Adventures in Applied Topology Series

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The Universe from the Perspective of Non-equilibrium Thermodynamics.

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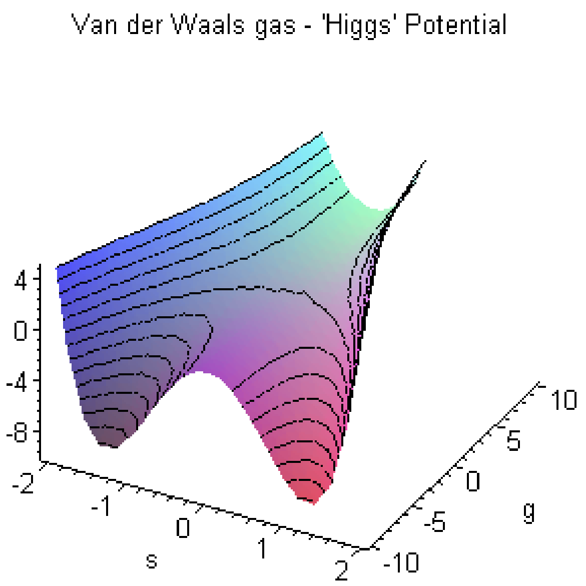
March 2, 2004



1. Cosmology and Thermodynamics

After measuring dynamical properties of stars in galaxies, star clusters and galactic clusters, astronomers are suggesting that our current understanding of the universe, based upon ponderable gravitating matter, does not appear to agree with the observations. In fact, their almost outrageous conjectures state that ordinary matter seems to account for only a few percent of the total matter in the universe, about 75% consists of dark energy, and about 25% consists of dark matter. Immediately, the theoretical physics community has jumped in with equally outrageous suggestions (similar to the abortive theoretical developments of the "fifth force") that such effects on a galactic scale are due to vacuum quantum fluctuations, to a mass inducing Higgs potential, to the tension of infinitesimal strings, to a cosmological constant modifications of the metrically based Einstein gravitational theory, gauge constraints and many other esoteric theoretical concoctions.

Yet could it be that there is a much simpler model that could account for the various "features" of the "new" Universe? In this article the idea is presented that the universe is a non-equilibrium thermodynamic system with topological features of a universal van der Waals gas. A topological perspective of non-equilibrium thermodynamics is based on following axioms:

1. Thermodynamic physical systems can be encoded in terms of a 1-form of covariant Action Potentials, $A_k(x, y, z, t)$, on a 4 dimensional abstract variety of ordered independent variables, $\{x, y, z, t\}$. The variety supports a abstract volume element $\Omega_4 = dx\hat{\ }dy\hat{\ }dz\hat{\ }dt$.

2. Thermodynamic processes are assumed to be encoded, to within a factor, $\rho(x, y, z, t)$, in terms of contravariant vector direction fields, $\mathbf{V}_4(x, y, z, t)$.

3. Continuous topological evolution of the thermodynamic system can be encoded in terms of Cartan's magic formula (see p. 122 in [Marsden 1994]). The Lie differential, when applied to a exterior differential 1-form of Action, $A = A_k dx^k$, is equivalent *abstractly* to the first law of thermodynamics.

4. The concept of Pfaff topological dimension permits the creation of thermo-

dynamic equivalence classes such that

$$\begin{aligned}
\textbf{Physical Systems} &: \text{ defined by the Pfaff dimension of } A && (1.1)\\
dA &= 0 \quad \textbf{Equilibrium} \text{ - Pfaff dimension 1} && (1.2)\\
A\hat{}dA &= 0 \quad \textbf{Isolated} \text{ - Pfaff dimension 2} && (1.3)\\
d(A\hat{}dA) &= 0 \quad \textbf{Closed} \text{ - Pfaff dimension 3} && (1.4)\\
dA\hat{}dA &\neq 0. \quad \textbf{Open} \text{ - Pfaff dimension 4.} && (1.5)
\end{aligned}$$

5. The Jacobian matrix of a thermodynamic non-equilibrium turbulent physical system of Pfaff dimension 4 yields a Cayley-Hamilton 4th order polynomial equation that is equivalent to the thermodynamic Phase-Potential function for the thermodynamic system.

6. The 4th order Phase function is deformably equivalent to a van der Waals gas.

It is to be noted that the method does not explicitly utilize metric or connection features of geometrical theories, but can yield relationships between "curvatures" that in include energy effects due to "string" or surface tension (mean linear curvature), Gauss quadratic curvature of metric based relativity, cubic curvature interaction and pressure effects (both negative and positive), and finally expansion and contraction fourth order curvature effects that account for irreversible dissipation.

2. A Cosmological Conjecture

Based upon the single assumption that the universe is a non-equilibrium thermodynamic system of Pfaff topological dimension 4 leads to a cosmology where the universe, at present, can be approximated in terms of the non-equilibrium states of a very dilute van der Waals gas near its critical point. The stars and the galaxies are the topological defects and coherent - but not equilibrium - structures of Pfaff topological dimension 3 in this non-equilibrium system of Pfaff topological dimension 4. The topological theory of the ubiquitous van der Waals gas leads to the concepts of negative pressure, string tension, and a Higgs potential as natural consequences of a topological point of view applied to thermodynamics. Perhaps of more importance is the fact that these concepts do not depend explicitly upon the geometric constraints of metric or connection, and yield a different perspective on the concept of gravity.

The original motivation for this conjecture is based on the classical theory of correlations of fluctuations presented in the Landau-Lifshitz volume on statistical mechanics [Landau 1958]. However, the methods used herein are not statistical, not quantum mechanical, and instead are based on Cartan's methods of exterior differential forms and their application to the topology of thermodynamic systems and their continuous topological evolution [RMK1991 b]. Landau and Lifshitz emphasized that real thermodynamic substances, near the thermodynamic critical point, exhibit extraordinary large fluctuations of density and entropy. In fact, these authors demonstrate that for an almost perfect gas near the critical point, the correlations of the fluctuations can be interpreted as a $1/r$ potential giving a $1/r^2$ force law of attraction. Hence, as a cosmological model, the almost perfect gas - such as a very dilute van der Waals gas - near the critical point yields a reason for both the granularity of the night sky and for the $1/r^2$ force law ascribed to gravitational forces between for massive aggregates.

In this article, a topological (and non statistical) thermodynamic approach is used to demonstrate how a four dimensional variety can support a turbulent, non-equilibrium, physical system with universal properties that are homeomorphic (deformable) to a van der Waals gas. The method leads to the necessary conditions required for the existence, creation or destruction of topological defect structures in such a non-equilibrium system. For those physical systems that admit description in terms of an exterior differential 1-form of Action potentials of maximal rank, a Jacobian matrix can be generated in terms of the partial derivatives of the coefficient functions that define the 1-form of Action. When expressed in terms of intrinsic variables, known as the similarity invariants, the Cayley-Hamilton 4 dimensional characteristic polynomial of the Jacobian matrix generates a universal phase equation. Certain topological defect structures can be put into correspondence with constraints placed upon those (curvature) similarity invariants generated by the Cayley-Hamilton 4 dimensional characteristic polynomial. These constraints define equivalence classes of topological properties.

The characteristic polynomial, or Phase function, can be viewed as representing a family of implicit hypersurfaces. The hypersurface has an envelope which, when constrained to a minimal hypersurface, is related to a swallowtail bifurcation set. The swallowtail defect structure is homeomorphic to the Gibbs surface of a van der Waals gas. Another possible defect structure corresponds to the implicit hypersurface surface defined by a zero determinant condition imposed upon the

Jacobian matrix. On 4 dimensional variety (space-time) , this non-degenerate hypersurface constraint leads to a cubic polynomial that always can be put into correspondence with a set of non-equilibrium thermodynamic states whose kernel is a van der Waals gas. Hence this universal topological method for creating a low density turbulent non-equilibrium media leads to the setting examined statistically by Landau and Lifschitz in terms of classical fluctuations about the critical point.

The turbulent non-equilibrium thermodynamic cosmology of a real gas near its critical point yields an explanation for:

1. The granularity of the night sky as exhibited by stars and galaxies.

2. The Newtonian law of gravitational attraction proportional to $1/\mathrm{r}^2$.

3. The expansion and irreversible dissipation of the universe (4th order curvature effects).

4. The possibility of domains of negative pressure (dark energy) due to a classical Higgs mechanism for aggregates below the critical temperature (3rd order curvature effects)

5. The possibility of domains where gravitational effects (2nd order Gauss curvature effects) appear to be related to entropy and temperature properties of the thermodynamic system.

6. The possibility of cohesion properties (dark matter) due to string or surface tension (1st order Mean curvature effects)

7. Black Holes (generated by Petrov Type D solutions in gravitational theory) are to be related to Minimal Surface solutions to the Universal thermodynamic 4th order Phase function.

All of the above are created without explicit use of the geometrical features of metric or connection.

In that which follows, emphasis is placed upon the thermodynamic topological features of non-equilibrium, turbulent van der Waals gas.

3. The Ubiquitous Topological van der Walls gas

The simplistic equation of state for an ideal (perfect) gas,

$$\text{Ideal Gas:} \quad P/RT = \rho = n/V, \tag{3.1}$$

does not encode certain thermodynamic features (phase transitions and critical point behavior) which are observable in "real" gases. It has been argued that the "real" gas consists of geometric "parts" that interact with one another, in contrast to an ideal gas, where geometric features (size and shape) of such "molecules" and their interactions had been ignored. Motivated by such ideas, van der Waals created, phenomenologically, an equation of state for "real" gases in terms of two parameters, a and b, which were introduced to encode the interaction and geometric size features of the "molecular" components. The resulting formula for an equation of state was cubic in the molar density, $\rho = n/V$.

$$\text{Van der Waals:} \quad P = \frac{\rho RT}{1 - b\rho} - a\rho^2, \tag{3.2}$$

$$\text{or:} \quad ab\rho^3 - a\rho^2 + \{RT + bP\}\rho - P = 0. \tag{3.3}$$

The formula has enjoyed remarkable success for qualitatively explaining the thermodynamic features of real gases. The formula represents an implicit surface in the space of variables, $\{P, T, \rho\}$. However, the development was phenomenological, and although motivated by the concept of microscopic "molecules", the fundamental properties were independent from the geometric size of its parts. As Sommerfeld has said

> "The atomistic, microscopic point of view is alien to thermodynamics. Consequently, as suggested by Ostwald, it is better to use moles rather than molecules." p. 11 [Sommerfeld 1964].

The ideal gas approximation has been found to be of utility to the study of agglomerates of parts that range from the geometric size of nuclei to the geometric size of stars. A major purpose of this section is to demonstrate the universality of the topological van der Waals gas to the study of condensates of all types of "parts" in non-equilibrium configurations.

By differentiating the Van der Waals equation of state with respect to the molar density, it can be determined that there exists a "critical point" on the hypersurface at which the values of the Pressure, Temperature and molar density take on values such that

$$\text{at the critical point, } P_c/T_c\rho_c = \text{ constant.} \tag{3.4}$$

When the thermodynamic variables are expressed in terms of dimensionless (reduced) variables, scaled in terms of their values at the critical point, the values of those parameters, a and b, which were used to model the geometric-interaction and geometric size features cancelled out. In this sense, the "renormalized" or "reduced" van der Waals equation of state became an element of equivalent class with universal topological properties (independent from scales), and independent from the size and interaction magnitudes of its component parts. In terms of the dimensionless variables,

$$\widetilde{P} = P/P_c,\ \widetilde{V} = V/V_c,\ \widetilde{T} = T/T_c,\ \rho = n/V, \tag{3.5}$$

the classic van der Waals equation may be considered as a cubic constraint on the space of variables $\{n; \widetilde{P}, \widetilde{T}, \widetilde{\rho}\}$ where $\widetilde{\rho} = n/\widetilde{V}$ is defined as the dimensionless molar density. The reduced universal van der Waals equation of state is given by the classic cubic expression,

$$\text{Classic Van der Waals equation} \tag{3.6}$$

$$\text{(in reduced coordinates)} \tag{3.7}$$

$$0 = \widetilde{\rho}^3 - 3\widetilde{\rho}^2 + \{(8\widetilde{T} + \widetilde{P})/3\}\widetilde{\rho} - \widetilde{P}. \tag{3.8}$$

This formula should be memorized, for it yields a direct connection of the van der Waals gas and a cubic polynomial. In the development that follows the formula will be related to the Cayley-Hamilton equation for a non-degenerate 4x4 Jacobian matrix of a non-equilibrium turbulent physical system.

