Irreversible Processes and the Torsion Current

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Abstract

For constrained physical systems that can be described by a 1-form of Action, the topological Pfaff dimension,, or class of the 1-form is either even or odd. Action 1-forms of odd Pfaff class generate contact manifolds of dimension 2n+1, and support an evolutionary path defined by the unique null eigenvector of the matrix of coefficients of the 2-form, dA. Such evolutionary extremal paths are conservative in a Hamiltonian sense, and describe processes which are thermodynamically reversible. Action 1-forms of even Pfaff class generate symplectic manifolds of dimension 2n+2, for which null eigenvectors (extremal fields) of the 2-form dA do not exist. Such symplectic manifolds do support a unique evolutionary path defined but the Torsion Current, a 2n+1 form constructed as AˆdAˆ..ˆdA. Evolution in the unique direction of the components of the Torsion Current is, in general, thermodynamically irreversible.

1. Introduction

This article is restricted to the study of :

1. those constrained physical systems that can be characterized mathematically by an Action integral (or a 1-form of Action, A, whose components define a covariant vector field under diffeomorphisms), and

2. those evolutionary processes that can be described by a contravariant vector density, $\rho\mathbf{V}$, or a current.

The Action 1-form will consist of two parts, one part representing the physical system in an idealized (isolated) mode and another part representing the interaction of any particular system with its environment. The part representing the interactions usually will be constructed from anholonomic differential forms, and their associated Lagrange multipliers. The objective herein is to develop a straight-forward non-statistical method for assessing thermodynamic information from the dynamical evolution of such physical systems. In particular, How can you determine if a given process, $\rho\mathbf{V}$, for a given physical system, A, is reversible or irreversible?

2. The Pfaff Class or the Topological Dimension

A specific 1-form of Action can be constructed from an ordered set of M functions, $A_m(x^k)$ and differentials, dx^m on a base space of M arbitrarily chosen independent variables, $\{x^1, ..., x^M\}$. A typical format in terms of coordinates then is given by the expression,

$$A = \sum_1^m A_m(x^k)dx^m. \tag{2.1}$$

The question arises as to the possible redundancy in this preliminary description of the physical system. In particular, it might be possible to find a lesser number of independent variables and covariant functions, $A_m(x^k)$, and still adequately describe the topological features of the 1-form, A. The implication is that there exists a $C1$ map, ϕ, from the original space of M independent variables to a space of N independent variables, with $N < M + 1$. On the N dimensional space, the 1-form of Action has a "canonical" format which is different depending on when $N = 2n + 1$, or $N = 2n + 2$. This number N is defined as the Pfaff Dimension.

To determine N, the procedure is to first construct, with exterior differential processes, the 2-form dA,and then to construct the elements of the Pfaff sequence, $\{A, dA, A\hat{}dA, dA\hat{}dA, ...\}$ using the exterior product. The elements of this sequence will terminate (become zero) at either $(dA)^{n+1}$ or $A\hat{}(dA)^n$. The value of the integer, n, determines the Pfaff topological dimension of the Action 1-form.

In short, the Pfaff Dimension determines the irreducible minimum number of functions that are required to describe the Action 1-form. If the Pfaff dimension is odd, 2n+1, then the 1-form of Action is said to generate a contact manifold. A canonical representation is of the form,

$$A = \sum_{k=1}^{n} p_k dq^k + d\tau \tag{2.2}$$

where the 2n+1 functions, $\{p_k,\ q^k, \tau\}$ are well defined functions of $\{x^1, ..., x^M\}$. If the Pfaff dimension is even, 2n+2, then the 1-form of Action is said to generate an exact symplectic manifold. This symplectic manifold of dimension 2n+2, is generated from a 2-form, which, by Stokes theorem, can not be compact without boundary. The 2-form represents either a manifold with boundary, or it is an open manifold, a perfect candidate for studying thermodynamic non-equilibrium systems.

A canonical representation for the 1-form of Action is of the form on the symplectic manifold is,

$$A = \sum_{k=1}^{n+1} p_k dq^k. \tag{2.3}$$

The key issue is that evolutionary behavior of the physical system, as represented by the 1-form of Action, on the contact manifold is different from the evolutionary behavior of the physical system on the symplectic manifold. The symplectic manifold of dimension 2n+2 is not compact without boundary.

