from vapor and liquid condensates. None of what has been said so far requires metric, or connection, or gauge theory.

From experiment one should expect wild fluctuations on molar density near the critical point. By comparison to the cubic polynomial for van der Waals gas, the eigenvalues, ξ, play the role of molar density. The Mean curvature M is related to a "length" as a conjugate variable, and is a measure of the "surface tension". The Gauss curvature G has an "area" as a conjugate variable, and is related to temperature. The cubic curvature A has a "volume" as a conjugate variable, and is related to pressure. The "energy balance" equation (to within a "molar" factor) becomes a sum of three parts,

$$\text{String\_tension} \quad + \text{ temperature} \quad - \text{Pressure}, \tag{6.274}$$

$$\text{Mean curvature} \quad + \text{ Gauss curvature} \quad - \text{Cubic curvature}. \tag{6.275}$$

Nowhere has there been imposed a restriction on negative values of pressure, or temperature, or surface tension. The divergence = 0 condition (Ricci curvature) is automatic, for the Pfaff dimension 4 topology.

For example, the pressure - molar density diagram for the van der Waals gas is repeated below:

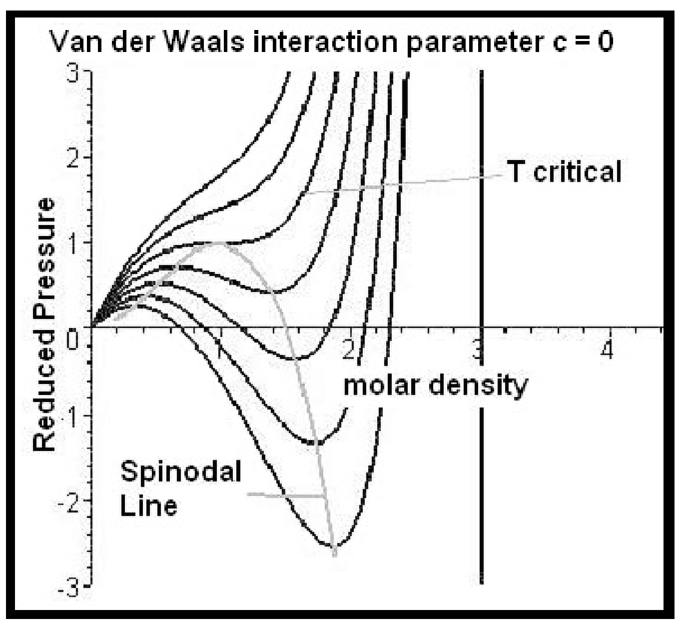


Figure 6.21 Note the region of possible negative pressure.

Note the region of "negative" pressure for the van der Waals gas, an effect known to most steam engine designers. No mention of vacuum fluctuations, or quantum effects, has been made.

Could it not be possible that dark matter is an effect due to bulk viscosity of a topologically four-dimensional physical system (the universe) evolving with a fixed point of expansion or rotation, and that dark energy is merely negative pressures associated with a non-equilibrium van der Waals gas?

Chapter 7
T0 OR NOT T0 ?

7.1 That is the topological question

This chapter was first published in 2010, and formulates my original thinking about the importance of Not-T0 topologies as applied to the thermodynamics of statistical ensembles of indistinguishable components (such as complex diffusion-wavelets, Bosons and/or Fermions, or Cartan Isotropic Spinors). In the next chapter this theme is extended in the form of a Category Theory of Thermodynamics that couples both particle and statistical thermodynamic systems of topological spaces and thermodynamic processes.

7.1.1 Introduction

Nowadays, the student of modern physics is trained to accept the jargon that there is a dualism between Waves and Newtonian Particles. A precise definition of the dualism is rarely formulated in simple terms. For example, in the domain of electromagnetism, the student is taught that electromagnetic waves (which are solutions to the Maxwell vector "wave equation") consist of an ensemble of wavelets called photons (a type of Boson that has integer spin); but, what exactly is a photon? This question continues to invoke a multitude of responses from optical engineers and physicists at the yearly SPIE conferences around the world [244].

Conjecture 28 *The difference between wavelets and particles can be found in terms of topological ideas. The ingredients of a wave topology consist of indistinguishable wavelets (Cartan Isotropic Spinors); the ingredients of a particle topology consist of distinguishable particles.*

Long ago I learned from my colleague, Professor Tom Hudson, at the University of Houston, a bit of extraordinary language insight. I will never forget a conversation we had, when he said :

"Waves transport energy and momentum without transporting mass."

The dual statement to be associated with particles is: Particles transport energy and momentum in the form of mass.

Thermodynamic systems are parts of an environment. These parts will undergo evolutionary processes due to different interactions with their environment.

1. Open thermodynamic systems admit exchange of both radiation (waves) and mass (particles) with their environment.

2. Closed thermodynamic systems admit the exchange of radiation (waves), but not mass (particles) with their environment.

3. Bounded isolated-equilibrium systems can *not* exchange radiation or mass with their environment (the Dewar bottle), but can have dynamic exchanges internally.

4. Pure Equilibrium systems are dynamically dead, internally and externally.

The objective of this chapter is to demonstrate how the fundamental topological properties of connectedness, continuity, topological dimension, evolutionary solution non-uniqueness (topological torsion), and especially the topological concept of object distinguishability, can furnish better insight into such things as the wave-particle duality, non-equilibrium thermodynamic systems, and irreversible processes. The statements above appear to be related to the mereology of inclusion, but when the subsets of the topologies of interest are expressed in terms of exterior differential forms and exterior differential form densities, the concept of exclusion dominates the concept of inclusion. In particular, the concept of Pfaff topological dimension will admit continuous topological evolution from higher to lower topological dimensions, where topological evolution from lower topological dimension to higher topological dimension is not continuous.

The approach herein will be limited to finite (but perhaps large) systems of distinguishable objects (defined as single systems, or particles), and finite systems of indistinguishable objects (defined as ensembles, or waves, or Cartan Isotropic Spinors). The restriction to finite topologies leads to three useful topological categories:

1. Metrizable discrete Hausdorff T2 topologies where all objects (particles) are distinguishable in terms of their *geometrical* evolutionary behavior (diffeomorphic covariance).

2. Non-equilibrium, Non-Metrizable Kolmogorov T0 (poset 3) topologies (which are Not T2) where all objects (particles) are distinguishable, in terms of their *topological* evolutionary behavior (homeomorphic covariance).

3. Statistical Not-T0 topologies that support thermodynamic (both macroscopic and microscopic Quantum Mechanical) systems where all objects (such as wavelet Bosons), as members of ”wave distributions” are indistinguishable.

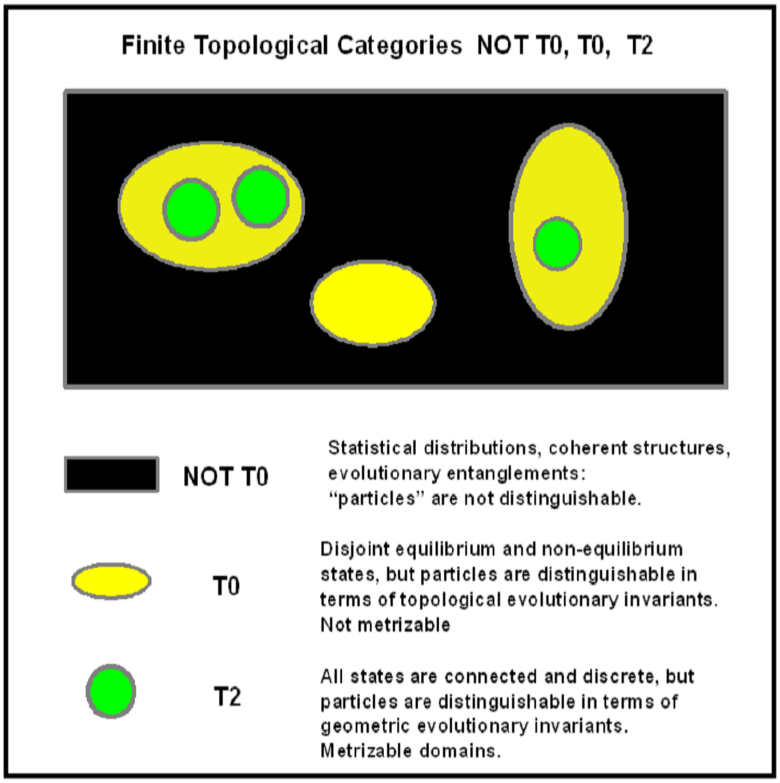


Figure 7.1 Finite Categories

Historical experimentation has recognized that thermodynamics consists of two parts: the thermodynamic system and the thermodynamic process. This idea will be discussed in terms of Category theory in the next chapter. Thermodynamic evolution describes how a system responds when subjected to various processes, and how the dynamical system reacts with its environment. Herein, the environment is presumed to be the universe itself, considered to be an Open topological space of Pfaff Topological Dimension 4. Evolution to Closed Pfaff Topological Dimension 3 spaces can be continuous, resulting in ubiquitous states that are homeomorphically deformed images of a Van der Walls gas.

A thermodynamic system (a finite topological space) of particles can be defined in terms of an exterior differential 1-form of Action per mole. Processes are defined in terms of a 4 dimensional direction field $\mathbf{V}_4$. Continuous topological evolution of a thermodynamic system can be defined in terms of a extended Homotopy propagator, called the Cartan-Lie differential, acting on the 1-form of Action. An extraordinary result was the recognition that the homotopic Lie differential encodes dynamically (and universally) the First Law of Thermodynamics, even though the direction field, $\mathbf{V}_4$, is not an element of a group. The use of the Lie differential to "define the equations of motion" is universally applicable to many familiar physical systems [1].

When the objects in the finite topological spaces are expressed in the language of exterior differential forms, a new feature is admissible. This feature involves the concept of exterior differential form densities, which are objects that are homogeneous of integer or fractal degree. These exterior differential form densities are topologically closed objects – as they are defined homogeneously in such a way that their limit sets are empty. The theorem is that such closed objects of indistinguishable objects (Boson ensemble wave distributions) can be integrated (mapped) into coherent collective structures that behave as distinguishable objects (single systems of particles). This possibility forms the basis of deRham cohomology theory, which makes a marriage of statistical indistinguishable objects (radiation distributions of Heat and Spins) and distinguishable objects (mass distributions of particles and angular momentum).

7.2 Connectedness

In the introduction to his book, R. Bott says [30]:

Remark 29 *"The most intuitively evident topological invariant of a space is the number of connected pieces into which it falls."*

A topological space is disconnected if it can be partitioned into two *disjoint* subsets (or more) which partition the topological space. In set theory these disjoint subsets are both open and closed. As has been demonstrated [294], [158], topologies that satisfy the Kolmogorov separation axiom, T0, such that closed singlets (particles) are distinguishable, can have both connected and disconnected realizations. In a topological space of 4 ingredients, X= {a,b,c,d}, there is one and only one T0 topological structure (out of 16 T0 possibilities) that is both disconnected and without segregated (isolated) closed singularities which have no boundary points and no limit points. This space has been given the name, the Kolmogorov-Cartan T0 poset 3 topology, and will serve as the basis for a topological perspective of both equilibrium and non-equilibrium thermodynamics. When the ingredients of the topology are expressed in terms of exterior differential forms, one component of the disconnection generates equilibrium systems of Pfaff Topological Dimension 2 or less, with zero topological torsion, and the second component of the disconnection generates non-equilibrium systems of Pfaff Topological Dimension 3 or more, with non-zero topological torsion. Evolution in the first component is uniquely integrable (in the classical sense), where evolution in the second component is not uniquely integrable.

The concepts of Topological Torsion, Pfaff Topological Dimension, and other properties of exterior differential systems have been discussed above in earlier chapters, and in more detail in the other monographs [1].

7.3 Continuity

7.3.1 Continuous Topological Evolution

A space is said to have a Topological Structure if it is possible to determine if a transformation of the space is continuous [85]. Continuous topological evolution is defined as a process which maps the closure of a topological space, on a domain, with a topology $\tau 1$, into the closure of a topological space, on a range, with a topology $\tau 2$. It is not necessary that such a continuous process preserves topology; e.g., $\tau 1 \neq \tau 2$. In fact, it has been demonstrated that topological change is a necessary part of continuous *irreversible* processes [294], the *arrow* of time [295], and the evolution between quantum states of different quantized values [1]. The continuous T0 processes can be between disconnected and connected topologies

A most important physical problem is how to fabricate the fundamental equations of continuous topological evolution in terms of a process, or a propagator, that is more general than the diffeomorphic, geometric, methods so prevalent in classical physics. The classic geometric methods involve kinematic constraints, conservation law invariants, balance equations, all of which are constrained by the universal dogma that unique initial data must determine (predict) unique final data. All of such classic methods used to construct the "equations of motion" impose topological constraints on a physical system and its dynamics. It must be admitted that the classic methods have useful and practical application. Within restricted domains of definition and certain limits of measurement accuracy, the classic theories appear to explain a wide range of experimental realizations. However, the classical methods are not able to describe topology change, the arrow of time, the non-uniqueness of emergent systems, nor the indistinguishability of quantum particles (Bosons and Fermions).

The solution to this problem is to be found in the concept of the Lie differential as a "extended" Homotopy propagator acting on topologies whose ingredient sets are defined in terms of the non-commuting algebra of exterior differential forms and exterior differential form densities.

7.3.2 The Lie differential and the Homotopy Propagator.

E. Cartan developed (1923) [44] and applied methods of exterior differential forms to various problems in physics. Cartan utilized a primitive version of a what is now known as a homotopy evolution operator, based upon groups of vector fields; he used the technique to develop "equations of motion" and their evolutionary solutions. Much later on, the homotopy evolution operator was redefined by Slebodzinski (1959-1963 in Polish, 1970 in English) [259] in terms of pseudo groups (no inverses, no identity elements) of vector fields and the exterior differential, d, acting on exterior differential forms. Slebodzinski defined this homotopy operator as the Lie derivative. In this article, the definition of the homotopy operator is extended to the concept of the Lie differential, which admits topological fluctuations, and leads ultimately to the explanation irreversible processes and the evolution of both distinguishable objects (particles) and indistinguishable objects (Boson-Fermion wave distributions and statistical ensembles).

One of the main features of the Cartan methods, using exterior differential forms, is that the concepts are independent from a choice of coordinates. Moreover, the Cartan methods can describe evolutionary integral invariants that do not depend upon geometric rigidity. About 1930-34, Van Dantzig demonstrated that Maxwell's partial differential equations were independent of a choice of coordinates [292]. In 1965-1980, it became obvious (to me) that Cartan's exterior differential forms were fundamental to a universal description of evolutionary processes. It particular, the early concept of the Lie derivative, originally based upon kinematic ideas ($d\mathbf{x}/dt$ - $\mathbf{V}$ = 0) of 1-parameter groups and fluid flows [**?**], could be used to advantage in describing a topological perspective of physics. This observation led to three publications which utilized the Lie derivative acting on exterior differential forms [200], [201], [209] to generate the "equations of motion". The first article utilized the Lie derivative and the theory of integral invariants to construct modifications of Hamilton's principle to include dissipative phenomena from a topological perspective. The Hamiltonian (and Lagrangian) equations of motion became special cases (extremal fields) of the more general Lie derivative point of view. The homotopic properties of the Lie derivative were recognized, early on, but not applied in full detail. The second article demonstrated how the Lie derivative could be used to describe irreversible dissipation in hydrodynamic flows. The third publication used the Lie derivative to generate *topological* change of closed integrals of exterior differential forms, and related this idea of the evolution of period integrals and topological change of Pfaff topological dimension. The quantum numbers generated by period integrals were related to Brower's degree of a map theorem.

Ultimately (1985-1991), it became apparent that the Lie derivative concept (based upon processes that were generators of 1-parameter groups) could be extended to include processes, $\mathbf{V}_4$, that were were not constrained to be 1-parameter groups. That is:

$$\text{Generalized Homotopy:} \qquad H \circ d + d \circ H, \qquad H = i(\rho\mathbf{V}), \tag{7.1}$$

$$\text{Extended Lie Differential} \ : \ L_{(\rho\mathbf{V})}\omega := i(\rho\mathbf{V})\circ\, d\circ\omega \quad + \quad d\circ\mathbf{V}_4\circ\,\omega), \tag{7.2}$$

$$\text{with a differential constraint} \ : \ d\mathbf{x} - \rho\mathbf{V}dt = \Delta \neq 0. \tag{7.3}$$

The Lie differential acting on exterior differential forms is in effect a homotopy propagator, which, when constrained to geometric settings, is known as the convective, or fisherman's derivative [8]. The exterior differential forms are not the usual sets for a classical topology, for they obey a non-commutative law of exterior multiplication. Application of the Lie differential method leads to a clean explanation of the arrow of time, a universal method of describing thermodynamic irreversibility, and the concept of emergence (far from equilibrium) to spaces far from equilibrium, as a result of a dissipative process. The latter result justifies Prigogene's conjectures [195].