The Cayley-Hamilton formula is of the type

$$\text{Cayley-Hamilton polynomial} = \xi^4 - X_M\xi^3 + Y_G\xi^2 - Z_A\xi^1 + T_K = 0. \tag{3.9}$$

If $T_K = 0$, then the Cayley-Hamiltonian equation becomes,

$$\text{Cayley-Hamilton polynomial} = (\xi^3 - X_M\xi^2 + Y_G\xi^1 - Z_A)\xi = 0, \tag{3.10}$$

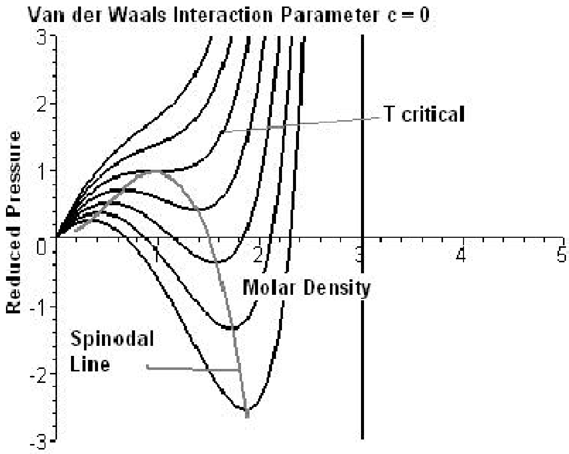


Figure 3.1:

and the coefficients become related to the curvatures of the implicit surface. The first (cubic) factor can be put into direct correspondence with the Classic van der Waals equation

$$\begin{aligned}
\xi &= \widetilde{\rho}, && (3.11)\\
\text{Mean Curvature } X_M &= 3, && (3.12)\\
\text{Gauss Curvature } Y_G &= \{(8\widetilde{T}+\widetilde{P})/3\} && (3.13)\\
\text{Cubic Curvature } Z_A &= \widetilde{P} && (3.14)
\end{aligned}$$

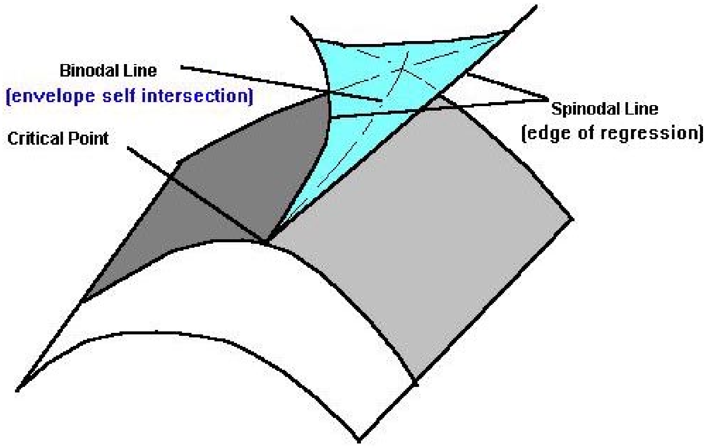
Forces and energies associated with the Mean curvature are typical of surface tension effects. It becomes apparent that forces and energies associated with the Cubic curvature represent the Pressures of interactions. The Gauss quadratic curvatures are dominated by temperature, with a pressure contribution.

A drawing of the implicit surface that portrays the van der Waals equation of state is given in the next figure.

Note that diagram displays a critical isotherm that separates a single phase (the gas) from the different topological domains that can be interpreted as liquids and vapor. The shape of the critical isotherm should be remembered, for above the critical isotherm, there exists a unique value for the pressure, and below the critical isotherm there is more than one value for the pressure. This feature represents a topological property of the van der Waals gas, and will have importance in the study of non-equilibrium systems. Of interest to cosmologists, the Pressure for the van der Waals gas, for values below the critical isotherm, can take on negative values. As will be shown below, the Phase function below the critical isotherm has the shape of a Higgs potential.

There exists a dual surface to the equation of state as defined by a Legendre transformation to the Gibbs function, $g = u - Ts + Pv$. The implicit surface defined by the Gibbs function (for a van der Waals gas) is not single valued, and appears as a deformation of a swallow tail bifurcation set. The actual Gibbs surface for the van der Waals gas can be numerically computed and is presented in the next Figure. An accurate drawing of the 3D Gibbs surface appears only occasionally in thermodynamic text books. Most presentations, if in 3D, are given by sketches, and not by actual computations. For example, in [Adkins, C,J. 1975] p.196, the Gibbs surface misses the fact that the spinodal line forms a cusp at the critical point. In the figure below, the salient features are displayed by numeric computation of the Gibbs surface for the van der Waals gas. Remarkably, the dual Gibbs surface displays the envelope features of the Universal Phase function. The cuspoidal critical point singularity, the winged cusp representing the Spinodal line, and the binodal self intersection are universal topological features of the discriminant (envelope) hypersurface. In the figure below, the white is where the temperature is above the critical isotherm and represents the pure gas. The other sectors are below the critical isotherm, and are influenced by the "Higgs" features of the Phase potential. The dark gray sector represents the fluid phase, and the light gray sector represents the vapor phase. The light blue sector represents the unstable mixed phase region.

Conjecture 3.1. *The topological features of the van der Waals gas are universal features (deformation invariants for all physical systems that admit a realization over 4D space-time variety. The van der Walls gas is one element of a topological equivalence class.*



A Universal Thermodynamic Swallowtail
The Gibbs surface of a Van der Waals gas

Historically the two implicit surfaces defined by the reduced van der Waals equation became quite useful to chemical engineers and led to the law of corresponding states. If properties of a gas near its critical point could be measured, then the law of corresponding states permitted estimates to be made for the properties of the gas by comparison to the universal van der Waals model. The topological results were independent of the geometric parameters of size, b, and interaction, a. In this and following sections, the universal topological features of the Phase function and the Gibbs surface of the generalized[1] van der Waals gas will be developed and applied to non-equilibrium systems. The "generalization" consists of adding a contribution to the reciprocal volume use in the interaction term. Recall that non-equilibrium requires that the Pfaff topological dimension of the Action 1-form is 3 or greater in certain regions. Non-equilibrium systems can exist in "stationary" states where there topological coherence properties are evolutionary invariants.

The principle (universal) topological defect structure of a van der Waals gas is the existence of a non-zero critical *point*. When expressed in terms of reduced coordinates, $\{\widetilde{P}, \widetilde{T}, \widetilde{\rho}\}$, the critical point of the implicit surface representing the equation of state, is where the reduced (dimensionless) functions all have the

[1] The "generalization" consists of adding a contribution to the reciprocal volume use in the interaction term.

common value unity. The topological significance of the critical isotherm, which passes through the critical point, has already been mentioned above. Another important topological defect structure is the existence of a spinodal *line*, of ultimate phase stability, consisting of two parts that meet in a cusp at the critical point. The spinodal line will be established by an edge of regression in the Gibbs surface. Yet another topological defect structure is exhibited by the binodal line, defining portions of a ruled *surface* representing the region of mixed phases. The binodal line can be described by a deformation of a pitchfork bifurcation emanating from the critical point, and line which outlines the domain of mixed phases. The domain of mixed phase is related to regions where the Pfaff topological dimension of the encoded physical system (the 1-form of Action) is at least 3. The domains of isolated single phase are related to regions where the Pfaff topological dimension is 2 or less.

A lot can be learned from the van der Waals example, for its features are experimentally verifiable. The universal qualities are obtained in terms of variables that represent deformations and non-equilibrium extensions of the van der Waals properties. The van der Waals internal energy is a Lagrangian (phase) function in terms of extensive variables. In the language of classical mechanics, the Lagrange function is a function of the base variables, q^k, and their first derivatives, v^k, or velocity "extensive" functions. A Legendre transformation leads to a Hamiltonian function in terms of intensive variables, the momenta, p_k. The classic van der Waals phase function defines a hypersurface in the space of extensive variables of entropy, S, volume, V, and energy, U. A Legendre transformation produces a "Gibbs-Hamiltonian" function of intensive variables, temperature T, pressure, P, and Gibbs free energy, $Gibbs$.

The zero sets of certain algebraic combinations of the similarity curvature invariants of these hypersurfaces define universal topological features of the physical system, which are of value to the study of both equilibrium and non-equilibrium systems. Rather that formulating the non-equilibrium universal phase equation in a phenomenological manner, it will be demonstrated that such a universal phase function can be generated as the Cayley-Hamilton polynomial equation of the Jacobian matrix for the 1-form of Action, A, that represents the physical system. The topological Pfaff dimension of A permits the delineation between those phase functions that represent non-equilibrium systems and those that do not.

The following subsections first will discuss the ideas associated with Extensive and Intensive variables. Then the classic van der Waals expression for a Phase equation will be used to define an internal energy surface in terms of intensive variables. A dual construction will be used to create the Gibbs energy in terms of intensive variables. The Gibbs surface is deformably (topologically) equivalent to the swallow-tail discriminant, or envelope of the classic phase equation. After this review of classical theory in the language of topological evolution, the theory will be extended to include non-equilibrium systems of the closed and open types.

3.1. Extensive and Intensive variables

Experiment has indicated that there are two species of variables in thermodynamic systems, Extensive variables such as Volume and Entropy $\{V$ and $S\}$ - which are additive quantities, and intensive variables, such as Pressure and Temperature $\{P$ and $T\}$ - which are not additive intensities. As Tisza points out [Tisza 1966], commenting on the geometrical approaches to thermodynamics,

> "It is remarkable that intrinsic subspace curvature properties can have any thermodynamic meaning, as metrical based geometries can not be used to distinguish between the two classes of intensive and extensive thermodynamic variables."

Be aware of the fact that physical theories that do not distinguish between extensive and intensive variables can have only a limited application to the problem of understanding thermodynamics. In electromagnetism, the electric and magnetic field intensities (**E** and **B**) are examples of intensive variables, and the electric and magnetic field excitations (**D** and **H**) are examples of extensive variables. It will be demonstrated below that such thermodynamic differences are related to the concepts of pair and impair exterior differential forms. Intensities will be related to covariant tensor fields (pair exterior differential forms -waves), and Extensive quantities will be related to contra-variant tensor densities (impair exterior differential forms - particles).

In the calculus of variations it is known that those extremal principles that are independent of parametric scales lead to projective geometries and Finsler spaces. Indeed, Chern [Chern 1944] has shown that the key assumption of a Finsler geometry is that the variational integrand be homogeneous of degree 1 in the variables of the tangent space, thereby forming, in his words, a “projectivized” tangent bundle. Finsler spaces are not well known, but have had a modest number of physical applications [Antonelli (1993)].

Conjecture 3.2. *The physical science of thermodynamics, based upon functions which are homogeneous of degree 1, is where the theory of projectivized Finsler spaces can be of practical application.*

A function, Θ, that is homogenous of degree 1 in the extensive variables (S,V,U), satisfies the Euler equation

$$V^k\partial\Theta/\partial V^j = S\partial\Theta/\partial S + V\partial\Theta/\partial V + U\partial\Theta/\partial U.. = \Theta, \tag{3.15}$$

and the scaling equation

$$\Theta(\lambda S, \lambda V, \lambda U) = \lambda\Theta((S,V,U). \tag{3.16}$$

The partial derivative coefficients in the Euler equation define the intensive variables. The test for homogeneity can be put into correspondence with Cartan's magic formula operating on functions. Let $\mathbf{R} = [S,V,U...]$ be a position vector in a space (of extensive variables), and let $L(S,V,...n)$ be a function on that space. Then

$$L_{(\mathbf{R})}\Theta = i(\mathbf{R})d\Theta = S\partial\Theta/\partial S + V\partial\Theta/\partial V + U\partial\Theta/\partial U.. \tag{3.17}$$

So if Θ is homogeneous of degree 1, then the evolution in the direction of the expansion (position) vector, $\mathbf{R}$, yields the result

$$\text{Homogeneous of degree 1}: \ L_{(\mathbf{R})}\Theta(\mathbf{R}) = \Theta(\mathbf{R}). \tag{3.18}$$

The application of integer and fractal homogeneous concepts are discussed in more detail in Chapter 7.