One of the remarkable features of Cartan's development of Hamiltonian mechanics is the result that the Hamiltonian evolutionary vector field on a contact manifold is, to within a factor, uniquely defined by the functional form of the Action integrand, and its first partial derivatives, with out ad-hoc additions, or forces, or constraints. Of all the possible evolutionary paths, the Hamiltonian vector field is the unique path which extremizes the Action integral in the sense of the Calculus of Variations. Much of theoretical physics has has been based on such idea, with great success in modeling experiment.

However, experience indicates that such Hamiltonian evolutionary processes are reversible in a thermodynamic sense, yielding a paradox that such methods apparently are in disagreement with the observation of aging. There seems to be no unique, non ad-hoc, way to describe irreversible processes. Part of the problem is that there seems to be no historical universal method or principle (known to this author) of connecting dynamical systems and thermodynamics, especially irreversible thermodynamics, that does not invoke some sort of statistical non-deterministic concept from the outset.

In this article a new, non-statistical, principle for understanding irreversible thermodynamic processes is offered. The work is motivated by the realization that while deterministic prediction of the functional form for any tensor field is impossible (for continuous processes in which the topology of the initial state is not the same as the topology of the final state), the concept of deterministic retrodiction has possible solutions for systems which simultaneously are only statistically predictable [4]. The concept of retrodictive determinism, however, is only possible on the cotangent domain of exterior differential forms, via the mathematical techniques of functional substitution and the pullback. Those physical systems for which the method is applicable must be capable of being defined in terms of differential forms. The simplest definitions of physical systems will be based upon the 1-form of Action.

The new possibility for understanding irreversible processes will be based upon the discovery that there exists a non-Hamiltonian, but unique - to within a factor - vector field, on non-compact symplectic manifolds, and this vector field is completely determined to within a factor by the physical system (that is the functional forms that make up the 1-form of Action), without ad-hoc additional assumptions. Moreover, evolution along these unique paths is irreversible in a thermodynamic sense.

The uniqueness of this vector field, defined as the topological torsion current, is perhaps its most crucial analytic feature. Recall it is the uniqueness of the Hamiltonian extremal field that gives it a philosophical basis as Nature's choice (for reversible processes). Similarly it is suggested that the uniqueness of the torsion current current establishes its philosophical position as Nature's choice for irreversible processes. Rather than defining a system of first order equations generated from a single Hamiltonian function, the Torsion current is inherently defined as a second order system of the Volterra-Hamilton type. Instead of leaving the Action stationary as a relative integral invariant, the Action is only conformally invariant relative to processes that proceed in the direction of the Torsion current. The thesis presented in this article is that it is this unique torsion current that describes irreversible dissipative evolution, in the same sense that Hamiltonian processes describe uniquely reversible processes.

To summarize: on a contact manifold generated by a 1-form of Action and of dimension 2n+1 there exists a unique "Conservative Hamiltonian Reversible" vector field that leaves the Action integral stationary, as a relative integral invariant. On a symplectic manifold generated by a 1-form of Action and of dimension 2n+2 there exists a unique "Irreversible Torsion Current" which leaves the Action integrand conformally invariant. It remains for experiment to determine if these unique irreversible processes generated by the Torsion current can be used to develop an understanding of irreversible dissipative systems in a practical way.

2.1. Cartan's Magic Formula

Cartan's magic formula [1],

$$L_{(\mathbf{V})}A = i(\mathbf{V})dA + d(i(\mathbf{V})A) = Q \tag{2.4}$$

serves as a bridge between the theories of dynamical systems, thermodynamics, irreversible processes, and the calculus of variations. Consider those physical systems that can be represented by a 1- form of Action, A, and those processes that can be represented by a dynamical system, $\mathbf{V}$. The Action 1-form is constructed from an ordered set of M functions, $A_m(x^k)$ and differentials, dx^m on a base space of M independent variables, $\{x^1, ..., x^M\}$. A typical format in terms of coordinates is given by the expression,

$$A = \sum_1^m A_m(x^k)dx^m \tag{2.5}$$

The covariant functions, $A_m(x^k)$, and their first partial derivatives, may be used to construct a topological basis on the domain of independent variables. In particular, the induced topological structure determines a minimum number of functions, N, required to express the topological features of the physical system. The implication is that there exists a $C1$ map, ϕ, from the base space of M variables to a space of N variables, with $N < M + 1$. On the N dimensional space, the 1-form of Action has a "canonical" format which is different depending on when $N = 2n + 1$, or $N = 2n + 2$. This number N is defined as the Pfaff Dimension.