7.4 Evolutionary non-uniqueness and Topological Torsion

7.4.1 Topological Thermodynamic Systems

Once the concept of a universal thermodynamic process had been formulated in terms of the homotopic Lie differential acting on exterior differential forms, the next problem was to formulate a precise universal topological description of the thermodynamic system, in terms of exterior differential forms. It must be admitted that the "discovery" of the universal topological thermodynamic theory was a moment of epiphany which took place about 1987. Prior to 1991 it was recognized that the elements, $\{A, dA, A\hat{}dA, dA\hat{}dA\}$, of the Pfaff Sequence generated by an exterior differential 1-form, A, of 4 ingredients, could be put into correspondence with a Kuratowski topological structure. In particular, the number of non-zero entries in the Pfaff Sequence could be used to define the irreducible Pfaff Topological Dimension* of the topology associated with a thermodynamic 1-form of Action, PTD(A).

The justification centered on the fact that exterior differential, d, and the Identity operator, $\mathbb{I}$, define a Kuratowski closure operator, $(\mathbb{I} \cup d)$. The exterior differential, d, acting on an exterior differential form, ω, generated the "limit points", $d\omega$, of the exterior differential form. The hypothesis could be validated by the derivation of the topological structure. With the algebraic help of Phil Baldwin, a post-doc at the University of Houston, the details of the topological structure were verified, and defined as the Cartan Topological Structure. The topological structure could be based on exterior differential forms of odd degree. The results were presented at the Santa Barbara conference in 1991 [12]. Ultimately, it was determined that any exterior differential 1-form could be could be used to construct a Kuratowski topology, with the topology dependent upon the Pfaff Topological Dimension, and the functional coefficients of the selected 1-form.

The important point I gleaned from the Cartan Topological Structure (based on the elements of the Pfaff Sequence of a 1-form) was that the topology of 4 ingredients so constructed was a *disconnected* topology without segregated points, but with two subsets that were both open and closed, and each with an empty topological boundary. One component of the disconnection was representative of uniquely integrable equilibrium thermodynamic systems, and the other, non-integrable, component was representative of non-equilibrium systems.

It was only much later that I was realized that the Cartan Topological Structure, based on exterior differential forms, was similar to the topological structure, based upon sets, and known to topologists as the poset 3 Kolmogorov T0 topology of 4 ingredients (See Appendix 1). There are 16 distinct Kolmogorov T0 topologies of 4 ingredients. Ten of the 16 topologies are connected topologies. Five of the 16 topologies are disconnected topologies, but these topologies have closure singletons (particles) that are segregated, in the sense that they have no boundary and no limit points. The poset 3 topology is the only disconnected T0 topology of 4 ingredients that does not have closure singletons which are segregated. Much of the story is presented in [294], where the universal topological structure was defined at first as the Cartan topology structure, and later on as the Kolmogorov-Cartan T0 topological structure, with the symbol KCT0.

The 6 "sequentially ordered poset" diagrams which are associated with the disconnected T0 topologies of 4 ingredients are displayed in the Figure below:

*Compare to the idea of essential parameters [73] or the class of a form [82].

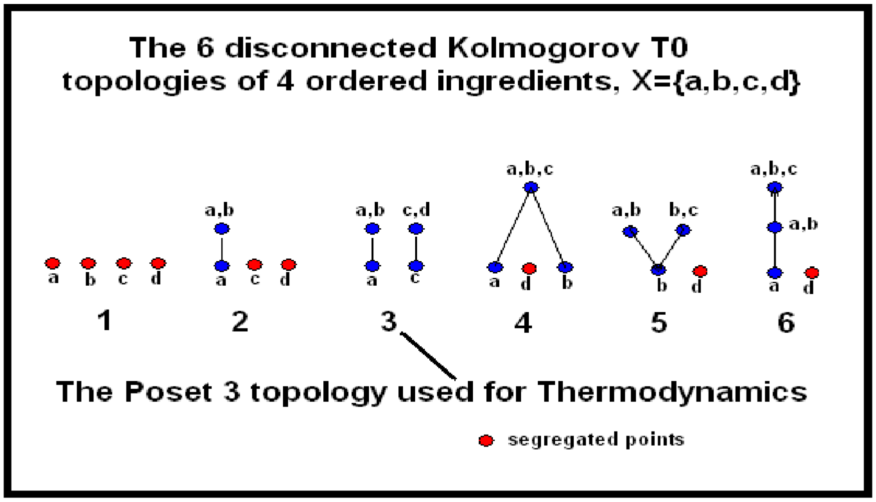


Figure 7.2 Disconnected T0 topologies

Note that the poset 3 case has no segregated sets that are disconnected singletons, but it does have two connected sets which are disjoint. The 10 other T0 topologies of 4 ingredients (not displayed) consist of sets connected in one tree component.

The topological properties of the Pfaff Sequence permit a correlation to be made between topologically different thermodynamic systems. As mentioned above, the Pfaff Topological Dimension of the 1-form of Action, A, is equal to the number of non-zero entries in the Pfaff sequence, $\{A, dA, A\hat{}dA, dA\hat{}dA\}$. The **PTD**(A) correlates with the 4 distinct (classical) categories of thermodynamic systems: equilibrium, "isolated" equilibrium, closed, and open systems.

.		**PTD and Thermodynamic Systems**	(7.4)
$\{A, 0, 0, 0\}$	PTD 1	**Equilibrium**	(7.5)
$\{A, dA, 0, 0\}$	PTD 2	**"Isolated" Equilibrium**	(7.6)
$\{A, dA, A\hat{}dA, 0\}$	PTD 3	**Closed**	(7.7)
$\{A, dA, A\hat{}dA, dA\hat{}dA\}$	PTD 4	**Open**	(7.8)

The suggestive topological definitions of the words – open, closed, isolated, and equilibrium – are technically a bit different from the classic thermodynamic definitions. Equilibrium and isolated equilibrium systems are uniquely integrable systems, equivalent to stationary and steady state compositions that do not exchange indistinguishable objects (radiation) or distinguishable objects (particle mass) with the environment. Closed thermodynamics systems exchange radiation with the environment, but not particles. Open thermodynamic systems can exchange both radiation and particles with the environment.

7.4.2 Frobenius Failure and Topological Torsion

It is remarkable, the solutions to a differential equation, $A = 0$, for a 1-form of PTD(A) = 2 (or less) can always be uniquely integrated (to within an integrating factor) for any number of independent geometrical arguments (dimensions). This result is known as the Frobenius theorem [82]. The same positive response of unique integrability is not true if PTD(A) > 2. This is essentially why the Newtonian 2-body problem is said to be uniquely soluble, but the 3-body problem is not. If PTD$(A) > 2$, and solutions to the differential equation, $A = 0$, exist, they are not unique. In differential geometry, such concepts of non-uniqueness lead to the theory of envelopes (as in Huygen wavelets, and the van der Waals equation of state) and edges of regression (which describe wakes in fluids, and the critical isotherm in thermodynamic equations of state). The bottom line is that the Dogma of unique initial data yielding unique final data does not apply to non-equilibrium thermodynamic, physical, systems with non-zero topological torsion.

However, the non-uniqueness of continuous topological evolution is *always* associated with a non-zero 3-form, $A\hat{}dA$, defined as the Topological Torsion 3-form. Note that a non-zero 3-form implies that there are at least 3 non-zero entries in the Pfaff Sequence. If the 4-form, $dA\hat{}dA$, is zero, then the irreducibly minimum number of functions required to describe the topological structure generated by the KCT0 topology is 3. If $dA\hat{}dA \neq 0$ then the irreducible number of functions required to describe the topological structure is 4. The number of coordinate (geometric) functions can be larger than the irreducible number of topological functions, which can have geometric coordinates as arguments.

Note that Eisenhart [73] describes continuous transformations as formulated terms of functions of $i = 1..n$ geometric variables and k parameters, $f^i = f^i(x^i; t^k)$. He states that it is possible to use the equations to reduce the number of parameters to an essential number m of parameters less than or equal to k, depending on the rank of a sequence of differential values. The Pfaff Sequence and the Pfaff Topological Dimension m is an equivalent formulation of the Eisenhart ideas concerning continuous semigroups of transformations with m essential parameters. The number of essential parameters defines the topological dimension of the problem.

For topologies of 4 ingredients with a non-zero differential volume element, (the Top Pfaffian) Ω_4, there is a correspondence between the coefficients of the 3-form $A\hat{}dA$ and the Homotopic 3-form, $H \circ \Omega_4$,

$$A\hat{}dA = H \circ \Omega_4 = i(\mathbf{T}) \circ \Omega_4. \tag{7.9}$$

That is, the functional coefficients of the 1-form, A, and the wedge product of A with the functional forms of the Kuratowski limit points, dA, create an intersection 3-form, whose components, $\mathbf{T}$, can be used as a (special) direction field of a thermodynamic process. This special direction field is orthogonal (transverse) to the component functions that define the 1-form of Action:

$$A\hat{}A\hat{}dA = (i(\mathbf{T})\mathbf{A}) \circ \Omega_4 = 0. \tag{7.10}$$

Vector direction fields that have this orthogonality property are said to define transverse direction fields, $i(\mathbf{T})\mathbf{A} = \mathbf{0}$, *associated* to the 1-form of Action. From a thermodynamic point of view, these direction fields define locally *adiabatic* processes. These locally adiabatic processes which are proportional to the topological torsion vector may, or may not, be reversible. To be irreversible, the Kuratowski limit point (divergence) of the topological torsion 3 form "current" must not vanish; hence the PTD of the 1-form, A, must be 4, and the thermodynamic system must be open in the sense that both "radiation heat" and "particles" can be exchanged with the environment.

7.5 Topological Thermodynamics and Indistinguishable Statistical Objects

The Axioms of Topological Thermodynamics which were based upon topologies of distinguishable objects are to be found in Chapter 1, and in more detail - with many examples and applications - in the earlier monographs of this series [1]. The major achievement is the correspondence between the Homotopic evolutionary operator, the Lie differential, and the First Law of Thermodynamics. A more abstract presentation of the details of the Lie differential can be found in [255].

In particular, the extended homotopy generated by the Lie differential permits evolutionary properties to be partitioned into three categories:

1. The Lie differential can describe evolutionary processes that preserve geometric properties as invariants (diffeomorphisms)

2. The Lie differential can also describe evolutionary processes that preserve topological properties as invariants (homeomorphisms), but would admit deformations such that geometric properties of size and shape are not invariant.

3. Finally (and remarkably) the Lie differential can describe continuous processes which are not homeomorphic and not diffeomorphic; such processes permit topological as well as geometric evolution.

The remarkable result is that if a Thermodynamic System can be defined by an exterior differential 1-form of Action (having particle like properties per unit source, mole or charge), with different Pfaff Topological Dimensions, then the Homotopy propagator in the form of the Lie differential acting on the exterior 1-form of Action (the thermodynamic system) represents an equation of continuous evolution – an equation which is abstractly equivalent to the First Law of Thermodynamics. The method works even though topology changes in a geometrically discrete manner, a result that directs us to the application of continuous topological evolution to problems in quantum mechanics. The fundamental property of quantum mechanics is that quantum objects (Bosons and Fermions) are indistinguishable.

Conjecture 30 *(2010) Can topological sets without separation axioms be included in a topological perspective of thermodynamics? (2011) Yes.*

It is apparent that the concept of heat is related to radiation, and that radiation is an artifact of quantum mechanical ensembles (Planck's radiation formula is independent from size, shape, or topology. In that which follows, the combination of Topologies without separation axioms will be added to the previous formulation of the Axioms of Thermodynamics.

7.5.1 Separation Axioms and Distinguishability

Definition 31 *(From Wikipedia) A finite topology on a set X is defined as a subset of P(X), the power set of X, which includes both* ■ *and X and is closed under finite intersections and finite unions.*

It is remarkable to note for finite spaces that there is a class of topologies for which all of the singlets are distinguishable; each singlet has a distinct, different, topological closure. Such topologies for which the all singlets are distinguishable will be defined as the Kolmogorov T0 "particle" topologies.

However, there is another class of finite topologies for which at least two of the singlets are indistinguishable. Such topologies for which at least two of the singlets are indistinguishable will be defined as the Not-T0 "wavelet, complex diffusion, or statistical" topologies. If the top set X consists of 4 "points", X={a,b,c,d}, then such domains support 17 different Not-T0 topologies, and 16 different T0 topologies. See the Appendix 1 of Vol6 for the Complete Lattice structures of both the particle T0 and the wavelet Not T0 topologies.

There are extremal topologies for each class. The T0 class contains a "discrete particle" topology where every subset of the top set, X, is both open and closed. The discrete topology is the only T0 topology which is metrizable.

The Not-T0 class contains an "indiscrete wavelet" topology where all subsets are neither open nor closed; but all subsets are dense in X. The "indiscrete wavelet" topology is defined by the Lattice Structure, LS = {{ },X}.

A classical result with physical application will be the description of which topologies in the two categories permit the subsets to be described in terms of exterior differential forms (or differential form densities). The Kuratowski limit point operator can be represented by the exterior differential, d, and the continuous evolution of the subsets can be described by the Homotopic formulation known as the Lie differential with respect to vector and spinor direction fields.

7.6 The Boundary of a Boundary of a subset is NOT always zero

To better understand the concepts of lattice and topological structures, a number of examples have been constructed and appear in Appendix 1. The presentation utilized a Maple program to execute the tedious computations, for finite, low dimensional topologies. The first versions were started in the latter part of 2009, and distributed to the contacts that I made at the Hacettepe Topology 2009 conference in Turkey. The program has been updated and extended. The computational results for over

100 Lattice Structures appear in the appear in the Appendix, and the maple code is available on request.

A study of the program results led to some surprises. Recall that about 1973, the buzz words "the boundary of a boundary† is zero" were used to inject topological thinking (Homology) into gravitational theories developed by Wheeler [311]. Now, the phrase appears often enough to be classified as Dogma! However, as a theorem for all topologies, it is not universally true:

$$Bnd\ (Bnd(S)) = 0 \text{ is not universally true.} \tag{7.11}$$

In fact, all subsets S (excluding the top set, X) of the non-metrizable $T0$ *connected* topologies have boundaries, but for any given boundary, the boundary of that boundary is NOT the empty set.

Theorem 32 $Bnd(Bnd(S)) \neq 0$ *if* S *is a subset of a finite connected T0 topology.*

Only the disconnected $T0$ topologies have subsets that have an empty boundary, and these subsets satisfy the theorem because the boundary of an empty boundary is empty.

A key property of finite Hausdorff $T2$ metrizable topologies is that all subsets have an empty boundary. So, in this sense of a $T2$, metric based topologies and differentiable manifolds the theorem is true, but only for equilibrium thermodynamic systems. The theorem is not valid for disconnected, not metrizable, $T0$ topologies, which, however, can be shown to encode non-equilibrium thermodynamics of particle systems.

It should be noted that all subsets of the "indiscrete" $Not - T0$ topology of two ingredients, $\tau = \{\{\ \}, X\}$, have boundaries which are dense in X, and satisfy the boundary of a boundary = 0 jargon.

$$\text{For } \tau = \{\{\ \}, X\} \tag{7.12}$$
$$Bnd(S) = X, \quad Bnd(X) = \{\ \}, \tag{7.13}$$
$$\text{hence } Bnd(Bnd(S)) = \{\ \}. \tag{7.14}$$

There are only two disconnected examples of the 17 Not-T0 topologies which have the property that $Bnd(Bnd(S)) = \{\ \}$, for all subsets. (See the Appendix 1)

Physically, waves (as solutions of systems of second order PDE's) are indistinguishable objects unless they are recognized by topological defects, such as shear discontinuities or shocks or fixed points, or collective superposition of indistinguishable states that represent domains of closed exterior differential form densities.

†The boundary operator is represented by the symbol ∂.

Conjecture 33 *There is a useful physical correspondence between the indiscrete collection of indistinguishable objects (waves of an ensemble, encoded in terms of Cartan Isotropic Spinors) in a Not T0 topology, and the discrete collection of distinguishable objects (encoded as singlet particles) in a T2 topology. The correspondence will involve closed integrals over collective coherent homogeneous domains of closed exterior differential form densities.*

The complete lattice structure for the extremal indiscrete topology of N ingredients $X = \{a, b, c, d...\}$ (which does not obey the T0 or T2 separation axioms) is given below:

Note that all subset boundaries (except for the top set X) and all subset closures (for the indiscrete topology) are dense in X. All subsets of the indiscrete topology are indistinguishable and have the same closure and boundary. All singlets are adherent points, and are torsion free, in that sense that the singletons have no intersection with their limit sets. Other examples can be constructed.

Conjecture 34 *Can Boson and Fermion Quantum properties of indistinguishability be encoded in terms of the indistinguishable subsets that have dense boundaries in Non-T0 topologies of topological dimension 4 or 3?*

Examples of both connected and disconnected $Not - T0$ topologies can be constructed which demonstrate subset examples where the boundary of a boundary does equal the empty set, and other examples where the boundary of a boundary does not equal the empty set.