It is also of some importance to note that most textbook treatments of thermodynamics agree with the idea that the phase function $\Theta(U,S,V,n...)$ must be homogenous of degree 1, but the formulas often presented for the ideal or van der Waals gas do not satisfy the Euler criteria of homogeneity. A correct formulation is presented below. It is a fact that any function can be made homogeneous of degree 1 by merely adding a new variable, and dividing, or renormalizing, each of the old variables by the new variable, and then multiplying the new function by the new variable. For example, consider $f(x,y,z)$. Then $F(x,y,z,s) = sf(x/s,y/s,z/s)$ is homogeneous of degree 1 in the variables, $\{x,y,z,s\}$. This same idea can be extended to vectors: for example, let

$$\mathbf{v} = [V^x, V^y, V^z] \Rightarrow \mathbf{V} = [V^x, V^y, V^z, \lambda(V^x, V^y, V^z)] \tag{3.19}$$

$$\Rightarrow \mathbf{J} = \boldsymbol{\lambda}\,[V^x/\lambda, V^y/\lambda, V^z/\lambda] \tag{3.20}$$

The key feature of the phase function of thermodynamics is that N, or n, “the number of parts, molecules, or phases” plays the role of the renormalization variable, designated as s in the example. For thermodynamics the expression

$$n\Theta(S/n, V/n, U/n..) = \Theta(S, V, U, n..) \tag{3.21}$$

is to be recognized as a function which is homogeneous of degree 1. The phase function is a projection from a space of 1 higher dimension (the coordinate n) to include the concept of multiple components or phases.

In classical mechanics it is also appreciated that there are two classes of variables: contravariant vectors such as the components of velocities, and covariant vectors, such as the components of a gradient field. Lagrangians are often functions of contravariant vector components (velocities, V^k), while Hamiltonians are functions of covariant variables (momenta, P_k).

3.2. Lagrangian-Hamiltonian features

In Hamiltonian-Lagrange formalism, the Legendre transformation is given as an equation constraining the Hamiltonian function to the Lagrange function in terms of a hyperbolic product of dual variables,

$$H + L = P_k V^k. \tag{3.22}$$

It follows that

$$\begin{aligned} P_k \partial H/\partial P_k - H &= -(P_k \partial L/\partial P_k - L), & (3.23) \\ V^k \partial H/\partial V^k - H &= -(V^k \partial L/\partial V^k - L) & (3.24) \end{aligned}$$

If L is homogeneous of degree 1 in the V^k then the last equation implies that either H is also Homogeneous of degree 1 in the V^k , or H is homogeneous of degree zero, and H equal to zero. It is to be noted that the Euler criteria for homogeneity of degree 1 in the V^k is given by the expression, $(V^k \partial L/\partial V^k - L) = 0$. The variational problem when the variational integrand, $L(\mathbf{v})$, is homogeneous of degree 1 in $\mathbf{v}$ is known as the Homogeneous Problem: $\mathbf{v}\partial L/\partial \mathbf{v} - L(\mathbf{v}) \Rightarrow 0$, which is precisely that constraint used in the theory of special relativity, the theory of minimal surfaces, and in Chern’s version of Finsler geometries built on projective connections.

Based on the concept that different thermodynamic phases represent topological properties, and that a phase change is to be recognized as a signature of a topological evolution, the basic ideas of projective differential geometry mentioned above were utilized [RMK 1990] to define certain topological properties of hydrodynamic flows (see section 3.6.1). Different domains of initial conditions for a given hydrodynamic flow could be associated with different phase regions of a thermodynamic substance. A specific example was given for a dynamical system in which the three dimensional flow explicitly induced the Gibbs free energy surface typical for the Van der Waals gas. It then was possible to determine that there were domains of initial conditions for which the system could be put into correspondence with the pure liquid, pure gas, or mixed phase regions of a two phase system. An unstable region would be in the domain which is interior to the spinodal line on the surface representing the equation of state. Although intuition implied that this correspondence was a universal result, no satisfactory argument was known at that time to substantiate the idea of universality, except in specific examples.

It this section it will be demonstrated that the observations described above are indeed universal concepts: Any dynamical system that can be described in terms of a non-linear C1 vector field in three variables can be associated with the thermodynamics of a Van der Waals gas. This universal behavior not only justifies the law of corresponding states in chemistry, but also yields explicit universal formulas (in terms of cross ratios of similarity invariants) to describe the limits of phase stability that are equivalent to the Spinodal Line and the Binodal line of two phase thermodynamic systems. In addition, the ideas lead to a well defined procedure for treating non-equilibrium thermodynamic systems as complex deviations from the real, or equilibrium, systems.

This claim of universality is not to be treated lightly. For example, it should be remarked that historically many authors, including Thom, have recognized that the cusp catastrophe generated by the cubic fold has many qualitative features of a Van der Waals gas. In 1977 Sewell noted a relationship between Legendre transformations and bifurcation theory, and clearly defined the relationship between the Gibbs free energy surface of a van der Waals gas and its relationship to the swallowtail catastrophe. However, the claims that catastrophe concepts have universal significance have been criticized sharply, both on method and style of presentation and specifically on the grounds that not all dynamical systems have a gradient representation. However, the analysis herein gives credence to some of Thom's claims of universality by demonstrating how the cusp singularity can be constructed in terms of any C1 three dimensional vector field, and the similarity invariants of its Jacobian matrix. Annulling individual similarity invariants (with respect to that special subset of projective transformations (equi-affine transformations) that preserve parallelism and perpendicularity) leads to local bifurcations, and constraining dimensionless cross ratios of similarity invariants leads to global bifurcation diagrams. Then using Sewell's result that the bifurcation set of the swallowtail singularity is related to the Legendre dual of the Cusp singularity completes the universal correspondence.

Others have attempted to use differential geometric methods to analyze thermodynamic systems, but almost always these attempts have tried to construct a suitable metric formalism. For example, Tisza mentions that Blashke attempted to deduce a differential geometry that would apply to the metric free Gibbs space, but with only limited success. In Blashke's geometry, the projective space was confined to the equi-affine group, which forces the shape matrix to be symmetric. Such equi-affine systems admit only real eigen values for the shape matrix, where the richness of non-equilibrium thermodynamics, and its possible application to the theory of dynamical systems, requires the existence of domains of both real and complex eigenvalues. In this section, a projective geometry without metric is presumed to be the natural basis for non-equilibrium, but reversible thermodynamics.

In the earlier work reported at the 1977 Aspen Conference on "New Frontiers in Thermodynamics", the Gibbs space used in deriving the shape matrix of the equilibrium "surface" was assumed to be a projective geometry of three dimensions, $(U/n, S/n, V/n\}$, on which the projective constraint was that given by the first law of thermodynamics

$$\omega = dU - Q + W = 0. \tag{3.25}$$

The presumption of classical thermodynamics is that the first law is locally equivalent to a Darboux representation

$$\omega = dU - TdS + PdV + \mu dn \ \Rightarrow 0. \tag{3.26}$$

From another point of view, the seven dimensional space $\{U, T, S, P, V, \mu, n\}$ is topologically constrained by an *exterior differential system* that consists of a function homogeneous of degree 1 in the variables, $\{U, S, V, n\}$,

$$\Theta(U, S, V, n) = n\Theta(U/n, S/n, V/n) \Rightarrow 0, \tag{3.27}$$

and the differential 1-form,

$$\omega = dU - TdS + PdV + \mu dn \Rightarrow 0. \tag{3.28}$$

The vanishing of the three form, $\omega\hat{}d\omega = 0$, insures that the 1-form, ω, is integrable in the sense of Frobenius, and the condition is at the foundation if Caratheodory's theory of equilibrium. The existence of this 1-form defines a non-standard, or Cartan surface, for which the shape matrix is not necessarily symmetric, and therefore can have complex eigenvalues. (See Chapter 8.) In equilibrium thermodynamics the additional constraint that ω is integrable implies the existence of a unique solution function, $n\Theta(U/n, S/n, V/n)$ which is homogeneous of degree 1 in the extensive variables $\{U, S, V, n\}$. The partial derivatives of the solution function with respect to the extensive variables, yield the thermodynamic intensities. Whether the 1-form ω is integrable or not, the vanishing of the 1-form constrains the projective shape matrix to be symmetric.

A point of departure is realized when the projective constraint is chosen such that the shape matrix admits complex eigenvalues. In all projective geometries, the fundamental invariants are constructed from six primitive cross ratios, two of which are bounded by negative infinity and zero, two of which are bounded by zero and one (the probability domain) and two of which are bounded by one and infinity. It will be demonstrated below how this signature of the three projective equivalence classes appear in the relationships that relate envelopes to bifurcations in projective space.

When the Gibbs primitive phase surface of the van der Waals gas is mapped to its dual by means of a Legendre transformation, the Spinodal line can be interpreted as an edge of regression in the dual surface of "Gibbs free energy". It is this clue that focuses attention on the theory of envelopes, for the edge of regression is a singularity in an enveloping surface [Struik1961]. It is apparent in the dual surface of Gibbs free energy that, in addition to the edge of regression, there exists another topological feature of singularity, a line of self intersection (which is not an intrinsic property that can be determined locally). This non-metrical feature of self intersection was interpreted as the Binodal line in the earlier work mentioned. Usually, the Binodal line is defined through a heuristic Maxwell construction on the PVT surface representing the equation of state. As Tisza states [Tisza 1966] in reference to the Maxwell procedure,

> "...a van der Waals gas (referring to the equation of state) does not constitute a fully defined thermodynamic system. A complete definition would include the specific heat as a function of say temperature and volume. ... In the concept of a Van der Waals gas a spurious interpolation (the Maxwell construction) through the instable range

(of the equation of state) is substituted for the missing (specific heat) information."

In differential geometry, the line of self intersection is a locus of singularities, and as such would offer a projective geometric definition of the Binodal line, without the heuristic Maxwell assumption. Although visually apparent in the equilibrium surface representing the Gibbs free energy, the differential geometry of the extrinsic Binodal line eluded algebraic formulation.

3.3. The Phase function for a van der Waals Gas

In the classical development of thermodynamics, the van der Waals gas is often used as a cornerstone example. However, the phase function, Θ, given in many textbook treatments is not explicitly homogeneous of degree 1 in the extensive variables. A homogeneously correct formulation, to within a constant, is given by the relation:

$$\Theta\{...S,V,n;U\} = n[e^{\frac{S}{nC_v}}(\frac{V}{n}-b)^{-\frac{R}{C_v}} - \frac{a}{(\frac{V}{n}+cb)} - \frac{U}{n}] \Rightarrow 0. \tag{3.29}$$

The constant b is a representative size of the "particles" that make up molar quantities of the gas. Currently, it is usual to consider the "molar" quantities to be microscopic molecules, but the molar quantities from a topological perspective can be any size, ranging from nuclei to stars. To repeat Sommerfeld's statement:

> "The atomistic, microscopic point of view is alien to thermodynamics. Consequently, as suggested by Ostwald, it is better to use moles rather than molecules." p. 11 [Sommerfeld 1964].

The constant a is representative of the interaction forces between the molar quantities. The term $a/(\frac{V}{n})^2$ has been described by Sommerfeld as representing the "forces (or energy) of cohesion" p. 58 [Sommerfeld 1964] . Note that a correction factor, cb, has been added to the historical collision term $a/(V/n) \rightarrow a/(V/n+cb)$ in order to account for the finite interaction size (or an effective scattering wavelength cb) of the interacting molar particles. The coefficient c can be adjusted to give a better fit of the van der Waals gas equation to the experimental data of $\Omega_c = (nRT_c/P_cV_c)$ at the critical point.