2.2. The Pfaff Class or the Topological Dimension

The Action 1-form on a domain of independent variables $\{x^1, ..., x^M\}$ may be used to define a (Cartan) topology on the domain. The first step is to compute the exterior derivative of the Action, dA. The second step is to construct, algebraically, the elements of the Pfaff sequence $\{A, dA, A\hat{}dA, dA\hat{}dA, ...\}$. Then elements of this sequence will terminate (become zero) at either $(dA)^{n+1}$ or $A\hat{}(dA)^n$. The value of the integer, n, determines the Pfaff topological dimension of the Action 1-form.

In short, the Pfaff Dimension determines the irreducible minimum number of functions that are required to describe the Action 1-form. If the Pfaff dimension is odd, 2n+1, then the 1-form of Action is said to generate a contact manifold. A canonical representation is of the form,

$$A = \sum_{k=1}^{n} p_k dq^k + d\tau \tag{2.6}$$

where the 2n+1 functions, $\{p_k,\, q^k, \tau\}$ are well defined functions of $\{x^1, ..., x^M\}$. If the Pfaff dimension is even, 2n+2, then the 1-form of Action is said to generate an exact symplectic manifold. This symplectic manifold of dimension 2n+2, by Stokes theorem, can not be compact without boundary. It is either a manifold with boundary, or it is an open manifold, a perfect candidate for studying thermodynamic non-equilibrium systems.

A canonical representation for the 1-form of Action is of the form on the symplectic manifold is,

$$A = \sum_{k=1}^{n+1} p_k dq^k. \tag{2.7}$$

The key issue is that evolutionary behavior of the physical system, as represented by the 1-form of Action, on the contact manifold is different from the evolutionary behavior of the physical system on the symplectic manifold. The symplectic manifold of dimension 2n+2 is not compact without boundary.

2.3. The First Law of Thermodynamics

When the term $i(\mathbf{V})dA$ in Cartan's magic formula is identified with the 1-form of virtual work, W, and the term $d(i(\mathbf{V})A)$ is identified with the Internal energy, $d(U)$, then Cartan's Magic formula is recognized as the equivalent to the first law of Thermodynamics:

$$L_{(\mathbf{V})} \int_a^b A = \int_b^a W + d(U) = \int_a^b Q \tag{2.8}$$

This equivalence of co-homology will be taken at face value in this article. Axiomatically, then, given a physical system, A, and a process, V, the evolution generates a 1-form of heat, Q. Processes for which $Q = 0$ are said to be locally adiabatic. Extremal processes which satisfy the equation $W = i(\mathbf{V})dA = 0$ are uniquely determined as the null eigenvector of the anti-symmetric matrix generated by $d\dot{A}$. For such processes the virtual work vanishes, but they need not be locally adiabatic. A process for which $W = i(\mathbf{V})dA = 0$ and $d(U) = d(i(\mathbf{V})A) = 0$ is locally adiabatic. Such processes are said to be defined by characteristic vector fields.

2.4. Irreversible Processes

Cartan's magic formula may be utilized to test if a dynamical system $\mathbf{V}$ represents an irreversible process. From calssical thermodynamics, a process is irreversible when the heat 1-form, Q, does NOT admit an integrating factor (the Temperature). From Frobenius, the 1-form Q does not admit an integrating factor iff

$$Q\hat{\ }dQ \neq 0. \tag{2.9}$$

Hence use Cartan's magic formula to compute

$$Q\hat{\ }dQ = L_{(\mathbf{V})}A\hat{\ }L_{(\mathbf{V})}dA \tag{2.10}$$

for a given physical system, A, and a given process, $\mathbf{V}$. If $L_{(\mathbf{V})}A\hat{\ }L_{(\mathbf{V})}dA \neq 0$, then the process represented by the dynamical system $\mathbf{V}$ is irreversible in a thermodynamic sense.