7.7 Cohomology is more useful to physics than Homology.

In physics, cohomology is of more use than homology, for the key operators in Cohomology theory are the limit point operator and the identity operator. In homology theory, the key operators are the boundary operator and the interior operator. The definitions of closure presume that for any subset, S, of a topology,

$$\{S \cup Limit\ points(S)\} = Closure(S) = \{Interior(S) \cup\ boundary(S)\}. \tag{7.15}$$

A principal objective of these monographs [1] has been to find a useful topological perspective that will give a universal foundation and better understanding of non-equilibrium thermodynamics and irreversible processes. It was discovered long ago (1985-1987) that Cartan's exterior differential was formally equivalent to the Kuratowski closure operator, and that a topology whose sets were exterior differential forms was a member of a disconnected topology. It was not until 2009 that it was realized that the Cartan disconnected topology of exterior differential forms was formally equivalent to the Kolmogorov T0, poset 3, finite topology of 4 ingredients. It was easy to demonstrate that, in terms of the Complete Topological Structure of the poset 3 topology, the limit points of any subset of differential form ingredients

X=$\{A, F = dA, H = A\hat{}F, K = dH\}$, is obtained by applying the exterior differential to that subset. In these terms, the limit of a limit set *always* vanishes for C2 differentiable sets:

$$d(d(S)) = 0. \tag{7.16}$$

For example, from a physics point of view, the electromagnetic Field intensity 2-form, $F = dA$, is formally the limit set of the 1-form, A, of electromagnetic potentials. In other words the electromagnetic **E** and **B** fields are limit points of the electromagnetic vector and scalar potentials! The "closure" of the Field intensities (dF = 0) establishes constraints, which lead to a system of partial differential equations, precisely equivalent to Maxwell's partial differential equations of electromagnetism. The property is universal and applies to fluids, plasmas, and any other thermodynamic system generated by a 1- form of Action, A.

In that which follows, by use of certain Complete topological structures as examples, it is possible to define a marriage between the Cartan exterior differential operator, d, and the Kuratowski closure operator. There is a unique T0 topology of 4 ingredients based on the Pfaff Sequence of p-forms, $\{\ A, F = dA, H = A\hat{}F,\ K = F\hat{}F = dH\}$ that demonstrates the equivalence between the exterior differential and the Kuratowski limit point operator. Similarly there is a unique T0 topology for finite T0 topological structures of 3 ingredients based on the Pfaff Sequence of p-forms, $\{A, F = dA, H = A\hat{}F\}$.

It is remarkable that the discrete T2 Hausdorff metrizable topology can also be used to demonstrate a correspondence between Cartan's exterior differential operator and the Kuratowski limit point operator, but now the ingredients (objects) of the T2 topology belong to the class of closed exterior differential form densities, $\widetilde{\omega}$. The differential form densities of interest are differential forms, ω, that admit integrating factors, $\rho(x^k)$, such that:

$$d\{\widetilde{\omega}\} = d\{\rho(x^k)\omega\} = d\rho\hat{}\omega + \rho d\omega = 0. \tag{7.17}$$

The "integrating factor", $\rho(x^k)$, can be chosen as a divisor that makes the exterior differential form homogeneous. For example, let

$$A = xdy - ydx, \tag{7.18}$$

$$\rho = 1/(ax^p \pm by^p)^{n/p}, \tag{7.19}$$

then

$$\widetilde{A} = \rho(xdy - ydx) = (xdy - ydx)/(ax^p \pm by^p)^{n/p} \tag{7.20}$$

$$\text{such that for } n = 2, \text{ and any a,b,p} \tag{7.21}$$

$$d\widetilde{A} = d\rho\hat{}\omega + \rho d\omega = 0. \tag{7.22}$$

For n=2, the 1-form density, $\widetilde{A}$, is homogeneous of degree 0. Scaling of all of the variables, x , dx and y dy, constant factor leads the same value of $\widetilde{A}$. It is precisely this condition that leads to the concept of the deRham period integrals, which are topological deformation invariants, and have been called topological quantum numbers. For electromagnetic systems:

$$\oint \widetilde{A} = n \cdot \hbar/e. \tag{7.23}$$

The integration is over a closed cycle (s), and is equal to zero if the integration chain is a boundary.

The same method of constructing homogeneous differential form densities can be used to construct the ingredients $\{\widetilde{A}, \widetilde{F}, \widetilde{H}, \widetilde{K}\}$. The result is that the Cartan topology of homogeneous differential form densities, $\{ \widetilde{A}, \widetilde{F}, \widetilde{H}, \widetilde{K}\}$, is equivalent formally to the complete topological structure of T2 Hausdorff topology. In addition, the exterior differential is a limit point generator, but the limit point set is the empty set.

Perhaps one of the most interesting of the closed set possibilities is given by the system of homogeneous 1-forms defined as

$$\widetilde{A} = (\Psi^* d\Psi - \Psi d\Psi^*)/(\{\Psi^*\}^2 + \{\Psi\}^2) \tag{7.24}$$

$$= (\Psi^* d\Psi - \Psi d\Psi^*)/(\Psi^* \Psi), \tag{7.25}$$

which is known as the probability current in Schroedinger-Copenhagen quantum mechanics. The new result is that the Hausdorff T2 topology can be used to define the domains of deRham cohomology which appear to be constrained to equilibrium based thermodynamic systems.

7.7.1 Comparisons of the Indiscrete and the Discrete topologies

It is remarkable that the "discrete (or extremal) topology" based on ALL of the elements of the power set of the top set, X=$\{x^k\}$, of any finite number of ingredients, $\{x^k\}$, is always Hausdorff T2 (geometrical and metrizable). All subsets of the power set are (Topologically) Torsion Free, isolated, and Segregated – no matter how many finite ingredients are in the top set, X. All subsets of the discrete topology have empty Boundaries, empty Limit sets, and all subsets are closed. The Topological Torsion free concept is the basis of isolated thermodynamic equilibrium. Hence, the Hausdorff T2 topology appears to be the domain of equilibrium, but not statistical or non-equilibrium thermodynamics, as the T0 separation axioms permit the "particles" to be distinguishable in terms of geometrical properties. The Wheeler idea that the boundary of a boundary is zero is true for all subsets of the discrete T2 metric based geometric topology. However there are Kolmogorov topologies where the boundary of a boundary is not zero.

X ={A,F,H,K}

LS ={{ },{A},{F},{H},{K},{A,F},{A,H},{A,K},{F,H},{F,K},{H,K},{A,F,H},{A,F,K},{A,H,K},{F,H,K},{A,F,H,K}}

The Lattice Structure, LS, generates a T2, METRIZABLE, Discrete, Topology, where all subsets are Distinguishable.
If A,F,H,K are homogenious differential form densities that are closed with respect to the exterior differential, then a correspondence is established between the T2 topology and exterior differential form densities.
The integrals of the differential form densities map to deRham period integrals, and quantum mechanics.

Is LS a topology = true, connected = false, Kolmogorov.T0 = true, Hausdorff.T2 = true

The COMPLETE Lattice Structure based on LS and all subsets of the Powet set of X

Subset S	Int(S)	Ext(S)	Bnd(S)	Clo(S)	Lim(S)	IsoSeg(S)	IsoLip(S)
{A}	{A}	{F, H, K}	{ }	{A}	{ }	{A}.Seg	{ }
{F}	{F}	{A, H, K}	{ }	{F}	{ }	{F}.Seg	{ }
{H}	{H}	{A, F, K}	{ }	{H}	{ }	{H}.Seg	{ }
{K}	{K}	{A, F, H}	{ }	{K}	{ }	{K}.Seg	{ }
{A, F}	{A, F}	{H, K}	{ }	{A, F}	{ }	{A, F}.Seg	{ }
{A, H}	{A, H}	{F, K}	{ }	{A, H}	{ }	{A, H}.Seg	{ }
{A, K}	{A, K}	{F, H}	{ }	{A, K}	{ }	{A, K}.Seg	{ }
{F, H}	{F, H}	{A, K}	{ }	{F, H}	{ }	{F, H}.Seg	{ }
{F, K}	{F, K}	{A, H}	{ }	{F, K}	{ }	{F, K}.Seg	{ }
{H, K}	{H, K}	{A, F}	{ }	{H, K}	{ }	{H, K}.Seg	{ }
{A, F, H}	{A, F, H}	{K}	{ }	{A, F, H}	{ }	{A, F, H}.Seg	{ }
{A, F, K}	{A, F, K}	{H}	{ }	{A, F, K}	{ }	{A, F, K}.Seg	{ }
{A, H, K}	{A, H, K}	{F}	{ }	{A, H, K}	{ }	{A, H, K}.Seg	{ }
{F, H, K}	{F, H, K}	{A}	{ }	{F, H, K}	{ }	{F, H, K}.Seg	{ }
{A, F, H, K}	{A, F, H, K}	{ }	{ }	{A, F, H, K}	{ }	{A, F, H, K}.Seg	{ }

Figure 7.3 The discrete extremal Kolmogorov T0 Topology of 4 ingredients (particles)

On the other hand, the "indiscrete" topology", based on only the top set X={x^k} of any finite number of ingredients, {x^k}, and the empty bottom set, { }, is always Not-T0 and connected. The indiscrete topology does not obey any separation axioms, so the closures of all subsets of the power set are indistinguishable; hence the indiscrete topology could be a topological basis for statistical thermodynamics of indistinguishable ingredients. All subsets (except the top set, X) of the power set of X have empty interiors and are dense in the Top Set X. All of the singlet subsets are torsion-free, but all doublets, triplets, etc. have non-zero torsion. These higher level sets are isolated in the sense that there are adherent points in their closures that are not in their limit sets. The boundaries of all subsets are equal to the top set, X, which has an empty boundary. Hence the Wheeler idea that the boundary of a boundary is zero can be valid for both the extremal "discrete" topology and the extremal "indiscrete" topology. The indiscrete topology is not metrizable with a finite norm. However, the use of a pseudometric (with zero norm) is applicable to the indiscrete topology. The indistinguishable ingredients can be interpreted in terms of Cartan Isotropic Spinors, which have a zero norm in terms of a finite distance function.

X ={A,F,H,K}

LS = { { }, {A, F, H, K} }

The Lattice Structure, LS, is a Not-T0, Indiscrete, Topology, all subsets are Indistinguishable.

Is LS a topology = true, connected = true, Kolmogorov.T0 = false, Hausdorff.T2 = false

The COMPLETE Lattice Structure based on LS and all subsets of the Powet set of X

Subset S	Int(S)	Ext(S)	Bnd(S)	Clo(S)	Lim(S)	IsoSeg(S)	IsoLip(S)
{A}	{ }	{ }	{A, F, H, K}	{A, F, H, K}	{F, H, K}	{A}	{ }
{F}	{ }	{ }	{A, F, H, K}	{A, F, H, K}	{A, H, K}	{F}	{ }
{H}	{ }	{ }	{A, F, H, K}	{A, F, H, K}	{A, F, K}	{H}	{ }
{K}	{ }	{ }	{A, F, H, K}	{A, F, H, K}	{A, F, H}	{K}	{ }
{A, F}	{ }	{ }	{A, F, H, K}	{A, F, H, K}	{A, F, H, K}	{ }	{A, F}
{A, H}	{ }	{ }	{A, F, H, K}	{A, F, H, K}	{A, F, H, K}	{ }	{A, H}
{A, K}	{ }	{ }	{A, F, H, K}	{A, F, H, K}	{A, F, H, K}	{ }	{A, K}
{F, H}	{ }	{ }	{A, F, H, K}	{A, F, H, K}	{A, F, H, K}	{ }	{F, H}
{F, K}	{ }	{ }	{A, F, H, K}	{A, F, H, K}	{A, F, H, K}	{ }	{F, K}
{H, K}	{ }	{ }	{A, F, H, K}	{A, F, H, K}	{A, F, H, K}	{ }	{H, K}
{A, F, H}	{ }	{ }	{A, F, H, K}	{A, F, H, K}	{A, F, H, K}	{ }	{A, F, H}
{A, F, K}	{ }	{ }	{A, F, H, K}	{A, F, H, K}	{A, F, H, K}	{ }	{A, F, K}
{A, H, K}	{ }	{ }	{A, F, H, K}	{A, F, H, K}	{A, F, H, K}	{ }	{A, H, K}
{F, H, K}	{ }	{ }	{A, F, H, K}	{A, F, H, K}	{A, F, H, K}	{ }	{F, H, K}
{A, F, H, K}	{A, F, H, K}	{ }	{ }	{A, F, H, K}	{A, F, H, K}	{ }	{A, F, H, K}

Figure 7.4 The indiscrete extremal Not-T0 Topology of 4 ingredients (waves)

It should be mentioned that the situation is more complicated for the finite Kolmogorov T0 topologies, where the Wheeler idea fails. The boundary of a boundary is NOT zero for all subsets of the T0 topologies based on a top set X=$\{x^k\}$ of any finite number of ingredients, $\{x^k\}$. The Kolmogorov topologies appear to be the domain of non-equilibrium, but not statistical, thermodynamics.

Conjecture 35 *Can the coexistence of "complex wavelet" topologies (QM) and non-metrizable "particle" topologies be used to formulate a marriage between Quantum concepts and gravitational fields.*

Chapter 8
THE CATEGORY THEORY OF TOPOLOGICAL THERMODYNAMICS.

8.1 Topological Spaces of exterior differential forms

8.1.1 Prologue

The Category theory of Topological Thermodynamics and Continuous Topological Evolution can be used as an abstract, but universal, foundation for understanding, among other things, scale invariance, chaos, emergence, self-similarity, non-equilibrium dynamical systems, irreversible non-deterministic processes, and other non-linear, and even statistical, phenomena. The abstract Category theory, as an exterior differential system, has dynamical solutions which can be put into correspondence with numerous physical experiments in different disciplines.

8.2 Introduction

Mathematicians consider a Category Theory, C, to be a collection of abstract objects, ob(C), constructed in terms of ingredients, with morphsims that map an object X to another object Y. For example, the abstract objects might be the vertices of a lattice structure, LS, and the morphisms might be defined abstractly as directed arrows from one vertex to another. Herein, the interest is focused on the abstract Category of objects and morphisms that represent Topological Thermodynamics and Continuous Topological Evolution. The objects are finite topological spaces of sets whose ingredients are exterior differential forms and/or differential form densities. The morphisms between sets are constrained to be classes of topologically continuous, but possibly non-invertible, maps. The coefficient functions of the exterior differential forms define abstract, and universal, thermodynamic systems. The topologically continuous morphisms define abstract thermodynamic processes. These morphisms include diffeomorphic processes that preserve geometry and metric scales, homeomorphic processes that preserve deformable topological features, which are scale invariant, and irreversible processes that permit topological change.

For finite topologies of 4 ingredients (as used in this article) there are 33 distinct lattice structures that generate different topologies. Sixteen of these topologies have T0 Kolmogorov separation axioms which can be used to distinguish singlet sets as "particles". One of these 16 is special in that it has stronger, T2, separation axioms that permit the special T2 Hausdorff topology to be metrizable. Real geometric topologies are Hausdorff T2. All subsets of the finite Hausdorff Topology are discrete, not connected, and both open and closed. The T2 separation axioms permit "particles" to be distinguished by geometrical processes. The other 15 T0 Topologies are not metrizable over the reals, but the T0 separation axioms permit particles to be distinguished by topological processes.

Seventeen of 33 distinct topologies of 4 ingredients do not support separation axioms and cannot be used to specify "distinguishable" particles. One of these Not-T0 topologies is special as it has properties that are conjugate to the geometric metrizable Hausdorff T2 topology. This special Not-T0 topology is the indiscrete topology, which has only two subsets, the empty set and the top set (of 4 ingredients). All subsets are connected, indistinguishable, and dense. Only the empty set and the top set are both open and closed.

Remarkably, the Not-T0 topologies have seen little practical application, yet such topologies appear to appropriate for the study of ensembles of indistinguishable parts (such as in topological statistics, and other complex wavelet solutions to Boson and Fermion systems). For example, Entropy will have two partitions, one due to distinguished particle systems and the other due to indistinguishable ensemble systems. But be aware: T0 topologies can not be used to faithfully represent ensemble systems, and Not-T0 topologies cannot faithfully represent particle systems. Particle systems are T0, wavelet systems are Not-T0.

The concept of a Category theory admits both topological partitions to be coexistent.

8.3 Particle Thermodynamic Systems

The subset ingredients of the topologies considered herein are assumed to be exterior differential forms, whose arguments are functions on differential varieties, $\varphi = \{x^k, dx^k\}$. In particular, consider the exterior differential 1-form, $A = A_k(x^m)dx^k$. Any 1-form, A, can be use to construct the Pfaff Sequence, PS(A)

$$\begin{aligned} PS(A) &= [A,\ dA,\ A\hat{}dA,\ dA\hat{}dA] = [A,\ F,\ H,\ K] \qquad (8.1) \\ & \quad A \text{ is a1-form, } F \text{ is a 2-form, } H \text{ is a 3-form, } K \text{ is a 4-form} \end{aligned}$$

by use of the exterior differential, d, and the exterior product, ˆ. The number of non-zero elements in the Pfaff Sequence determines the Pfaff Topological Dimension, PTD(A) , (or irreducible class) of the exterior differential system. The PTD determines the minimum number of functions and differentials need to locally define the topological properties generated by the 1-form of Action. For symplicity this presentation considers a maximum PTD(A) = 4.

Using the fact that the exterior differential, d, and the Identity, define a Kuratowski closure operator, $Kcl = \mathbb{I} \cup d$, it becomes apparent that the exterior differential of a p-form, Σ, defines the limit set, $d\Sigma$, as a p+1 form. Hence the complete topological structure can be constructed from a 1-form of Action, A. The Kuratowski basis for the topology is composed of the elements in the Pfaff Sequence:

$$\text{Kuratowski Basis} = \{A, Kcl(A), A\hat{}dA, Kcl(A\hat{}dA)\} = \{A, A \cup F, H, H \cup K\}. \quad (8.2)$$

The Open sets of the Cartan - Kuratowski topology are generated from the basis by composing all unions of the basis sets. For 4 ingredients, the Cartan-Kuratowski topology becomes recognizable as the equivalent of the disconnected Kolmogorov T0 (poset 3) topology (2009). In passing, it should be noted that the ubiquitous claim of homology theory that the boundary of a boundary is zero is false. The T0 poset 3 topological structure is one of many counter examples.