This equation for $\Theta\{S,V,n;U\}$ satisfies the Euler condition for homogeneity of degree 1, with respect to the *extensive* variables $\{S,V,n;U\}$:

$$U\partial\Theta/\partial U + V\partial\Theta/\partial V + S\partial\Theta/\partial S + n\partial\Theta/\partial n - \Theta = 0. \tag{3.30}$$

The partial derivatives the phase function, Θ, with respect to the extensive variables may be used to define *intensive* variables, (P, T, μ),

$$(P = -\partial\Theta/\partial V,\ T = \partial\Theta/\partial S,\ \mu = -\partial\Theta/\partial n, \beta = -\partial\Theta/\partial U). \tag{3.31}$$

From the phase function (3.29), partial differentiation yields:

$$T = \frac{\partial}{\partial S}(\Theta) = (e^{\frac{S}{nC_v}}(\frac{V}{n} - b)^{-\frac{R}{C_v}})/C_v \tag{3.32}$$

$$P = -\frac{\partial}{\partial V}(\Theta) = \frac{nRT}{V - bn} - a\frac{n^2}{(V + cbn)^2}. \tag{3.33}$$

Differentiating P with with respect to V yields

$$\partial P/\partial V = -\frac{nRT}{(-V + bn)^2} + 2a\frac{n^2}{(V + cbn)^3}, \tag{3.34}$$

and differentiation again leads to

$$\partial^2 P/\partial V^2 = -2\frac{nRT}{(-V + bn)^3} - 6a\frac{n^2}{(V + cbn)^4}. \tag{3.35}$$

The classic argument to determine the critical point sets these relations to zero. The values of the thermodynamic variables at the critical point are:

$$V_c = bn(2c + 3), \qquad T_c = \frac{8a/27}{bR(c + 1)}, \quad P_c = \frac{a/27}{b^2(c + 1)^2} \tag{3.36}$$

which leads to the universal constant, Ω_c, independent from the geometrical parameters $\{a, b\}$:

$$\Omega_c = nRT_c/(P_cV_c) = 8\frac{c + 1}{2c + 3}. \tag{3.37}$$

For the van der Waals gas ($c = 0$), Ω_c=1/.375, but for many real gases, the experimental value is closer to Ω_c=1/.27. This result is in good agreement with the value of $c = 4$. The value of $1/\Omega_c$ versus the coefficient c is displayed in the following figure:

For the classic van der Waals gas ($c = 0$), a reduced equation of state can be obtained in terms of the dimensionless variables, scaled by their values at the critical point.

$$0 = \widetilde{\rho}^3 - 3\widetilde{\rho}^2 + \{(8\widetilde{T} + \widetilde{P})/3\}\widetilde{\rho} - \widetilde{P}.$$

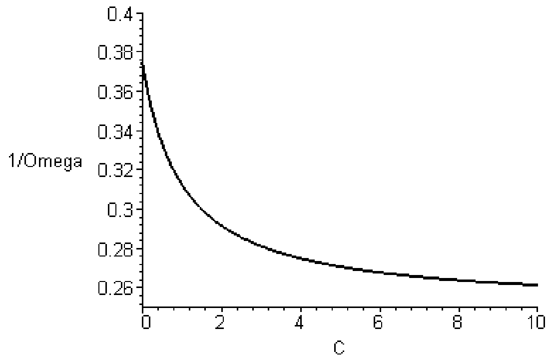


Figure 3.2:

The classic van der Waals formula leads to a critical isotherm that topologically separates the pure gas phase from those regions that admit liquid, or vapor, co-existent mixed phases. The shape of the critical isotherm is given in the figure below. It is a topological invariant and is to be recognized by its distinctive shape.

For arbitrary coefficient c, the cubic formula for the reduced equation of state is of the format

$$(8\widetilde{T}c^3 + (27 + 8\widetilde{T} + \widetilde{P})c^2 + 54c + 27)(\rho/(2c+3))^3 \tag{3.38}$$
$$+((-27 - \widetilde{P} + 16\widetilde{T})c^2 + (-54 + 16\widetilde{T} + 2\widetilde{P})c - 27)(\rho/(2c+3))^2 \tag{3.39}$$
$$+((-2\widetilde{P} + 8\widetilde{T})c + 8\widetilde{T} + \widetilde{P})(\rho/(2c+3)) - \widetilde{P} \Rightarrow 0. \tag{3.40}$$

For the special case $c = 1/2$ the cubic formula mimics the classical Virial expansion where the pressure is expanded in powers of the density:

$$\widetilde{P} = +3\widetilde{T}\rho + 1/16(-243/4 + 12\widetilde{T} + 3/4\widetilde{P})\rho^2 + 1/64(3T + 1/4\widetilde{P} + 243/4)\rho^3. \tag{3.41}$$

The $(\widetilde{P}, \rho)$ projection of the implicit surface for $c = 0$ jas been given above, and the projection for $c = 4$ is generated in the following figure. It is obvious that the two figures are deformably equivalent.

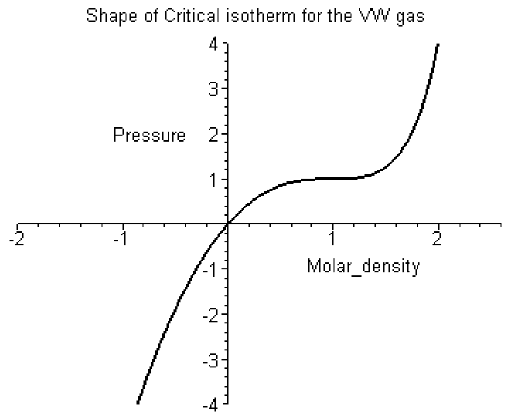
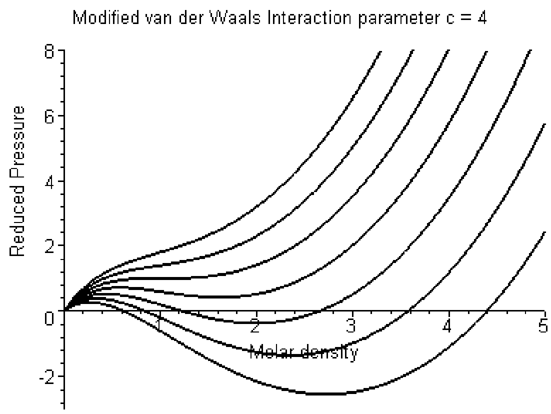


Figure 3.3:



The reader is encouraged to pick out the critical isotherm, which is a deformation of the critical isotherm for the van der Waals gas with c = 0 (see the previous figure for the reduced pressure at c = 0). It is also true that the Gibbs function for c=4 is a deformation equivalent to the previous figure that describes the swallow tail surface of the van der Waals (c=0) gas.

3.4. The Jacobian Matrix of the Action 1-form.

In most of the previous work, the Cartan topological methods of exterior differential forms have emphasized the antisymmetric features of a physical system, especially through the anti-symmetric matrix that encodes the 2-form dA. The more geometric formulation of the van der Waals gas, as described in the previous section, can also be obtained from the symmetrical differential properties of the coefficients of 1-form of Action, A. It will be assumed that the 1-form of Action, A, that encodes the physical system, is of Pfaff topological dimension 4, except on certain subspaces of the 4 independent variables.

The idea to be exploited in that which follows is that the Jacobian matrix $\mathbb{J} = [\partial A_k/\partial x^j]$ of partial derivative functions, created from the coefficients of the 1-form of Action, satisfies a Cayley-Hamilton matrix polynomial equation, and a complex algebraic polynomial equation in terms of the eigenvalues, ξ, of the Jacobian matrix.

$$\text{Cayley Hamilton polynomial} = \xi^4 - X_M\xi^3 + Y_G\xi^2 - Z_A\xi^1 + T_K = 0. \quad (3.42)$$

The coefficients of the polynomials $\{X_M, Y_G, Z_A, T_K\}$ are invariant with respect to similarity transformation of the Jacobian matrix, and in this sense the method is universal. The Jacobian matrix contains both symmetric and anti-symmetric components, where the 2-form, dA, emphasizes the anti-symmetric features of the partial derivatives of the 1-form coefficients. The symmetric components are more related to geometric properties of the physical system.

It is assumed that this characteristic equation, as a polynomial of 4th degree, is in effect a Universal Thermodynamic Phase function, $\Theta(x, y, z, t;\ \xi)$:

$$\text{Cayley-Hamilton polynomial} = \Theta(x, y, z, t;\ \xi) = 0. \quad (3.43)$$

The Phase function is distinct for different choices of the 1-form of Action, A, but all such Phase functions are related to the deformation equivalence classes that include the classic van der Waals gas. The Universal Phase function defines a family of implicit hypersurfaces in the space of "universal" coordinates defined in terms of the similarity invariants, $\{X_M, Y_G, Z_A, T_K\}$. It will be demonstrated how and when the similarity invariants can be related to "curvatures" of the universal implicit hypersurface. However, no metric is used explicitly to define the "curvatures".

The non-equilibrium extensions of the van der Waals gas (of Pfaff topological dimension 4) are encoded in the third and fourth order similarity invariants, Z_A and T_K, and the possibilities that the polynomial can have complex roots. When these third and fourth order similarity invariants vanish, the Universal Phase function describes isolated or equilibrium versions of the Universal van der Waals gas (of Pfaff topological dimension ≤ 2). It is this universality that gives credence to the idea that the 4 dimensional universe could be represented as a non-equilibrium van der Waals gas near its critical point [RMK2004 b].

In the special isolated-equilibrium cases, the topological features of a universal thermodynamic critical point, and a spinodal line of ultimate phase stability have realizations in terms of topological constraints on the phase function implicit hypersurface that represents the universal equilibrium van der Waals gas. When written in terms of curvatures it can be demonstrated that the zero set of the quadratic similarity invariant (the Gauss curvature) represents the spinodal line, or the edge of regression in the Gibbs surface, of a van der Waals gas. The thermodynamic critical point occurs when both the Mean curvature and the Gauss curvature of the equilibrium surface vanish. These concepts will be extended to the non-equilibrium systems in that which follows.

3.5. The details of Universal Characteristic Phase Function

The 1-form of Action, used to encode a physical system, contains other useful topological information, as well as geometric information. Consider the details of an open thermodynamic system generated by a 1-form of Action, A, of Pfaff topological dimension 4. The component functions of the Action 1-form can be used to construct a 4x4 Jacobian matrix of partial derivatives, $[\mathbb{J}_{jk}] = [\partial(A)_j/\partial x^k]$. In general, this Jacobian matrix will be a 4 x 4 matrix that satisfies a 4th order Cayley-Hamilton characteristic polynomial equation, $\Theta(x, y, z, t;\ \xi) = 0$, with 4 perhaps complex roots representing the 4 perhaps complex eigenvalues, ξ_k, of the Jacobian matrix.

$$\Theta(x, y, z, t;\ \xi) = \xi^4 - X_M\xi^3 + Y_G\xi^2 - Z_A\xi^1 + T_K \Rightarrow 0. \tag{3.44}$$

The Cayley-Hamilton polynomial equation defines a family of implicit functions in the space of variables $X_M(x, y, z, t), Y_G(x, y, z, t), Z_A(x, y, z, t), T_K(x, y, z, t)$. The functions $X_M(x, y, z, t), Y_G(x, y, z, t), Z_A(x, y, z, t), T_K(x, y, z, t)$ are the similarity invariants of the Jacobian matrix. If the eigenvalues are distinct, then the similarity invariants are given by the expressions:

$$X_M = \xi_1 + \xi_2 + \xi_3 + \xi_4 = Trace\,[\mathbb{J}_{jk}], \tag{3.45}$$
$$Y_G = \xi_1\xi_2 + \xi_2\xi_3 + \xi_3\xi_1 + \xi_4\xi_1 + \xi_4\xi_2 + \xi_4\xi_3, \tag{3.46}$$
$$Z_A = \xi_1\xi_2\xi_3 + \xi_4\xi_1\xi_2 + \xi_4\xi_2\xi_3 + \xi_4\xi_3\xi_1, \tag{3.47}$$
$$T_K = \xi_1\xi_2\xi_3\xi_4 = \det\,[\mathbb{J}_{jk}]. \tag{3.48}$$

Each coefficient can be related to a symmetric function of the curvatures of the implicit surface. The coefficient X_M is a linear function of the eigenvalues, and can be related to the Mean (linear) curvature of the implicit surface. From the theory of strings and surface tension, the X_M term is - in a sense - a linear deformation contribution to the "energy" of the system. The coefficient Y_G can be related to the Gauss (quadratic) curvature of the system, and is related to an area deformation contribution. The coefficient Z_A can be related to the Interaction (Cubic) curvature of the system, and is related to a volume deformation contribution (a Pressure) to the "energy". The last term T_K is a quartic contribution and can be related to an expansion or contraction of the 4 dimensional volume element.