For Extremal processes, $dL_{(\mathbf{V})}A = L_{(\mathbf{V})}dA = di(\mathbf{V})dA + ddU = 0 + ddU = dQ = 0$; for Symplectic-Helmholtz processes, $d(i(\mathbf{V})dA) = 0$,and therefore all such processes (which include Hamiltonian processes) are thermodynamically reversible. On an odd dimensional contact manifold , the extremal Hamiltonian process is uniquely defined in terms of the functions that make up the definition of the physical system. On a even dimensional system, symplectic Hamiltonian fields can exist, but they are not uniquely defined interms of the functions that make up the definition of the physical system. None of these Helmholtz-Symplectic-Hamiltonian methods, on contact or symplectic manifolds, can be used to describe thermodynamic irreversibility.

However, there does exist a vector field that satisfies that criteria of thermodynamic irreversibility. Moreover, this vector field is uniquely defined to within a factor by the functions that make up the definition of the physical system. It is unique for a given physical system, but only if the 1-form that defines the physical system generates an even dimensional non-compact symplectic manifold.

2.5. Calculus of Variations

From the definitions given above, a process acting on a physical system is defined in terms of a contravariant vector field, $\mathbf{V} = V^m(x^k)$. Those vector fields, $\mathbf{V}$, such that $L_{(\mathbf{V})} \int_a^b A \Rightarrow 0$ are equivalent to those paths in the calculus of variations that leave the Action integral stationary [2]. For boundary conditions such that $[i(\mathbf{V})A]_a^b = 0$, or for integrals over a closed integration chain, the processes that leave the Action integral stationary (a relative integral invariant) must satisfy the equation,

$$i(\mathbf{V})dA = 0. \tag{2.11}$$

When the 1-form $W = i(\mathbf{V})dA$, is defined as the virtual work, it becomes apparent that the stationary condition is equivalent to the d'Alambert principle, which requires that the virtual work must vanish [3]. The vector fields that satisfy $W = i(\mathbf{V})dA = 0$ are defined as "exrtremal" vector fields, relative to the Action A. Note that the extremal vector fields are uniquely defined only to within a factor. For if $\mathbf{V}$ is an extremal field with respect to the physical system, then so is $\rho(x^k)\mathbf{V}$ an extremal field. Only the paths, or vector lines, are determined by the extremal system, a feature typical of projective geometries.

However, the extremal vector field exists uniquely only for physical systems that induce an odd dimensional Pfaff topology. The extremal field has 2n+1 components.

2.6. Hamiltonian Mechanics

A vector field is said to be a Hamiltonian vector field if there exists a generating function such that the 2n conjugate components of the vector field satisfy the ordinary differential equations,

$$dq/\{\partial\Theta(p,q)/\partial p\} = dp/\{-\partial\Theta(p,q)/\partial q\} \tag{2.12}$$

On domains of topological dimension equal to 2n+1 (contact manifolds), Cartan has demonstrated that their exists a unique solution, $\mathbf{V} \to \mathbf{E}$, such that the virtual work vanishes, $W = i(\mathbf{E})dA = 0$, and that unique solution has a Hamiltonian representation, with a Hamiltonian function, $\Theta \Rightarrow H(p,q,t)$ that can be a domain constant, $dH = 0$. [4].

Cartan obtained this idea by prolonging the classic phase space 1-form to include time on a footing similar to the coordinates. He chose for his Action, the 1-form given by the expression,

$$A = p_k dq^k + ds \Rightarrow p_k dq^k + H(p_k, q^k, t)dt \qquad (2.13)$$

The Pfaff dimension is 2n+1, and the Action A is said to generate a contact manifold. The 2-form, dA, is of rank 2n, and as a anti-symmetric matrix of functions has a unique null eigen vector. It is this null eigenvector that describes (to within a factor) the unique extremal Hamiltonian field. The key issue is that this unique evolutionary process, $\mathbf{V} \to \mathbf{E}$, is determined to within a factor, only by the functional form of the specific 1-form of the Action, and its partial derivatives; e. g., it is determined by the choice of a specific physical system. For C2 differentiable functions, the 2-form, dA, and all of its products are evolutionary invariants. $L_{(\mathbf{E})}dA = 0$. The 2n dimensional submanifold has an invariant volume, a result that forms the basis of Liouville's theorem.