The Kuratowski representation leads to the lattice structure of open sets:

$$LS = \{\{\}, \{A\}, \{H\}, \{A, F\}, \{A, H\}, \{H, K\}, \{A, F, H\}, \{A, H, K\}, \{A, F, H, K\}\} \quad (8.3)$$

X = {A,F,H,K}   poset 3

LS = { { }, {A}, {H}, {A, F}, {A, H}, {H, K}, {A, F, H}, {A, H, K}, {A, F, H, K} }

This Lattice Structure generates the unique T0 topology based on X, which represents the marriage between Cartan's Closure based upon Exterior Differential forms, and the Kuratowski Closure based upon sets. The marriage recognizes that the exterior differential, d, operating on exterior differential forms is a limit point generator. All subsets are Distinguishable.

Is LS a topology = true,   connected = false,   Kolmogorov.T0 = true,   Hausdorff.T2 = false

The COMPLETE Lattice Structure for the Powet set of X

Subset S	Int(S)	Ext(S)	Bnd(S)	Clo(S)	Lim(S)	IsoSeg(S)	IsoLip(S)
{A}	{A}	{H, K}	{F}	{A, F}	{F}	{A}	{ }
{F}	{ }	{A, H, K}	{F}	{F}	{ }	{ }	{ }
{H}	{H}	{A, F}	{K}	{H, K}	{K}	{H}	{ }
{K}	{ }	{A, F, H}	{K}	{K}	{ }	{ }	{ }
{A, F}	{A, F}	{H, K}	{ }	{A, F}	{F}	{ }	{F}
{A, H}	{A, H}	{ }	{F, K}	{A, F, H, K}	{F, K}	{A, H}	{ }
{A, K}	{A}	{H}	{F, K}	{A, F, K}	{F}	{A, K}	{ }
{F, H}	{H}	{A}	{F, K}	{F, H, K}	{K}	{F, H}	{ }
{F, K}	{ }	{A, H}	{F, K}	{F, K}	{ }	{ }	{ }
{H, K}	{H, K}	{A, F}	{ }	{H, K}	{K}	{ }	{K}
{A, F, H}	{A, F, H}	{ }	{K}	{A, F, H, K}	{F, K}	{A, H}	{F}
{A, F, K}	{A, F}	{H}	{K}	{A, F, K}	{F}	{ }	{F}
{A, H, K}	{A, H, K}	{ }	{F}	{A, F, H, K}	{F, K}	{A, H}	{K}
{F, H, K}	{H, K}	{A}	{F}	{F, H, K}	{K}	{ }	{K}
{A, F, H, K}	{A, F, H, K}	{ }	{ }	{A, F, H, K}	{F, K}	{ }	{F, K}

Figure 8.1 Kuratowski T0 format for poset 3 N=4

Following E. Cartan's method (called Cartan's magic formula by Marsden), the "laws of motion" that describe the evolution of the T2 or T0 "particle" topologies will be generated by applying the continuous topological Lie differential (not derivative), $L_{(\mathbf{V}_4)}$, to the exterior differential 1-form of Action, A, that encodes the system:

$$\text{Cartan's magic formula} : L_{(\mathbf{V}_4)}A = i(\mathbf{V}_4)dA + d(i(\mathbf{V}_4)A) \Rightarrow Q. \tag{8.4}$$

The direction field, $\mathbf{V}4 = [\mathrm{V}^1(\mathrm{x}^m), \mathrm{V}^2(\mathrm{x}^m), \mathrm{V}^3(\mathrm{x}^m), \mathrm{V}^4(\mathrm{x}^m)]$, combined with the Lie differential, represents a physical process, that maps $A \Rightarrow Q$.

With a change of notation,

$$L_{(\mathbf{V}_4)}A = W + dU = Q, \tag{8.5}$$

$$\text{Work 1-form } W = i(\mathbf{V}_4)dA, \tag{8.6}$$

$$\text{Heat 1-form } Q = L_{(\mathbf{V}_4)}A, \tag{8.7}$$

$$\text{internal energy } U = i(\mathbf{V}_4)A, \tag{8.8}$$

Cartan's Magic formula becomes recognizable as a topological, universal, expression of the First Law of Thermodynamics – a derivation from first principles of a category theory. The system of partial differential equations given below represents a statement of deRham Cohomology.

Cartan's First Fundamental Partial Differential System

$$Q - W = dU. \tag{8.9}$$

The 1-form of heat, Q, can also generate a topology which need not be homeomorphic to the topology generated by A. Hence Cartan's magic formula can describe topological change, indeed. In all cases all 1-forms are not uniquely integrable if their PTD is 3 or 4. These domains of non-integrability are equivalent to domains of non-equilibrium thermodynamics. The Cartan magic formula works for equilibrium systems (PTD < 3) and non-equilibrium systems PTD >2). The direction field, $\mathbf{V}_4$, representing a process can be an element of a differential semi-group, or an element of a complex isotropic, macroscopic, spinor space. Processes are dissipatively thermodynamically irreversible if $dQ\hat{}dQ \neq 0$, and thermodynamically reversible if $Q\hat{}dQ = 0$.

These particle aspects of continuous topological evolutions have been well documented in numerous publications over the years, starting in 1962. The research is summarized with many many examples in six monographs (http://www.lulu.com/kiehn). Free pdf files are available by email request (rkiehn2352@aol.com).

8.4 Universal System Correlations

It is remarkable that starting from any exterior 1-form of Action, $A(\mathrm{x}^m,\mathrm{dx}^m)$, it is possible to deduce a topological category of particle systems that universally obey the First Law of thermodynamics. In addition, it is possible to construct a 4 x 4 Jacobian *correlation* matrix from the functional coefficients that define the 1-form of Action, A:

$$[\mathbb{J}_{correlation}] = [\mathbb{J}_{jk}] = [\partial A_j(x^m)/\partial x^k] \tag{8.10}$$

The correlation matrix (of maximal rank 4) can be used to define a thermodynamic system equation of state, in terms of the Cayley-Hamilton characteristic polynomial,

$$\Theta(\mathbb{J}_{jk}) = \xi^4 - X_M\xi^3 + Y_G\xi^2 - Z_A\xi + T_K. \tag{8.11}$$

In the above formula, the set $[\xi 1,\xi 2,\xi 3,\xi 4]$ are the local eigenvalues of the matrix, $[\mathbb{J}_{jk}]$, and the set $[\mathrm{X}_M,\mathrm{Y}_G,\mathrm{Z}_A,\mathrm{T}_K]$ define the similarity invariant coefficients, when evaluated at the point $\{x,y,z,t\}$.

$$\begin{aligned}
\text{Linear } \mathrm{X_M} &= Trace[\mathbb{J}_{jk}] = \xi 1 + \xi 2 + \xi 3 + \xi 4 && (8.12)\\
\text{Quadratic } \mathrm{Y_G} &= \xi 1\xi 2 + \xi 2\xi 3 + \xi 3\xi 1 + \xi 1\xi 4 + \xi 2\xi 4 + \xi 3\xi 4 && (8.13)\\
\text{Cubic } \mathrm{Z_A} &= \xi 1\xi 2\xi 3 + \xi 2\xi 3\xi 4 + \xi 3\xi 1\xi 4 + \xi 1\xi 2\xi 4 && (8.14)\\
\text{Quartic } \mathrm{T_K} &= \det[\mathbb{J}_{jk}] = \xi 1\xi 2\xi 3\xi 4 && (8.15)
\end{aligned}$$

The similarity coefficients can be related to the linear, quadratic, cubic and quartic curvatures of the implicit surface, $\Theta = 0$, interpreted as a parametric hypersurface in the 4D space of variables $[\mathrm{X_M},\mathrm{Y_G}, \mathrm{Z_A}, \mathrm{T_K}]$.

If $\mathrm{T_K} = 0$, the determinant of the Jacobian correlation matrix vanishes, hence the Pfaff Topological Dimension, PTD$(A) < 4$. The Cayley-Hamilton equation for the singular correlation matrix becomes,

$$\text{Singular Cayley-Hamilton polynomial} = (\xi^3 - X_M\xi^2 + Y_G\xi - Z_A)\xi = 0. \tag{8.16}$$

It follows that there is a single eigenvalue which is zero, and the 4th order polynomial has a cubic factor. The cubic factor can be put into direct correspondence with the Classic van der Waals equation of state, and by change of notation,

$$\begin{aligned}
\xi &= \widetilde{\rho}, && (8.17)\\
\mathrm{X_M} &\Rightarrow 3, && (8.18)\\
\mathrm{Y_G} &\Rightarrow (8\widetilde{T} + \widetilde{P})/3, && (8.19)\\
\mathrm{Z_A} &= \widetilde{P},
\end{aligned}$$

yields the expression for the scaled van der Waals gas about the critical point. Forces and energies associated with the Linear curvature are typical of surface tension effects.

It becomes apparent that forces and energies associated with the Cubic curvature represent the pressures of interactions. The Gauss quadratic curvatures are dominated by temperature, with a pressure contribution.

The Projective dimension of the correlation matrix can be determined by the rank of the Jacobian matrix, or by the zero sets of the similarity invariants. If only one eigenvalue is zero, then $\mathrm{T_K} = 0$, and there is a projection from 4D to 3D. If in addition, $\mathrm{Z_A} = 0$, then two eigenvalues are zero, and there is a projection from 4D to 2D. Recall that non-equilibrium systems are determined by domains of Pfaff Topological Dimension equal to 3 or 4. The remarkable result, demonstrated by examples, is that there are situations where the Pfaff dimension is 4, and the Projective dimension is 3, and other situations where the Pfaff dimension is 4, but the Projective dimension is 2. These results are entwined with the idea that there are correlations constructed from 1-forms with integrating factors such that the determinant of the correlation can be made to vanish. On the other hand their are (N-1)-forms, or currents, with integrating factors that can make the divergence (trace of a collineation) vanish.

8.5 The universal van der Waals gas

A $\widetilde{P}, \widetilde{\rho}$ projection of the implicit universal van der Waals surface is given in Figure 8.2

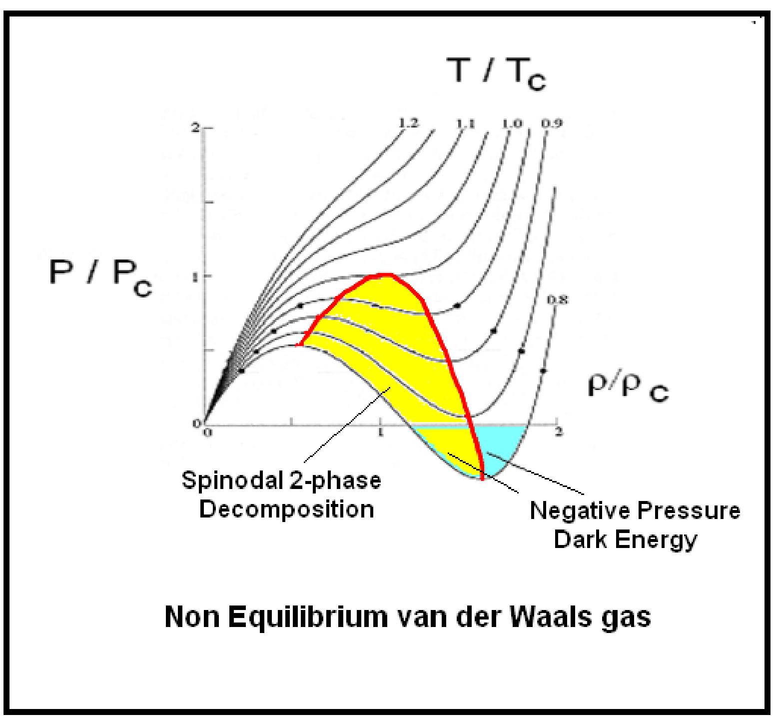


Figure 8.2 Negative Pressure in a van der Waal's gas

The diagram displays a critical isotherm that separates a single phase (the gas) from the different topological domains that can be interpreted as liquids and vapor.

Note that the word "vapor" is used for the phase between the critical isotherm and the Spinodal line, for molar densities below the critical point value. The word "gas" is used to describe the phase above the critical isotherm. Note that the word "liquid" is used for the phase between the critical isotherm and the Spinodal line, for molar densities above the critical point value. The difference between vapor and gas is not appreciated in the historical literature.

The molal roots to the characteristic polynomial represented by a van der Waals gas are complex above the critical isotherm and real below the critical isotherm in the vapor region. This result can be attributed to the eigendirection fields associated with Jacobian matrix of the Action 1-form, A, used to encode the thermodynamic system. Above the critical isotherm, the eigendirection fields consist of one real vector, and two complex spinors. Below the critical isotherm, the eigendirection fields consist of three real vectors.

The shape of the critical isotherm should be remembered, for above the critical isotherm, there exists a unique value for the pressure, and below the critical isotherm there is more than one value for the pressure. This feature represents a topological property of the van der Waals gas, and will have importance in the study of non-equilibrium systems. Of interest to cosmologists who are interested in dark energy and negative pressure, note that the pressure for the van der Waals gas, for values below the critical isotherm, can take on negative values. The Phase function below the critical isotherm has the shape of a quartic Higgs potential. There exists a dual surface to the equation of state as defined by a Legendre transformation to the Gibbs function, $g = u - Ts + Pv$. The implicit surface defined by the Gibbs function (for a van der Waals gas) is not single valued, and appears as a deformation of a swallowtail bifurcation set. The actual Gibbs surface for the van der Waals gas can be numerically computed and is presented in Figure 8.3.

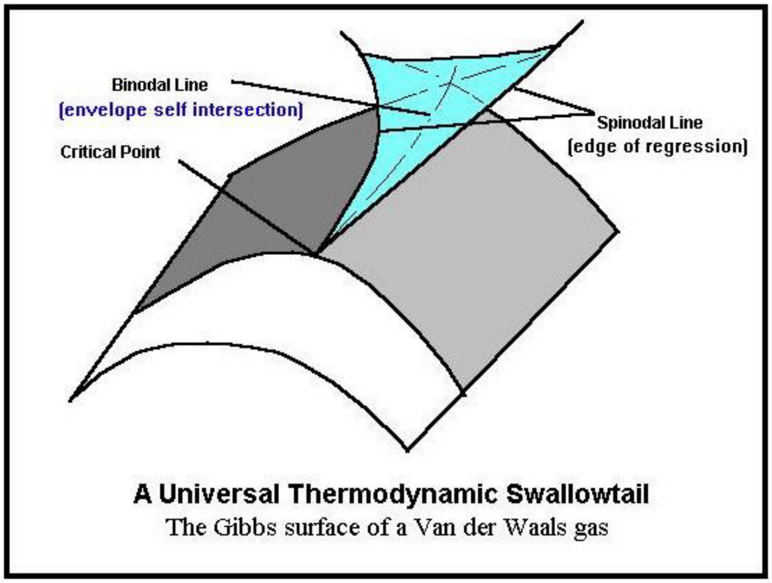


Figure 8.3 Gibbs surface for a van der Waals gas

Remark 36 *The topological features of the van der Waals gas are universal features (deformation invariants) for all physical systems that admit a realization over 4D space-time variety. The van der Waals gas is one element of a topological equivalence class.*

8.5.1 Topological Torsion

Not only is it possible to use the functional form of the 1-form of Action to deduce a Topology, *and* an equation of state that matches the induced topology, it is also possible to deduce a universal Vector process field, $\mathbf{T_4}$, using the equality,

$$A\hat{\ }dA = i(\mathbf{T_4})[dx\hat{\ }dy\hat{\ }dz\hat{\ }dt] \quad (8.20)$$

$$= T^x dy\hat{\ }dz\hat{\ }dt - T^y dx\hat{\ }dz\hat{\ }dt + T^z dx\hat{\ }dy\hat{\ }dt - T^t dx\hat{\ }dy\hat{\ }dz. \quad (8.21)$$

When the Action 1-form for a physical system is of Pfaff dimension 4, there exists a unique direction field, $\mathbf{T}_4$, defined as the Topological Torsion 4-vector, that can be evaluated *entirely* in terms of those component functions of the 1-form of Action which define the physical system. To within a factor, this direction field has the four coefficients of the 3-form $A\hat{}dA$, with the following properties:

Properties of the Topological Torsion vector $\mathbf{T}_4$

$$\begin{aligned}
i(\mathbf{T}_4)\Omega_4 &= i(\mathbf{T}_4)[dx\hat{}dy\hat{}dz\hat{}dt] = A\hat{}dA, && (8.22)\\
L_{(\mathbf{T}_4)}A &= \sigma\ A, && (8.23)\\
W &= i(\mathbf{T}_4)dA = Q = \sigma\ A, && (8.24)\\
W\hat{}dW &= Q\hat{}dQ = \sigma^2 A\hat{}dA \neq 0 \quad \text{A, W, Q are not integrable} && (8.25)\\
i(\mathbf{T}_4)W &= 0, \quad \text{Work is always transversal} && (8.26)\\
i(\mathbf{T}_4)Q &= 0, \quad \text{the Process is adiabatic when Q transversal} && (8.27)\\
U &= i(\mathbf{T}_4)A = 0, && (8.28)\\
dA\hat{}dA &= (2!)\ \sigma\ \Omega_4. && (8.29)
\end{aligned}$$

Note that a $\mathbf{T}_4$ process is locally adiabatic, as $i(\mathbf{T}_4)Q = 0$.