Symbolically, multiply the phase function by u/ξ^4 and and consider u/ξ to be a length deformation, $\delta Length$, u/ξ^2 to be an area deformation, $\delta Area$, u/ξ^3 to be a volume deformation, δVol, and u/ξ^4 to be an space-time expansion deformation, δExp_xyzt. The suggestive formula becomes

$$u - M \cdot \delta Length + G \cdot \delta Area - A \cdot \delta Vol + K \cdot \delta Exp\_xyzt \tag{3.49}$$

$$\text{or by comparison with a van der Waals gas, the coefficients are} \tag{3.50}$$

$$M \approx surface\_tension \tag{3.51}$$

$$G \approx Temperature \tag{3.52}$$

$$A \approx \Pr essure \tag{3.53}$$

$$K \approx xyzt - Expansion \tag{3.54}$$

Automatically, the phase function incorporates string or surface tension effects through M, a mean curvature. Gravity effects due to the Gauss curvature G are "area" related. From the idea that the entropy of a gravitational black hole is related to an area, and the fact that the phase formula for a van der Waals gas implies that G is dominated by the temperature (see eq(3.11), the universal phase formula suggests that the idea of gravity (and the Gauss curvature) is a temperature -entropic concept, contributing energy of the type TS. The phase formula for a van der Waals gas implies that the A coefficient is related to Pressure (which can be both negative or positive), and the energy contribution is of the type PV. The last term represents a xyzt expansion, which from the topological theory of thermodynamics presented above can be related to irreversible dissipation.

The similarity invariants may be considered as a coordinate map from the original variety of independent variables, $\{x, y, z, t\} \Rightarrow \{X_M, Y_G, Z_A, T_K\}$. When the similarity invariants are treated as generalized coordinates, then the characteristic polynomial becomes a Universal Phase function, and will be used to encode universal thermodynamic properties.

3.5.1. The reduced Phase function

There exists a well known transformation of complex variable which will reformulate the characteristic polynomial. Substitute $\xi = s + M/4$. The result is a new "reduced" Phase polynomial of the form

$$\begin{aligned}
\Phi(x, y, z, t;\ s) &= s^4 + gs^2 - as + k = 0. && (3.55)\\
g &= (-3/2X_M^2/8 + Y_G) && (3.56)\\
a &= (X_M/2)^3 - Y_G(X_M/2) + Z_A) && (3.57)\\
k &= T_K - Z_A(X_M/4) + Y_G(X_M/4)^2 - 3(X_M/4)^4. && (3.58)
\end{aligned}$$

Consider the reduced Phase formula, and its derivatives with respect to the family parameter, s.

$$\begin{aligned}
\Phi &= s^4 + gs^2 - as + k = 0, && (3.59)\\
&\therefore\ k = -(s^4 + gs^2 - as), && (3.60)\\
\Phi_s &= 4s^3 + 2gs - a = 0 && (3.61)\\
&\therefore\ a = 4s^3 + 2gs && (3.62)\\
\Phi_{ss} &= 12s^2 + 2g = 0 && (3.63)
\end{aligned}$$

Replacing the parameter a in the equation for k yields

$$k = s^2(3s^2 + g). \qquad (3.64)$$

A plot of the equation is given below

If g = 0 represents the "critical point", then for g (~temperature) values below the critical point, the function k is a polynomial of 4th degree, but above the "critical temperature" the function k is quadratic. It is evident that below the critical isotherm, the "expansion" term k can both negative and positive values. The formula for the 4D expansion coefficient can have positive or negative values. The quadratic "potential" is reminiscent of the "Higgs" potential in relativistic field theories. Note that these properties have been obtained without explicit use of a metric or connection.

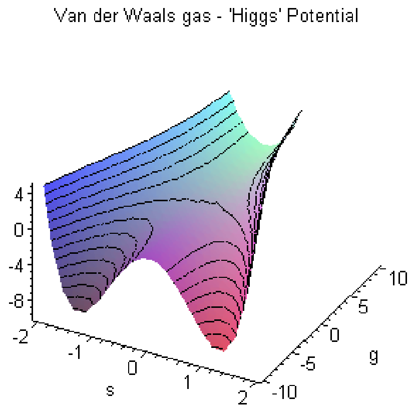
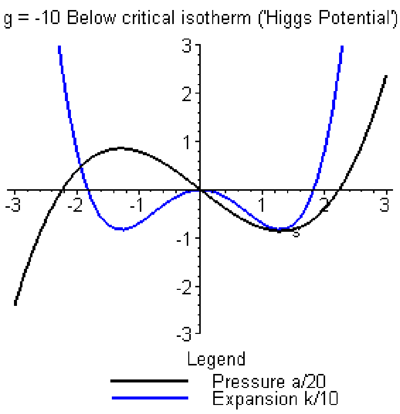
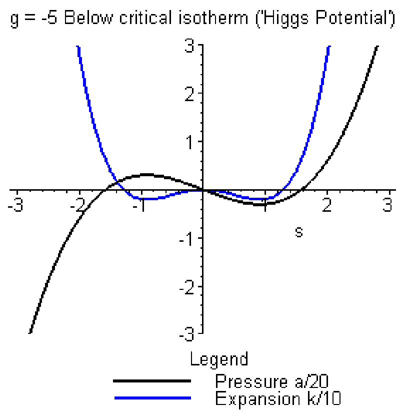
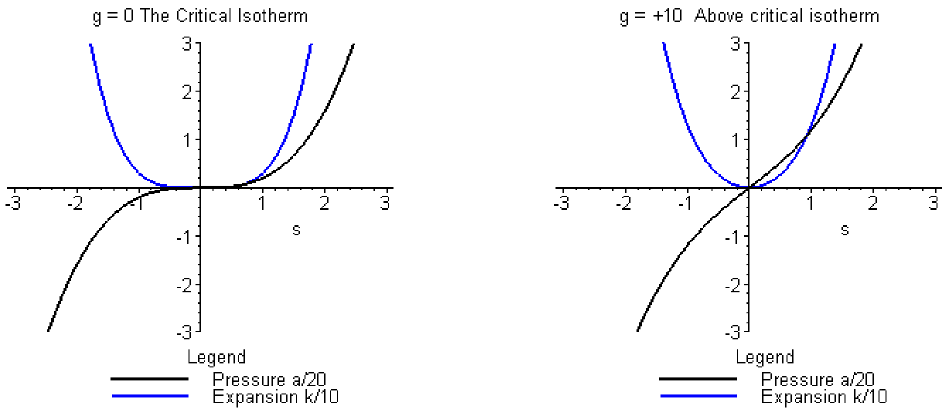


Figure 3.4:

From the van der Waals theory, the first partial derivative of the classic phase function yields the Pressure. For the universal Phase polynomial the pressure is determined by the equation $\Phi_s = 0$, Indeed, the formula $a = 4s^3 + 2gs$ yields the universal equation for a (the Pressure) in terms of the molar density "s". A plot of a (Pressure) versus s (molar density) at fixed g (temperature) gives the familiar cubic shape, deformably equivalent to the van der Waals gas. For the critical temperature ($g = 0$) the shape of the critical isotherm is exactly the same as for the critical isotherm of the van der Waals gas. Both k (Expansion in blue) and a (Pressure in black) are presented in the following diagrams as constant g (~temperature) slices above and below the critical point.







The topology of the quartic phase (potential) function is separated by a critical isotherm into two sectors. For temperatures below the critical temperature, the quartic formula yields a Higgs-like sector where expansion properties k are negative, and where liquid and vapor phases can coexist. Above the critical temperature the 4th order expansion properties k are positive, and the sector has lost its Higgs-like properties.

3.5.2. The Binodal line

The zero sets of the reduced Pressure (a) occur only for temperatures below the critical point and are described by the solution formula, $0 = 4s^3 + 2gs$. Along with the solution $s = 0$ coming from the second factor in the general phase formula for zero k, $\Phi = s^4 + gs^2 - as = 0$, a plot of the zeros of the Pressure in the $s - g$ plane yield the Binodal line as a pitchfork bifurcation, with the transition occurring at the critical temperature. From the van der Waals gas model, the Binodal line delineates the single phase from the mixed phase regions. The Pitchfork is essentially the line of zero first partial derivatives of the Higgs sector of the universal phase function[2]. These Pitchfork features are readily seen in the previous figure giving a 3D version of the Higgs - van der Waals gas potential.

[2] This result appears to be the first non-phenomenological derivation of the binodal line.

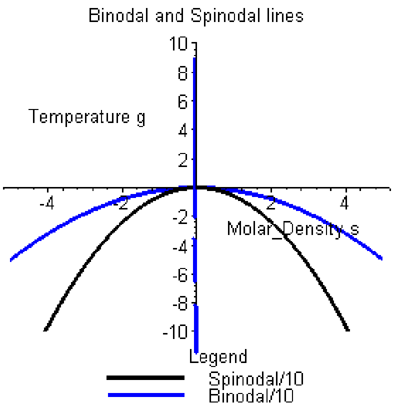


Figure 3.5:

3.5.3. The Spinodal Line

A second piece of topological information can be obtained from those points where the partial derivative of the pressure vanishes. These points are given by solutions to the equation $\Phi_{ss} = 12s^2 + 2g = 0$. Again only for temperatures g below the critical point will the formula give a set of points that describes classically what has been called the Spinodal line. In van der Waals theory the spinodal line defines the "limit" of single phase stability and can only be realized transiently, in the absence of fluctuations. Both spinodal line (blue) and the binodal line (black) are plotted in the next figure

3.6. Minimal surfaces

The Universal Phase function, Θ, may be considered as a family of hypersurfaces in the 4 dimensional space, $\{X_M, Y_G, Z_A, T_K\}$ with a complex family (order) parameter, ξ. Moreover, it should be realized that the Universal Phase Function is a holomorphic function, $\Theta = \phi + i\chi$ in the complex variable $\xi = u + iv$. That is

$$\Theta(X_M, Y_G, Z_A, T_K;\ \xi) \Rightarrow \phi + i\chi, \tag{3.65}$$

where

$$\phi = u^4 - 6u^2v^2 + v^4 - X_M(u^2 - 3v^2)u + Y_G(u^2 - v^2) - Z_A u + T_K \tag{3.66}$$
$$\chi = 4(u^2 - v^2)uv - X_M(3u^2 - v^2)v + 2Y_G uv - Z_A v. \tag{3.67}$$

As such, in the 4D space of two complex variables, $\{\phi + i\chi, u + iv\}$, according to the theorem of Sophus Lie, any such holomorphic function produces a pair of conjugate *minimal* surfaces in the 4 dimensional space $\{\phi, \chi, u, v\}$. It follows that there exist a sequence of maps,

$$\{x, y, z, t\} \Rightarrow \{X_M, Y_G, Z_A, T_K\} \Rightarrow \{\phi, \chi, u, v\} \tag{3.68}$$

such that the family of hypersurfaces can be decomposed into a pair of conjugate minimal surface components. The criteria for a minimal surface is equivalent to the idea that $X_M = 0$. By suitable renormalization, the similarity invariant X_M is equivalent to the Mean Curvature of the hypersurface.