Such a unique Hamiltonian vector field does not exist on the even dimensional symplectic manifold, because there do not exist null eigenvectors of the 2n+2 anti-symmetric matrix of coefficients, dA. Hamiltonian vector fields can exist on the 2n+2 symplectic manifold, but they need not be uniquely defined, and moreover they depend on functions exterior to those required to define the physical system and its 1-form of Action. It will be demonstrated below, however, that there does exist a unique evolutionary process on a symplectic manifold, a process that is completely determined by the physical system, e.g., the functional form of the 1-form of Action. The 2n+1 volume element is not an evolutionary invariant, unless the Hamiltonian function is indpendent of time.

On the contact manifold the evolution of the 2n+1 volume element $Vol = A\hat{}dA... = (h - p\partial h/\partial p)dp\hat{}dq\hat{}dt$ satisfies the equation,

$$\begin{aligned} L_{(\mathbf{E})}Vol &= d(i(\mathbf{E})A)\hat{}dA = (dpv + pdv - dh)\hat{}(dp\hat{}dq - dh\hat{}dt) \qquad (2.14) \\ &= -dh\hat{}dp\hat{}dq = -(\partial h/\partial t)dt\hat{}dp\hat{}dt \end{aligned}$$

The 2n volume of the phase space sub-manifold is invariant, but the 2n+1 contact volume element is not - if the hamiltonian is time dependent. There is no "braiding" for the torsion current is a pseudo scalar.

2.7. Symplectic systems

The topology of the domain can be determined by computing the dimension of the Pfaff sequence for any given 1-form of Action. If the Pfaff dimension is 2n+2, then the 1-form of Action creates a non-compact symplectic manifold. In contrast to the contact manifold when the Pfaff dimension is odd (2n+1), the symplectic manifold (2n+2) does not admit solutions to the equation $W = i(\mathbf{V})dA = 0$. On a symplectic manifold, it is possible to generate Hamiltonian processes that leave the Action integral stationary, but such processes are usually not unique, and depend not only on the physical system, but also on the process itself. Such Hamiltonian-like processes, and their extensions to Symplectic processes ($L_{(\mathbf{V})}dA = dQ = 0$), are always reversible. A most interesting class of such processes are generated from the Bernoulli-Helmholtz equation of evolution,

$$W = i(\mathbf{V})dA = d\Theta, \qquad d\Theta \neq 0 \tag{2.15}$$

The function Θ is known as the Bernoulli-Casimir function, and is an invariant along any evolutionary path, $L_{(\mathbf{V})}d\Theta = d(i(\mathbf{V})i(\mathbf{V})dA) + i(\mathbf{V})dd\Theta = 0$. However the function Θ cannot be the same in all nearby neighborhoods for then it would be a constant and the RHS of the above equation would vanish. (As is well known in hydrodynamics, the Bernoulli function will vary from streamline to streamline.) The Bernoulli function cannot have a zero gradient on the sympletic manifold, except as a topological defect where the symplectic manifold of topological dimension 2n+2 becomes a contact manifold of topological dimension 2n+1.

On the symplectic manifold, the Bernoulli-Casimir function (if given) serves as the the generating function for a Hamiltonian vector field, but the Bernoulli-Casimir function cannot be a domain constant, as $d\Theta \neq 0$. It is this fact that leads to (Recall that the Hamiltonian-Energy function associated with a contact manifold can be a domain constant, $dH = 0$).

Consider the volume element $Vol = dA\hat{}...dA$. Then the evolution of the volume element satisfies the equation,

$$L_{(\mathbf{V})}Vol = 0, \tag{2.16}$$

which vanishes

Consider the 2n+1 form $Z = A\hat{}...dA$ on the 2n+2 symplectic manifold. Then the evolution of Z satisfies the equation,

$$L_{(\mathbf{V})}Z = d(i(V)A + \Theta)\hat{}(dA)^n,$$

which is exact and admits the period integrals,when the RHS vanishes.