8.5.2 A Physical System with Topological Torsion

The Topological Torsion vector vanishes when the Pfaff Topological Dimension of A is 2 or less. Note that the Frenet-Serret geometric torsion of a space curve vanishes when the geometric dimension is 2 or less. It is this analog dependence on dimension 3 or more that led to the name "Topological Torsion" for the 3-form $A\hat{}dA$. Solution uniqueness is lost when the Topological Torsion vector is not zero. In 4D, the 3-form $A\hat{}(dA)$ has been defined as the Topological Torsion 3-form. The Torsion current depends only on the system (the Action) and not upon a process. The divergence of this Torsion current is proportional to the measure of the 4D volume, that defines the symplectic space, and cannot be zero on the symplectic domain. The components of the Topological Torsion vector $\mathbf{T}_4$ generate what is called the "subsidiary Pfaffian system" by Forsyth

For purposes of more rapid comprehension, consider a 1-form of Action,

$$A(x^k, dx^k) = A_x dx + A_y dy + A_z dz - \phi dt,$$

with an exterior differential, dA, and a notation that admits an electromagnetic interpretation

$$\begin{aligned}
\mathbf{E} &= -\partial\mathbf{A}/\partial t - \nabla\phi\\
\mathbf{B} &= \nabla\times\mathbf{A}
\end{aligned}$$

The explicit format of the Electromagnetic Topological Torsion 4-vector, $\mathbf{T}_4$ becomes,

$$\begin{aligned}
\mathbf{T}_4 &= -[\mathbf{E}\times\mathbf{A} + \mathbf{B}\phi,\ \mathbf{A}\circ\mathbf{B}], && (8.30)\\
A\hat{}dA &= i(\mathbf{T}_4)\Omega_4, && (8.31)\\
&= T_4^x dy\hat{}dz\hat{}dt - T_4^y dx\hat{}dz\hat{}dt\\
&\quad + T_4^z dx\hat{}dy\hat{}dt - T_4^t dx\hat{}dy\hat{}dz, && (8.32)\\
dA\hat{}dA &= 2(\mathbf{E}\circ\mathbf{B})\ \Omega_4 = K\Omega_4, && (8.33)\\
&= \{\partial T_4^x/\partial x + \partial T_4^y/\partial y + \partial T_4^z/\partial z + \partial T_4^t/\partial t\}\ \Omega_4. && (8.34)
\end{aligned}$$

When the divergence of the topological torsion vector is not zero, $\sigma = (\mathbf{E} \circ \mathbf{B}) \neq 0$, and A is of Pfaff dimension 4, W is of Pfaff dimension 4, and Q is of Pfaff dimension 4. The process generated by $\mathbf{T}_4$ is thermodynamically irreversible, as

$$Q\hat{}dQ = L_{(\mathbf{T}_4)}A\hat{}L_{(\mathbf{T}_4)}dA = \sigma^2 A\hat{}dA \neq 0. \tag{8.35}$$

The evolution of the volume element relative to the irreversible process $\mathbf{T}_4$ is given by the expression,

$$L_{(\mathbf{T}_4)}\Omega_4 = i(\mathbf{T}_4)d\Omega_4 + d(i(\mathbf{T}_4)\Omega_4) \tag{8.36}$$

$$= 0 + d(A\hat{}dA) = 2(\mathbf{E} \circ \mathbf{B})\ \Omega_4. \tag{8.37}$$

Hence, the differential volume element Ω_4 is expanding or contracting depending on the sign and magnitude of $\mathbf{E} \circ \mathbf{B}$, a useful fact when topological thermodynamics is applied to cosmology. The irreversible dissipation induced by a $\mathbf{T}_4$ process can be compared to a bulk viscosity coefficient. A cosmology on 4D can have an expanding volume element, Ω_4, but with embedded 3D defect structures (the galaxies) which are not "expanding".

8.6 Statistical thermodynamic Systems

The Not-T0 topologies are not constrained by separation axioms. Some, if not all, of the singleton subsets are indistinguishable. The claim is made herein that such Not-T0 partitions are applicable to the understanding of complex wave, diffusion, and, more generally, statistical systems, all of which are composed of indistinguishable ingredients. Equations that describe continuous topological evolution of Not-T0 topologies are constructed in terms of the continuous topological operator, the Lie Differential, acting on the sets of exterior differential form densities. A number of examples will be described below.

Historically, for finite systems, the properties of the metrizable T2 Hausdorff, disconnected, discrete topology have been used to construct geometric models of thermodynamic systems of distinguishable particles in equilibrium. The subsets of the T2 topological structure imply that the boundary of a subset boundary, ∂S, vanishes, $\partial\partial S = 0$. However, it is easy to demonstrate that the theorem fails for non-equilibrium systems of distinguishable particles which utilize the non-metrizable, disconnected, T0 poset 3 topology.

Extending a suggestion of Chern, the Top set, X, of classical set theory, will be replaced by the Top Pfaffian, μ, which is a differential 4-form density on a variety of 4D topological dimensions.

$$X \Rightarrow \mu = \rho(x^k)\Omega(dx^k) = \rho(x^k)\Omega(dx\hat{}dy\hat{}dz\hat{}du). \tag{8.38}$$

The exterior differential of the Top Pfaffian, μ, vanishes:

$$d\mu = 0. \tag{8.39}$$

The Lattice Structures of the Power set of X of 4 ingredients will produce 17 topologies that are not T0, and 16 topologies that are T0. All of the 17 Not-T0 topologies have indistinguishable subsets, some of which are dense, and others which are not dense. The boundary of a boundary theorem is true for those indistinguishable subsets that are dense, but not true for those subsets that are not dense.

8.6.1 The Indiscrete Not-T0 Topology

For simplicity, attention will be focused on the simplest of the Not-T0 lattice structures, the indiscrete Topology of 4 ingredients which has only two subsets, { }, and {X}. In Lattice Structure notation,

$$\text{LS} = \{\{\ \}, \{X\}\}$$

X ={A,F,H,K}

LS = { { }, {A, F, H, K} }

The Lattice Structure, LS, is a Not-T0, Indiscrete, Topology, all subsets are Indistinguishable.

Is LS a topology = true, connected = true, Kolmogorov.T0 = false, Hausdorff.T2 = false

The COMPLETE Lattice Structure based on LS and all subsets of the Powet set of X

Subset S	Int(S)	Ext(S)	Bnd(S)	Clo(S)	Lim(S)	IsoSeg(S)	IsoLip(S)
{A}	{ }	{ }	{A, F, H, K}	{A, F, H, K}	{F, H, K}	{A}	{ }
{F}	{ }	{ }	{A, F, H, K}	{A, F, H, K}	{A, H, K}	{F}	{ }
{H}	{ }	{ }	{A, F, H, K}	{A, F, H, K}	{A, F, K}	{H}	{ }
{K}	{ }	{ }	{A, F, H, K}	{A, F, H, K}	{A, F, H}	{K}	{ }
{A, F}	{ }	{ }	{A, F, H, K}	{A, F, H, K}	{A, F, H, K}	{ }	{A, F}
{A, H}	{ }	{ }	{A, F, H, K}	{A, F, H, K}	{A, F, H, K}	{ }	{A, H}
{A, K}	{ }	{ }	{A, F, H, K}	{A, F, H, K}	{A, F, H, K}	{ }	{A, K}
{F, H}	{ }	{ }	{A, F, H, K}	{A, F, H, K}	{A, F, H, K}	{ }	{F, H}
{F, K}	{ }	{ }	{A, F, H, K}	{A, F, H, K}	{A, F, H, K}	{ }	{F, K}
{H, K}	{ }	{ }	{A, F, H, K}	{A, F, H, K}	{A, F, H, K}	{ }	{H, K}
{A, F, H}	{ }	{ }	{A, F, H, K}	{A, F, H, K}	{A, F, H, K}	{ }	{A, F, H}
{A, F, K}	{ }	{ }	{A, F, H, K}	{A, F, H, K}	{A, F, H, K}	{ }	{A, F, K}
{A, H, K}	{ }	{ }	{A, F, H, K}	{A, F, H, K}	{A, F, H, K}	{ }	{A, H, K}
{F, H, K}	{ }	{ }	{A, F, H, K}	{A, F, H, K}	{A, F, H, K}	{ }	{F, H, K}
{A, F, H, K}	{A, F, H, K}	{ }	{ }	{A, F, H, K}	{A, F, H, K}	{ }	{A, F, H, K}

Figure 8.4 The Extremal Indiscrete Not T0 Topology

The topology is the indiscrete, but connected, Not-T0 topology. All subsets of the power set of X are dense and indistinguishable. The boundary of the boundary theorem is true for the connected indiscrete topology, suggestively similar to the result obtained for the discrete, disconnected, T2 topology.

The continuous topological Cartan-Lie differential applied to Top Pfaffian yields the equation of continuous topological evolution:

$$L_{(\mathbf{V}_4)}\mu = \kappa(x^k)\ \mu. \tag{8.40}$$

This coefficient expression, $\kappa(x^k)$, has been called the "bulk dissipation, or self-similarity, or fractal, or homogeneity, or Entropic" coefficient. If the function, $\kappa(x^k)$, vanishes, the density, μ, is a local evolutionary volume invariant. If $\kappa = +1$, the evolution is homogeneous of degree 1, a necessary requirement for positive definite additivity of indistinguishable Bosons and statistical ingredients. If $\kappa = -1$, it is conjectured that the system describes indistinguishable (mod Spin) Fermions. If $\kappa(x^k) > 0$, then the 4D universe is expanding. Evaluating the formula of continuous topological evolution, above, yields the expression,

$$L_{(\mathbf{V}_4)}\mu = i(\mathbf{V}_4)d\mu + d[i(\mathbf{V}_4)\mu] = 0 + d[i(\mathbf{V}_4)\mu] \Rightarrow \kappa(x^k)\ \rho(x^k)\ \Omega(dx^k). \tag{8.41}$$

Factorization of this formula yields a necessary PDE equation that describes a 4D divergence relation with inhomogeneous source terms:

$$div_4(\mathbf{V}_4) = [\kappa(x^k)\ - \{\Sigma_k(V^k \circ \partial(ln\rho)/\partial x^k)\}]. \tag{8.42}$$

Realizations of this ubiquitous partial differential system of continuous topological evolution can be recognized as the inhomogeneous wave equation or diffusion equation, the fractal minimal surface equation, the Klein-Gordan equation, the Schroedinger equation, the Hopf conjugate Spinor equation, the Landau-Ginsburg equation of superconductivity, and the Petrov blackhole equations of relativity theory. The term in brackets, $\{\}$, represents a generalization of the non-linear Eikonal expression, whose zero sets represent propagating singularities and discontinuities in the Maxwell theory of electromagnetic waves, and the Raleigh-Taylor wave crests so cherished by surfers, and the von Karmon mushroom cloud singularities seen in liquid and plasma fluids.

8.7 Continuous Topological Evolution of N-form densities

The Top Pfaffian is defined in terms of a differential volume element, $\Omega(dx^k) = dx\hat{}dy\hat{}dz\hat{}dt\}$, and a density coefficient $\rho(x^k)$, that may or may not be a constant:

The exterior differential 4-form density,

the Top Pfaffian, μ (8.43)

$$\mu \;=\; \rho(x^k)\Omega(dx^k) = \rho(x^k)\ \Omega(dx^k) = \rho(x,y,z,t)\{dx\hat{}dy\hat{}dz\hat{}dt\}. \tag{8.44}$$

Following the success and experience of continuous topological evolution of the distinguishable topologies, it is presumed that the evolutionary "equations of motion" of the indiscrete topology of indistinguishable sets are deduced by applying the continuous topological Lie differential operator $L_{(\mathbf{V}_4)}$ (relative to the process, $\mathbf{V}_4$) to the 4-form density, μ.

For a process, $\mathbf{V}_4(x^k)$, assumed to be of the form,

$$\textbf{Process}: \; \mathbf{V}_4(x^k) = [V^x(x^k), V^y(x^k), V^z(x^k), V^t(x^k)], \tag{8.45}$$

the application of the continuous topological Lie differential operator to Top Pfaffian produces the equation,

$$L_{(\mathbf{V}_4)}\mu = \kappa(x^k)\ \mu. \tag{8.46}$$

As μ is the Top Pfaffian, the only possible continuous topological values are function multiples $\kappa(x^k)$ of μ; hence

$$L_{(\mathbf{V}_4)}\mu = \kappa(x^k)\ \mu = \kappa(x^k)\ \rho(x^k)\ \Omega(dx^k). \tag{8.47}$$

The function, $\kappa(x^k)$, is called the homogeneity coefficient, which can take on values equal to zero, or greater or less than zero, $\kappa = (0, +1, -1)$. If $\kappa = 0$, then the evolution of the Top Pfaffian is an invariant of the process. If If $\kappa = +1$, the evolution is homogenous of degree 1, a requirement for positive definite additivity of indistinguishable Bosons. If $\kappa = -1$,it is conjectured that the system describes indistinguishable (Mod Spin) Fermions.

As demonstrated below special case of interest is when

$$\kappa(x^k) = ln(\rho + \delta), \tag{8.48}$$

and δ takes in values

$$\delta = \{0, +1, -1\}. \tag{8.49}$$

Remark 37 *Does the process, $L_{(\mathbf{V}_4)}$, cause the differential 4-form volume element, $\Omega(dx^k)$, to expand, or contract, or remain invariant? Can the expansion of the volume element compensate for a change in density?*

8.7.1 Continuous Topological Evolution of the Not T0 indiscrete Topology

Using Cartan's magic formula to define the continuous topological evolution operator, the evolution of the Top Pfaffian becomes:

$$\begin{aligned} L_{(\mathbf{V}_4)}\mu &= (L_{(\mathbf{V}_4)}\rho(x^k))\ \Omega(dx^k) + \rho(x^k)(L_{(\mathbf{V}_4)}\Omega(dx^k) && (8.50)\\ &= (V^k \circ \partial\rho/\partial x^k)\Omega(dx^k) + \rho(x^k)d(i(\ \mathbf{V}_4)\Omega(dx^k)) && (8.51)\\ &= \{\Sigma_k(V^k \circ \partial(ln\rho)/\partial x^k) + \partial V^k/\partial x^k)\ \}\mu = \kappa\ \mu. && (8.52)\end{aligned}$$

Note that the evolution of the differential 4-form volume element, $\Omega(dx^k)$, can be described in terms of the limit points, dC, of a current 3-form, C:

$$
\begin{aligned}
C &= i(\mathbf{V}_4)\Omega(dx^k) && (8.53)\\
&= [V^x\hat{}dy\hat{}dz\hat{}dt - V^y\hat{}dx\hat{}dz\hat{}dt && (8.54)\\
&\quad +V^z\hat{}dx\hat{}dy\hat{}dt - V^t dx\hat{}dy\hat{}dz], && (8.55)\\
L_{(\mathbf{V}_4)}\Omega(dx^k) &= d(C) = div_4(\mathbf{V}_4)\Omega(dx^k) && (8.56)\\
div_4(\mathbf{V}_4) &= \text{Trace}(\mathbb{J}) && (8.57)\\
\mathbb{J} &= [\partial V^m/\partial x^n] = \text{Jacobian matrix} && (8.58)
\end{aligned}
$$

It follows that the continuous topological Evolution of the Top Pfaffian density 4-form, μ, is governed by the equation:

$$L_{(\mathbf{V}_4)}\mu = \{\Sigma_k(V^k \circ \partial(ln\rho)/\partial x^k) + div_4(\mathbf{V}_4))\ \}\mu = \kappa\ \mu. \tag{8.59}$$

Note that if the density is not zero, $\mu \neq 0$, then the universal evolutionary equation above has a common factor, μ, which - when removed - leads to a partial differential system, that must vanish universally. By re-arranging terms to emphasize the divergence component, the 2nd Fundamental Partial Differential system becomes:

Cartan's 2nd Fundamental Partial Differential system:

$$\mathbf{div}_4(\mathbf{V}_4) = \kappa - \{\Sigma_k(V^k \circ \partial(ln\rho)/\partial x^k)\}, \tag{8.60}$$

It is this differential system, **PDE II**, that defines the Second Fundamental Equation of indiscrete continuous topological Evolution.