3.6.1. Envelopes

The theory of implicit hypersurfaces focuses attention upon the possibility that the Universal Phase function has an envelope. The existence of an envelope depends upon the possibility of finding a simultaneous solution to the two implicit surface equations of the family:

$$\Theta(x, y, z, t;\ \xi) = \xi^4 - X_M\xi^3 + Y_G\xi^2 - Z_A\xi + T_K \Rightarrow 0. \tag{3.69}$$

$$\partial\Theta/\partial\xi = \Theta_\xi = 4\xi^3 - 3X_M\xi^2 + 2Y_G\xi - Z_A \Rightarrow 0. \tag{3.70}$$

For the envelope to be smooth, it must be true that $\partial^2\Theta/\partial\xi^2 = \Theta_{\xi\xi} \neq 0$, and that the exterior 2-form, $d\Theta\hat{}d\Theta_\xi \neq 0$ subject to the constraint that the family parameter is a constant: $d\xi = 0$. The envelope as a smooth hypersurface does not exist unless both conditions are satisfied (see Chapter 7).

The envelope is determined by the discriminant of the Phase Function polynomial, which as a zero set is equal to a universal hypersurface, $DISC\Theta \Rightarrow 0$, in the 4 dimensional space of similarity variables $\{X_M, Y_G, Z_A, T_K\}$. This function can be written in terms of the similarity "coordinates" (suppressing the subscripts) as :

Discriminant (envelope) (3.71)

of the Universal Phase Function Θ (3.72)

$$\begin{aligned} DISC\Theta &= 18X^3ZYT - 27Z^4 + Y^2X^2Z^2 - 4Y^3X^2T + 144YX^2T^2 && (3.73)\\ &+18XZ^3Y - 192XZT^2 - 6X^2Z^2T + 144TZ^2Y - 4X^3Z^3 && (3.74)\\ &-27X^4T^2 - 4Y^3Z^2 + 16Y^4T - 128Y^2T^2 + 256T^3 - 80XZY^2T && (3.75)\end{aligned}$$

Figure 3.6:

The discriminant (envelope) has eliminated the family order parameter, ξ. An alternate formulation describes the discriminant of the Reduced Phase phase function.

$$\textbf{Discriminant (envelope) of the} \tag{3.76}$$
$$\textbf{Reduced Universal Phase Function } \Theta_{reduced} \tag{3.77}$$
$$DISC\Theta_{reduced} \; : \; = -27a^4 + 4(-g^2 + 36k)ga^2 + 16k(4k - g^2)^2 \tag{3.78}$$

The hypersurface defined by the discriminant of the phase function G yields the (symmetrized) version of the universal swallow tail hypersurface. A plot (in terms of the coordinates $(X - 1 = g,\ Y = a,\ Z = k)$ is given in the following Figure.

It is apparent that the van der Waals gas is a deformation of the universal swallowtail hypersurface formed as the envelope of the reduced phase function, $\Theta_{reduced}$.

Remarkably, choosing the constraint condition in terms of the hypothetical condition that the Mean Curvature vanishes, $X_M \Rightarrow 0$, leads to a domain in the 4D space where the reduced discriminant defines a universal swallow tail surface homeomorphic (deformable) to the Gibbs surface of a van der Waals gas (subscripts suppressed):

$$\text{Minimal Surface} \; : \; \textbf{Universal Swallowtail Envelope } \; X_M \Rightarrow 0 \tag{3.79}$$
$$DISC\Theta_{(X_M=0)} = -27Z^4 + 144TZ^2Y - 4Y^3Z^2 + 16Y^4T - 128Y^2T^2 + 256T^3 \tag{3.80}$$
$$\approx DISC\Theta_{reduced} \Rightarrow 0.. \tag{3.81}$$

It must be remembered that this Minimal surface is a hypersurface in the space of Pfaff topological dimension 4. Examples are given in that which follows.

Another choice would be to constrain the envelope such that it resides in a domain where the 1-form of Action is of Pfaff topological dimension 3. The physical system is closed, but it is not necessarily in equilibrium. An equilibrium or isolated physical system consists of a single topological component, or phase (the Cartan topology is a connected topology). Domains where the Pfaff topological dimension represent mixed phases imply more than 1 topological component, and are to be associated with regions where the Pfaff topological dimension is ≥ 3. The case of Pfaff dimension 3 would correspond to regions where the 3-from of Topological Torsion is not zero (the Cartan topology becomes a disconnected topology - See Chapter 4). Such closed domains correspond to the situation where the determinant of the 4×4 Jacobian matrix vanishes. That is, set $T_K = 0$, to obtain the reduced (or constrained) envelope:

$$\text{Mixed Phases} \quad : \quad \textbf{Pfaff Dimension 3 Envelope -} \ (\text{Set } T_K = 0) \tag{3.82}$$

$$DISC_{(T_K=0)} = -27Z^2 + (-4X^3 + 18XY)Z + X^2Y^2 - 4Y^3 \Rightarrow 0. \tag{3.83}$$

In other words, the Gibbs function for a van der Waals gas is a universal idea associated with minimal hypersurfaces, $X_M = 0$, of thermodynamic systems of Pfaff topological dimension 4. The similarity coordinate T_K plays the role of the Gibbs free energy, in terms of the Pressure ($\tilde{\ } Z_A$) and the Temperature ($\tilde{\ } Y_G$). The Spinodal line as a limit of phase stability, and the critical point are ideas that come from the study of a van der Waals gas, but herein it is apparent that these concepts are universal topological concepts that remain invariant with respect to deformations. The universal formulas for such constraints are presented in the next section. The universal formulas for such constraints are presented in the next section. It is important to recognize that the development of a universal non-equilibrium van der Waals gas has not utilized the concepts of metric, connection, statistics, relativity, gauge symmetries, or quantum mechanics.

3.6.2. The Edge of Regression and Self Intersections

The envelope is smooth as long as $\partial^2\Theta/\partial\Psi^2 = \Theta_{\xi\xi} \neq 0$, and that the exterior 2-form, $d\Theta\hat{\ }d\Theta_\xi \neq 0$ subject to the constraint that the family parameter is a constant: $d\xi = 0$. If $d\Theta\hat{\ }d\Theta_\xi \neq 0$, but $\Theta_{\xi\xi} = 0$, then the envelope has a self intersection singularity. If $d\Theta\hat{}d\Theta_\xi = 0$, but $\Theta_{\xi\xi} \neq 0$, there is no self intersection, and no envelope.

If the envelope exists, further singularities are determined by the higher order partial derivatives of the Universal Phase function with respect to ξ.

$$\partial^2\Theta/\partial\xi^2 = \Theta_{\xi\xi} = 12\xi^2 - 6X_M\xi + 2Y_G. \tag{3.84}$$

$$\partial^3\Theta/\partial\xi^3 = \Theta_{\xi\xi\xi} = 24\xi - 6X_M \tag{3.85}$$

When $\partial^3\Theta/\partial\xi^3 = \Theta_{\xi\xi\xi} \neq 0$, and $d\Theta\hat{}d\Theta_\xi\hat{}d\Theta_{\xi\xi} \neq 0$, the envelope terminates in a edge of regression. The edge of regression is determined by the simultaneous solution of $\Theta = 0, \Theta_\xi = 0$ and $\Theta_{\xi\xi} = 0$. For the minimal surface representation of the Gibbs surface for a van der Waals gas, the edge of regression defines the Spinodal line of ultimate phase stability. The edge of regression is evident in the Swallowtail figure (Figure 2.1) describing the Gibbs function for a van der Waals gas.

If $\Theta_{\xi\xi\xi} = 0$, then for $X_M = 0$, it follows that $Y_G = 0$, $Z_A = 0$, $T_K = 0$, which defines the critical point of the Gibbs function for the van der Waals gas. In other words, the critical point is the zero of the 4-dimensional space of similarity coordinates.

If $\Theta_{\xi\xi} = 0$, then for $X_M = 0$ the envelope has a self intersection. It follows from $\Theta_{\xi\xi} = 0$, that $\xi^2 = -Y_G/6$, which when substituted into

$$\Theta_\xi = 4\xi^3 + 2Y_G\xi - Z_A \Rightarrow 0, \tag{3.86}$$

yields the

$$\textbf{Universal Gibbs Edge of Regression}: Z_A^2 + Y_G^3(8/27) = 0, \tag{3.87}$$

which defines the Spinodal line, of the minimal surface representation for a universal non-equilibrium van der Waals gas, in terms of "similarity" coordinates.

Within the swallow tail region the "Gibbs" surface has 3 real roots; outside the swallow tail region there is a unique real root. The edge of regression furnished by the Cardano function defines the transition between real and imaginary root structures. The details of the universal non-equilibrium van der Waals gas in terms of envelopes and edges of regression with complex molal densities or order parameters will be presented elsewhere. These systems are not equilibrium systems for the Pfaff dimension is not 2. Of obvious importance is the idea that the a zero value for both Z_G and T_K are required to reduce the Pfaff dimension to 2, the necessary condition for an equilibrium system.

3.6.3. Ginsburg Landau Currents

With a change of notation ($\xi \Rightarrow \Psi$), the Universal Phase function can be solved for the determinant of the Jacobian matrix, which is equal to the similarity invariant T_K,

$$T_K = -\{\Psi^4 - X_M\Psi^3 + Y_G\Psi^2 - Z_A\Psi\}. \tag{3.88}$$

The similarity invariant T_K represents the determinant of the Jacobian matrix. All determinants are in effect N - forms on the domain of independent variables. All N-forms can be related to the exterior derivative of some N-1 form or current, J. Hence

$$dJ = T_K\Omega_4 = (div\mathbf{J} + \partial\rho/\partial t)\Omega_4 = -(\Psi^4 - X_M\Psi^3 + Y_G\Psi^2 - Z_A\Psi)\Omega_4. \tag{3.89}$$

For currents of the form

$$\mathbf{J} = grad\ \Psi, \tag{3.90}$$

$$\rho = \Psi, \tag{3.91}$$

the Universal Phase function generates the universal Ginsburg Landau equations

$$\nabla^2\Psi + \partial\Psi/\partial t = -(\Psi^4 - X_M\Psi^3 + Y_G\Psi^2 - Z_A\Psi). \tag{3.92}$$

3.7. Singularities as defects of Pfaff dimension 3

The family of hypersurfaces can be topologically constrained such that the topological dimension is reduced, and/or constraints can be imposed upon functions of the similarity variables forcing them to vanish. Such regions in the 4 dimensional topological domain indicate topological defects or thermodynamic changes of phase. It is remarkable that for a given 1-form of Action there are an infinite number rescaling functions, λ, such that the Jacobian matrix $\left[\mathbb{J}_{jk}^{scaled}\right] = [\partial(A/\lambda)_j/\partial x^k]$ is singular (has a zero determinant). For if the coefficients of any 1-form of Action are rescaled by a divisor generated by the Holder norm,

$$\textbf{Holder Norm:}\ \ \lambda = \{a(A_1)^p + b(A_2)^p + c(A_3)^p + e(A_4)^p\}^{m/p}, \tag{3.93}$$

then the rescaled Jacobian matrix

$$\left[\mathbb{J}_{jk}^{scaled}\right] = [\partial(A/\lambda)_j/\partial x^k] \tag{3.94}$$

will have a zero determinant, for any index p, any set of isotropy or signature constants, a, b, c, e, if the homogeneity index is equal to unity: $m = 1$. This homogeneous constraint implies that the similarity invariants become projective invariants, not just equi-affine invariants. Such species of topological defects can have the image of a 3-dimensional implicit characteristic hypersurface in space-time:

$$\textbf{Singular hypersurface in 4D}: \quad \det[\partial(A/\lambda)_j/\partial x^k] \Rightarrow 0 \tag{3.95}$$

The singular fourth order Cayley-Hamilton polynomial of $[\mathbb{J}_{jk}]$ then will have a cubic polynomial factor with one zero eigenvalue.