2.8. Volterra Mechanics

On the symplectic domain there does exist a unique vector field that is determined (to within a factor) by the physical system alone (that is, from the functional form of the 1-form of Action). That unique vector field is defined as the Torsion Vector field, $\mathbf{T}$, and remarkably it describes irreversible processes according to the test given above.

$$Q\hat{}dQ = L_{(\mathbf{T})}A\hat{}L_{(\mathbf{T})}dA \neq 0. \tag{2.17}$$

The vector may derived from the equation

$$W = i(\mathbf{T})dA = \Gamma A \tag{2.18}$$

where Γ is a function that is determined by the measure of the symplectic domain of volume Vol which is generated by the Pfaff sequence.

$$A\hat{}dA\hat{}dA...\hat{}dA = i(\mathbf{T})Vol \tag{2.19}$$

Evolutionary processes in the direction of the Torsion vector satisfie the equation of conformal invariance,

$$L_{(\mathbf{T})}A = \Gamma A \neq 0 \tag{2.20}$$

and do not leave the action integral stationary.

For an Action 1-form that starts with a Lagrange function $L(\mathbf{x}, \mathbf{v}, t)$, evolution along the Torsion vector is generated by a dynamical system of the Volterra-Hamilton second order format:

$$d\mathbf{x}/d\tau = \mathbf{v} \tag{2.21}$$

$$d\mathbf{v}/d\tau = \mathbf{T} \tag{2.22}$$

Systems which are integrable do not have a non-zero Torsion vector. Hence integrable systems of Pfaff dimension 2 are acceleration free (in the sense that there exists coordinates for which the accelerations vanish.)

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3. Example: The Sliding Bowling Ball

3.1. The Observation

Consider a bowling ball given an initial amount of translational energy and rotational energy. Assume the angular momentum and the linear momentum are orthogonal to themselves and also orthogonal to the ambient gravitational field. Then place the bowling ball, subject to these initial conditions, in contact with the bowling alley. Initially, it is observed that the ball slips or skids, dissipating its linear and angular momentum, until the No-Slip condition is achieved. Note that it is possible for the angular momentum or the linear momentum to change sign during the irreversible phase of the evolution. The dynamical system representing the evolutionary process is irreversible until the No-Slip condition is reached. Thereafter, the dynamical system is reversible, and momentum is conserved.

3.2. The Analysis

Assume that the physical system may be represented by a 1-form of Action constructed from a Lagrange function:

$$L = L(x, \theta, v, \omega, t) = \{\beta m(\lambda\omega)^2/2 + mv^2/2\} \tag{3.1}$$

The constants are: m=mass, $\beta =$ moment of inertial factor (2/5 for sphere), $\lambda =$ effective "radius" of the object, the moment of inertia $= \beta m\lambda^2$.

Let the topological constraints be defined anholonomically by the Pfaffian system:

$$\{dx - vdt\} \Rightarrow 0, \qquad \{d\theta - \omega dt\} \Rightarrow 0, \quad \{dx - \lambda d\theta\} \Rightarrow 0 \tag{3.2}$$

Define the constrained 1-form of Action as

$$A = L(x, \theta, v, \omega, t)dt + p\{dx - vdt\} + l\{d\theta - \omega dt\} + ms\{\lambda d\theta - dx\} \tag{3.3}$$

where $\{p, l, s\}$ are Lagrange multipliers. Rearrange the variables to give (in the language of optimal control theory) a pre-Hamiltonian action:

$$A = (-p - ms)dx + (l + \lambda ms)d\theta - \{pv + l\omega - L\}dt. \tag{3.4}$$

It is apparent that the Pfaff dimension of this Action 1-form is 2n+2 = 6. The Action defines a symplectic manifold of dimension 6.