8.7.2 The Wave Equation

For example assume that

$$\left(x^k\right) = \{x, y, z, s = ict\} \tag{8.61}$$

$$\rho = \Psi\left(x^k\right), \quad \varepsilon\mu c^2 = 1, \tag{8.62}$$

$$grad_4(\rho) = [\partial\Psi/\partial x, \partial\Psi/\partial y, \partial\Psi/\partial z, (\sqrt{\varepsilon\mu}/i)\partial\Psi/\partial t] = \mathbf{V}_4(x^k) \tag{8.63}$$

then the Fundamental Partial Differential system becomes:

$$div_4(\mathbf{V}_4) = \partial^2\Psi/\partial x^2 + \partial^2\Psi/\partial y^2 + \partial^2\Psi/\partial z^2 - \varepsilon\mu\partial^2\Psi/\partial t^2 \tag{8.64}$$

$$div_4(\mathbf{V}_4) = \{\kappa - (1/\Psi)\{(\partial\Psi/\partial x)^2 + (\partial\Psi/\partial y)^2 + (\partial\Psi/\partial z)^2 - \varepsilon\mu(\partial\Psi/\partial t)^2\}\} \tag{8.65}$$

$$div_4(\mathbf{V}_4) = k - (1/\Psi)\{Eikonal\ PDE\}. \tag{8.66}$$

This set of equations is to be recognized as the inhomogeneous wave equations (with the non-linear quadratic PDE known as the Eikonal constraint.). If the RHS is zero, then the result is the linear, second order, Wave equation. The wave speed, c, is determined from the formula, $\varepsilon\mu c^2 = 1$.

If the RHS = MΨ then the Partial Differential System generates the Klein-Gordan equation, with the constant M related to "mass" in the Klein-Gordan theory.

8.7.3 The Diffusion Equation

Consider a possible solution to the continuous topological evolution of the indiscrete topology generated by the Top Pfaffian, μ, in terms the process, $\mathbf{V}_4$:

$$\mathbf{V}_4 = [V^k] = [\partial\Psi/\partial x, \partial\Psi/\partial y, \partial\Psi/\partial z, -D\Psi], \tag{8.67}$$

$$div_4(\mathbf{V}_4) = \partial^2\Psi/\partial x^2 + \partial^2\Psi/\partial y^2 + \partial^2\Psi/\partial z^2 - D\partial\Psi/\partial t \tag{8.68}$$

Substitution of the example expression for $div_4(\mathbf{V}_4)$ leads to the equation,

$$\begin{aligned} & \partial^2\Psi/\partial x^2 + \partial^2\Psi/\partial y^2 + \partial^2\Psi/\partial z^2 - D\partial\Psi/\partial t \\ = & \{\kappa - \Sigma_k(V^k \circ \partial(ln\rho)/\partial x^k)\} \end{aligned}$$

This equation is to be recognized as the inhomogeneous Diffusion equation, where the terms on the RHS can be interpreted as interactions, or source terms. The solutions to the diffusion equation consists of indistinguishable distributions. If the diffusion coefficient is imaginary, $D \rightarrow \hbar/i$, then the equation of continuous topological evolution has the format of the Schroedinger equation.

8.7.4 The Minimal Surface equation

Consider a possible solution to the continuous topological evolution of the indiscrete topology generated by the Top Pfaffian, μ, in terms the process direction field, $\mathbf{V}_4/\lambda_H$, where λ_H is a Holder norm, with arbitrary constant coefficients, $\{a, b, c, e\}$ and constant exponents, (p, h):

$$\lambda_H = [a(V^x)^p + b(V^y)^p + c(V^z)^p + \varepsilon(V^s)^p]^{h/p} \tag{8.69}$$

$$\mathbb{J} = [(\partial \mathbf{V}_4/\lambda_H)/\partial x^n] = \text{Jacobian matrix} \tag{8.70}$$

$$div_4(\mathbf{V}_4) = Trace[\mathbb{J}] \tag{8.71}$$

The Holder norm can be used to adjust the homogeneity properties of the "rescaled" process direction field. The Jacobian matrix always supports a Hamilton-Jacobi characteristic polynomial, $Ch[Jac(\mathbf{V}_4/\lambda_H)] = \Theta$, of the form

$$\Theta = \eta^4 - M\eta^3 + G\eta^2 - A\eta + K = 0, \tag{8.72}$$

where the coefficients $\{M, G, A, K\}$ are the similarity coefficients, invariant with respect to similarity transformations of the Jacobian Matrix, $\mathbb{J}$.

The zero set of the characteristic polynomial, $\Theta = 0$, defines a hypersurface family (with parameter, η) in the 4 dimensional space whose coordinates are $\{M, G, A, K\}$. It is easily demonstrated (with Maple) that the similarity coefficients depend upon a choice for the homogeneity index h in the Holder Norm, λ_H.

$$\text{Mean Curvature: } M = (4-h)/\lambda_H = Trace\ [Jac(\mathbf{V}_4/\lambda_H)], \tag{8.73}$$

$$\text{Gauss Curvature: } G = (6-3h)/\lambda_H^2, \text{ gravitational energy} \tag{8.74}$$

$$\text{Cubic Curvature: } A = (4-3h)/\lambda_H^3 = Adoint[Jac(\mathbf{V}_4/\lambda_H)]\text{Interaction energy} \tag{8.75}$$

$$\text{Quartic Curvature: } K = (1-h)/\lambda_H^4 = det[Jac(\mathbf{V}_4/\lambda_H)], \text{ Irreversible Dissipation.} \tag{8.76}$$

The Hamilton-Jacobi characteristic polynomial, $\Theta = 0$, can be interpreted as a universal thermodynamic equation of state that describes the continuous topological Evolution of the indiscrete topology, $Not - T0$.

$$\Theta = (\eta\lambda_H - 1)^3 (\eta\lambda_H - 1 + h) = 0. \tag{8.77}$$

When the mean curvature vanishes, $M \Rightarrow 0$, the Hypersurface has a normal field with zero divergence, and a homogeneity index, $h = 4$.

When there is no irreversible Dissipation, $K \Rightarrow 0$, and the homogeneity index is finite, $h = 1$. In this situation, the universal equation of state has three non-zero eigen values, and one zero eigenvalue, that mimics the equation of state for a van der Walls gas.

8.7.5 Dissipation interaction

Assume that the volume element on space-time is proportion to the volume element created by the differentials of the components of the process:

$$\Omega(dV^k) \Rightarrow \rho\,\Omega(dx^k) = det\ [\mathbb{J}]\ dx\hat{}dy\hat{}dz\hat{}dt \tag{8.78}$$

$$[\mathbb{J}] = [\mathbb{J}_n^m] = [\partial V^m/\partial x^n], \tag{8.79}$$

$$\rho = det\ [\mathbb{J}]. \tag{8.80}$$

$$\text{and define } A_k = grad_{\mathbf{4}}\ \rho = grad_{\mathbf{4}}\ (det\ [\mathbb{J}]) \tag{8.81}$$

Then the Fundamental Differential system becomes

$$div_4(\mathbf{V}_4) = \{\kappa\ - (V^k \circ A_k)/det\ [\mathbb{J}]\} \tag{8.82}$$

$$\text{where } \mathbb{J} = [\partial V^k/\partial x^n] = \text{Jacobian matrix} \tag{8.83}$$

$$div_4(\mathbf{V}_4) = \text{Trace}(\mathbb{J}). \tag{8.84}$$

The term $(V^k \circ A_k)$ plays the role of an interaction dissipation coefficient, which vanishes if the innerproduct is zero, $(V^k \circ A_k) = 0$. The interaction energy vanishes if either the gradient of the density distribution (determinant of the process Jacobian) is zero, or the gradient of the density distribution (determinant of the process Jacobian) is orthogonal to the process. If $(V^k \circ A_k)$ is not zero, then a thermodynamic argument indicates that such processes are irreversible.

8.7.6 The Characteristic Polynomial of Homogeneous Processes

The process direction fields can include homogeneous structures, where the process directions are renormalized by divisors that are functions of the coefficients of the process. A particularly useful divisor is given by the Holder norm, λ_H:

$$\mathbf{V}_4 = [V^k/\lambda_H], \tag{8.85}$$
$$\lambda_H = [a(V^x)^p + b(V^y)^p + c(V^z)^p + \varepsilon(V^s)^p]^{h/p} \tag{8.86}$$
$$C = i(V^k/\lambda_H)dx\hat{}dy\hat{}dz\hat{}dt = i(V^k/\lambda_H)\Omega(dx^k). \tag{8.87}$$

The coefficients $\{a, b, c, \varepsilon, p, h)$ are presumed to be constant. The coefficient, h, is defined to be the homogeneity index In 4D, the characteristic polynomial of $[Jac(\mathbf{V}_4/\lambda_H)]$ is of 4th degree, where for (possibly complex) eigen values, γ, the polynomial (by the Cayley Hamilton theorem) generates the hypersurface,

$$Ch[Jac(\mathbf{V}_4/\lambda_H)] = \gamma^4 - M\gamma^3 + G\gamma^2 - A\gamma + K = 0, \tag{8.88}$$
$$= \gamma^4 - (4-h)\gamma^3/\lambda_H + (6-3h)\gamma^2/\lambda_H^2 - (4-3h)\gamma/\lambda_H^3 + (1-h)/\lambda_H^4. \tag{8.89}$$
$$= (\gamma\lambda_H - 1)^3 (\gamma\lambda_H - 1 + h) = 0. \tag{8.90}$$

For different values of the homogeneity index, h, the Holder norm, λ_H, creates different homogeneity criteria. It is easily demonstrated (with Maple) that

$$\text{Mean Curvature: } M = (4-h)/\lambda_H = Trace\ [Jac(\mathbf{V}_4/\lambda_H)], \tag{8.91}$$
$$\text{Gauss Curvature: } G = (6-3h)/\lambda_H^2, \tag{8.92}$$
$$\text{Cubic Curvature: } A = (4-3h)/\lambda_H^3, \tag{8.93}$$
$$\text{Quartic Curvature: } K = (1-h)/\lambda_H^4, \tag{8.94}$$

So, as mentioned above, for h = 4, the trace of $Ch[Jac(\mathbf{V}_4/\lambda_H)]$ vanishes and the Mean Curvature of the hypersurface is zero. Hence the homogeneous process is a minimal surface. The 4-divergence of the process $\mathbf{V}_4/\lambda_H$ is then zero, and the volume element is invariant for such all such homogeneous processes (h=4). If the Mean curvature is not zero, then divergence of the homogeneous process depends upon the Holder norm, λ_H.

$$L_{(\mathbf{V}_4)}\rho\Omega = \rho\ d(i(V^k/\lambda_H)\Omega\} + (L_{(\mathbf{V}_4)}\rho)\ \Omega = \tag{8.95}$$
$$\text{where } \mathbb{J}=[\partial(V^m/\lambda_H)/\partial x^n] = \text{Jacobian matrix of } \mathbf{V}_4 \tag{8.96}$$
$$\{div_4(\mathbf{V}_4)\} = \Psi = (4-h)/\lambda_H \neq 0. \tag{8.97}$$

is well defined for any h, and for all forms of the Holder norm. When h > 4 the 4D volume is contracting; when h < 4, the 4D volume is expanding, due to the homogeneous process $\mathbf{V}_4/\lambda_H$.

Now return to the "diffusion" format,

$$\mathbf{V}_4 = [\partial\psi/\partial x, \partial\psi/\partial y, \partial\psi/\partial z, D\psi], \quad \rho\mathbf{V}_4 = \mathbf{V}_4/\lambda_H \quad , \tag{8.98}$$

$$\lambda_H = [a(\partial\psi/\partial x)^p + b(\partial\psi/\partial y)^p + c(\partial\psi/\partial z)^p + \varepsilon(D\psi)^p]^{h/p}. \tag{8.99}$$

Then the divergence of $\mathbf{V}_4/\lambda_H$ then yields a modified diffusion equation:

$$div_4(\mathbf{V}_4/\lambda_H) = (div_4\mathbf{V}_4)/\lambda_H - \mathbf{V}_4 \circ grad(\lambda_H)/(\lambda_H)^2 = (4-N)/\lambda_H, \tag{8.100}$$

$$= (div_4\mathbf{V}_4) - \mathbf{V}_4 \circ grad_4(\lambda_H)/(\lambda_H) = (4-h)/\lambda_H, \tag{8.101}$$

$$-(\varepsilon D)\partial\psi/\partial t = \partial^2\psi/\partial x^2 + \partial^2\psi/\partial y^2 + \partial^2\psi/\partial z^2 - (4-h) - \mathbf{V}_4 \circ grad_4(\ln\rho). \tag{8.102}$$

If the Characteristic Polynomial defines a minimal surface, $M = 0$, for $h = 4$, then the Gauss curvature is negative for any signature, indicating the surface is unstable. The volume element is invariant, but the Minimal Surface is unstable. Suppose that $N \leq 2$; then the Gauss curvature is positive, indicating that the Hypersurface generated by the Characteristic polynomial is stable, but it is not a Minimal surface. However, the Volume element is expanding.

Conjecture 38 *Is the Expansion of the universe required to stabilize the Hypersurface generated by the Characteristic polynomial?*

8.7.7 Ginsburg Landau Currents

With a change of notation ($\eta \Rightarrow \Psi$), the Universal Phase function (eq. 8.72) can be solved for the determinant of the Jacobian matrix, which is equal to the similarity invariant T_K,

$$T_K = -\{\Psi^4 - X_M\Psi^3 + Y_G\Psi^2 - Z_A\Psi\}. \tag{8.103}$$

The similarity invariant, T_K, represents the determinant of the Jacobian matrix. All determinants are in effect N-forms on the domain of independent variables. All N-forms can be related to the exterior differential of some (N-1)-form or current, J. Hence

$$dJ = T_K\Omega_4 = (div\mathbf{J} + \partial\rho/\partial t)\Omega_4 = -(\Psi^4 - X_M\Psi^3 + Y_G\Psi^2 - Z_A\Psi)\Omega_4. \tag{8.104}$$

For currents of the form,

$$[\mathbf{J}, \rho] = [grad\ \Psi, -D\Psi], \tag{8.105}$$

the Universal Phase function generates the format of the universal Ginsburg Landau equations of superconductivity:

$$\nabla^2\Psi - D\partial\Psi/\partial t = -(\Psi^4 - X_M\Psi^3 + Y_G\Psi^2 - Z_A\Psi). \tag{8.106}$$

It is also possible to construct conjugate minimal surfaces (see page187 in [270]) using the holomorphic function, Φ,

$$\Phi = A\ ln\rho + B + C\rho + D\rho^2, \tag{8.107}$$

which couples indistinguishable sets to Gibbs entropy, $A\ ln\rho + C\rho$, and Mandelbrot term $B \pm D\rho^2$, familiar to chaotic systems.

8.8 Topological Torsion again

It is remarkable that any 1-form of action, A, can be used to define a topology, and an equation of state via the Jacobian matrix of the coefficients. Perhaps even more remarkable is the fact that the system 1-form, A, also can be used to define a unique non-zero 3-form current, H = A ˆdA, associated only with non-equilibrium systems. This current 3-form is defined as the topological torsion 3-form and does not exist in equilibrium systems. The Topological Torsion vector, T4, (not V4) is a 4 component direction field that generates the topological torsion 3-form by contraction with the space-time volume element:

$$i(T4)\Omega(dx^k) = i(T4)dx\hat{}dy\hat{}dz\hat{}dt = A\ \hat{}\ dA. \tag{8.108}$$

The Topological Torsion vector is an associated vector with respect to 1-forms of Action, A, Work, W, and the Heat, Q. That is i(T4)A = 0 , i(T4)W = 0, i(T4)Q = 0, hence the process, T4, is transversely adiabatic. The equation of homotopic evolution relative to T4 becomes:

$$L(T4)A = W + 0 = \sigma \ A = Q. \tag{8.109}$$

The 4 divergence of the Topological Torsion vector, T4 (which is equivalent to the Kuratowski closure of the 3-form, A ˆdA) determines the Space-Time Irreversible Dissipation Coefficient σ.

$$d(A\hat{}dA) = dA\hat{}dA = \sigma \ (dx\hat{}dy\hat{}dz\hat{}dt). \tag{8.110}$$

On regions where $\sigma \neq 0$, the PTD(A) is 4. The thermodynamic system is not in equilibrium, as

$$L(T4)A\hat{} \ L(T4)dA = Q \ \hat{} \ dQ = (-\sigma)^2 \ A \ \hat{} \ dA \neq 0. \tag{8.111}$$

Then it seems reasonable to define

$$(\sigma)^2 \simeq \text{Entropy Production Rate}, \tag{8.112}$$

and the process in the direction of the Torsion vector T4 is thermodynamically irreversible.

Examples demonstrate that to reduce irreversible dissipation and turbulence in a Plasma, minimize $\sigma =$ (E $\circ$ B). To reduce irreversible dissipation and turbulence in a Navier Stokes fluid minimize $\sigma =$ (a $\circ$ ϖ), where

$$(a \circ \varpi) = \{gradP/\rho + \mu_B \ graddivV - \nu \ V \ x \ \nabla^2 V\} \circ \varpi \tag{8.113}$$

and μ_B =bulk viscosity, ν = shear viscosity, a = acceleration, ϖ = vorticity.

Note that turbulent solutions imply $\sigma =$ (a $\circ$ ϖ) $\neq 0$, thereby proving that there exist solutions to the Navier-Stokes equations that are thermodynamically irreversible, and thereby turbulent.

The fact that any differential 1-form can be used to define a topological space came to my attention about 1987, after some 20 years of studying Cartan's utilization of differential forms and their application to the theory of integral invariants. I defined this topological space as the disconnected Cartan topology of exterior differential forms. However, it was not until 2009, when I attended my first conference for professional topologists at Haceteppe, Turkey, that I learned that the Cartan topology of 4 differential form ingredients was exactly equivalent to a Kuratowski representation of the disconnected Kolmogorov T0 (poset 3) topology, based upon the power set of 4 exterior differential forms as ingredients.