For example, consider the simple case where the determinant of the Jacobian vanishes: $T_K \Rightarrow 0$. Then the Phase function becomes

$$\textbf{Universal Equation of State}: \quad \Theta(\{X_M, Y_G, Z_A, T_K = 0\};\ \xi) \tag{3.96}$$

$$= \xi(\xi^3 - X_M\xi^2 + Y_G\xi - Z_A) \Rightarrow 0 \tag{3.97}$$

The space has been topologically reduced to 3 dimensions (one eigen value is zero), and the zero set of the resulting singular Universal Phase function becomes a universal cubic equation that is homeomorphic to the cubic equation of state for a van der Waals gas.

When the rescaling factor λ is chosen such that $p = 2, a = b = c = 1, m = 1$, then the Jacobian matrix, $[\mathbb{J}_{jk}]$, is equivalent to the "Shape" matrix for an implicit hypersurface in the theory of differential geometry. (See Chapter 8.) Recall that the homogeneous similarity invariants can be put into correspondence with the linear Mean curvature, $X_M \Rightarrow C_M$, the quadratic Gauss curvature, $Y_G \Rightarrow C_G$, and the cubic Adjoint curvature, $Z_A \Rightarrow C_A$, of the hypersurface. The characteristic cubic polynomial can be put into correspondence with a nonlinear extension of an ideal gas *not necessarily* in an equilibrium state.

3.8. The Adjoint Current and Topological Spin

From the singular Jacobian matrix, $\left[\mathbb{J}_{jk}^{scaled}\right] = [\partial(A/\lambda)_j/\partial x^k]$, it is always possible to construct the Adjoint matrix as the matrix of cofactors transposed:

$$\text{Adjoint Matrix}: \left[\widehat{\mathbb{J}}^{kj}\right] = adjoint \left[\mathbb{J}_{jk}^{scaled}\right] \tag{3.98}$$

When this matrix is multiplied times the rescaled covector components, the result is the production of an adjoint current,

$$\text{Adjoint current}: \left|\widehat{\mathbf{J}}^k\right\rangle = \left[\widehat{\mathbb{J}}^{kj}\right] \circ |\mathbf{A}_j/\lambda\rangle \tag{3.99}$$

It is remarkable that the construction is such that the Adjoint current 3-form, if not zero, has zero divergence globally:

$$\widehat{J} = i(\widehat{\mathbf{J}}^k)\Omega_4 \tag{3.100}$$

$$d\widehat{J} = 0. \tag{3.101}$$

From the realization that the Adjoint matrix may admit a non-zero globally conserved 3-form density, or current, $\widehat{J}$, it follows abstractly that there exists a 2-form density of "excitations", $\widehat{G}$, such that

$$\textbf{Adjoint current} : \widehat{J} = d\widehat{G}. \tag{3.102}$$

$\widehat{G}$ is not uniquely defined in terms of the adjoint current, for $\widehat{G}$ could have closed components (gauge additions $\widehat{G}_c$, such that $d\widehat{G}_c = 0$), which do not contribute to the current, $\widehat{J}$.

From the topological theory of electromagnetism [RMK 2004] [RMK 1999 b] there exists a fundamental 3-form, $A\hat{}G$, defined as the "topological Spin" 3-form,

$$\textbf{Topological Spin 3-form} : A\hat{}G. \tag{3.103}$$

The exterior derivative of this 3-form produces a 4-form, with a coefficient energy density function that is composed of two parts:

$$d(A\hat{}G) = F\hat{}G - A\hat{}\widehat{J}. \tag{3.104}$$

The first term is twice the difference between the "magnetic" and the "electric" energy density, and is a factor of 2 times the Lagrangian usually chosen for the electromagnetic field in classic field theory:

$$\textbf{Lagrangian Field energy density} : F\hat{}G = \mathbf{B} \circ \mathbf{H} - \mathbf{D} \circ \mathbf{E} \tag{3.105}$$

The second term is defined as the "interaction energy density"

$$\textbf{Interaction energy density} : A\hat{}\widehat{J} = \mathbf{A} \circ \widehat{\mathbf{J}} - \rho\phi. \tag{3.106}$$

For the special (Gauss) choice of integrating denominator, λ with $(p = 2, a = b = c = 1, m = 1)$ it can be demonstrated that the Jacobian similarity invariants are equal to the classic curvatures:

$$\{X_M, Y_G, Z_A, T_K\} \Rightarrow \{C_{M(mean\_linear)}, C_{G(gauss\_quadratic)}, C_{A(adjoint\_cubic)}, 0\}. \tag{3.107}$$

It can be demonstrated that the interaction density is exactly equal to the Adjoint curvature energy density:

$$\textbf{Interaction energy } A\hat{}\widehat{J} = C_A\ \Omega_4 \quad \text{(The Adjoint Cubic Curvature)}. \tag{3.108}$$

The conclusion reached is that a non-zero interaction energy density implies the thermodynamic system is not in an equilibrium state.

However, it is always possible to construct the 3-form, $\widehat{S}$:

$$\textbf{Topological Spin 3-form} : \widehat{S} = A\hat{}\widehat{G} \tag{3.109}$$

The exterior derivative of this 3-form leads to a cohomological structural equation similar the first law of thermodynamics, but useful for non-equilibrium systems. This result, now recognized as a statement applicable to non-equilibrium thermodynamic processes, was defined as the "Intrinsic Transport Theorem" in 1969 [RMK 1969] :

$$\textbf{Intrinsic Transport Theorem (Spin)} : d\widehat{S} = F\hat{}\widehat{G} - A\hat{}\widehat{J} \tag{3.110}$$

$$\textbf{First Law of Thermodynamics (Energy)} : dU = Q - W \tag{3.111}$$

If one considers a collapsing system, then the geometric curvatures increase with smaller scales. If Gauss quadratic curvature, C_G, is to be related to gravitational collapse of matter, then at some level of smaller scales a term cubic in curvatures, C_H, would dominate. It is conjectured that the cubic curvature produced by the interaction energy effect described above could prevent the collapse to a black hole. Cosmologists and relativists apparently have ignored such cubic curvature effects associated with non-equilibrium thermodynamic systems

3.9. Non-Equilibrium Examples.

In order to demonstrate content to the thermodynamic topological theory, two algebraically simple examples are presented below. (The algebra can become tedious for the rescaled Action 1-forms. A Maple program has been written compute the various terms. See HopfPhase.mws and Holder4d.mws.) The first corresponds to a Jacobian characteristic equation that has a cubic polynomial factor, and hence can be identified with a van der Waals gas. The second example exhibits the features associated with a Hopf bifurcation, where the characteristic equation has a quadratic factor with two pure imaginary roots, and two null roots. The third example demonstrates how a bowling ball, given initial angular momentum and energy, skids and/or slips changing its angular momentum and kinetic energy irreversibly via friction effects, until the dynamics is such that the ball rolls with out slipping. Once that "excited" state is reached, and topological fluctuations are ignored, the motion continues without dissipation. The system is in an excited state far from equilibrium.

3.9.1. Example 1: van der Waals properties from rotation and contraction

In this example,the Action 1-form is presumed to be of the form

$$A_0 = a(ydx - xdy) + b(tdz + zdt). \tag{3.112}$$

The 1-form of Potentials depends on the coefficients a and b. The similarity invariants of the Jacobian matrix , $\mathbb{J}\left[(A_0)\right]$, formed from A_0, are:

$$\begin{aligned}
\text{Based on the 1-form } A_0 &: & (3.113)\\
X_M &= 0, & (3.114)\\
Y_G &= a^2 - b^2 & (3.115)\\
Z_A &= 0 & (3.116)\\
T_K &= -a^2b^2 & (3.117)
\end{aligned}$$

The eigen values of the Jacobian matrix are global constants: $\pm b, \pm\sqrt{-1}a$.

If the 1-form of Action is rescaled by the Gauss map

$$\begin{aligned}
A_0 &\Rightarrow A = A_0/\sqrt{(ax)^2 + (ay)^2 + (bz)^2 + (bt)^2} & (3.118)\\
r^2 &= (ax)^2 + (ay)^2 + (bz)^2 + (bt)^2 & (3.119)
\end{aligned}$$

then the Jacobian matrix becomes the equivalent of the shape matrix, and the similarity invariants of the shape matrix are related to the curvatures of the implicit Phase hypersurface:

$$\begin{aligned}
\text{Based on the 1-form } A &: \text{Gauss map scaling} & (3.120)\\
\text{Linear Mean curvature} &: C_M = -2b^3tz/(r^2)^{3/2} & (3.121)\\
\text{Quadratic Gauss curvature} &: C_G = -a^2b^2\{(x^2+y^2) - (z^2+t^2)\}/(r^2)^2 & (3.122)\\
\text{Cubic Adjoint curvature} &: C_A = -2a^2b^3tz/(r^2)^{5/2} & (3.123)\\
\text{Quartic Curvature :} &\quad C_K = 0. & (3.124)
\end{aligned}$$

The Determinant (4th order curvature) vanishes by construction of the renormalization in terms of the Gauss map. This null result does not mean the Pfaff dimension of A is less than 4 globally, but the constraint defines a singular set upon which there is a closed Current. This current is the Adjoint current of the previous section.

However, the rescaled 1-form A is still of Pfaff dimension 4 and has a non-zero topological torsion 3-form and a non-zero topological torsion 4 form:

$$Top\_Torsion = 2ab \cdot [0,0,-z,t]/(r^2) \tag{3.125}$$

$$\text{Pfaff Dimension 4} : dA\hat{\ }dA = 4b^3a(t^2-z^2)/(r^2)^2\ \Omega_4 \tag{3.126}$$

The Gauss map permits the construction of the "Adjoint conserved current", which combined with the components of the Action 1-form yield an interaction energy density exactly equal to the cubic curvature C_A.

$$\text{Adjoint Current} : \mathbf{J}_s = ([x,y,z,t])\ /(r^2)^2, \tag{3.127}$$

$$\text{interaction energy density} : \mathbf{A}\circ\mathbf{J}_s - \rho\phi = C_A. \tag{3.128}$$

The rescaled Jacobian matrix has 1 zero eigen value and 3 non-zero eigenvalues. Hence, the cubic polynomial will yield an interpretation as a van der Waals gas. The Adjoint current represents a contraction in space-time, while the flow associated with the 1-form has a rotational component about the z axis.

3.9.2. Example 2: A Hopf 1-form

In this example,the Hopf 1-form is presumed to be of the form

$$A_0 = a(ydx - xdy) + b(tdz - zdt). \tag{3.129}$$

The 1-form of Potentials depends on the chirality coefficients a and b. There are two cases corresponding to left and right handed "polarizations": $a = b$ or

$a = -b$. The results of the topological theory are :

$$\begin{aligned} \text{Based on the 1-form } A_0 &:& & (3.130)\\ X_M &=& 0, & (3.131)\\ Y_G &=& a^2 + b^2 & (3.132)\\ Z_A &=& 0 & (3.133)\\ T_K &=& a^2 b^2 & (3.134)\\ \text{Eigenvalues} &:& \pm\sqrt{-1}b, \pm\sqrt{-1}a & (3.135)\\ \text{Torsion Current} &=& [x, y, z, t]ab, & (3.136)\\ \text{Parity} &=& 4ab & (3.137) \end{aligned}$$

The 4 eigenvalues come in two imaginary pairs. The elements of each pair are equal and opposite in sign.

What is remarkable for this Action 1-form is that both the linear similarity invariant X_M and the cubic similarity invariant Z_A of the implicit phase hypersurface in 4D vanish, for any real values of a or b. The quadratic similarity invariant is non-zero, positive real and is equal to $a^2 = b^2$. The quartic similarity invariant T_K is non-zero, positive real and is equal to $a^2 b^2$. The 1-form also supports a Topological Torsion current, with a non-zero divergence.