For simplicity, assume initially that the Lagrange multipliers (momenta) are defined canonically; e.g.,

$$p = \partial L/\partial v \Rightarrow mv, \qquad l = \partial L/\partial \omega \Rightarrow \beta m \lambda^2 \omega \tag{3.5}$$

which implies that

$$A = (mv - ms)dx + (\beta m \lambda^2 \omega + \lambda s)d\theta - \{-mv^2/2 - \beta m(\lambda\omega)^2/2\}dt. \tag{3.6}$$

The volume element of the symplectic manifold is given by the expression

$$6Vol = 6m^3\beta\lambda^2\{v - \lambda\omega\}dx\hat{}d\theta\hat{}dv\hat{}d\omega\hat{}ds\hat{}dt = dA\hat{}dA\hat{}dA \tag{3.7}$$

The symplectic manifold has a singular subset upon which the Pfaff dimension of the Action 1-form is 2n+1 = 5. The constraint for such a contact manifold is precisely the no-slip condition:

$$\{v - \lambda\omega\} \Rightarrow 0 \tag{3.8}$$

On the 5 dimensional contact manifold there exists a unique extremal (Hamiltonian) field which (to within a projective factor) defines the conservative reversible part of the evolutionary process. As this unique extremal vector satisfies the equation

$$i(\mathbf{V})dA = 0, \tag{3.9}$$

it is easy to show that dynamical systems defined by such vector fields must be reversible in the thermodynamic sense. (As $dQ = d(i(\mathbf{V})dA) = 0$ for all Hamiltonian or symplectic processes, $Q\hat{}dQ = 0$)

However, on the 6 dimensional symplectic manifold, there does not exist a unique extremal field, nor a unique stationary field, that can be used to define the dynamical system. The symplectic manifold does support vector fields, **S**, that leave the Action integral invariant, but these vector fields are not unique in the sense that that they depend on an arbitrary gauge addition to the 1-form of Action that may be required to satisfy initial conditions.

There does exist a unique torsion field (or current) defined (to within a projective factor, σ) by the 6 components of the 5 form,

$$Torsion = A\hat{}dA\hat{}dA \tag{3.10}$$

This unique vector, **T**, independent of gauge additions, has the properties that

$$\mathcal{L}_{(\mathbf{T})}A = \Gamma \cdot A \quad and \quad i(\mathbf{T})A = 0. \tag{3.11}$$

This "Torsion" vector field statisfies the equation

$$\mathcal{L}_{(\mathbf{T})}A\hat{}\mathcal{L}_{(\mathbf{T})}dA = Q\hat{}dQ \neq 0. \tag{3.12}$$

Hence a dynamical system having a component constructed from this unique Torsion vector field becomes a candidate to describe the initial irreversible decay of angular momentum and kinetic energy.

Solving for the components of the Torsion vector for the bowling ball problem leads to the (unique) decaying dynamical system:

$$dv/dt = m^3\beta\lambda^2\{-\beta\lambda^2\omega^2 - 2\lambda v\omega + v^2\} \tag{3.13}$$

$$d\omega/dt = m^3\lambda^2\{-\beta\lambda^2\omega^2 + 2\beta\lambda v\omega + v^2\} \tag{3.14}$$

$$ds/dt = m^3\beta\lambda^2\{-\beta\lambda^2\omega^2 - v^2 - 2(\lambda\omega - v)s\} \tag{3.15}$$

This is a Volterra system generated on a Finsler space. (See Antonelli)

It is to be noted that the non-canonical "symplectic momentum" variables, defined by inspection from the constrained 1-form of Action lead to the momentum map:

$$P_x \doteq m(v - s), \qquad P_\theta \doteq m(\beta\lambda^2\omega + s\lambda). \tag{3.16}$$

Substitution in terms of the momentum variables leads to the canonical form (Zhitomirski) for the 1-form of Action:

$$A = P_x dx + P_\theta d\theta - H dt \tag{3.17}$$

where H is an independent variable on the 6-dimensional manifold. The H map is given by the expression for energy where v and ω are eliminated in terms of the P_x and the P_θ.

$$H = (mv^2/2 + \beta m(\lambda\omega)^2/2) \Rightarrow 1/2m[(P_x/m + s)^2 + \beta\lambda(\frac{P_\theta/m\lambda - s}{\beta\lambda})^2] \tag{3.18}$$

Note that $v = \partial H/\partial P_x$ and $\omega = \partial H/\partial P_\theta$. Each component of "canonical momenta" decays with the same rate in the canonical domain.