8.9 Emergence of Topological Defects

The Category theory of Topological Thermodynamics yields examples explaining how continuous irreversible dissipative processes can form emergent Closed topological defect structures, far from equilibrium, in an Open thermodynamic environment of PTD

= 4. These examples give formal justification to Progogine's conjectures of dissipative emergence. Continuous topological evolution can describe the irreversible evolution on an "Open" symplectic non-equilibrium domain of Pfaff dimension 4, with evolutionary irreversible orbits being attracted to a contact "Closed" non-equilibrium domain of Pfaff dimension 3, with an ultimate decay to the "Isolated-Equilibrium" domain of Pfaff dimension 2 or less (an integrable Caratheodory surface). The experimental creation of Falaco Solitons in a swimming pool are visual examples of the thermodynamic emergence concept.

(see http://www22.pair.com/csdc/pdf/falaco85o.pdf)

Chapter 9

A SUMMMARY OF SIGNIFICANT RESULTS

As stated above, the Category of Topological Thermodynamics and Continuous Topological Evolution can be used as an abstract, but universal, method for understanding, among other things, scale invariance, chaos, emergence, self-similarity, dynamical systems, other non-linear, and even statistical, phenomena. The abstract categorical foundation, as an exterior differential system, admits solutions which can be put into correspondence with numerous physical experiments, in different physical disciplines.

• The method explains topological distinctions between equilibrium PTD(A) < 3 (integrable) and non-equilibrium PTD(A) > 2 (non-integrable) thermodynamic systems of distinguishable particles, based upon the concept of Pfaff Topological Dimension.

• The method explains topological distinctions between thermodynamically reversible, integrable processes (Q ˆdQ = 0) and irreversible non-integrable (Q ˆ dQ $\neq 0$) processes.

• The method explains topological distinctions between thermodynamic systems based on distinguishable sets (T0 or T2 topologies), and thermodynamic systems based upon statistical distributions of indistinguishable sets (Not-T0 topologies).

• The method explains that Entropy can consist of two simultaneous components: one component is based on distinguishable sets (massive particles) and the other component is based upon indistinguishable sets (radiation).

• The method demonstrates that the First Law of Thermodynamics is a statement of cohomology, where the difference of the two (not necessarily integrable) 1-forms of Heat Q and Work W are equal to an exact differential of internal energy, dU.

• The method demonstrates that the decay of turbulence can be described in terms of a topologically continuous process, but the creation of turbulence can not.

• The method indicates that the top down nested sequence of Open, Closed, Isolated, and Equilibrium thermodynamic domains is in 1-1 correspondence with the Pfaff Topological Dimension [4,3,2,1] of the 1-form, A, used to create the Cartan topological structure.

• The method suggests that the environment of the universe is not a vacuum (a void empty set PTD(A) = 0), but is best described as the non-integrable Open system of PTD(A) = 4, in which domains of PTD(A) < 4 are distinguishable as topological defects in the environment.

• Non-Equilibrium open systems of Pfaff Topological Dimension 4 (PTD(A) = 4) can decay (in finite time) to Closed systems of Pfaff Topological Dimension 3 (PTD(A) = 3), thereby justifying Prigogine's conjecture of emergence by means of dissipative, but topologically continuous, processes. These metastable PTD(A) = 3 Closed states are topological defects in the Open systems of PTD(A) = 4.

• Topological change is a necessary condition for thermodynamic irreversibility.

• The constraints of topological continuity and non-homeomorphic processes establish a logical basis for thermodynamic irreversibility and the arrow of time.

• Any synergistic system of parts defines a topology such that the Category of Topological Thermodynamics is applicable to economic systems, political systems, as well as to biological systems. Such systems admit long-lived states far from equilibrium (adulthood). However, the theory can be used to design subtle perturbations to such systems such that their metastable lifetimes can be extended or destroyed.

• C2 Continuous Topological Evolution permits irreversible processes, for which, $Q \hat{} dQ \neq 0$. Segmented C1 processes approximating smooth C2 processes can be reversible, $Q \hat{} dQ = 0$.

• On odd-dimensional spaces, sequential C1 (translational) processes can be thermodynamically reversible, while intransitive C2 processes (rotation and expansion with a fixed point) can be thermodynamically irreversible.

• The relativistic Twin Paradox is resolved if the evolutionary paths of each twin are defined by processes that cause (unequal) topological change. If there is no topological change, or if each twin suffers the same topological change, there is no disparate biological aging.

• Adiabatic processes are transverse to the heat 1-form, $(i(\rho V)Q) = 0$. Adiabatic processes need not be quasi-static, and can be reversible or irreversible. Topological Transitions can occur in finite (often short) time.

• The work 1-form, W, is always transverse to the 4D process direction field, ρV4, but the heat 1-form, Q, may or may not be transverse, The heat 1-form, Q can have longitudinal components in the direction of the process, corresponding to irreversible dissipation.

• Engineers should be guided by the universal concept of minimizing longitudinal heat in order to improve efficiency. For fluids, this idea can be translated to reducing vorticity components in the direction of fluid accelerations (by using wing tip tabs on commercial jets).

• For non-equilibrium systems, the 3-form of Topological Torsion (an N-1 form current) is not zero: $A \hat{} dA = i(T4)\, dx\hat{}dy\hat{}dz\hat{}dt \neq 0$. The Topological Torsion vector, T4, is deduced intrinsically from the 1-form A that encodes the thermodynamic system; ρT4 can be used as a direction field that describes a unique process current density.

• For PTD(A) = 3 "closed" thermodynamic systems, the process current has zero divergence, and the 4D volume element is a conformal invariant (any ρ). This result is the space-time extension of the Liouville theorem that preserves the phase-space volume element in classical theory.

• For a PTD(A) = 4 "open" thermodynamic systems, the Topological Torsion vector does not have zero divergence, and so the process current ρT4 may not be closed for arbitrary ρ (that is, the divergence of the process current is not zero).

• The combination of continuity and non-equilibrium, PTD(Q)>2, requires a causal direction to the arrow or time.

• PTD(A) = 3 domains can have a local basis in terms of one complex Spinor pair with complex conjugate eigenvalues, and one real vector with eigenvalue zero. The eigenvalue 0 state can represent metastable (long-lived) configurations far from equilibrium. Such domains are locally "contact" manifolds.

• A key artifact of non-equilibrium is the existence of Topological Torsion current 3-forms, Topological Spin current 3-forms and Topological Adjoint current 3-forms. All of these current 3-forms are similar to the Amperian charge current 3-form of electromagnetic theory, but are related to different species of dissipative phenomena, which occur only in non-equilibrium systems. The dissipation coefficients are related to the non-zero divergences of the vector coefficients of the various 3-forms. For example, in electromagnetic systems, the dissipation coefficient is proportional to E ∘ B;

(see http://www22.pair.com/csdc/download/ebookvol4.pdf) ;

in hydrodynamics, the dissipation coefficient is called "Bulk Viscosity":

(see http://www22.pair.com/csdc/download/ebookvol3.pdf).

• PTD(W) = 4 domains can have a local basis in terms of two complex Spinor pairs. Locally, such domains are symplectic manifolds. In such symplectic domains there exist local density distributions (sub domains), ρ, such that the divergence of any process current is zero in that sub domain. Such sub domains are metastable contact manifolds. In other words, the symplectic domain can contain defect structures in the form of contact sub domains. It can be demonstrated in terms of continuous topological evolution that such local density distributions, which define a "stationary" state, can emerge as a topological defect in a PTD(W) = 4 system, by means of a dissipative processes.

(See http://www22.pair.com/csdc/pdf/falaco85o.pdf)

• The Adjoint Current, Jadjoint, like the Torsion Current, can be constructed entirely from the topological features of the thermodynamic system as determined by the functional coefficients of the 1-form of Action, A. The Adjoint Current also admits an infinity of integrating factors. However, when the 1-form coefficients, A_k are divided by a Holder Norm, λ, of homogeneity index one (known as the Gauss map), then the components of the Adjoint Current are defined as |Jadjoint> = [Jacobian(A_k/λ)] ∘ |(A_k/λ)>. The 3-form of the adjoint current is defined as Jadjoint = i(Jadjoint) dxˆdyˆdzˆdt, and has zero divergence globally, and a pre-image 2-form that is not closed, Gadjoint, such that dGadjoint = Jadjoint. It can be demonstrated that the adjoint current is related to the cubic curvature of the shape matrix, leading to the idea that the source of electromagnetic charge is related to cubic curvatures, similar to the idea that mass is related to quadratic curvatures.

• Examples of thermodynamic systems can be given to demonstrate that the conjectured format of the London Current of superconductivity, where J = χ A, can be deduced as an emergent consequence of the Topological Theory of Thermodynamics.

(see http://www22.pair.com/csdc/download/ebookvol5.pdf)

• Examples can generate a Spin current 3-form, S = A ˆ G, where, formally, the Spin current is proportional to the Lorentz force (the space-time components of the Work 1-form, W). This is a new interpretation of an old result, J = σ (E + V x B), which is Ohm's law. The new part is due to the idea that the dissipation is due to Spin Currents and the transport of collective spins, A ˆ G.

• The topological structure of wave density domains of Pfaff dimension 2 or less creates a connected, but not necessarily simply connected topology. Evolutionary predictive solution uniqueness is possible.

• The topological structure of particle domains of Pfaff dimension 3, or more, creates a disconnected topology of multiple components. If solutions to a particular evolutionary problem exist, then the solutions are not unique. Envelope solutions, such as Huygen wavelets and propagating tangential discontinuities (called signals, or wakes) are classic examples of solution non-uniqueness. Topological Torsion is an artifact of non-uniqueness, and must be non-zero if a hydrodynamic system exhibits turbulence.

• All Hamiltonian, Symplectic-Bernoulli and Helmholtz processes are thermodynamically reversible. In particular, the work 1-form, W, created by Hamiltonian processes is of Pfaff Topological Dimension 1 or less. In all reversible cases the Work 1-form is closed, dW = 0. The evolutionary equations in such cases are time reversal invariant.

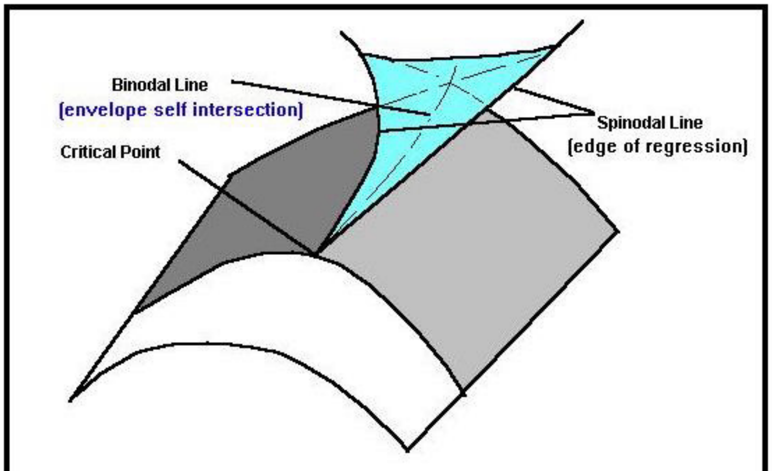
• The assumption of uniqueness of evolutionary particle solutions (which implies the Pfaff Topological Dimension of the thermodynamic system be equal to 2 or less), or the assumption of homeomorphic evolution, have imposed constraints upon classical mechanics that eliminate any time asymmetry, and the existence of isotropic spinors.

• The Lie differential acting on differential forms is not necessarily the same as a linear affine covariant differential acting on differential forms. The covariant differential always defines an adiabatic process, where the Lie differential does not.

• The particle view is based upon the 1-form of action whose coefficients, $[A_m]$, admit a correlation Jacobian matrix $[\partial A_m/\partial x^n]$.

• The statistical point of view is based upon a current N-1 form whose coefficients, $[C^m]$, admit a collineation Jacobian matrix, $[\partial C^m/\partial x^n]$, with a trace equal to a divergence of the Current.

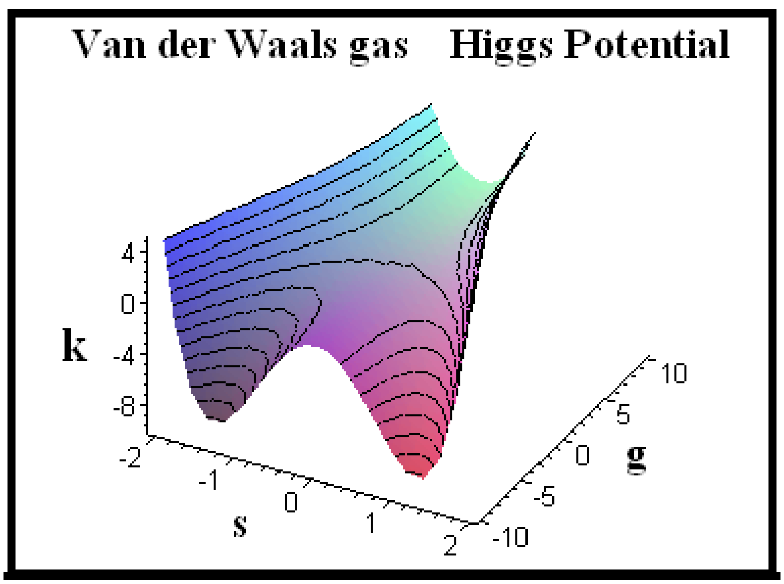
• On spaces of PTD(A) = 4, the correlation Jacobian matrix has a characteristic Cayley-Hamilton polynomial that defines an quartic equation of state in terms of similarity invariants. The characteristic polynomial produces an implicit hypersurface function that can have envelopes and edges of regression in the format of the Gibbs function for a (universal and deformable) van der Waals gas.



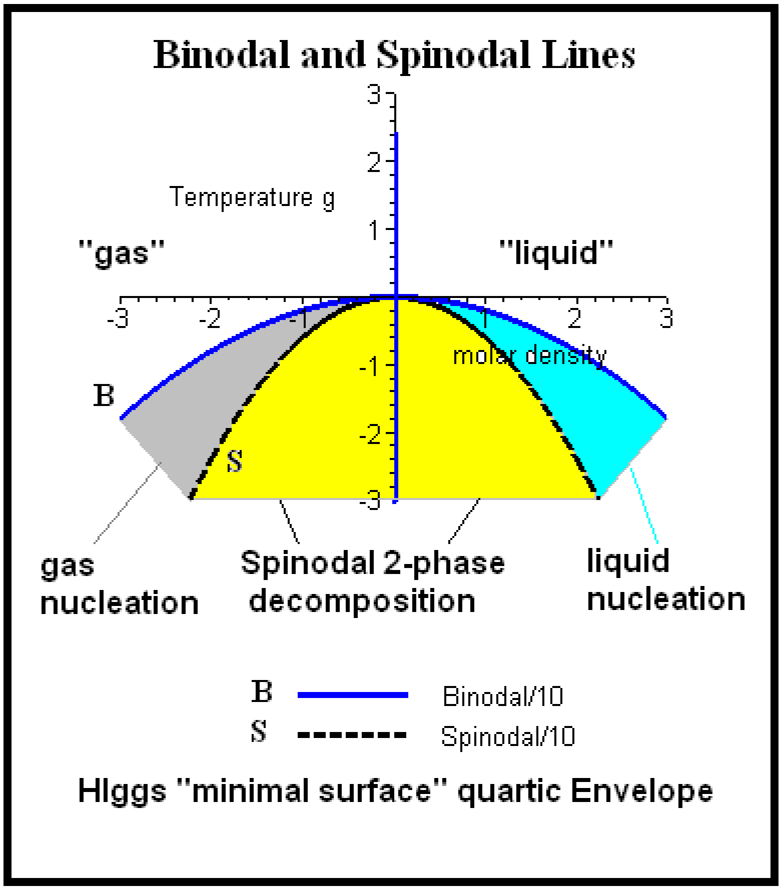
A Universal Thermodynamic Swallowtail
The Gibbs surface of a Van der Waals gas

(see http://www22.pair.com/csdc/graphics/univgibb.jpg

• The correlation can be mapped into a reduced characteristic polynomial representing a quartic Universal Thermodynamic Phase function, with an envelope, which, below the critical point, has features of a Higgs potential.



• The reduced polynomial yields universal analytic expressions for the critical point, and the binodal and spinodal lines, in terms of the similarity invariants. The same technique can be applied to dynamical systems.

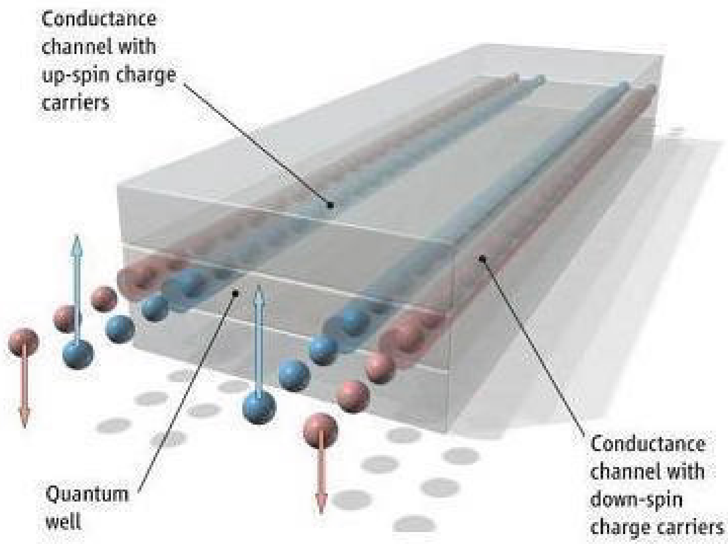


• The collineation can have complex eigenvalues and eigenvectors, even though the maximal rank matrix elements are real. These complex eigenvectors have a zero norm, and are generators of conjugate minimal surfaces, that have both statistical and fractal components.