However, if the 1-form A_0 is scaled by the Gauss map, the resulting Hopf implicit surface is a single 4D imaginary *minimal* two dimensional hyper surface in 4D and has two non-zero imaginary curvatures, but a positive Gauss curvature! This a most unusual result, for the usual 2D minimal surface has equal and opposite real curvatures, with a negative Gauss curvature.

$$\begin{aligned} \text{Based on the 1-form } A &:& \text{Gauss map scaling} & (3.138)\\ \lambda^2 &=& (ax)^2 + (ay)^2 + (bz)^2 + (bt)^2 & (3.139)\\ r &=& \sqrt{x^2 + y^2 + z^2 + t^2)} & (3.140)\\ \text{Linear Mean curvature} &:& C_M = 0 & (3.141)\\ \text{Quadratic Gauss curvature} &:& C_G = +a^2 b^2 \{r^2\}/(\lambda^2)^2 & (3.142)\\ \text{Cubic Adjoint curvature} &:& C_A = 0 & (3.143)\\ \text{Quartic Curvature} &:& C_K = 0 & (3.144)\\ \text{Eigenvalues} &:& [0, 0, +\sqrt{-1}, -\sqrt{-1}](abr/\lambda^2). & (3.145) \end{aligned}$$

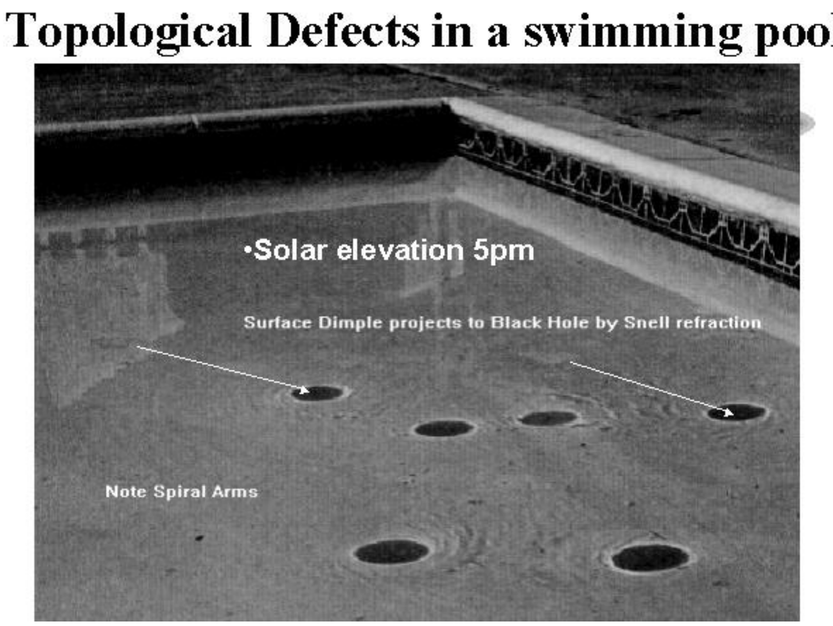
Strangely enough the charge-current density induced by the Adjoint current is not zero, but it is proportional to the Topological Torsion vector that generates the 3 form $A\hat{}F$. The topological Parity 4 form is not zero, and depends on the sign of the coefficients a and b. In other words the 'handedness' of the different 1-forms determines the orientation of the normal field with respect to the implicit surface. It is known that a process described by a vector proportional to the topological torsion vector in a domain where the topological parity is non-zero $4ba/(x^2 + y^2 + z^2 + t^2) \neq 0$ is thermodynamically irreversible.

4. The Falaco Soliton - A Topological String in a Swimming Pool

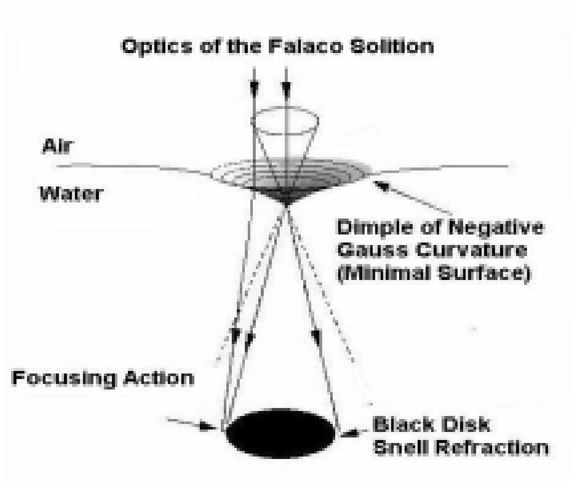
4.1. Visual Topological Defects in a Fluid

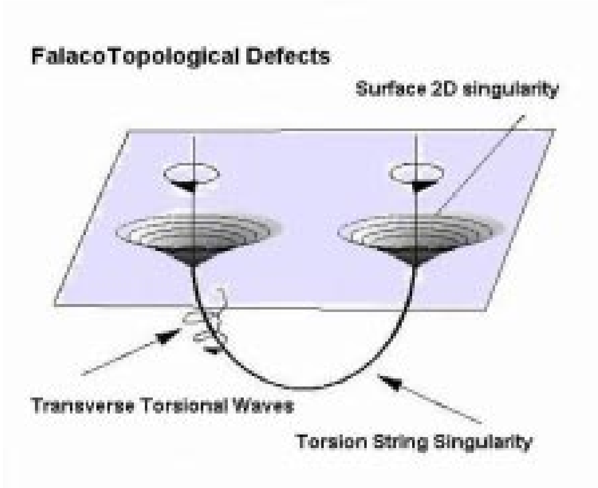
Although of importance to the cosmological concept of a universe expressible as a low density (non-equilibrium) van der Waals gas near its critical point, the factorization of the Jacobian characteristic polynomial into a cubic is not the only cosmological possibility. Of particular interest is the factorization that leads to a Hopf bifurcation. In this case the characteristic determinant vanishes, the Adjoint cubic curvature vanishes, the mean curvature vanishes (indicating a minimal surface), but the Gauss curvature is positive, and the two remaining eigenvalues of the characteristic polynomial are pure imaginary conjugates. Such results indicate rotations or oscillations (as in the chemical Brusselator reactions) and the possibility of spiral concentration or density waves on such minimal surfaces. Such structures at a cosmological level would appear to explain the origin of spiral arm galaxies. The Hopf type minimal surfaces of positive Gauss curvature do not represent thermodynamic equilibrium systems, for their curvatures, although two in number, are pure imaginary. The molal density distributions (or order parameters) are complex.

The idea that stars could be long lived topological defects at a cosmological level gains credence at the macroscopic level by the creation of Falaco Solitons (topological defects) in a swimming pool.



See the photo taken by D. Radabaugh of the 3 pairs of Falaco Solitons created in a swimming pool in the late afternoon. The lighting and optics enables the dimpled surface structures to be seen. Note the vestiges of mushroom spirals in the surface structures around each pair. The surface spiral arms can be enhanced by spreading chalk dust on the free surface of the pool. The photo demonstrates the existence of Falaco Solitons, a few minutes after creation. The kinetic energy and the angular momentum initially given to a pair of Rankine vortices (of positive Gauss curvature) created in the free surface of water quickly decay into dimpled, locally unstable, singular surfaces (of negative Gauss curvature) that have an extraordinary lifetime of more than 15 minutes in a still pool. The Falaco Soliton is observed by means of the unique optics of Snell refraction from a surface of negative Gauss curvature. The dimpled surface is almost a minimal surface, and the projection to the floor of the pool is almost conformal, preserving the circular appearance of the black disc, independent from the angle of solar incidence.





A remarkable feature of the Falaco Soliton [RMK 1986] is that it consists of a pair of two dimensional topological defects, in a surface of discontinuity, which apparently are connected by means of a topological singular thread. It is conjectured that the tension in the singular thread provides the force that maintains the pair of dimpled structures. The equilibrium mode for the free surface is that it be flat, without dimples. The solitons are representative of non-equilibrium long lived structures. If dye drops are injected into the water, the dye particles will execute a *torsional* wave motion that oscillates up and down until the dye maps out the thread singularity (a circular arc) that connects the two vertices of the Falaco Soliton. The singular thread acts as a guiding center for the torsion waves. If the thread is severed, the endcap singularities disappear almost immediately, and not diffusively. The long lifetime of the Falaco Soliton is due to this global stabilization of the connecting string singularity.

4.2. Falaco Solitons as Landau Ginsburg structures in micro, macroscopic and cosmological systems

The Falaco experiments demonstrate that such topological defects are available at all scales. The Falaco Solitons consist of spiral "vortex defect" structures (analogous to CGL theory) on a two dimensional minimal surface, one at each end of a 1-dimensional "vortex line" or thread (analogous to GPG theory). Remarkably the topological defect surface structure is locally unstable, as the surface is of negative Gauss curvature. Yet the pair of locally unstable 2-D surfaces is *globally* stabilized by the 1-D line defect attached to the "vertex" points of the minimal surfaces.

For some specific physical systems it can be demonstrated that period (circulation) integrals of the 1-form of Action potentials, A, lead to the concept of "vortex defect lines". The idea is extendable to "twisted vortex defect lines" in three dimensions. The "twisted vortex defects" become the spiral vortices of a Complex Ginsburg Landau (CGL) theory , while the "untwisted vortex lines" become the defects of Ginzburg-Pitaevskii-Gross (GPG) theory [Tornkvist 1997].

In the macroscopic domain, the experiments visually indicate "almost flat" spiral arm structures during the formative stages of the Falaco solitons. In the cosmological domain, it is suggested that these universal topological defects represent the ubiquitous "almost flat" spiral arm galaxies. Based on the experimental creation of Falaco Solitons in a swimming pool, it has been conjectured that M31 and the Milky Way galaxies could be connected by a topological defect thread [RMK 1986]. Only recently has photographic evidence appeared suggesting that galaxies may be connected by strings.

At the other extreme, the rotational minimal surfaces of negative Gauss curvature which form the two endcaps of the Falaco soliton, like quarks, apparently are confined by the string. If the string (whose "tension" induces global stability of the unstable endcaps) is severed, the endcaps (like unconfined quarks in the elementary particle domain) disappear (in a non-diffusive manner). In the

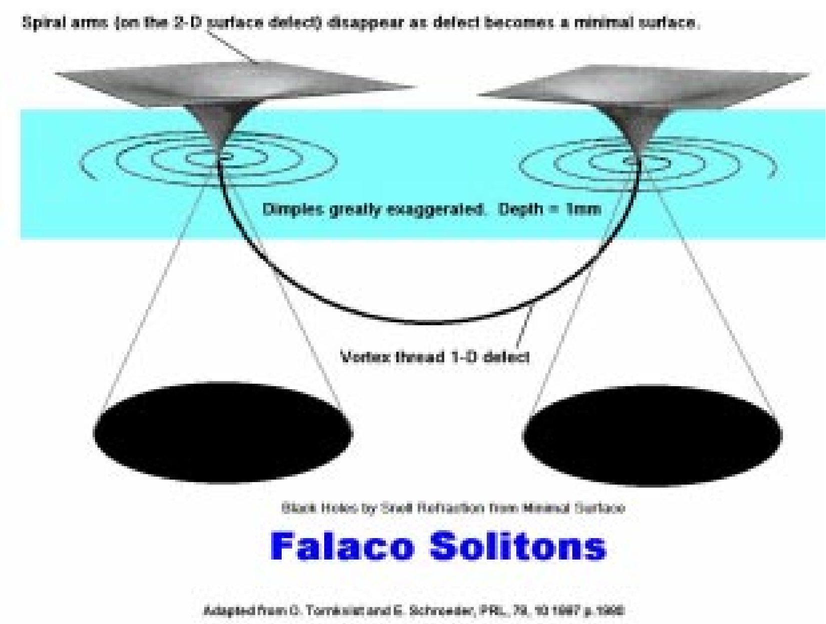

Spiral arms (on the 2-D surface defect) disappear as defect becomes a minimal surface.
Dimples greatly exaggerated. Depth = 1mm
Vortex thread 1-D defect
Black Holes by Snell Refraction from Minimal Surface
Falaco Solitons
Adapted from O. Tornkvist and E. Schroeder, PRL, 79, 10 1997 p.1908

Figure 4.1:

Figure 4.2:

microscopic electromagnetic domain, the Falaco soliton structure offers an alternate, topological, pairing mechanism on a Fermi surface, that could serve as an alternate to the Cooper pairing in superconductors.

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