• The Jacobian matrix can have complex eigenvalues and eigenvectors, even though the matrix elements are real. These complex eigenvectors have a zero norm, and are generators of conjugate minimal surfaces. Cartan defined such sets as Isotropic Spinors (Majorana Spinors, not Dirac Spinors). Pairs of non-co-linear Spinors define an area, but the "norm" of each Spinor is Zero!

• The hypersurface minimal surface can be generated by a holomorphic function that includes both the Gibbs entropy and a Mandelbrot fractal germ, $\Theta = (\text{zln z-z}) + (\text{a-bz}^2)$. The third partial derivative of Θ leads to conjugate pairs of minimal surfaces. and the Mandelbrot germ vanishes. All functional iterates remain holomorphic and hence generate minimal surfaces with fractal boundaries.

• Topological fluctuations can be induced by processes that have components in terms of thermodynamic macroscopic Spinors. Thermodynamic Macroscopic Spinors are non-zero complex eigenvectors with complex eigenvalues (and zero quadratic form) of the antisymmetric 2-form (or matrix) representing the "Limit



Topological Spin Currents

Picture Courtesy of C. Bickel / Science

Figure 1

Points", dA, of the 1-form of Action, A. Such Macroscopic Spinors are capable of representing minimal surface conjugate pairs.

• Topological Insulators correspond to Impedances defined in terms of quantized Topological Spin, and Spin pairs coupled via Falaco Solitons. If the divergence of the Spin 4-vector vanishes, the Spin Current is time reversal invariant.

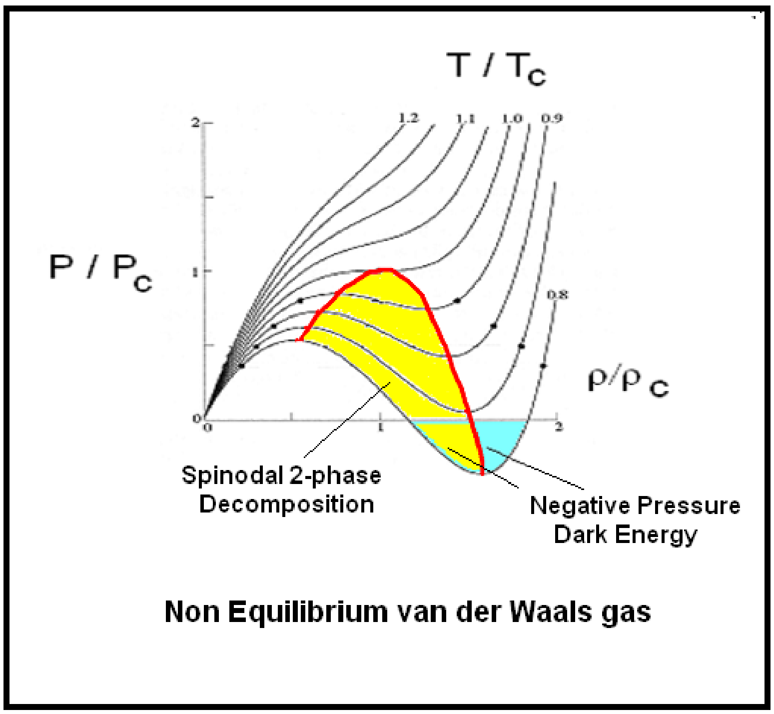
• The thermodynamic processes that lead to self-similarity of a Current 3-form L(J) C $= \sigma$ C can generate fractals and holographic effects where small neighborhoods replicate the whole, approximately. The homogeneity coefficient is the trace of the Jacobian Collineation: $\sigma = \text{Trace}\ [\partial \mathrm{C}^m/\partial \mathrm{x}^n]$, or the divergence of the Process vector field.

• Following Landau, it is conjectured that a turbulent non-equilibrium thermodynamic cosmology can be constructed in terms of a dilute van der Waals gas near its critical point. The conjecture yields an explanation for:

a The granularity of the night sky as exhibited by stars and galaxies is due to density fluctuations near the critical point, and the Newtonian law of gravitational attraction proportional to $1/\mathrm{r}^2$ is due to a correlation between fluctuations (due to Landau).

b The conformal expansion of the universe as an irreversible phenomenon-associated with Quartic similarity invariants in the thermodynamic phase function, and conformally related to dissipative effects.

c The possibility of domains of negative pressure (explaining what has recently been called "dark energy") could be due to a classical "Higgs" mechanism for aggregates below the critical temperature.



Non Equilibrium van der Waals gas

d The possibility of domains of negative temperature (explaining what has recently been called "dark matter") could be due to macroscopic collective states of ordered spins. The conjecture is that Positive temperature radiates, Negative temperature does not. The conjecture is that black holes could be negative temperature states of collective spins.

e The possibility of domains where gravitational effects (quadratic similarity invariants, or 2nd order Gauss curvature effects) appear to be related to entropy and temperature properties of the thermodynamic system, and where cubic curvature effects could impede gravitational collapse.

f Black Holes (generated by Petrov Type D solutions in gravitational theory) are related to Minimal Surface solutions to the Universal thermodynamic 4th order Phase function.

Chapter 10
EPILOGUE: WAVES VS PARTICLES.

(*December 2011 The following consists of a set of preliminary results of a work in progress*)

The thermodynamic states are of two topological species: Those finite Particle-like T0 topologies and those finite Wave-like Not-T0 topologies.

1. The Particle-like T0 topologies have separation axioms such that all singlet subsets are *Distinguishable, with distinct closures.*

2. The Complex Wave-like Not-T0 topologies do not have separation axioms, so that some, if not all, of the singlet subsets are *Indistinguishable, and have identical closures.*

The Category theory of Thermodyamics admits the coexistence of the two species of topological objects.

In order to be independent from a choice of coordinates, the particle-like objects can be encoded as collections of exterior differential 1-forms, $A(x^m, dx^m)$, and the wave-like objects can be encoded as collections of exterior differential N-form densities, $\mu(x^m, dx^m)$. The 1-forms can generate T0 topological spaces of Pfaff Topological Dimension 4. N-form densities can generate Not-T0 topologies without separation axioms such as the trivial Sierpinsky topology, or the indiscrete topology. The N-form densities are closed.

As demonstrated in a previous chapter, the Continuous Topological evolution of an N-form density leads to many of the complex diffusion, or wave equations used in modern physics. However, the student of modern physics is trained to accept the jargon that there is a dualism between waves and particles. A precise definition of the dualism is rarely formulated in simple terms. For example, in the domain of electromagnetism, the student is taught that electromagnetic waves (which are solutions to the Maxwell "wave equations") consist of objects called photons (a type of Boson that has integer spin); but, what exactly is a photon? This question continues to invoke a multitude of responses from optical engineers and physicists at the yearly SPIE conferences around the world [244].

Waves are collections of indistinguishable ingredients, and particles are collections of distinguishable ingredients. Waves transport energy and momentum without transporting mass. Particles transport energy and momentum in the form of mass.

The conclusion indicated by the six monographs of this series is that Waves are artifacts of selected Not-T0 topologies, where Particles are artifacts of selected T0 topologies.

Yet it is commonplace in the scientific community to ignore these topological distinctions.

The newly recognized concept is that the Not-T0 topological class appears to be the appropriate domain for describing indistinguishable objects of a statistical ensembles. The fundamental equations of thermodynamic evolution in terms of exterior differential form densities lead to complex wave - diffusion PDE,s, such as the Schroedinger (Hamiltonian) equation over odd dimensional systems, and Cartan Spinor equations over even dimensional systems. For Non-Equilibrium systems of distinguishable particles, it appears that a Kuratowski representation of a Kolmogorov, disconnected topology are required. In category theory these different topological objects can coexist. The concepts of "Statistical" thermodynamics are distinct from "Particle" thermodynamics, and the two concepts can not be equilibrated as they belong to different topological species. Therefor, there must be a statistical component of Entropy distinct from, but coexistent with, a mechanical component of Entropy.

Conjecture 39 *The holy Grail of finding a marriage between Quantum Mechanics and Gravity will require the recognition that Quantum Mechanics (of indistinguishable Cartan isotropic spinors) is based upon connected Not-T0 topologies and that Gravity (of distinguishable particles) is based upon disconnected T0 or T2 topologies.*

Chapter 11
REFERENCES, ADDENDA, AND APPENDIX 1

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lished on my web site as http://www22.pair.com/csdc/pdf/cologne.pdf. The work, however, was accepted for presentation at the Helsinki conference on the Foundations of Quantum Mechanics, but due to other commitments I could not attend the Helsinki meeting. It was not until 1986, with the recognition of the Falaco Solitons, that the interesting mapping oddity took on a more important character. The basic ideas were presented later in Florida,1989, but all with out my appreciation of Bohm's ideas about quantum mechanics. The update work appeared as http://www22.pair.com/csdc/pdf/bohmplus.pdf.

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324. I attended a conference in Copenhagen in late June 2009, where the venue was to include the new ideas in thermodynamics. I wanted to present an outline of the topological ideas contained within this monograph. After the meeting commenced, the organizers changed the scope of the conference (!), and my work on topological thermodynamics, and the work of others on other new ideas, were not permitted to be presented. Those of us with new ideas were told that the only topic that was admissible was "how to teach the concept of entropy". I have attended more than 60 conferences and meetings over the years, but this was the first time that the venue was changed - unilaterally - after the meeting had started, and the participants had arrived! Egad.

11.2 Acknowledgments, Comments and Index

I must thank David Radabaugh, at one time a student, then a colleague, and always a friend. Not only did he supply the first photos in 1986 of the Falaco Solitons as topological defects in a fluid, but also he spent many painstaking hours editing, suggesting changes, and checking cross references, and all the other things that I hate to do. Thanks, David.

Index

11.3 Symbols

The notation used in this monograph is:

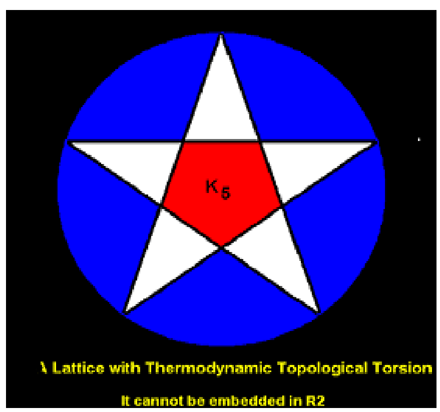
L	Lagrange function.
$L_{(V)}$	Lie differential with respect to V.
H	Hamiltonian function (typically $\mathsf{H} = pV - \mathsf{L}$).
H	The 3-form of Topological Torsion, $A\hat{}F$ (not necessarily equal to Frenet Torsion, or Affine Torsion).
K	The 4-form of Topological Parity.
A	The 1-form of system Action (can be path dependent).
F	The exact 2-form of field intensities $F = dA$ (coefficients are covariant tensors wrt diffeomorphisms).
G	The non-exact 2-form of field quantities (coefficients are contravariant tensor densities wrt diffeomorphisms).
J	The exact 3-form of charge current densities.
W	The 1-form of Work (can be non-exact and path dependent).
Q	The 1-form of Heat (can be non-exact and path dependent).
PTD PD	Pfaff Topological Dimension

Take care to distinguish the Lagrange function (L sans serif) from the Lie differential, $L_{(V)}$, and the Hamiltonian Function (H sans serif) and the 3-form of Topological Torsion, H.

11.4 About the Cover Picture

In the study of the Topological Thermodynamics one of the most interesting features is the correspondence between non-equilibrium systems and the metric free concepts of Topological Torsion and Topological Spin. The classic concept of Frenet-Serret torsion is related to Topological Torsion, but Frenet Torsion depends upon metric properties, in contrast to Topological Torsion which does not. Engineers would say that an elastic rod exhibits torsion if the rod of elastic material is twisted. However, when the constraints of the twist are relaxed, the rod returns to its original state - which is torsion free. Such concepts lead to a description of what is called affine torsion, based upon metric properties and their differentials. An open space curve with Frenet torsion and two endpoints (a 1D string) in R3 with can be embedded (perhaps with deformation) into R2. A more complicated problem occurs with lattices of points connected with deformable curves (defined by curvilinear arrows). Such things are called categories.

The lattice points and their deformable curvilinear connections can be sketched into a planar drawing. The question that can be asked, is the planar graph an embedding? That is, can the lattice with its connections be drawn in the plane such that no two curves of connection intersect?



Kuratowski demonstrated that there are two simple configurations, named K5 and K3,3, that cannot be embedded in R2; and moreover, that if any connected lattice contains either of the these simple configurations, then the more complex structure could not be embedded in the plane. That is, the lattices K5 and K3,3 are irreducibly 3 dimensional. The pentagon shape is a realization of K5

This irreducible property is one of the characteristics of Topological Torsion – a system with Topological Torsion is irreducibly of three topological dimensions or more. A key engineering feature is that if a differential system has Topological Torsion, it does not yield unique predictive solutions for given initial conditions. These non-metrizable features of Topological Torsion also apply to the study of chiral molecules. If a closed lattice structure (a connected lattice) has a Mobius Band with and odd number (3 or greater) of shunts, the lattice cannot be embedded in the plane. This leads to the idea of Topological Chirality. useful to the Chemistry of chiral molecules.

11.5 About the Author

Professor R. M. Kiehn, B.Sc. 1950, Ph.D. 1953 (Physics, Course VIII, MIT), started his career working (during the summers) at MIT, and then at the Argonne National Laboratory on the Navy's nuclear powered submarine project. Argonne was near his parents home in the then small suburban community known as Elmhurst, Illinois. At Argonne, Dr. Kiehn was given the opportunity to do nuclear experiments using Fermi's original reactor, CP1. The experience stimulated an interest in the development of nuclear energy. After receiving the Ph. D. degree as the Gulf Oil Fellow at MIT, Dr. Kiehn went to work at Los Alamos, with the goal of designing and building a plutonium powered fast breeder reactor, a reactor that would produce more fissionable fuel then it consumed. He was instrumental in the design and operation of LAMPRE, the Los Alamos Molten Plutonium Reactor Experiment. He also became involved with diagnostic experiments on nuclear explosions, both in Nevada on shot towers above ground, and in the Pacific from a flying laboratory built into a KC-135 jet tanker. He is one of the diminishing number of people still alive who have witnessed atmospheric nuclear explosions.

Dr. Kiehn has written patents that range from AC ionization chambers, plutonium breeder reactor power plants, to dual polarized ring lasers and down-hole oil exploration instruments. He is active, at present, in creating new devices and processes, from the nanometer world to the macroscopic world, which utilize the features of Non-Equilibrium Systems and Irreversible Processes, from the perspective of Continuous Topological Evolution.

Dr. Kiehn left Los Alamos in 1963 to become a professor of physics at the University of Houston. He lived about 100 miles from Houston on his Pecan Orchard - Charolais Cattle ranch on the banks of the San Marcos river near San Antonio. As a pilot, he would commute to Houston, and his classroom responsibilities, in his Cessna 172 aircraft. He was known as the "flying professor".

He is now retired, as an "emeritus" professor of physics, and lives in a small villa at the base of Mount Ventoux in the Provence region of southeastern France. He maintains an active scientific website at

(http://www.cartan.pair.com).

11.6 Other Volumes in the Series

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Volume 7 Selected Publications.

Maple programs

Many of the examples used to describe the theoretical concepts of non-equilibrium systems involve overwhelming algebra and calculus computations. These computations can be made tractable by the use of symbolic mathematics programs such as Maple. As Maple permits the rapid visualization of many different cases, it has been my experience that unsuspected trends can be deduced, leading to unsuspected new theorems. A number of useful Maple programs have been compiled and can be downloaded from http://www22.pair.com/csdc/maple

11.7 Extensions and Ideas

At the time of writing of this volume (2009-2010), research investigations utilizing the techniques of non-equilibrium hydrodynamics are being continued in the areas of:

Is the universe expanding **and** rotating?
Can dual polarized ring lasers detect the chirality of the universe?
Potential Vorticity and Rossby waves
Limit cycles on the geosphere and frontal patterns in weather.
Polarization and Chirality in Wake formation.
Vortex Drag Reduction via Minimal Surface design.
Odd-dimensional topological effects, Flux (circulation) vs. Torsion Helicity.
Magnus Forces, Metastable states, Topological phase changes in fluids.
Cavitation and condensation. Fractal effects.
The River Model of Cosmology.
Spinors and topological fluctuations in hydrodynamics.
Affine Diffusion, translational shear vs. non-Affine rotations and expansions.
Zeno's paradox as a Spinor limit.
Not-T0 Topologies that yield MB, Boson and Fermion distributions.

11.8 Appendix 1: Lattice Structures